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ON p-GEOMETRICALLY CONVEX SETS AND FUNCTIONS

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Abstract The geometrically convex function is a generalization of convex functions and a useful tool to discover and prove inequalities. In present paper, the authors improve the concept of geometrically convex sets, define the p-geometrically convex sets, the p-geometrically convex hulls, and the p-geometrically convex combinations, and give several equivalent conditions for judging p-geometrically convex sets. On this basis, the authors introduce the concept of p-geometrically convex functions and study its properties and invariant property under several operations. Moreover, the authors establish some new integral inequalities of p-geometrically convex functions.

Keywords Geometrically convex function, p-geometrically convex function, p-geometrically convex set, p-geometrically convex hull, epigraph.

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1. Introduction

Geometrically convex functions are a parallel concept of convex functions and are given by a transformation or an inequality, which generalize the theory and method of majorized inequalities for convex functions. As early as 1988 and 1990, Li [10] and Lucht [11] discussed some properties of geometrically convex functions by virtue of a transformation, but the concept was formally put forward by Gronau and Matkowski [7] in 1994. In 2000, Niculescu [15] gave the criteria of judging geometrically convex functions and several specific functions with geometric convexity, and established the Popoviciu type inequalities. In the same year, Finol and Wójtowicz [6] obtained the differential criterion of judging geometrically convex functions. In 2002, Yang [24] studied some matrix inequalities for geometrically convex functions, as well as Wu [21] gave a judgment method of geometrically convex functions and established inequalities of the Jensen type for geometrically convex functions. In 2004, Zhang [27] defined the geometrically convex sets and the Schur-geometrically convex functions, and established inequalities of the Hadamard type for geometrically convex functions. In 2013, Song [19] established integral inequalities of the Jensen type for geometrically convex functions and their weighted forms. In recent years, generalized geometrically convex functions, such as m-geometrically convex functions, (α, m)-geometrically convex functions, s-geometrically convex functions, and h-geometrically convex functions, have been proposed and investigated in the references [2,22,25]. Geometrically convex functions are a useful tool to discover and prove inequalities, and many classical inequalities are

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obtained using the convexity feature, see [1,5,9,12–14], for example. These inequalities play an important role in game theory, optimality theory, physics, engineering, economics, and many other fields.

In [17], the definition of convex functions was given by the convexity of epigraphs of the functions. In [16], Peck generalized the concept of convex sets and introduced the concept of p-convex sets. In [18], Sevda et. al. defined p-convex functions using p-convexity of epigraphs of the functions and researched some properties of this kind of functions. In [20], the equivalent definition of geometrically convex functions was given by the geometric convexity of epigraphs of the functions and the judgment theorems and some properties of geometric convex functions were obtained. On more recent results and generalizations for p-convex sets and p-convex functions, please see the papers [23, 26] and the references therein.

The parameter p in p-convexity plays a critical role in extending and refining the classical theory of convexity, offering both conceptual and practical value. By allowing p to deviate from 1, it can move beyond the limitations of standard convexity to create a unified framework that encompasses a richer spectrum of functional behaviors. In analysis, p-convexity provides a unifying lens for generalizing inequalities like the Hermite–Hadamard and Simpson inequalities [3], and in information theory, it underpins flexible divergence measures such as the Rényi entropy [4]. Thus, varying p away from 1 is not merely a mathematical generalization but also a practical tool that bridges abstract theory to real-world applications across economics, engineering, and data science.

The following portions of this work are inspired by the aforementioned findings. In Section 3, the concept of p-geometrically convex sets is introduced and its judgment criterions and some fundamental characterizations are investigated. In Section 4, p-geometrically convex function is defined by the p-geometric convexity of epigraphs of the function. Its equivalent judgment theorems are given by converting between analytical and geometric methods. Furthermore, an example and some operational properties of p-geometrically convex functions are discussed. In Section 5, some new integral inequalities for p-geometrically convex functions are established. In Section 6, some conclusions and new ideas for future research are given.

2. Preliminaries

Throughout this paper, we use the notations $\mathbb{R}^n$, $\mathbb{R}^n_0$, and $\mathbb{R}^n_+$ to denote the sets of all n-dimensional real vectors, all n-dimensional non-negative real vectors, and all n-dimensional positive real vectors, respectively.

Definition 2.1 ([27]). Let $x = (x_1, x_2, \dots, x_n) \in U \subseteq \mathbb{R}^n_+$, $y = (y_1, y_2, \dots, y_n) \in U \subseteq \mathbb{R}^n_+$, and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$. If $x^\alpha y^\beta = \left(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \dots, x_n^\alpha y_n^\beta\right) \in U$, then U is said to be a geometrically convex set, where we use the operations $x^\alpha = (x_1^\alpha, x_2^\alpha, \dots, x_n^\alpha)$ and $xy = (x_1y_1, x_2y_2, \dots, x_ny_n)$.

Definition 2.2 ([7]). Let $U \subseteq \mathbb{R}^n_+$ be a geometrically convex set and $f : U \to \mathbb{R}_+$. For $x, y \in U$ and $\lambda \in [0, 1]$, if $f\left(x^\lambda y^{1-\lambda}\right) \le [f(x)]^\lambda [f(y)]^{1-\lambda}$, then $f(x)$ is said to be a geometrically convex function.

Definition 2.3 ([16, 18]). Let $U \subseteq \mathbb{R}^n$ and $0 < p \le 1$. For $x, y \in U$ and $\lambda, \mu \ge 0$ with $\lambda^p + \mu^p = 1$, if $\lambda x + \mu y \in U$, then U is said to be a p-convex set.

Definition 2.4 ([17]). Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$. Then the set

$$\{(x, \nu) \in \mathbb{R}^{n+1} : x \in U, \nu \in \mathbb{R}, \nu \geq f(x)\}$$

is said to be the epigraph of f and is denoted by epi(f).

Definition 2.5 ([18]). Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$. If

$$\operatorname{epi}(f) = \big\{(x, \nu) \in \mathbb{R}^{n+1} : x \in U, \nu \in \mathbb{R}, \nu \geq f(x)\big\}$$

is a p-convex set, then f is called a p-convex function.

3. On p-geometrically convex sets

In this section, we define the notion of p-geometrically convex sets and study its basic properties.

Definition 3.1. Let $U \subseteq \mathbb{R}^n_+$ and $0 < p \leq 1$. For each $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, if $x^\lambda y^\mu \in U$, then U is called a p-geometrically convex set in $\mathbb{R}^n_+$.

Remark 3.1. When $p = 1$ in Definition 3.1, we deduce the definition of geometrically convex sets defined in [27].

Example 3.1. For $0 < p \leq 1$, the set $U = \{(x, y) \in \mathbb{R}^2_+ : |\ln x|^p + |\ln y|^p \leq 1\}$ is a p-geometrically convex set in $\mathbb{R}^2_+$. But it is neither a p-convex set nor a geometrically convex set in $\mathbb{R}^2_+$. In fact, for $(x_1, y_1), (x_2, y_2) \in U$ and $\lambda, \mu \geq 0$ with $\lambda^p + \mu^p = 1$, we have

$$\begin{aligned}
\big|\ln\big(x_1^\lambda x_2^\mu\big)\big|^p + \big|\ln\big(y_1^\lambda y_2^\mu\big)\big|^p &= |\lambda \ln x_1 + \mu \ln x_2|^p + |\lambda \ln y_1 + \mu \ln y_2|^p \\
&\leq (\lambda |\ln x_1| + \mu |\ln x_2|)^p + (\lambda |\ln y_1| + \mu |\ln y_2|)^p \\
&\leq \lambda^p(|\ln x_1|^p + |\ln y_1|^p) + \mu^p(|\ln x_2|^p + |\ln y_2|^p) \\
&\leq \lambda^p + \mu^p \\
&= 1.
\end{aligned}$$

Thus, we obtain

$$(x_1, y_1)^\lambda (x_2, y_2)^\mu = \big(x_1^\lambda x_2^\mu, y_1^\lambda y_2^\mu\big) \in U,$$

that is, U is a p-geometrically convex set in $\mathbb{R}^2_+$.

But taking $p = \frac{1}{2}$, $(x_1, y_1) = (\mathrm{e}^{-1}, 1) \in U$, $(x_2, y_2) = (1, \mathrm{e}^{-1}) \in U$, if $\lambda = \mu = \frac{1}{4}$, then

$$\left|\ln \frac{1+\mathrm{e}}{4\,\mathrm{e}}\right|^{1/2} + \left|\ln \frac{1+\mathrm{e}}{4\,\mathrm{e}}\right|^{1/2} = 2.07\ldots > 1$$

means

$$\lambda(x_1, y_1) + \mu(x_2, y_2) = \left(\frac{1+\mathrm{e}}{4\,\mathrm{e}}, \frac{1+\mathrm{e}}{4\,\mathrm{e}}\right) \notin U;$$

if $\lambda = \frac{1}{2}$, then

$$\big|\ln \mathrm{e}^{-1/2}\big|^{1/2} + \big|\ln \mathrm{e}^{-1/2}\big|^{1/2} = \sqrt{2} > 1$$

means

$$(x_1, y_1)^\lambda (x_2, y_2)^{1-\lambda} = \big(\mathrm{e}^{-1/2}, \mathrm{e}^{-1/2}\big) \notin U.$$

Therefore, U is neither a $\frac{1}{2}$-convex set nor a geometrically convex set in $\mathbb{R}^2_+$.

Theorem 3.1. *Let $U \subseteq \mathbb{R}_+^n$ and $0 < p \le 1$. Then U is a p-geometrically convex set if and only if $\ln U$ is a p-convex set, where $\ln U = \{\ln x : x \in U\}$.*

Proof. If $U \subseteq \mathbb{R}_+^n$ is a p-geometrically convex set, then for each $x, y \in U$ and $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$, we can obtain $x^\lambda y^\mu \in U$, which means

$$\lambda u + \mu v = \ln\bigl(x^\lambda y^\mu\bigr) \in \ln U,$$

where $u = \ln x$, $v = \ln y$, and $u, v \in \ln U$. That is, $\ln U$ is a p-convex set.

Conversely, for $u, v \in \ln U$ and $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$, if $\ln U$ is a p-convex set, then $x, y \in U$ and

$$\lambda u + \mu v = \ln\bigl(x^\lambda y^\mu\bigr) \in \ln U,$$

where $u = \ln x$ and $v = \ln y$. Consequently, we acquire $x^\lambda y^\mu \in U$, that is, U is a p-geometrically convex set. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set and $0 < p \le 1$. Then the closure $\overline{U}$ of U is also a p-geometrically convex set.*

Proof. Let $x, y \in \overline{U}$. If $x, y \notin U$, then there exists two point sequences $\bigl\{x^{(n)}\bigr\}_{n=1}^\infty$ and $\bigl\{y^{(n)}\bigr\}_{n=1}^\infty$ in U such that

$$x^{(n)} \to x, \quad y^{(n)} \to y, \quad \text{and} \quad \bigl(x^{(n)}\bigr)^\lambda \bigl(y^{(n)}\bigr)^\mu \to x^\lambda y^\mu$$

as $n \to \infty$, where $\lambda, \mu \ge 0$ and $\lambda^p + \mu^p = 1$. Since U is a p-geometrically convex set, then $\bigl(x^{(n)}\bigr)^\lambda \bigl(y^{(n)}\bigr)^\mu \in U$, and then $x^\lambda y^\mu \in \overline{U}$. That means $\overline{U}$ is a p-geometrically convex set.

If $x, y \in U$, then $U = \overline{U}$, the conclusion is clear. The proof of Theorem 3.2 is complete. □

Theorem 3.3. *Let $U_i \subseteq \mathbb{R}_+^n$ for $i = 1, 2, \dots, m$ be p-geometrically convex sets and $0 < p \le 1$. Then $\bigcap_{i=1}^m U_i$ is also a p-geometrically convex set.*

Proof. For $x, y \in \bigcap_{i=1}^m U_i$, it is clear that $x, y \in U_i$ for $i = 1, 2, \dots, m$ and $U_i \subseteq \mathbb{R}_+^n$ for $i = 1, 2, \dots, m$ are p-geometrically convex sets. Accordingly, for $\lambda, \mu \ge 0$ and $\lambda^p + \mu^p = 1$, we acquire that $x^\lambda y^\mu \in U_i$ for $i = 1, 2, \dots, m$, that is, $x^\lambda y^\mu \in \bigcap_{i=1}^m U_i$. The proof of Theorem 3.3 is complete. □

Definition 3.2. Let $U \subseteq \mathbb{R}_+^n$ and $0 < p \le 1$. The intersection of all p-geometrically convex sets containing U is said to be a p-geometrically convex hull and is denoted by $\operatorname{PGConv}(U)$.

Remark 3.2. It is clear that $\operatorname{PGConv}(U)$ is the unique and smallest p-geometrically convex set containing U.

Definition 3.3. Let $U \subseteq \mathbb{R}_+^n$, $0 < p \le 1$, and $\lambda_1, \lambda_2, \dots, \lambda_m \ge 0$ with $\sum_{i=1}^m \lambda_i^p = 1$. For $x_1, x_2, \dots, x_m \in U$, the product of vectors $\prod_{i=1}^m x_i^{\lambda_i}$ is called a p-geometrically convex combination of $x_1, x_2, \dots, x_m$.

Theorem 3.4. *Let $U \subseteq \mathbb{R}_+^n$ and $0 < p \le 1$. Then U is a p-geometrically convex set if and only if it contains the p-geometrically convex combination of any elements in U.*

Proof. The sufficiency is obvious. In what follows, we will inductively prove the necessity.

Suppose that U is a p-geometrically convex set, then for $x_1, x_2 \in U$ and $\lambda_1, \lambda_2 \ge 0$ such that $\lambda_1^p + \lambda_2^p = 1$, we have $x_1^{\lambda_1} x_2^{\lambda_2} \in U$.

Suppose that the conclusion is correct for $m-1$ vectors, then for $x_1, x_2, \ldots, x_m \in U$ and $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ such that $\sum_{i=1}^m \lambda_i^p = 1$, let $x = \prod_{i=1}^m x_i^{\lambda_i}$. In this case, it is clear that $\lambda_i^p < 1$ for all $i \in \{1, 2, \ldots, m\}$. Therefore, we can choose

$$\lambda_i' = \frac{\lambda_i}{\left(1-\lambda_1^p\right)^{1/p}}, \quad i = 2, 3, \ldots, m.$$

Then, we arrive at

$$\sum_{i=2}^m (\lambda_i')^p = \sum_{i=2}^m \frac{\lambda_i^p}{1-\lambda_1^p} = 1.$$

Accordingly, by the induction hypothesis, we obtain

$$y = \prod_{i=2}^m x_i^{\lambda_i'} = \prod_{i=2}^m x_i^{\lambda_i/(1-\lambda_1^p)^{1/p}} \in U.$$

Then

$$x = y^{(1-\lambda_1^p)^{1/p}} x_1^{\lambda_1} \in U.$$

The proof of Theorem 3.4 is complete. □

Theorem 3.5. *Let $U \subseteq \mathbb{R}_+^n$ and $0 < p \leq 1$. Then* PGConv(U) *is composed of p-geometrically convex combinations of all elements in U.*

Proof. Let C be the set of p-geometrically convex combinations of all elements in U. It is obvious that $C \subseteq \mathrm{PGConv}(U)$.

For $x, y \in C$, there exist $x_i, y_j \in U$ for $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, m$ such that

$$x = \prod_{i=1}^{\ell} x_i^{\lambda_i} \quad \text{and} \quad y = \prod_{i=1}^m y_i^{\mu_i},$$

where $\lambda_i, \mu_j \geq 0$ such that $\sum_{i=1}^{\ell} \lambda_i^p = 1$ and $\sum_{j=1}^m \mu_j^p = 1$. Then for $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, we have

$$\lambda^p \sum_{i=1}^{\ell} \lambda_i^p + \mu^p \sum_{j=1}^m \mu_j^p = 1.$$

Accordingly, we obtain

$$x^\lambda y^\mu = \left(\prod_{i=1}^{\ell} x_i^{\lambda_i}\right)^\lambda \left(\prod_{i=1}^m y_i^{\mu_i}\right)^\mu \in C,$$

which means that C is a p-geometrically convex set. By the relation $U \subseteq C$, we arrive at the relation $\mathrm{PGConv}(U) \subseteq C$. The proof of Theorem 3.5 is complete. □

Example 3.2. Consider the set $U = \left\{\left(\frac{1}{2}, 1\right), \left(1, \frac{1}{2}\right)\right\}$. In Figure 1, the set of p-convex combinations of the points $\left(\frac{1}{2}, 1\right)$ and $\left(1, \frac{1}{2}\right)$ for each $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ is showed. The corresponding PGConv(U) set for each $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ is the region under the corresponding curve.

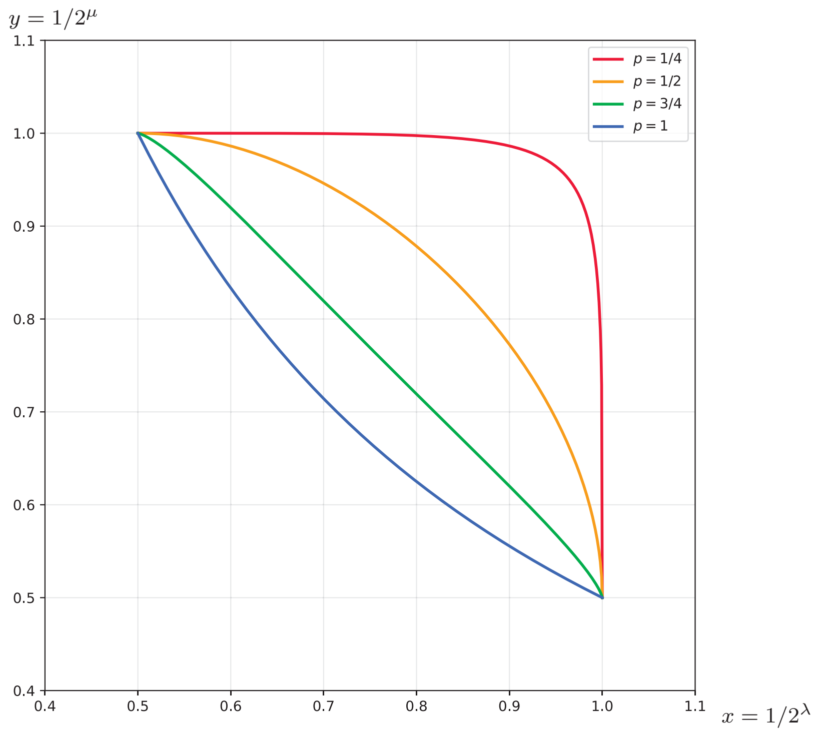

Figure 1. The p-convex combinations of $U = \left\{\left(1/2^\lambda, 1/2^\mu\right) : \lambda, \mu \in [0,1], \lambda^p + \mu^p = 1\right\}$.

4. On p-geometrically convex functions

In this section, we introduce the notion of p-geometrically convex functions and investigate its basic properties.

Definition 4.1. Let $f : U \subseteq \mathbb{R}_+^n \to \mathbb{R}_+$ and $0 < p \le 1$. If

$$\operatorname{epi}(f) = \left\{(x,\nu) \in \mathbb{R}_+^{n+1} : x \in U, \nu \in \mathbb{R}_+, f(x) \le \nu\right\}$$

is a p-geometrically convex set, then f is said to be a p-geometrically convex function. If the negative $-f$ is a p-geometrically convex function, then f is said to be a p-geometrically concave function.

Theorem 4.1. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set, $0 < p \le 1$, and $f : U \to \mathbb{R}_+$. Then f is a p-geometrically convex function if and only if the inequality*

$$f\left(x^\lambda y^\mu\right) \le [f(x)]^\lambda [f(y)]^\mu \tag{4.1}$$

is valid for all $x, y \in U$ and $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$.

Proof. If $f : U \to \mathbb{R}_+$ is a p-geometrically convex function, then $\operatorname{epi}(f)$ is a p-geometrically convex set. This means that the relation

$$(x,\alpha)^\lambda (y,\beta)^\mu = \left(x^\lambda y^\mu, \alpha^\lambda \beta^\mu\right) \in \operatorname{epi}(f)$$

is valid for all $(x,\alpha), (y,\beta) \in \operatorname{epi}(f)$ and $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$. Accordingly, we have

$$f\left(x^\lambda y^\mu\right) \le \alpha^\lambda \beta^\mu. \tag{4.2}$$

Since the inequality (4.2) is true for all α, β such that $(x, \alpha), (y, \beta) \in \operatorname{epi}(f)$, it is also true for $f(x) = \alpha$ and $f(y) = \beta$. Thus, the inequality (4.1) holds.

Conversely, if the inequality (4.1) holds for all $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, then we acquire $f(x) \leq \alpha$ and $f(y) \leq \beta$ for $(x, \alpha), (y, \beta) \in \operatorname{epi}(f)$. From (4.1), we can deduce $f\big(x^\lambda y^\mu\big) \leq \alpha^\lambda \beta^\mu$. Consequently, we arrive at

$$(x, \alpha)^\lambda (y, \beta)^\mu = \big(x^\lambda y^\mu, \alpha^\lambda \beta^\mu\big) \in \operatorname{epi}(f).$$

Hence, it follows that $\operatorname{epi}(f)$ is a p-geometrically convex set. Equivalently, it follows that f is a p-geometrically convex function. The required proof is complete. □

Similarly, we can prove the following result.

Theorem 4.2. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set, $0 < p \leq 1$, and $f : U \to \mathbb{R}_+$. Then f is a p-geometrically concave function if and only if the inequality*

$$f\big(x^\lambda y^\mu\big) \geq [f(x)]^\lambda [f(y)]^\mu$$

is valid for $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$.

Remark 4.1. Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set, $0 < p \leq 1$, and $f : U \to \mathbb{R}_+$. Then f is a p-geometrically convex function if and only if the inequality

$$f\big(x^\lambda y^{(1-\lambda^p)^{1/p}}\big) \leq [f(x)]^\lambda [f(y)]^{(1-\lambda^p)^{1/p}} \tag{4.3}$$

holds for $x, y \in U$ and $\lambda \in [0, 1]$.

When $p = 1$ in (4.3), we obtain the definition of geometrically convex functions.

Example 4.1. For $0 < p \leq 1$ and $x \in \mathbb{R}_+$, the function $f(x) = \mathrm{e}^{-|\ln x|^p}$ is p-geometrically convex, but it is neither a p-convex function nor a geometrically convex function in $\mathbb{R}_+$. In fact, for $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, it is clear that

$$0 < \lambda + \mu \leq \lambda^p + \mu^p = 1.$$

When $\lambda + \mu = 1$, by virtue of the concavity of the function $|x|^p$ for $0 < p < 1$, we have

$$|\lambda \ln x + \mu \ln y|^p \geq \lambda |\ln x|^p + \mu |\ln y|^p.$$

When $0 < \lambda + \mu < 1$, let $\lambda' = \frac{\lambda}{\lambda + \mu}$ and $\mu' = \frac{\mu}{\lambda + \mu}$, then

$$\begin{aligned} |\lambda \ln x + \mu \ln y|^p &= (\lambda + \mu)^p |\lambda' \ln x + \mu' \ln y|^p \\ &\geq (\lambda + \mu)^p (\lambda' |\ln x|^p + \mu' |\ln y|^p) \\ &= (\lambda + \mu)^{p-1} (\lambda |\ln x|^p + \mu |\ln y|^p) \\ &\geq \lambda |\ln x|^p + \mu |\ln y|^p. \end{aligned}$$

By using the monotonicity of the exponential function, we have

$$f\big(x^\lambda y^\mu\big) = \mathrm{e}^{-|\ln x^\lambda y^\mu|^p} = \mathrm{e}^{-|\lambda \ln x + \mu \ln y|^p} \leq \mathrm{e}^{-\lambda |\ln x|^p - \mu |\ln y|^p} = [f(x)]^\lambda [f(y)]^\mu.$$

Thus $f(x)$ is a p-geometrically convex function in $\mathbb{R}_+$.

But, taking $p=\frac{1}{2}$, $x=\mathrm{e}$, and $y=\mathrm{e}^{-1}$, for $\lambda=\mu=\frac{1}{4}$, we obtain

$$f(\lambda x+\mu y)=\exp\left(-\sqrt{\left|\ln\frac{1+\mathrm{e}^2}{4\,\mathrm{e}}\right|}\right)\ge\frac{\mathrm{e}^{-1}}{2}=\lambda f(x)+\mu f(y),$$

for $\lambda=\frac{1}{2}$, we acquire

$$f\big(x^\lambda y^{1-\lambda}\big)=1\ge\mathrm{e}^{-1}=[f(x)]^\lambda[f(y)]^{1-\lambda}.$$

As a result, the function $f(x)$ is neither a $\frac{1}{2}$-convex function nor a geometrically convex function in $\mathbb{R}_+$.

Example 4.2. Consider the function $f(x)=\mathrm{e}^x$ for $x\in\mathbb{R}_+$. It is clear that $f(x)$ is both convex and geometrically convex. But it is not a p-geometrically convex function for $0<p\le1$. In fact, taking $0<p<1$, $x=y=1$, and $\lambda,\mu>0$ with $\lambda^p+\mu^p=1$, we have

$$0<\lambda+\mu<\lambda^p+\mu^p=1.$$

Accordingly, it follows that

$$f\big(x^\lambda y^\mu\big)=f(1)=\mathrm{e}>\mathrm{e}^{\lambda+\mu}=[f(x)]^\lambda[f(y)]^\mu.$$

This means that $f(x)=\mathrm{e}^x$ for $x\in\mathbb{R}_+$ is not p-geometrically convex for $0<p\le1$.

Theorem 4.3. *Let $U\subseteq\mathbb{R}_+^n$ be a p-geometrically convex set, $0<p\le1$, and $f:U\to\mathbb{R}_+$. Then f is a p-geometrically convex function if and only if*

$$f\big(x^{1/2^{1/p}}y^{1/2^{1/p}}\big)\le[f(x)]^{1/2^{1/p}}[f(y)]^{1/2^{1/p}}\quad\text{and}\quad f(x^\alpha)\le[f(x)]^\alpha$$

for $x,y\in U$ and $\alpha\in\mathbb{R}$.

Proof. The necessity is obvious. We now prove the sufficiency.

Let $\lambda,\mu\ge0$ such that $\lambda^p+\mu^p=1$. For $x,y\in U$, we have

$$f\big(x^\lambda y^\mu\big)\le\big[f\big(x^{2^{1/p}\lambda}\big)\big]^{1/2^{1/p}}\big[f\big(y^{2^{1/p}\mu}\big)\big]^{1/2^{1/p}}\le[f(x)]^\lambda[f(y)]^\mu.$$

The proof is thus complete. □

Theorem 4.4. *Let $U\subseteq\mathbb{R}_+^n$ be a p-geometrically convex set, $0<p\le1$, and $f:U\to\mathbb{R}_+$ be a p-geometrically convex function. Then the inequality*

$$f\left(\prod_{i=1}^m x_i^{\lambda_i}\right)\le\prod_{i=1}^m[f(x_i)]^{\lambda_i}\tag{4.4}$$

holds for $x_1,x_2,\dots,x_m\in U$ and $\lambda_1,\lambda_2,\dots,\lambda_m\ge0$ such that $\sum_{i=1}^m\lambda_i^p=1$.

Proof. When $m=2$, the proof is trivial.

Suppose that the inequality (4.4) holds for some $m=k\ge2$. When $m=k+1$, take $x_1,x_2,\dots,x_{k+1}\in U$ and let $x=\prod_{i=1}^{k+1}x_i^{\lambda_i}$, where $\lambda_1,\lambda_2,\dots,\lambda_{k+1}\ge0$ with $\sum_{i=1}^{k+1}\lambda_i^p=1$. Then at least one among $\lambda_1,\lambda_2,\dots,\lambda_{k+1}$ is less than 1. Without loss of generality, assume $0<\lambda_{k+1}<1$, then $\sum_{i=1}^k\lambda_i^p=1-\lambda_{k+1}^p$. Hence, there exists a number $0<\lambda^*<1$ such that $\sum_{i=1}^k\lambda_i^p=(\lambda^*)^p$, which means $\sum_{i=1}^k\big(\frac{\lambda_i}{\lambda^*}\big)^p=1$. Thus, we arrive at

$$f(x)=f\left(\left[\prod_{i=1}^k x_i^{\lambda_i/\lambda^*}\right]^{\lambda^*}x_{k+1}^{\lambda_{k+1}}\right)\le\left[f\left(\prod_{i=1}^k x_i^{\lambda_i/\lambda^*}\right)\right]^{\lambda^*}[f(x_{k+1})]^{\lambda_{k+1}}\le\prod_{i=1}^{k+1}[f(x_i)]^{\lambda_i}.$$

The required proof is complete. □

Definition 4.2. Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set, $0 < p \le 1$, and $f : U \to \mathbb{R}_+$. Then the set

$$\inf\{\mu \in \mathbb{R}_+ | (x, \mu) \in \mathrm{PGConv}(\mathrm{epi}(f))\}$$

is called the p-geometrically convex hull of f and denoted by $\mathrm{PGConv}(f)$.

Theorem 4.5. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set, $0 < p \le 1$, and $f : U \to \mathbb{R}_+$. Then f is a p-geometrically convex function if and only if $f = \mathrm{PGConv}(f)$.*

Proof. If f is a p-geometrically convex function, then it is clear that

$$f(x) \ge \mathrm{PGConv}(f), \quad x \in U. \tag{4.5}$$

For $(x, \mu) \in \mathrm{PGConv}(\mathrm{epi}(f))$, there exists $(x_i, \mu_i) \in \mathrm{epi}(f)$ for $i = 1, 2, \dots, m$ such that

$$(x, \mu) = \prod_{i=1}^{m} (x_i, \mu_i)^{\lambda_i} = \left(\prod_{i=1}^{m} x_i^{\lambda_i}, \prod_{i=1}^{m} \mu_i^{\lambda_i} \right),$$

where $\lambda_1, \lambda_2, \dots, \lambda_m \ge 0$ with $\sum_{i=1}^{m} \lambda_i^p = 1$. From the p-geometric convexity of f, we find that

$$f(x) = f\left(\prod_{i=1}^{m} x_i^{\lambda_i} \right) \le \prod_{i=1}^{m} [f(x_i)]^{\lambda_i} \le \prod_{i=1}^{m} \mu_i^{\lambda_i} = \mu,$$

that is

$$f(x) \le \inf\big\{\mu | (x, \mu) \in \mathrm{PGConv}(\mathrm{epi}(f))\big\} = \mathrm{PGConv}(f). \tag{4.6}$$

Combining (4.6) with (4.5) yields $f(x) = \mathrm{PGConv}(f)$ for all $x \in U$.

If $f(x) = \mathrm{PGConv}(f)$, then f is a p-geometrically convex function, due to $\mathrm{PGConv}(\mathrm{epi}(f))$ is a p-geometrically convex set. The proof is complete. □

Theorem 4.6. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set and $0 < p \le 1$. If $f_i : U \to \mathbb{R}_+$ is a p-geometrically convex function for $i = 1, 2, \dots, m$, then*

$$f(x) = \prod_{i=1}^{m} [f_i(x)]^{a_i}, \quad a_i \in \mathbb{R}_+$$

for $i = 1, 2, \dots, m$ is also a p-geometrically convex function.

Proof. For $x, y \in U$ and $\lambda, \mu \ge 0$ such that $\lambda^p + \mu^p = 1$, we have

$$f\big(x^\lambda y^\mu\big) = \prod_{i=1}^{m} \big[f_i\big(x^\lambda y^\mu\big)\big]^{a_i} \le \prod_{i=1}^{m} [f_i(x)]^{\lambda a_i} [f_i(y)]^{\mu a_i} = [f(x)]^\lambda [f(y)]^\mu.$$

The required proof is complete. □

Theorem 4.7. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set and $0 < p \le 1$. If $f_i : U \to \mathbb{R}_+$ is a p-geometrically convex function for $i = 1, 2, \dots, m$, then $f(x) = \max_{1 \le i \le m}\{f_i(x)\}$ is also a p-geometrically convex function on U.*

Proof. For $x, y \in U$ and $\lambda, \mu \ge 0$ with $\lambda^p + \mu^p = 1$, there exists $t \in \{1, 2, \dots, m\}$ such that

$$f\big(x^\lambda y^\mu\big) = \max_{1 \le i \le m} \big\{f_i\big(x^\lambda y^\mu\big)\big\}$$

$$
\begin{aligned}
&= f_t\big(x^\lambda y^\mu\big)\\
&\le [f_t(x)]^\lambda [f_t(y)]^\mu\\
&\le \Big[\max_{1\le i\le m}\{f_i(x)\}\Big]^\lambda \Big[\max_{1\le i\le m}\{f_i(y)\}\Big]^\mu\\
&= [f(x)]^\lambda [f(y)]^\mu.
\end{aligned}
$$

The required proof is thus complete. □

Theorem 4.8. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set and $0 < p \le 1$. If $f : U \to \mathbb{R}_+$ is a p-geometrically convex function and $f \ge 1$, then the local minimum of f is a global minimum.*

Proof. Suppose that $x^* \in U$ is the local minimum point of f, but not the global minimum point of f, then there exists some $y \in U$ such that $f(y) \le f(x^*)$. Since $f : U \to \mathbb{R}_+$ is a p-geometrically convex function, we obtain

$$
f\big((x^*)^\lambda y^\mu\big) \le [f(x^*)]^\lambda [f(y)]^\mu \le [f(x^*)]^{\lambda^p} [f(y)]^{\mu^p} = f(x^*)\left[\frac{f(y)}{f(x^*)}\right]^{\mu^p} \le f(x^*)
$$

for $\lambda, \mu \ge 0$ with $\lambda^p + \mu^p = 1$. This leads to a contradiction of x^* being local minimum point. The proof is complete. □

Theorem 4.9. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set and $0 < p \le 1$. If $f : U \to \mathbb{R}_+$ is a p-geometrically convex function and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing p-geometrically convex function, then the composite $g \circ f : U \to \mathbb{R}_+$ is a p-geometrically convex function.*

Proof. For $x, y \in U$ and $\lambda, \mu \ge 0$ with $\lambda^p + \mu^p = 1$, we have

$$
g \circ f\big(x^\lambda y^\mu\big) = g\big(f\big(x^\lambda y^\mu\big)\big) \le g\big(f^\lambda(x) f^\mu(y)\big) \le [g(f(x))]^\lambda [g(f(y))]^\mu = [g \circ f(x)]^\lambda [g \circ f(y)]^\mu.
$$

Consequently, the composite $g \circ f$ is a p-geometrically convex function. □

Theorem 4.10. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set and $0 < p \le 1$. If $f : U \to \mathbb{R}_+$ is a p-geometrically convex function and $t = \inf_{x\in U} f(x)$ with $t \ge 1$, then $E = \{x \in U | f(x) = t\}$ is a p-geometrically convex set.*

Proof. For $x, y \in E$ and $\lambda, \mu \ge 0$ with $\lambda^p + \mu^p = 1$, we have

$$
t \le f\big(x^\lambda y^\mu\big) \le [f(x)]^\lambda [f(y)]^\mu = t^\lambda t^\mu \le t^{\lambda^p} t^{\mu^p} = t.
$$

This means that $x^\lambda y^\mu \in E$. Thus E is a p-geometrically convex set. The proof is complete. □

Theorem 4.11. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set, $0 < p \le 1$, and $f : U \to \mathbb{R}_+$. If f is a p-geometrically convex function, then the upper level set*

$$
L_{\le\alpha} = \{x \in U | f(x) \le \alpha\}
$$

is a p-geometrically convex set, where $\alpha \in \mathbb{R}_+$ and $\alpha \ge 1$. If f is a p-geometrically concave function, then the lower level set

$$
L_{\ge\alpha} = \{x \in U | f(x) \ge \alpha\}
$$

is a p-geometrically convex set, where $\alpha \in \mathbb{R}_+$ and $0 < \alpha \le 1$.

Proof. For $\alpha \geq 1$, from the p-geometrically convexity of f, we derive

$$f\big(x^{\lambda}y^{\mu}\big) \leq [f(x)]^{\lambda}[(f(y)]^{\mu} \leq \alpha^{\lambda}\alpha^{\mu} \leq \alpha^{\lambda^p}\alpha^{\mu^p} = \alpha,$$

where $x, y \in L_{\leq\alpha}$ and $\lambda, \mu \geq 0$ with $\lambda^p + \mu^p = 1$. This means $x^{\lambda}y^{\mu} \in L_{\leq\alpha}$, that is, $L_{\leq\alpha}$ is a p-geometrically convex set.

Similarly, for the case $0 < \alpha \leq 1$, the proof can be carried out in an analogous manner. □

Theorem 4.12. *Let $U \subseteq \mathbb{R}_+^n$ be a p-geometrically convex set, $0 < p \leq 1$, and $f : U \to \mathbb{R}_+$ be a p-geometrically convex function, then the inequality* (4.1) *holds for all $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda^p + \mu^p \leq 1$ if and only if $0 < f(1) \leq 1$.*

Proof. If the inequality (4.1) holds for $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda^p + \mu^p \leq 1$, taking $\lambda = \mu = 0$ yields $f(1) \leq 1$.

Conversely, if $0 < f(1) \leq 1$, let $\gamma = \lambda^p + \mu^p$. If $\gamma = 0$, then $\lambda = \mu = 0$, the inequality (4.1) obviously holds. If $0 < \gamma \leq 1$, let $\alpha = \lambda\gamma^{-1/p}$ and $\beta = \mu\gamma^{-1/p}$, then we have $\alpha^p + \beta^p = \frac{\lambda^p}{\gamma} + \frac{\mu^p}{\gamma} = 1$. Accordingly, we obtain

$$\begin{aligned}
f\big(x^{\lambda}y^{\mu}\big) &= f\big(x^{\alpha\gamma^{1/p}}y^{\beta\gamma^{1/p}}\big)\\
&\leq \big[f\big(x^{\gamma^{1/p}}\big)\big]^{\alpha}\big[f\big(y^{\gamma^{1/p}}\big)\big]^{\beta}\\
&\leq [f(x)]^{\alpha\gamma^{1/p}}[f(y)]^{\beta\gamma^{1/p}}[f(1)]^{(\alpha+\beta)(1-\gamma)^{1/p}}\\
&\leq [f(x)]^{\alpha\gamma^{1/p}}[f(y)]^{\beta\gamma^{1/p}}\\
&= [f(x)]^{\lambda}[f(y)]^{\mu}.
\end{aligned}$$

The required proof is complete. □

5. Inequalities of p-geometrically convex functions

We now start out to establish several integral inequalities for p-geometrically convex functions.

Theorem 5.1. *Let $U \subseteq \mathbb{R}_+$ be a p-geometrically convex set, $f : U \to \mathbb{R}_+$ be a p-geometrically convex function, and $0 < p \leq 1$. Then for any $a, b \in U$, we have*

$$\int_0^1 \ln f\big(a^{t^{1/p}}b^{(1-t)^{1/p}}\big)\,\mathrm{d}t \leq \frac{p}{p+1}[\ln f(a) + \ln f(b)].$$

Proof. Since f is p-geometrically convex, then for any $a, b \in U$, we have

$$\ln f\big(a^{t^{1/p}}b^{(1-t)^{1/p}}\big) \leq t^{1/p}\ln f(a) + (1-t)^{1/p}\ln f(b). \tag{5.1}$$

Integrating on both sides of the inequality (5.1) over $t \in [0, 1]$ leads to the desired results. □

Remark 5.1. Under the conditions of Theorem 5.1, taking $p = 1$ and $a \neq b$ reduces to

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(x)}{x}\,\mathrm{d}x \leq \frac{\ln f(a) + \ln f(b)}{2}.$$

Theorem 5.2. *Let $U \subseteq \mathbb{R}_+$ be a p-geometrically convex set, $f : U \to \mathbb{R}_+$ be a p-geometrically convex function, and $0 < p \leq 1$. Then for any $a, b \in U$, we have*

$$\int_0^1 \ln f\Big((ab)^{[t^{1/p}+(1-t)^{1/p}]/2^{1/p}}\Big)\,\mathrm{d}t \leq \frac{1}{2^{1/p-1}}\int_0^1 \ln f\big(a^{t^{1/p}}b^{(1-t)^{1/p}}\big)\,\mathrm{d}t \leq \frac{p}{1+p}\frac{\ln f(a) + \ln f(b)}{2^{1/p-1}}.$$

Proof. By using the p-geometrically convexity of f and the monotonicity of the logarithmic function, for any $a,b\in U$, we have

$$\begin{aligned}\ln f\Big((ab)^{[t^{1/p}+(1-t)^{1/p}]/2^{1/p}}\Big)&\le\frac{1}{2^{1/p}}\big[\ln f\big(a^{t^{1/p}}b^{(1-t)^{1/p}}\big)+\ln f\big(a^{(1-t)^{1/p}}b^{t^{1/p}}\big)\big]\\&\le\frac{1}{2^{1/p}}\big[t^{1/p}+(1-t)^{1/p}\big][\ln f(a)+\ln f(b)].\end{aligned}$$

Integrating this inequality over $t\in[0,1]$ leads to the desired results. □

Remark 5.2. Under the conditions of Theorem 5.2, taking $p=1$ and $a\ne b$ reduces to

$$\ln f\big(\sqrt{ab}\,\big)\le\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(x)}{x}\,\mathrm{d}x\le\frac{\ln f(a)+\ln f(b)}{2}.$$

Theorem 5.3. *Let $U\subseteq\mathbb{R}_+$ be a p-geometrically convex set, $f:U\to[1,\infty)$ be a p-geometrically convex function, and $0<p\le1$. Then for any $a,b\in U$, we have*

$$\begin{aligned}\int_0^1 f\Big((ab)^{[t^{1/p}+(1-t)^{1/p}]/2^{1/p}}\Big)\,\mathrm{d}t&\le\int_0^1\big[f\big(a^{t^{1/p}}b^{(1-t)^{1/p}}\big)f\big(a^{(1-t)^{1/p}}b^{t^{1/p}}\big)\big]^{1/2^{1/p}}\,\mathrm{d}t\\&\le\int_0^1 f\big(a^{t^{1/p}}b^{(1-t)^{1/p}}\big)\,\mathrm{d}t\\&\le L(f(a),f(b))\\&\le\frac{f(a)+f(b)}{2},\end{aligned}$$

where $L(u,v)$ is the logarithmic mean defined for $u,v>0$ by

$$L(u,v)=\begin{cases}\dfrac{u-v}{\ln u-\ln v}, & u\ne v,\\ u, & u=v.\end{cases}$$

Proof. Since f is p-geometrically convex, then for any $x,y\in U$, we have

$$f\big(\sqrt[2^{1/p}]{xy}\,\big)\le\sqrt[2^{1/p}]{f(x)f(y)}\le\frac{f(x)+f(y)}{2}. \tag{5.2}$$

Letting $x=a^{t^{1/p}}b^{(1-t)^{1/p}}$ and $y=a^{(1-t)^{1/p}}b^{t^{1/p}}$ with $t\in[0,1]$ in (5.2) results in

$$\begin{aligned}f\Big((ab)^{[t^{1/p}+(1-t)^{1/p}]/2^{1/p}}\Big)&\le\big[f\big(a^{t^{1/p}}b^{(1-t)^{1/p}}\big)f\big(a^{(1-t)^{1/p}}b^{t^{1/p}}\big)\big]^{1/2^{1/p}}\\&\le\frac{f\big(a^{t^{1/p}}b^{(1-t)^{1/p}}\big)+f\big(a^{(1-t)^{1/p}}b^{t^{1/p}}\big)}{2}\\&\le\frac{[f(a)]^{t^{1/p}}[f(b)]^{(1-t)^{1/p}}+[f(a)]^{(1-t)^{1/p}}[f(b)]^{t^{1/p}}}{2}\\&\le\frac{[f(a)]^t[f(b)]^{1-t}+[f(a)]^{1-t}[f(b)]^t}{2}\\&\le\frac{f(a)+f(b)}{2}.\end{aligned}$$

Integrating this inequality over $t\in[0,1]$ leads to the desired inequalities. □

Remark 5.3. Under the conditions of Theorem 5.3, taking $p = 1$ and $a \neq b$ reduces to

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x}\sqrt{f(x) f\left(\frac{ab}{x}\right)}\,\mathrm{d}x \\ &\le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x \\ &\le L(f(a), f(b)) \\ &\le \frac{f(a)+f(b)}{2}, \end{aligned}$$

which is in [8, Theorem 1].

6. Conclusions

In this paper, we presented the concepts of p-geometrically convex set and p-geometrically convex function, and studied their fundamental characterizations and some operational properties. As applications, some new integral inequalities of p-geometrically convex functions were established. The obtained results may be seen as the generalizations and extensions of previous work in [7, 8, 16, 18, 20, 27]. Specially, some of the results in [7, 8, 20, 27] are the cases of our results when $p = 1$.

There are still a lot of works to be done on this topic. For example, the analytical and topological properties and the generalizations of various classical inequalities for p-geometrically convex functions need to be investigated. The completion of these works will provide new ideas and new content for the study of convexity theory and enrich the convexity theory system.

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