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BEHAVIOR OF SOLUTIONS TO A COUPLED KIRCHHOFF-TYPE PARABOLIC SYSTEM WITH SINGULAR POTENTIAL AND LOGARITHMIC NONLINEARITY*

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Abstract In this paper, we apply the modified potential well method and the variational method to study the long-time behaviors of solutions to a coupled Kirchhoff-type parabolic system with singular potential and logarithmic nonlinearity. By classifying the initial energy ($J(u_0, v_0) < d,\ = d,\ > d$), we obtain global existence and finite-time blow-up of solutions. Noting that the value of the potential well depth d is very small such that it is difficult to calculate precisely, by the concavity method, we also discuss finite time blow-up of solutions independent of d. Furthermore, we derive new threshold criteria for extinction and non-extinction phenomena of solutions, and obtain the threshold time for the extinction phenomenon under some appropriate conditions.

Keywords Coupled Kirchhoff-type parabolic system, global existence, blow up, extinction, non-extinction.

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1. Introduction

In 1883, Kirchhoff [32] studied the transverse oscillations of a stretched string and derived a hyperbolic equation:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where the parameters have the following physical interpretations:

ρ: The initial axial tension; P_0: The initial tension;

h: The area of cross-section; E: The Young modulus of the material;

L: The length of the string.

Motivated by the mathematical structure of this equation, Lions [36] investigated its high dimensional manifestations under the action of an external force $f(x,t)$, specifically analyzed the

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case with parameter:

$$\varepsilon u_{tt}^{\varepsilon}+u_t^{\varepsilon}-M\left(\int_{\Omega}|\nabla u^{\varepsilon}|^2dx\right)\Delta u^{\varepsilon}=f\left(x,t\right), \tag{1.1}$$

which was regarded as an extension of the D'Alembert wave equation for free vibrations of elastic strings, and also constituted a Kirchhoff-type parabolic equation. Notably, the Kirchhoff-type parabolic equation $u_t-M\left(\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f\left(x,t\right)$ as a specific nonlocal diffusion equation can be used to describe the motion of a nonstationary fluid or gas in a nonhomogeneous and anisotropic medium [27]. For other nonlocal diffusion equations in many physical, chemical and biological phenomena, we also refer to [9, 18, 20, 51] and references cited therein. The study of Kirchhoff-type parabolic equation has attracted huge attention in recent years. When $\varepsilon=0$ in (1.1), Chipot, Valente and Caffarelli [13] studied a Kirchhoff-type parabolic equation:

$$u_t-M\left(\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f\left(x\right), x\in\Omega, t>0,$$

where $u_0\in H_0^1\left(\Omega\right)$, $f\left(x\right)\in L^2\left(\Omega\right)$, the nonlocal diffusion coefficient $M\left(\tau\right)$ is a C^1 function satisfying $0<a_0\leq M\left(\tau\right)\leq b_0$ with a_0, b_0 are positive constants. The authors obtained the existence, uniqueness and asymptotic behavior of weak solutions using the functional analysis method introduced by Lions [36]. When the nonlocal diffusion coefficient $M\left(\int_{\Omega}|\nabla u|^2dx\right)$ is replaced by $M\left(\int_{\Omega}g\left(x\right)udx\right)$ in the above Kirchhoff-type parabolic equation with $g\left(x\right)\in L^2\left(\Omega\right)$, this problem lacks a Lyapunov functional. Zheng and Chipot [57] employed some other methods instead of using dynamical systems results to investigate the asymptotic behavior of solutions as $t\to\infty$. Moreover, for the case $g(x)=1$ in the nonlocal coefficient, Chipot and Lovat [12] studied the asymptotic behavior of solutions under suitable conditions. In addition, as an extension of the above Kirchhoff-type parabolic equation, Almeida, Antontsev and Duque [3] studied a nonlocal degenerate parabolic problem:

$$u_t-\left(\int_{\Omega}u^2dx\right)^{\gamma}\Delta u=f\left(x,t\right), x\in\Omega, t>0,$$

the authors proved the existence, uniqueness and asymptotic behaviors of global weak solutions when $\gamma\geq 0$ and local weak solutions when $\gamma<0$, respectively. Recently, researchers have studied a class of Kirchhoff-type equations with logarithmic nonlinearity. The logarithmic nonlinearity is widely employed to describe various natural phenomena in physics due to its unique mathematical structure, such as in optics [10], inflation cosmology [6], quantum mechanics [15], and nuclear physics [7, 21]. Sert and Shmarev [45] considered the Dirichlet problem for the p-Laplace equation of Kirchhoff-type with the non-local logarithmic nonlinearity:

$$u_t-M\left(\int_{\Omega}|\nabla u|^pdx\right)\Delta_p u=\ln\left(\int_{\Omega}|u|^2dx\right)|u|^{q(x,t)-2}u, x\in\Omega, t>0,$$

where $\Delta_p u=div\left(|\nabla u|^{p-2}\nabla u\right)$, $p>1$. The authors studied the phenomena of blow-up or vanishing of solutions in a finite time, and found the upper bound for the blow-up moment. Tan and Yang [50] investigated a nonlocal Kirchhoff-type diffusion problem involving the singular potential and logarithmic nonlinearity:

$$\frac{u_t}{|x|^{2s}}+M\left([u]_s^2\right)L_K u+L_K u_t=|u|^{p-2}u\ln u,\ x\in\Omega, t>0,$$

where L_K represents the nonlocal integro-differential operator, $[u]_s^2$ represents the Gagliardo seminorm of u, $0 < s < \min\left\{1, \frac{n}{2}, \frac{n(p-2)}{p}\right\}$, $M(\tau) \geq h\tau^{\eta-1}$ for two constants $h > 0$, $\eta > 1$. The authors presented an analysis of the finite-time blow-up, including a new blow-up criterion, the blow-up rate, and bounds on the blow-up time. When $s = 0$, Guo et al. [24] discussed the finite time blow-up criterion, the blow-up rate, and the bounds of blow-up time. For more related research regarding existence and asymptotic behaviors of solutions on single Kirchhoff-type equation, please refer to [1, 26, 34, 43, 46, 48, 52, 54] and the references cited therein.

In research on research on Kirchhoff-type parabolic systems, Medekhelz, Boulaaras and Guefaifia [40] investigated a Kirchhoff-type parabolic system with multiple parameters:

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = F_1(x,u,v), x \in \Omega, t > 0, \\ v_t - N\left(\int_\Omega |\nabla v|^2 dx\right)\Delta v = F_2(x,u,v),\ x \in \Omega, t > 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and $M,\ N : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions. When

$$F_1(x,u,v) = \lambda_1\alpha(x) f(v) + \mu_1\beta(x) h(u),\ F_2(x,u,v) = \lambda_2\gamma(x) g(u) + \mu_2\eta(x)\tau(v),$$

where $\alpha,\ \beta,\ \gamma,\ \eta \in C\left(\bar{\Omega}\right)$ and $\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2$ are nonnegative parameters. The authors employed the sub-super solutions method to prove the existence of weak positive solutions for system (1.2). When $F_1(x,u,v) = \lambda u^\alpha v^\gamma,\ F_2(x,u,v) = \lambda u^\delta v^\beta$ with $0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ \delta,\ \gamma > 0$ and $\theta = (1-\alpha)(1-\beta) - \gamma\delta > 0$ for each $\lambda > 0$, Boulaaras [11] obtained the existence of weak positive solutions for system (1.2) on a bounded domain under symmetry conditions.

Almeida, Antontsev and Duque [2] considered a non-linear coupled system of reaction–diffusion non-local type:

$$\begin{cases} u_t + M(l_1(u), l_2(v))\Delta u + \varsigma_1|u|^{p-2}u = f_1(x,t), x \in \Omega, t > 0, \\ v_t + N(l_1(u), l_2(v))\Delta v + \varsigma_2|v|^{p-2}v = f_2(x,t),\ x \in \Omega, t > 0, \end{cases}$$

where $\varsigma_1,\ \varsigma_2 > 0,\ r > 1,\ f_1,\ f_2 \in L^2\left(0,T;L^2(\Omega)\right)$, and $l_i : L^2(\Omega) \to \mathbb{R}\,(i = 1,2)$ is a continuous linear form. The authors studied the existence, uniqueness, long-time behavior and localization properties of solutions.

Au, Kirane and Tuan [4] considered a parabolic system with nonlinear–nonlocal diffusion terms:

$$\begin{cases} u_t + M\left(\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}, \left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}}\right)\Delta u = f_1(x,t), x \in \Omega, t > 0, \\ v_t + N\left(\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}, \left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}}\right)\Delta v = f_2(x,t),\ x \in \Omega, t > 0, \end{cases}$$

the authors obtained the existence, uniqueness and regularity of solutions of the regularized problem by introducing two regularization methods.

Cömert and Piskin [14] considered a nonlinear initial-boundary value problem concerning a Kirchhoff-type parabolic system with logarithmic source terms:

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx\right)\Delta u - \Delta u_t = |u|^{q-2} u \ln |u|\,, x\in\Omega, t>0,\\ v_t - M\left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx\right)\Delta v - \Delta v_t = |v|^{q-2} v \ln |v|\,,\ x\in\Omega, t>0,\end{cases}$$

where $M(\tau)=\alpha+\beta\tau^\gamma,\ \alpha\ge 1,\ \beta\ge 0,\ \gamma>0$ with $q\ge 2\gamma+2$. The authors used the potential method to prove the existence of global weak solutions and obtained the exponential energy decay estimate of solutions under certain conditions. Furthermore, concerning the existence and multiplicity of solutions for the Kirchhoff-type systems and coupled systems, we refer to [8, 23, 29, 39, 47, 53] and the references cited therein.

Inspired by the above research work. In this paper, we consider the following initial-boundary value problem to a coupled Kirchhoff-type parabolic system with singular potential and logarithmic nonlinearity:

$$\begin{cases} \dfrac{u_t}{|x|^s} - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = v^p u^{p-1}\ln|uv|\,, x\in\Omega, t>0,\\ \dfrac{v_t}{|x|^s} - M\left(\int_\Omega |\nabla v|^2 dx\right)\Delta v = u^p v^{p-1}\ln|uv|\,, x\in\Omega, t>0,\end{cases} \tag{1.3}$$

subject to boundary conditions

$$u=v=0,\quad x\in\partial\Omega,\ t>0,$$

and initial data

$$u(x,0)=u_0,\ v(x,0)=v_0,\quad x\in\Omega,$$

where $0<s\le 2$, $\Omega\subset\mathbb{R}^N\,(N\ge 2)$ is a bounded domain with smooth boundary $\partial\Omega$, and the nonlocal diffusion coefficient $M(\tau)=a+b\tau$ with $a,\ b$ are positive parameters, it is worth noting that τ can be taken as $\tau=\int_\Omega|\nabla u|^2dx$ or $\tau=\int_\Omega|\nabla v|^2dx$, respectively. We note that the valuable work of Cömert and Piskin [14] on a Kirchhoff-type parabolic system with logarithmic source terms has inspired our current study. However, unlike the coupled nonlocal diffusion term $M\left(\int_\Omega|\nabla u|^2dx+\int_\Omega|\nabla v|^2dx\right)\Delta u$ considered in [14], the problem (1.3) investigated in this paper adopts an independent form of nonlocal diffusion. Specifically, the coefficients $M\left(\int_\Omega|\nabla u|^2dx\right)$ and $M\left(\int_\Omega|\nabla v|^2dx\right)$ in each equation depend only on the gradient of the respective solution, rather than on the superposition of the gradients of both solutions. This independent nonlocal structure better aligns with certain physical contexts where the diffusion processes are inherently independent. Meanwhile, the coupling in our model is realized solely through the logarithmic source terms $v^pu^{p-1}\ln|uv|$ and $u^pv^{p-1}\ln|uv|$, not via the diffusion terms. This design concentrates the coupling effect entirely on the nonlinear interaction, thereby introducing a series of novel mathematical challenges: The logarithmic coupling terms satisfy neither the usual monotonicity condition nor the Ambrosetti–Rabinowitz condition. Their interplay with the singular potential $\frac{1}{|x|^s}$ further complicates the energy estimates and asymptotic analysis. Consequently, this work differs fundamentally from [14] in both the model structure and the analytical methodology, providing a new framework for studying coupled parabolic systems with such nonlocal, singular, and logarithmic nonlinearities. For the research on this system, there exist the following three difficulties:

• The nonlocal diffusion term $M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u$ is different from the usually degenerate term $u^p\Delta u$, which studied by Fila and Winkler [17] in $\mathbb{R}^N$. We can not derive an upper bound for decay rate of the weak solutions of problem (1.3) by using the Gagliardo-Nirenberg type inequality as in [17].

• Comparing with Chipot, Valente and Caffarelli [13], Han and Li [27], Ugur [52], Guo et al. [24], the major difficulty in this paper is the coupled logarithmic nonlinearity terms $v^p u^{p-1}\ln|uv|$ and $u^p v^{p-1}\ln|uv|$ have unique structure that are different from the coupled polynomial nonlinearity terms $|v|^{p+1}|u|^{p-1}u$ and $|u|^{p+1}|v|^{p-1}v$, which satisfy neither the monotonicity condition nor the Ambrosetti-Rabinowitz condition. Moreover, the classical logarithmic Sobolev inequality (see [22, 28, 33, 35, 42])

$$p\int_\Omega |u|^p \ln\frac{|u|}{\|u\|_{L^p(\Omega)}}dx + \frac{n}{p}\ln\left(\frac{p\mu e}{nl_p}\right)\int_\Omega |u|^p dx \le \mu\int_\Omega |\nabla u|^p dx$$

is no longer applicable to deal with $v^p u^{p-1}\ln|uv|$ and $u^p v^{p-1}\ln|uv|$, which brings some difficulties for ensuring the compactness of Euler-Lagrange functional associated with problem (1.3).

• The singular potential $\frac{1}{|x|^s}$ makes the methods from [16, 30, 49] inapplicable for analyzing finite-time blow-up of solutions with arbitrary initial energy levels, extinction and non-extinction behavior of solutions in finite time.

In this paper, the research results of this paper are based on the framework of the modified potential well theory. This method was initially proposed by Sattinger and Payne [41], Sattinger [44], and later modified by Liu [37], Liu and Zhou [38], Gazzola and Weth [19], Xu and Su [55] and so on. Based on this framework, we overcome these difficulties by employing some new techniques such as the Hardy-Sobolev inequality, the Cauchy-Schwarz inequality and polynomial expansion technique. We apply the modified potential well method and the variational method to study the long-time behaviors of solutions to a coupled Kirchhoff-type parabolic system with singular potential and logarithmic nonlinearity. By classifying the initial energy ($J(u_0, v_0) < d,\ = d,\ > d$), we obtain global existence and finite-time blow-up of solutions. Noting that the value of the potential well depth d is very small such that it is difficult to calculate precisely, by the concavity method, we also discuss finite-time blow-up of solutions independent of d. Furthermore, we derive new threshold criteria for extinction and non-extinction phenomena of solutions, and obtain the threshold time of extinction phenomenon under some appropriate conditions.

This paper is organized as follows. In Section 2, we introduce essential notations, definitions, and preliminary lemmas that are fundamental to our main results. In Section 3, we establish the global existence and a new algebraic decay estimate for weak solutions of weak solutions to problem (1.3) when $J(u_0, v_0) < d$, and demonstrate that weak solutions blow up in finite time and give the corresponding upper bound of the blow-up time. In Section 4, we extend the previously established results from subcritical initial energy $J(u_0, v_0) < d$ to the critical initial energy $J(u_0, v_0) = d$. In Section 5, we demonstrate finite-time blow-up of weak solutions under arbitrarily high initial energy $J(u_0, v_0) > d$, which extends prior results for subcritical initial energy $J(u_0, v_0) < d$ and critical initial energy $J(u_0, v_0) = d$. Here, the absence of potential well depth constraints invalidates classical methods, leading to the breakdown of manifold invariance. To resolve this, we devise new lemmas to the functionally replace invariant manifold technique. Subsequently, in Section 6, by the concavity method, we prove the blow-up properties of weak solutions to problem (1.3) under different energy conditions without relying on the potential well depth d, and estimate the upper bound of the blow-up time. Finally, Section 7 derives new threshold criteria for extinction and non-extinction phenomena of weak solutions to problem (1.3), and under some appropriate conditions, we also obtain the critical threshold time for the extinction phenomenon.

2. Perliminaries

Throughout this paper, we equip $L^p(\Omega)$ with the norm

$$\|u\|_{L^p(\Omega)} := \|u\|_p = \begin{cases} \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}, 1 \le p < +\infty, \\ ess\sup_{x\in\Omega} |u(x)|, \quad p = +\infty, \end{cases}$$

and equip $H_0^1(\Omega)$ with the norm $\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_2$. We equip $L^2(\Omega)$ with the inner product

$$(\cdot,\cdot)_2 := \int_\Omega uv dx.$$

And we define

$$N^* := \begin{cases} \dfrac{N}{N-2}, N \ge 3, \\ +\infty, \quad N = 2. \end{cases}$$

We introduce two definitions as follows:

Definition 2.1. A function $(u,v) = (u(x,t), v(x,t))$ is called a weak solution of problem (1.3) if

$$(u,v) \in L^\infty\left(0,T;H_0^1(\Omega)\right) \times L^\infty\left(0,T;H_0^1(\Omega)\right),$$
$$\left(\frac{u_t}{|x|^{\frac{s}{2}}}, \frac{v_t}{|x|^{\frac{s}{2}}}\right) \in L^2\left(0,T;L^2(\Omega)\right) \times L^2\left(0,T;L^2(\Omega)\right),$$

satisfying $u(x,0) = u_0,\ v(x,0) = v_0$, and

$$\left(\frac{u_t}{|x|^s}, \phi\right)_2 + \left(a + b\|\nabla u\|_2^2\right)(\nabla u, \nabla\phi)_2 = \left(v^p u^{p-1}\ln|uv|, \phi\right)_2, \tag{2.1}$$
$$\left(\frac{v_t}{|x|^s}, \varphi\right)_2 + \left(a + b\|\nabla v\|_2^2\right)(\nabla v, \nabla\varphi)_2 = \left(u^p v^{p-1}\ln|uv|, \varphi\right)_2, \tag{2.2}$$

for any $(\phi,\varphi) \in H_0^1(\Omega) \times H_0^1(\Omega)$, *a.e.* $t \in [0,T]$. Moreover, the weak solution (u,v) satisfies the following energy equality

$$\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right) d\tau + J(u,v) = J(u_0,v_0),\ t \in [0,+\infty). \tag{2.3}$$

Definition 2.2. Let T be maximal existence time of a weak solution (u,v) of problem (1.3).
(i) If (u,v) exists for $0 \le t < +\infty$, then $T = +\infty$, the weak solution is global existence.

(ii) If there exists a $t_0 \in [0, +\infty)$ such that (u, v) exists for $0 \le t < t_0$, but does not exist at $t = t_0$, then $T = t_0$, the weak solution is local existence and blows up in finite time, namely,

$$\lim_{t \to T^-} \int_0^t \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau = +\infty.$$

(iii) A weak solution (u, v) is called extinction in finite time if the maximal existence time T is finite and

$$\lim_{t \to T^-} \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) = 0.$$

Since problem (1.3) has an important relation to the stationary problem

$$\begin{cases} -\left(a + b \|\nabla u\|_2^2\right) \Delta u = v^p u^{p-1} \ln |uv|, \, x \in \Omega, \\ -\left(a + b \|\nabla v\|_2^2\right) \Delta v = u^p v^{p-1} \ln |uv|, \, x \in \Omega, \\ u = v = 0, \qquad\qquad\qquad\qquad\qquad x \in \partial\Omega, \end{cases}$$

hence we can define the energy functional

$$\begin{aligned} J(u, v) :=& \frac{a}{2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + \frac{b}{4} \left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4 \right) + \frac{1}{p^2} \|uv\|_p^p \\ & - \frac{1}{p} \int_\Omega u^p v^p \ln |uv| \, dx, \end{aligned} \tag{2.4}$$

and the Nehari functional

$$I(u, v) := a \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + b \left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4 \right) - 2 \int_\Omega u^p v^p \ln |uv| \, dx. \tag{2.5}$$

From (2.4) and (2.5), we obtain $J(u, v)$ and $I(u, v)$ satisfy the following relation

$$\begin{aligned} J(u, v) = & \frac{1}{2p} I(u, v) + \frac{a(p-1)}{2p} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + \frac{b(p-2)}{4p} \left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4 \right) \\ & + \frac{1}{p^2} \|uv\|_p^p. \end{aligned} \tag{2.6}$$

By virtue of the Nehari functional (2.5), we can define a Nehari manifold

$$\mathcal{N} := \left\{ (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{0, 0\} \middle| \, I(u, v) = 0, \|\nabla u\|_2 \neq 0, \|\nabla v\|_2 \neq 0 \right\},$$

which separates the two unbounded sets:

$$\mathcal{N}_+ := \left\{ (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{0, 0\} \middle| \, I(u, v) > 0 \right\}$$

and

$$\mathcal{N}_- := \left\{ (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{0, 0\} \middle| \, I(u, v) < 0 \right\}.$$

Furthermore, the potential well and its corresponding stable set W and unstable set V are defined respectively by

$$\mathrm{W} := \left\{ (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \middle| \, J(u, v) < d, I(u, v) > 0 \right\} \cup \{0, 0\},$$

$$\mathrm{V} := \left\{ (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \middle| J(u,v) < d, I(u,v) < 0 \right\},$$

where d represents the depth of the potential well and d is defined by

$$d := \inf_{(u,v)\in\mathcal{N}} J(u,v).$$

In order to consider the weak solution at high initial energy level, we introduce some notions:

$$\begin{aligned}
&J^c := \left\{ (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, J(u,v) < c \right\}, \\
&\mathcal{N}_c := \mathcal{N} \cap J^c \\
&\quad = \left\{ (u,v) \in \mathcal{N} \middle| \frac{a(p-1)}{2p} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + \frac{b(p-2)}{4p} \left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4 \right) \right. \\
&\qquad \left. + \frac{1}{p^2} \|uv\|_p^p < c \right\},
\end{aligned}$$

and for all $c > d$,

$$\lambda_c := \inf \left\{ \frac{1}{2} \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) | (u,v) \in \mathcal{N}_c \right\},$$

$$\Lambda_c := \sup \left\{ \frac{1}{2} \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) | (u,v) \in \mathcal{N}_c \right\}.$$

It is obvious that λ_c, Λ_c have the following monotonicity properties:

$$c \mapsto \lambda_c \;\; \text{noincreasing}, \qquad \mathrm{c} \mapsto \Lambda_{\mathrm{c}} \;\; \text{nodecreasing}.$$

If the weak solution (u,v) is global existence, we define the ω–limit set of (u_0,v_0) by

$$\omega(u_0,v_0) := \bigcap_{t\geq 0} \overline{\{(u(s), v(s)) : s \geq t\}}.$$

The following several lemmas play an important role in this paper.

Lemma 2.1 ([35]). *Let Ψ be a positive continous function. Then there exists a suitable small positive constant σ such that*

$$\begin{cases} \Psi^p \ln \Psi \leq \dfrac{\mathrm{e}^{-1}}{\sigma} \Psi^{p+\sigma}, & \Psi \geq 1, \\ |\Psi^p \ln \Psi| \leq (\mathrm{e}p)^{-1}, & 0 < \Psi < 1. \end{cases}$$

Lemma 2.2 ([31]). *Assume that $H(t)$ is a positive and twice-differentiable function, satisfying the following inequality*

$$H''(t) H(t) - (1+\varsigma) \left[H'(t) \right]^2 \geq 0, \forall t \geq t_0 \geq 0,$$

where $\varsigma > 0$. If $H(t_0) > 0$, $H'(t_0) > 0$, then $H(t) \to \infty$ as $t \to t_* \leq t^* = t_0 + \frac{H(t_0)}{\varsigma H'(t_0)}$.

Lemma 2.3 ([25]). *Suppose that $h > 0$, $\alpha > 0$, $\beta > 0$ and $\upsilon(t)$ is a nonnegative and absolutely continuous function satisfying*

$$\upsilon'(t) + \alpha\upsilon^h(t) \geq \beta.$$

Then, for $0 < t < +\infty$, it holds

$$\upsilon(t) \geq \min\left\{\upsilon(0), \left(\frac{\beta}{\alpha}\right)^{\frac{1}{h}}\right\}.$$

Lemma 2.4 (Hardy-Sobolev inequality, [5]). *Let $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, where $2 \leq k \leq N$, and $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. For given λ and s satisfying $1 < \lambda < N$, $0 \leq s \leq \lambda$, $s < k$, and $\delta(s, N, \lambda) = \frac{\lambda(N-s)}{N-\lambda}$, there exists a constant $B(s, N, \lambda, k) > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^\delta}{|y|^s} dx \leq B\left(\int_{\mathbb{R}^N} |\nabla u(x)|^\lambda dx\right)^{\frac{N-s}{N-\lambda}}, \ \forall u \in W^{1,\lambda}\left(\mathbb{R}^N\right).$$

Remark 2.1. In particular, we set $k = N$ in Lemma 2.4, which implies $x = y \in \mathbb{R}^N$. We define $u(x) = 0$ for $x \in \mathbb{R}^N \backslash \Omega$, thus,

$$\int_{\Omega} \frac{|u(x)|^\delta}{|x|^s} dx \leq B\left(\int_{\Omega} |\nabla u(x)|^\lambda dx\right)^{\frac{N-s}{N-\lambda}}, \ \forall u \in W^{1,\lambda}(\Omega).$$

If $\delta = \frac{\lambda(N-s)}{N-\lambda} = 2$, then, by the $0 < s \leq 2$, $N \geq 2$, and the Rellich-Kondrachov theorem, we have

$$\int_{\Omega} \frac{|u(x)|^2}{|x|^s} dx \leq B\left(\int_{\Omega} |\nabla u(x)|^{\frac{2N}{N-s+2}} dx\right)^{\frac{N-s+2}{N}} \leq C_B \|\nabla u\|_2^2, \tag{2.7}$$

where $C_B = BB_1$, B_1 is the optimal embedding constant of the embedding $H_0^1(\Omega) \hookrightarrow W_0^{1,\frac{2N}{N-s+2}}(\Omega)$. Due to the boundedness of Ω, there exists a sufficiently large $L > 0$ such that $\Omega \subset B_L(0)$, and $|x| \leq L$ for all $x \in \Omega$. Hence, we can introduce the cut-off function

$$\rho_n := \min\left\{|x|^{-s}, n\right\}, \ n \in \mathbb{N}^+$$

to handle the singular potencial $\frac{1}{|x|^s}$ in problem (1.3). Then, it follows from (2.7) that

$$\min\left\{L^{-s}, n\right\} \|u\|_2^2 \leq \int_{\Omega} \rho_n |u|^2 dx \leq C_B \|\nabla u\|_2^2, \ \forall n \in \mathbb{N}^+.$$

Lemma 2.5. *The depth d of the potential well W on $\mathcal{N}$ is positive for all $p > 2$.*

Proof. Since $(u, v) \in \mathcal{N}$, we obtain $I(u, v) = 0$, namely

$$\begin{aligned} a\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + b\left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4\right) &= 2\int_{\Omega} u^p v^p \ln|uv|\, dx \\ &= 2\int_{\Omega_1} u^p v^p \ln|uv|\, dx + 2\int_{\Omega_2} u^p v^p \ln|uv|\, dx, \end{aligned} \tag{2.8}$$

where $\Omega_1 = \{x \in \Omega |\, |uv| < 1\}$, $\Omega_2 = \{x \in \Omega |\, |uv| \geq 1\}$. By virtue of Lemma 2.1 and the Young inequality, it follows from (2.8) that

$$\begin{aligned}
&2\int_{\Omega_1} u^p v^p \ln|uv|\,dx + 2\int_{\Omega_2} u^p v^p \ln|uv|\,dx \\
\leq\ & 2\int_{\Omega_2} u^p v^p \ln|uv|\,dx \\
\leq\ & \frac{1}{\mathrm{e}\sigma}\|u\|_{2(p+\sigma)}^{2(p+\sigma)} + \frac{1}{\mathrm{e}\sigma}\|v\|_{2(p+\sigma)}^{2(p+\sigma)} \\
\leq\ & \frac{1}{\mathrm{e}\sigma} S^{2(p+\sigma)}\|\nabla u\|_2^{2(p+\sigma)} + \frac{1}{\mathrm{e}\sigma} S^{2(p+\sigma)}\|\nabla v\|_2^{2(p+\sigma)},
\end{aligned} \tag{2.9}$$

where S is the optimal constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{2(2+\sigma)}(\Omega)$, σ is a sufficiently small positive constant introduced in Lemma 2.1 to handle the logarithmic nonlinearity, and subsequent occurrences of σ serve a similar purpose. We can deduce from (2.8) and (2.9) that

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \geq \left(\frac{a\mathrm{e}\sigma}{S^{2(p+\sigma)}}\right)^{\frac{1}{p-1+\sigma}} > 0. \tag{2.10}$$

Since $(u,v) \in \mathcal{N}$, it follows from (2.6) and (2.10) that

$$J(u,v) \geq \frac{a(p-1)}{2p}\left(\frac{a\mathrm{e}\sigma}{S^{2(p+\sigma)}}\right)^{\frac{1}{p-1+\sigma}} > 0,$$

which implies $d = \inf\limits_{(u,v)\in\mathcal{N}} J(u,v) > 0$. The proof of Lemma 2.5 is complete. $\square$

Lemma 2.6. *Let $2 < p < N^*$, for any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{(0,0)\}$, and assume that $g(\lambda) = J(\lambda u, \lambda v)$ for all $\lambda > 0$. Then,*

(i) $\lim\limits_{\lambda\to 0^+} g(\lambda) = 0$ *and* $\lim\limits_{\lambda\to+\infty} g(\lambda) = -\infty$*;*

(ii) $g(\lambda)$ *increasing on* $(0,\lambda_*)$*, decreasing on* $(\lambda_*, +\infty)$ *and attains the maximum at* λ_**;*

(iii)

$$I(\lambda u, \lambda v) = \lambda g'(\lambda) \begin{cases} > 0, 0 < \lambda < \lambda_*, \\ = 0, \quad \lambda = \lambda_*, \\ < 0, \quad \lambda > \lambda_*. \end{cases}$$

Proof. We define a fibering map

$$g : \lambda \mapsto J(\lambda u, \lambda v),\ \lambda > 0,$$

namely,

$$\begin{aligned}
g(\lambda) = J(\lambda u, \lambda v) =& \frac{a\lambda^2}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{b\lambda^4}{4}\left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4\right) \\
&+ \left(\frac{\lambda^{2p}}{p^2} - \frac{2\lambda^{2p}}{p}\ln\lambda\right)\|uv\|_p^p - \frac{\lambda^{2p}}{p}\int_\Omega u^p v^p \ln|uv|\,dx,
\end{aligned}$$

which implies (i) in Lemma 2.6 holds. By a straightforward calculation, we obtain

$$g'(\lambda) = a\lambda\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + b\lambda^3\left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4\right) - 4\lambda^{2p-1}\ln\lambda\,\|uv\|_p^p$$

$$- 2\lambda^{2p-1}\int_\Omega u^p v^p \ln|uv|\,dx.$$

We define $j(\lambda) := \lambda^{-1}g'(\lambda)$, then,

$$\begin{aligned} j'(\lambda) =& 2b\lambda\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right) - 8(p-1)\lambda^{2p-3}\ln\lambda\,\|uv\|_p^p - 4\lambda^{2p-3}\|uv\|_p^p \\ & - 4(p-1)\lambda^{2p-3}\int_\Omega u^p v^p\ln|uv|\,dx. \end{aligned}$$

We continue to define $z(\lambda) := \lambda^{-1}j'(\lambda)$, then,

$$\begin{aligned} z'(\lambda) :=& - 16(p-1)(p-2)\lambda^{2p-5}\ln\lambda\,\|uv\|_p^p - 8(p-1)\lambda^{2p-5}\|uv\|_p^p \\ & - 8(p-1)(p-2)\lambda^{2p-5}\int_\Omega u^p v^p\ln|uv|\,dx \\ <& 0. \end{aligned}$$

Since $\lim_{\lambda\to 0^+} z(\lambda) = 2b\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right) > 0$ and $\lim_{\lambda\to+\infty} z(\lambda) = -\infty$, $z'(\lambda) < 0$ implies that $z(\lambda)$ is strictly decreasing for $\lambda > 0$ and there exists a $\hat\lambda \in (0,+\infty)$ such that

$$\begin{cases} j'(\lambda) > 0, \; j(\lambda) \text{ is increasing on } \lambda\in\left(0,\hat\lambda\right), \\ j'(\lambda) = 0, \; j(\lambda) \text{ arives its maximum value at } \lambda{=}\hat\lambda, \\ j'(\lambda) < 0, \; j(\lambda) \text{ is decreasing on } \lambda\in\left(\hat\lambda,+\infty\right). \end{cases}$$

On the other hand, since $\lim_{\lambda\to 0^+} j(\lambda) > 0$ and $j(\lambda) := \lambda^{-1}g'(\lambda)$, similar to the above analysis, there exists a λ_* such that (ii) and (iii) hold, respectively. The proof of Lemma 2.6 is complete. □

Lemma 2.7. *Assume that* $2 < p < N^*$, $(u,v) \in H_0^1(\Omega)\times H_0^1(\Omega)\setminus\{0,0\}$ *with* $I(u,v) < 0$. *Then,*

$$I(u,v) < 2p\left[J(u,v) - d\right]. \tag{2.11}$$

Proof. Based on Lemma 2.6, there exists a $\lambda_* \in (0,1)$ such that $I(\lambda_* u, \lambda_* v) = 0$. Setting

$$\begin{aligned} h(\lambda) &= 2pJ(\lambda u,\lambda v) - I(\lambda u,\lambda v), \\ &= a(p-1)\lambda^2\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right) + \frac{b}{2}(p-2)\lambda^4\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right) + \frac{2}{p}\lambda^{2p}\|uv\|_p^p, \end{aligned}$$

then, by a direct calculation, we obtain

$$h'(\lambda) = 2a(p-1)\lambda\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right) + 2b(p-2)\lambda^3\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right) + 4\lambda^{2p-1}\|uv\|_p^p > 0.$$

So $h(\lambda)$ is strictly increasing for $\lambda > 0$. Hence, by $0 < \lambda_* < 1$, we can get $h(1) > h(\lambda^*)$, namely,

$$2pJ(u,v) - I(u,v) > 2pJ(\lambda_* u,\lambda_* v) - I(\lambda_* u,\lambda_* v) \geq 2pd > 0,$$

which implies (2.11) holds. The proof of Lemma 2.7 is complete. □

Lemma 2.8. *Let $2 < p < N^*$, $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $0 < J(u_0, v_0) < d$.*
(i) *If $I(u_0, v_0) > 0$, then the weak solution (u, v) of problem (1.3) belongs to W for all $t \in [0, T)$;*
(ii) *If $I(u_0, v_0) < 0$, then the weak solution (u, v) of problem (1.3) belongs to V for all $t \in [0, T)$.*

Proof. (i) Let (u, v) be a weak solution of problem (1.3), T be the maximal existence time of (u, v). By using Lemma 2.5 and $I(u_0, v_0) > 0$, $J(u_0, v_0) < d$, then $(u_0, v_0) \in \mathrm{W}$.

Next, we assert that $(u, v) \in \mathrm{W}$ for all $t \in [0, T)$. If it is false, then according to the continuing of $I(u, v)$, there exists a $t_0 \in (0, T)$ such that $(u(t_0), v(t_0)) \in \partial\mathrm{W}$, i.e. $J(u(t_0), v(t_0)) = d$ or $I(u(t_0), v(t_0)) = 0$, $\|u(t_0)\|_2 \neq 0$, $\|v(t_0)\|_2 \neq 0$. According to (2.3), we know that $J(u(t_0), v(t_0)) \neq d$. The contradiction can be derived. On the other hand, if $I(u(t_0), v(t_0)) = 0$, $\|u(t_0)\|_2 \neq 0$, $\|v(t_0)\|_2 \neq 0$, by the definition of d, we have

$$d = \inf_{(u,v)\in\mathcal{N}} J(u, v) \leq J(u(t_0), v(t_0)).$$

Thus, the contradiction can be derived from (2.3).

(ii) This conclusion can be proved by the same method in (i), we omit it here. □

3. Subcritical initial energy $J(u_0, v_0) < d$

In this section, we prove the global existence of weak solutions of problem (1.3), and then give a new algebraic decay estimate for weak solution. Furthermore, we show that the weak solution blows up in finite time, and we also obtain the corresponding upper bound of time.

Theorem 3.1. *Let $2 < p < N^*$ and $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Assume that $J(u_0, v_0) < d$, $I(u_0, v_0) \geq 0$. Then the weak solution (u, v) of problem (1.3) is global existence. Furthermore, the global weak solution (u, v) is algebraic decay, namely,*

$$\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2 \leq \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right) e^{-2aC_B^{-1}\left\{1-\left[\frac{d}{J(u_0,v_0)}\right]^{\frac{1}{p}-1}\right\}t}, \ t \in [0, +\infty). \tag{3.1}$$

Proof. Firstly, we prove the global existence of weak solution of problem (1.3) by the Galerkin method, the proof will be divided into four steps.

Step 1. Approximation problem. We can construct the Galerkin approximate solutions $(u_m(x, t), v_m(x, t))$ of problem (1.3) for a positive integer m, we now take a Galerkin basis $\{w_j\}_{j=1}^m$ in the Sobolev space $H_0^1(\Omega)$ and define a finite dimensional space

$$V_m = \mathrm{Span}\{w_1, w_2, \cdots, w_m\},$$

then we construct the approximate solution $(u_m(x, t), v_m(x, t))$ as

$$\begin{cases} u_m(x, t) = \displaystyle\sum_{j=1}^m g_{mj}(t) w_j(x), \\ v_m(x, t) = \displaystyle\sum_{j=1}^m h_{mj}(t) w_j(x), \end{cases}$$

satisfying

$$(\rho_n u_{mt}, w_j)_2 + a(\nabla u_m, \nabla w_j)_2 + b\|\nabla u_m\|_2^2 (\nabla u_m, \nabla w_j)_2 = \left(v_m^p u_m^{p-1} \ln|u_m v_m|, w_j\right)_2, \tag{3.2}$$

$$\left(\rho_n v_{mt}, w_j\right)_2 + a(\nabla v_m, \nabla w_j)_2 + b\left\|\nabla v_m\right\|_2^2 \left(\nabla v_m, \nabla w_j\right)_2 = \left(u_m^p v_m^{p-1} \ln\left|u_m v_m\right|, w_j\right)_2, \tag{3.3}$$

and we choose $g_{mj} = g_{mj}(0)$, $h_{mj} = h_{mj}(0)$ such that $u_m(x,0) = u_{0m} = \sum_{j=1}^{m} g_{mj} w_j(x)$ and $v_m(x,0) = v_{0m} = \sum_{j=1}^{m} h_{mj} w_j(x)$, which makes

$$\left\|\sum_{j=1}^{m} g_{mj} w_j(x) - u_0\right\|_{H_0^1(\Omega)} \to 0, \text{ as m} \to +\infty, \tag{3.4}$$

$$\left\|\sum_{j=1}^{m} h_{mj} w_j(x) - v_0\right\|_{H_0^1(\Omega)} \to 0, \text{ as m} \to +\infty. \tag{3.5}$$

According to the standard Picard iteration method in ordinary differential equations, we can find a positive T such that $(g_{mj}(t), h_{mj}(t)) \in C^1([0,T]) \times C^1([0,T])$, and thus,

$$\left(u_m(x,t), v_m(x,t)\right) \in C^1\left([0,T], H_0^1(\Omega)\right) \times C^1\left([0,T], H_0^1(\Omega)\right).$$

Step 2. Priori estimates. Multiplying (3.2) and (3.3) by $\frac{d}{dt} g_{mj}(t)$ and $\frac{d}{dt} h_{mj}(t)$, respectively, summing for j from 1 to m, integrating with respect to time variable on $[0,t]$, we have

$$\int_0^t \left(\left\|\frac{u_{m\tau}}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_{m\tau}}{|x|^{\frac{s}{2}}}\right\|_2^2\right) d\tau + J(u_m, v_m) = J(u_{0m}, v_{0m}), \ t \in [0,T]. \tag{3.6}$$

By (3.4) and (3.5), we get $J(u_{0m}, v_{0m}) \to J(u_0, v_0)$ as $m \to +\infty$. Since $J(u_0, v_0) < d$, then it follows from (3.6) that

$$\int_0^t \left(\left\|\frac{u_{m\tau}}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_{m\tau}}{|x|^{\frac{s}{2}}}\right\|_2^2\right) d\tau + J(u_m, v_m) < d, \ t \in [0,T], \tag{3.7}$$

for large enough m.

From (3.4), (3.5) and $(u_0, v_0) \in \mathrm{W}$, it yields that $(u_{0m}, v_{0m}) \in \mathrm{W}$ for sufficiently large m. When $t \in [0,T]$, we can conclude $(u_m, v_m) \in \mathrm{W}$ by Lemma 2.8 (i). From (2.6), we have

$$\begin{aligned} J(u_m, v_m) =& \frac{1}{2p} I(u_m, v_m) + \frac{a(p-1)}{2p}\left(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2\right) \\ &+ \frac{b(p-2)}{4p}\left(\|\nabla u_m\|_2^4 + \|\nabla v_m\|_2^4\right) + \frac{1}{p^2}\|u_m v_m\|_p^p. \end{aligned} \tag{3.8}$$

Since $(u_m, v_m) \in W$ for large enough m and combine with (3.7) and (3.8), it obtains that

$$\begin{aligned} &\int_0^t \left(\left\|\frac{u_{m\tau}}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_{m\tau}}{|x|^{\frac{s}{2}}}\right\|_2^2\right) d\tau + \frac{a(p-1)}{2p}\left(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2\right) \\ &+ \frac{b(p-2)}{4p}\left(\|\nabla u_m\|_2^4 + \|\nabla v_m\|_2^4\right) + \frac{1}{p^2}\|u_m v_m\|_p^p < d, \end{aligned} \tag{3.9}$$

which implies

$$\|\nabla u_m\|_2^2+\|\nabla v_m\|_2^2<\frac{2pd}{a\,(p-1)}, \tag{3.10}$$

$$\int_0^t\left(\left\|\frac{u_{m\tau}}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_{m\tau}}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau<d. \tag{3.11}$$

From (3.11) and $t\in[0,T]$, we know that it implies $T=+\infty$.

Step 3. Passage to the limit. By (3.10) and (3.11), there exists a (u,v) and a subsequence of $(\{u_m\}_{m=1}^{\infty},\{v_m\}_{m=1}^{\infty})$ which we still denoted by $(\{u_m\}_{m=1}^{\infty},\{v_m\}_{m=1}^{\infty})$ such that when $m\to\infty$,

$$\begin{aligned}
&u_m\overset{*}{\rightharpoonup}u,\ \text{ in } \mathrm{L}^{\infty}\left(0,+\infty;\mathrm{H}_0^1(\Omega)\right),\\
&v_m\overset{*}{\rightharpoonup}v,\ \text{ in } \mathrm{L}^{\infty}\left(0,+\infty;\mathrm{H}_0^1(\Omega)\right),\\
&u_{mt}\to u_t,\ \text{ strongly in } \mathrm{L}^{2}\left(0,+\infty;\mathrm{L}^2(\Omega)\right),\\
&v_{mt}\to v_t,\ \text{ strongly in } \mathrm{L}^{2}\left(0,+\infty;\mathrm{L}^2(\Omega)\right).
\end{aligned}$$

Though the Aubin-Lions-Simon theorem, we can know that

$$\begin{aligned}
&\int_\Omega v_m^p u_m^{p-1}\ln|u_mv_m|\,dx\to\int_\Omega v^pu^{p-1}\ln|uv|\,dx,\ \textit{a.e.} \text{ in } \Omega\times[0,+\infty),\\
&\int_\Omega u_m^p v_m^{p-1}\ln|u_mv_m|\,dx\to\int_\Omega u^pv^{p-1}\ln|uv|\,dx,\ \textit{a.e.} \text{ in } \Omega\times[0,+\infty).
\end{aligned}$$

On the other hand, by a direct calculation, we deduce from Lemma 2.2, the Hölder inequality and the Young inequality that

$$\begin{aligned}
&\int_\Omega\left(v_m^pu_m^{p-1}\ln|u_mv_m|\right)^{\frac{p}{p-1}}dx\\
=&\int_{\Omega_1}\left(v_m^pu_m^{p-1}\ln|u_mv_m|\right)^{\frac{p}{p-1}}dx+\int_{\Omega_2}\left(v_m^pu_m^{p-1}\ln|u_mv_m|\right)^{\frac{p}{p-1}}dx\\
\le&\int_\Omega\left[\frac{1}{\mathrm{e}\,(\mathrm{p}-1)}|v_m|\right]^{\frac{p}{p-1}}dx+\int_\Omega\left(\frac{1}{\mathrm{e}\sigma}|u_m|^{p-1+\sigma}|v_m|^{p+\sigma}\right)^{\frac{p}{p-1}}dx\\
=&[\mathrm{e}\,(\mathrm{p}-1)]^{-\frac{p}{p-1}}\|v_m\|_{\frac{p}{p-1}}^{\frac{p}{p-1}}+(\mathrm{e}\sigma)^{-\frac{p}{p-1}}\int_\Omega\left(|u_m|^{\frac{p(p-1+\sigma)}{p-1}}|v_m|^{\frac{p(p+\sigma)}{p-1}}\right)^{\frac{p}{p-1}}dx\\
\le&[\mathrm{e}\,(\mathrm{p}-1)]^{-\frac{p}{p-1}}\|v_m\|_{\frac{p}{p-1}}^{\frac{p}{p-1}}+(\mathrm{e}\sigma)^{-\frac{p}{p-1}}\|u_m\|_{\frac{2p(p-1+\sigma)}{p-1}}^{\frac{p(p-1+\sigma)}{p-1}}\|v_m\|_{\frac{2p(p+\sigma)}{p-1}}^{\frac{p(p+\sigma)}{p-1}}\\
\le&[\mathrm{e}\,(\mathrm{p}-1)]^{-\frac{p}{p-1}}S_1^{\frac{p}{p-1}}\|\nabla v_m\|_2^{\frac{p}{p-1}}\\
&+(\mathrm{e}\sigma)^{-\frac{p}{p-1}}S_2^{\frac{p(p-1+\sigma)}{p-1}}S_3^{\frac{p(p+\sigma)}{p-1}}\|\nabla u_m\|_2^{\frac{p(p-1+\sigma)}{p-1}}\|\nabla v_m\|_2^{\frac{p(p+\sigma)}{p-1}}\\
\le&C,
\end{aligned} \tag{3.12}$$

where S_1 is the optimal constant of the embedding $H_0^1(\Omega)\hookrightarrow L^{\frac{p}{p-1}}(\Omega)$, S_2 is the optimal constant of the embedding $H_0^1(\Omega)\hookrightarrow L^{\frac{2p(p-1+\sigma)}{p-1}}(\Omega)$, S_3 is the optimal constant of the embedding

$H_0^1(\Omega) \hookrightarrow L^{\frac{2p(p+\sigma)}{p-1}}(\Omega)$, the positive real number σ such that $0 < \frac{2p(p-1+\sigma)}{P-1} < N^*,\ 0 < \frac{2p(p+\sigma)}{P-1} < N^*$. Similar to (3.12), we can derive that

$$\int_\Omega \left(u_m^p v_m^{p-1} \ln |u_m v_m|\right)^{\frac{p}{p-1}} dx \leq C.$$

Though the Aubin-Lions theorem, we have

$$\int_\Omega v_m^p u_m^{p-1} \ln |u_m v_m|\, dx \xrightarrow{*} \int_\Omega v^p u^{p-1} \ln |uv|\, dx, \text{ in } \mathrm{L}^\infty\left(0, +\infty; \mathrm{L}^2(\Omega)\right),$$

$$\int_\Omega u_m^p v_m^{p-1} \ln |u_m v_m|\, dx \xrightarrow{*} \int_\Omega u^p v^{p-1} \ln |uv|\, dx, \text{ in } \mathrm{L}^\infty\left(0, +\infty; \mathrm{L}^2(\Omega)\right).$$

Following the method of [30], and passing to the limit in (3.2), (3.3) and using $\lim_{n\to+\infty} \rho_n = |x|^{-s}$, we obtain

$$\left(\frac{u_t}{|x|^s}, w_j\right)_2 + a(\nabla u, \nabla w_j)_2 + b\|\nabla u\|_2^2 (\nabla u, \nabla w_j)_2 = \left(v^p u^{p-1} \ln |uv|, w_j\right)_2, \tag{3.13}$$

$$\left(\frac{v_t}{|x|^s}, w_j\right)_2 + a(\nabla v, \nabla w_j)_2 + b\|\nabla v\|_2^2 (\nabla v, \nabla w_j)_2 = \left(u^p v^{p-1} \ln |uv|, w_j\right)_2. \tag{3.14}$$

Next we show that (u, v) is a weak solution of problem (1.3). Setting a positive integer k and $m \geq k$ such that

$$\begin{cases} \phi(x,t) = \sum_{j=1}^k g_j(t) w_j(x), \\ \varphi(x,t) = \sum_{j=1}^k h_j(t) w_j(x). \end{cases}$$

Multiplying (3.13) and (3.14) by $g_j(t)$ and $h_j(t)$, respectively, and summing for j from 1 to k, we obtain

$$\left(\frac{u_t}{|x|^s}, \phi\right)_2 + a(\nabla u, \nabla \phi)_2 + b\|\nabla u\|_2^2 (\nabla u, \nabla \phi)_2 = \left(v^p u^{p-1} \ln |uv|, \phi\right)_2,$$
$$\left(\frac{v_t}{|x|^s}, \varphi\right)_2 + a(\nabla v, \nabla \varphi)_2 + b\|\nabla v\|_2^2 (\nabla v, \nabla \varphi)_2 = \left(u^p v^{p-1} \ln |uv|, \varphi\right)_2,$$

for *a.e.* $t \in [0, +\infty)$ and any $(\phi, \varphi) \in H_0^1(\Omega) \times H_0^1(\Omega)$, and $u(x,0) = u_0,\ v(x,0) = v_0$. Then from Definition 2.2, (u, v) is a global weak solution of problem (1.3).

Step 4. Algebraic decay. Finally, we show the algebraic decay of the global weak solution (u, v) of problem (1.3). Combining (2.3), (2.6) and $(u, v) \in \mathrm{W}$, we have

$$\frac{a(p-1)}{2p}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{b(p-2)}{4p}\left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4\right) + \frac{1}{p^2}\|uv\|_p^p \leq J(u,v) \leq J(u_0, v_0).$$

We can find a $\lambda_* > 1$ from Lemma 2.6 such that $I(\lambda_* u, \lambda_* v) = 0$ and

$$\lambda_*^{2p}\left(\frac{a(p-1)}{2p}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{b(p-2)}{4p}\left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4\right) + \frac{1}{p^2}\|uv\|_p^p\right) \geq J(\lambda_* u, \lambda_* v) \geq d,$$

which implies

$$\lambda_* \geq \left[\frac{d}{J\left(u_0, v_0\right)}\right]^{\frac{1}{2p}} > 1. \tag{3.15}$$

Since $I\left(\lambda_* u, \lambda_* v\right) = 0$, thus

$$\begin{aligned} I\left(\lambda_* u, \lambda_* v\right) =& a{\lambda_*}^2\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+b{\lambda_*}^4\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right) \\ & -2{\lambda_*}^{2p}\ln\lambda_*\|uv\|_2^2-2{\lambda_*}^{2p}\int_\Omega u^p v^p \ln|uv|\,dx \\ =& a\left({\lambda_*}^2-{\lambda_*}^{2p}\right)\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+a\left({\lambda_*}^4-{\lambda_*}^{2p}\right)\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right) \\ & -2{\lambda_*}^{2p}\ln\lambda_*\|uv\|_2^2+{\lambda_*}^{2p} I(u,v) \\ =& 0. \end{aligned} \tag{3.16}$$

It follows from (3.16) that

$$I(u,v) \geq a\left[1-\frac{1}{\lambda_*^{2(p-1)}}\right]\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right). \tag{3.17}$$

Then by (3.15) and (3.17), we get

$$\begin{aligned} I(u,v) &\geq a\left\{1-\left[\frac{d}{J\left(u_0, v_0\right)}\right]^{\frac{1}{p}-1}\right\}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right) \\ &\geq aC_B^{-1}\left\{1-\left[\frac{d}{J\left(u_0, v_0\right)}\right]^{\frac{1}{p}-1}\right\}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right). \end{aligned} \tag{3.18}$$

From (3.18), we obtain

$$\begin{aligned} \frac{d}{dt}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right) &= -2I(u,v) \\ &\leq -2aC_B^{-1}\left\{1-\left[\frac{d}{J\left(u_0, v_0\right)}\right]^{\frac{1}{p}-1}\right\}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right). \end{aligned} \tag{3.19}$$

By (3.19) and the Gronwall inequality, (3.1) is obtained. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let $2 < p < N^*$ and $\left(u_0, v_0\right) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Assume that $J\left(u_0, v_0\right) < d$, $I\left(u_0, v_0\right) < 0$. Then the weak solution (u,v) of problem (1.3) blows up in finite time, namely, there exists a $T > 0$ such that*

$$\lim_{t\to T^-}\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau = +\infty, \tag{3.20}$$

and an upper bound of blow up time is estimated as

$$T \leq \frac{(2p-1)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)}{2p(p-1)^2\left[d - J\left(u_0, v_0\right)\right]}. \tag{3.21}$$

Proof. The proof of Theorem 3.2 will be divided into two steps.

Step 1. Blow-up in finite time. We first show the weak solution (u,v) of problem (1.3) blows up in finite time by contradiction. Assume that a weak solution (u,v) of problem (1.3) is global existence, namely, the maximum existence time $T = +\infty$. Setting $G(t) = \int_0^t \left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right) d\tau$, then

$$G'(t) = \left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2, \tag{3.22}$$

$$G''(t) = \frac{d}{dt}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right) = 2\left(\frac{u_t}{|x|^s}, u\right)_2 + 2\left(\frac{v_t}{|x|^s}, v\right)_2 = -2I(u,v). \tag{3.23}$$

Since (u,v) satisfies (2.3), by (2.6) and (3.23), we get

$$\begin{aligned}
G''(t) &= -4pJ(u,v) + 2a(p-1)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + b(p-2)\left(\|\nabla u\|_2^4 + \|\nabla v\|_2^4\right) + \frac{4}{p}\|uv\|_p^p \\
&\geq 4p\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right) d\tau - 4pJ(u_0,v_0) + 2a(p-1)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\
&\geq 4p\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right) d\tau - 4pJ(u_0,v_0) + 2a(p-1)C_B^{-1}G'(t).
\end{aligned} \tag{3.24}$$

It is worth noting that

$$\begin{aligned}
&\left\{\int_0^t \left[\left(\frac{u_\tau}{|x|^s}, u\right)_2 + \left(\frac{v_\tau}{|x|^s}, v\right)_2\right] d\tau\right\}^2 \\
=&\left[\int_0^t \frac{1}{2}\frac{d}{d\tau}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right) d\tau\right]^2 \\
=&\left[\frac{1}{2}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right) - \frac{1}{2}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\right]^2 \\
=&\frac{1}{4}\left\{\left[G'(t)\right]^2 + \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^2 - 2G'(t)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\right\},
\end{aligned}$$

which implies

$$\left[G'(t)\right]^2 = 4\left\{\int_0^t \left[\left(\frac{u_\tau}{|x|^s}, u\right)_2 + \left(\frac{v_\tau}{|x|^s}, v\right)_2\right] d\tau\right\}^2 + 2G'(t)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)$$

$$-\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^2. \tag{3.25}$$

Combining (3.24) and (3.25), we get

$$\begin{aligned}
&G(t)\,G''(t)-p\big[G'(t)\big]^2\\
\geq&4p\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\\
&-4p\left\{\int_0^t\left[\left(\frac{u_\tau}{|x|^s},u\right)_2+\left(\frac{v_\tau}{|x|^s},v\right)_2\right]d\tau\right\}^2-4pJ(u_0,v_0)\,G(t)+2a(p-1)\,C_B^{-1}G'(t)\,G(t)\\
&-2pG'(t)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)+p\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^2.
\end{aligned} \tag{3.26}$$

By virtue of the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\\
&-\left\{\int_0^t\left[\left(\frac{u_\tau}{|x|^s},u\right)_2+\left(\frac{v_\tau}{|x|^s},v\right)_2\right]d\tau\right\}^2\\
\geq&\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\\
&-\left[\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2\right)d\tau\right]^2\\
\geq&\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\\
&-\left(\int_0^t\sqrt{\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2}\sqrt{\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2}\,d\tau\right)^2\\
\geq&0.
\end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.26), and combining with $\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^2\geq 0$, there holds

$$\begin{aligned}
&G(t)\,G''(t)-p\big[G'(t)\big]^2\\
\geq&-4pJ(u_0,v_0)\,G(t)+2a(p-1)\,C_B^{-1}G'(t)\,G(t)-2pG'(t)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right).
\end{aligned} \tag{3.28}$$

We can rewrite (3.28) as

$$G(t)G''(t)-p\big[G'(t)\big]^2 \ge \big[a(p-1)C_B^{-1}G'(t)-4pJ(u_0,v_0)\big]G(t) + \left[a(p-1)C_B^{-1}G(t)-2p\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\right]G'(t). \quad (3.29)$$

From $J(u_0,v_0)<d$, $I(u_0,v_0)<0$, it yields $(u_0,v_0)\in \mathrm{V}$. If $(u_0,v_0)\in \mathrm{V}$, then $(u,v)\in \mathrm{V}$ by contradiction. If it is false, let (u,v) leaves V at time t_0, then there exists a sequence $\{t_n\}$ such that $I(u(t_n),v(t_n))\le 0$ when $t_n\to t_0$. By the lower semicontinuity of $H_0^1(\Omega)$, we obtain

$$I(u(t_0),v(t_0))\le \lim_{n\to\infty}\inf I(u(t_n),v(t_n))\le 0.$$

Since $(u(t_0),v(t_0))\notin \mathrm{V}$, then we have $I(u(t_0),v(t_0))=0$. By the similar method, there holds $J(u(t_0),v(t_0))=d$. However, by the definition of d, it leads to a contradiction

$$d=\inf_{(u,v)\in\mathcal{N}}J(u,v)\le J(u_0,v_0)<d.$$

So $(u,v)\in \mathrm{V}$ provided $(u_0,v_0)\in \mathrm{V}$. By (2.11), for $I(u_0,v_0)<0$, one has

$$-2I(u,v)>4p\left[d-J(u,v)\right], \quad (3.30)$$

and together with (2.3), (3.23), we have

$$G''(t)=-2I(u,v)>4p\left[d-J(u,v)\right]\ge 4p\left[d-J(u_0,v_0)\right]>0. \quad (3.31)$$

It follows from (3.31) that for any $t\ge 0$,

$$G'(t)\ge 4p\left[d-J(u_0,v_0)\right]t+G'(0)\ge 4p\left[d-J(u_0,v_0)\right]t, \quad (3.32)$$

$$G(t)\ge 2p\left[d-J(u_0,v_0)\right]t^2+G(0)=2p\left[d-J(u_0,v_0)\right]t^2. \quad (3.33)$$

Combining (3.32) and (3.33) for sufficiently large t, there holds

$$a(p-1)C_B^{-1}G'(t)-4pJ(u_0,v_0)>0, \quad (3.34)$$

$$a(p-1)C_B^{-1}G(t)-2p\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)>0. \quad (3.35)$$

By (3.29)-(3.35), we get $G(t)G''(t)-p[G'(t)]^2>0$. Then,

$$\big[G^{1-p}(t)\big]''=(1-p)G^{-p-1}(t)\left\{G(t)G''(t)-p\big[G'(t)\big]^2\right\}<0,$$

there exists a finite time $T>0$ such that

$$\lim_{t\to T^-}G^{1-p}(t)=\lim_{t\to T^-}\frac{1}{\left[\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\right]^{p-1}}=0,$$

which implies

$$\lim_{t\to T^-}\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau=+\infty,$$

this contradicts with (u,v) is a global weak solution of problem (1.3), hence (u,v) blows up in finite time.

Step 2. Upper bound estimate of blow-up time. For any $T^*\in(0,T)$, we define an auxiliary function $M:[0,T^*]\to R$ by

$$M\left(t\right)=\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau+(T^*-t)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)+\beta(t+\alpha)^2,$$

where $t\in[0,T^*]$, $\frac{\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2}{2(p-1)\beta}<\alpha<+\infty,\ 0<\beta\le\frac{2p[d-J(u_0,v_0)]}{2p-1}$. Through a direct calculation, we have

$$\begin{aligned}M'\left(t\right)&=\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2-\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)+2\beta\left(t+\alpha\right)\\&=\int_0^t\frac{d}{d\tau}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau+2\beta\left(t+\alpha\right)\\&=2\int_0^t\left[\left(\frac{u_\tau}{|x|^s},u\right)_2+\left(\frac{v_\tau}{|x|^s},v\right)_2\right]d\tau+2\beta\left(t+\alpha\right),\end{aligned}\tag{3.36}$$

$$M''\left(t\right)=2\left(\frac{u_t}{|x|^s},u\right)_2+2\left(\frac{v_t}{|x|^s},v\right)_2+2\beta=-2I\left(u,v\right)+2\beta.\tag{3.37}$$

By virtue of (2.2) and (3.37), we obtain

$$\begin{aligned}M''\left(t\right)&\ge4p\left[d-J\left(u,v\right)\right]+2\beta\\&\ge4p\left[d-J\left(u_0,v_0\right)\right]+4p\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau+2\beta.\end{aligned}\tag{3.38}$$

By (3.36), the Cauchy-Schwarz inequality and the Hölder inequality, we have

$$\begin{aligned}&\left[M'\left(t\right)\right]^2\\&=4\left\{\int_0^t\left[\left(\frac{u_t}{|x|^s},u\right)_2+\left(\frac{v_t}{|x|^s},v\right)_2\right]d\tau+\beta\left(t+\alpha\right)\right\}^2\\&\le4\left[\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2\right)d\tau+\beta\left(t+\alpha\right)\right]^2\\&\le4\left\{\left[\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\right]^{\frac{1}{2}}\left[\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\right]^{\frac{1}{2}}+\beta\left(t+\alpha\right)\right\}^2\end{aligned}\tag{3.39}$$

$$\le 4\left[\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau+\beta\right]\left[\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau+\beta(t+\alpha)^2\right]$$

$$\le 4M\left(t\right)\left[\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau+\beta\right]. \tag{3.40}$$

Combining (3.38) and (3.39), there holds

$$M\left(t\right)M''\left(t\right)-p\big[M'\left(t\right)\big]^2\ge M\left(t\right)\left\{4p\left[d-J\left(u_0,v_0\right)\right]-2\left(2p-1\right)\beta\right\}. \tag{3.41}$$

Because of $0<\beta\le\frac{2p[d-J(u_0,v_0)]}{2p-1}$, so by (3.41),

$$M\left(t\right)M''\left(t\right)-p\big(M'\left(t\right)\big)^2>0.$$

Hence by Lemma 2.2 for $t_0=0$, we can deduce that

$$T^*\le\frac{\beta\alpha^2}{2\left(p-1\right)\beta\alpha-\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)}. \tag{3.42}$$

In order to estimate the upper bound of blow up time, we consider a minimizing problem

$$T^*\le\inf_{(\alpha,\beta)\in\Phi}\frac{\beta\alpha^2}{2\left(p-1\right)\beta\alpha-\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)}, \tag{3.43}$$

where

$$\Phi=\left\{(\beta,\alpha)\,\middle|\,0<\beta\le\frac{2p\left[d-J\left(u_0,v_0\right)\right]}{\left(2p-1\right)},\ \frac{\left(2p-1\right)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)}{4p\left(p-1\right)\left[d-J\left(u_0,v_0\right)\right]}\le\alpha<+\infty\right\}.$$

Minimizing the right hand side of (3.43) for $(\alpha,\beta)\in\Phi$, we obtain

$$T^*\le\inf_{\alpha\in\left[\frac{(2p-1)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)}{4p(p-1)[d-J(u_0,v_0)]},+\infty\right)}\frac{2p\alpha^2\left[d-J\left(u_0,v_0\right)\right]}{4p\left(p-1\right)\alpha\left[d-J\left(u_0,v_0\right)\right]-\left(2p-1\right)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)}$$

$$=\frac{\left(2p-1\right)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)}{2p(p-1)^2\left[d-J\left(u_0,v_0\right)\right]}. \tag{3.44}$$

Hence (3.21) holds from (3.44) by virtue of arbitrariness of $T^*\to T$. The proof of Theorem 3.2 is complete. □

4. Critical initial energy $J(u_0, v_0) = d$

In this section, we generalize the previous section results from the subcritical initial energy $J(u_0, v_0) < d$ to the critical initial energy $J(u_0, v_0) = d$.

Theorem 4.1. *Let $2 < p < N^*$ and $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Assume that $J(u_0, v_0) = d$, $I(u_0, v_0) > 0$. Then the weak solution (u, v) of problem (1.3) is global existence.*

Proof. For $m = 2, 3, \cdots$, we define $\varsigma_m := 1 - \frac{1}{m}$, $(u_{0m}, v_{0m}) := (\varsigma_m u_0, \varsigma_m v_0)$, then $0 < \varsigma_m < 1$ and $\varsigma_m \to 1$ as $m \to +\infty$. Therefore, we consider the following problem:

$$\begin{cases} \dfrac{u_{mt}}{|x|^s} - \left(a + b\|\nabla u_m\|_2^2\right)\Delta u_m = v_m^p u_m^{p-1} \ln|u_m v_m|, x \in \Omega, \quad t > 0, \\ \dfrac{v_{mt}}{|x|^s} - \left(a + b\|\nabla v_m\|_2^2\right)\Delta v_m = u_m^p v_m^{p-1} \ln|u_m v_m|, \ x \in \Omega, \quad t > 0, \\ u_m(x,t) = v_m(x,t) = 0, \qquad x \in \partial\Omega, t > 0, \\ u_m(x,0) = u_{0m}, v_m(x,0) = v_{0m}, \qquad x \in \Omega. \end{cases} \tag{4.1}$$

According to $I(u_0, v_0) > 0$ and Lemma 2.6, there exists a $\lambda_* > 1$ such that $I(\lambda_* u_0, \lambda_* v_0) = 0$. Then, from $\varsigma_m < 1 < \lambda_*$, we have

$$I(u_{0m}, v_{0m}) = I(\varsigma_m u_0, \varsigma_m v_0) > 0, \ J(u_{0m}, v_{0m}) = J(\varsigma_m u_0, \varsigma_m v_0) < J(u_0, v_0) = d.$$

By Theorem 3.1, for each m in problem (4.1), there exists a corresponding global solution $(u_m, v_m) \in L^\infty\left(0, +\infty; H_0^1(\Omega)\right) \times L^\infty\left(0, +\infty; H_0^1(\Omega)\right)$, and its initial data

$$\begin{aligned} u_m(0) = u_{0m} \to u_0, \ \text{in } H_0^1(\Omega) \ \text{as } m \to +\infty, \\ v_m(0) = v_{0m} \to v_0, \ \text{in } H_0^1(\Omega) \ \text{as } m \to +\infty. \end{aligned}$$

Therefore, there exists $(u_m, v_m) \in \mathrm{W}$ for all $t \in [0, +\infty)$,

$$\begin{aligned} \left(\frac{u_{mt}}{|x|^s}, \phi\right)_2 + \left(a + b\|\nabla u_m\|_2^2\right)(\nabla u_m, \nabla\phi)_2 = \left(v_m^p u_m^{p-1}\ln|u_m v_m|, \phi\right)_2, \\ \left(\frac{v_{mt}}{|x|^s}, \varphi\right)_2 + \left(a + b\|\nabla v_m\|_2^2\right)(\nabla v_m, \nabla\varphi)_2 = \left(u_m^p v_m^{p-1}\ln|u_m v_m|, \varphi\right)_2, \end{aligned}$$

and

$$\int_0^t \left(\left\| \frac{u_{m\tau}}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_{m\tau}}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau + J(u_m, v_m) = J(u_{0m}, v_{0m}) < d.$$

From (2.6), we can derive

$$\begin{aligned} J(u_m, v_m) =& \frac{1}{2p} I(u_m, v_m) + \frac{a(p-1)}{2p}\left(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2\right) + \frac{b(p-2)}{4p}\left(\|\nabla u_m\|_2^4 + \|\nabla v_m\|_2^4\right) \\ &+ \frac{1}{p^2}\|u_m v_m\|_p^p, \end{aligned}$$

and thus we can obtain

$$\int_0^t \left(\left\| \frac{u_{m\tau}}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_{m\tau}}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau + \frac{a(p-1)}{2p}\left(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2\right)$$

$$+\frac{b\left(p-2\right)}{4p}\left(\left\|\nabla u_m\right\|_2^4+\left\|\nabla v_m\right\|_2^4\right)+\frac{1}{p^2}\left\|u_m v_m\right\|_p^p<d,$$

for all $t\in[0,+\infty)$. By the same processes as the proof of Theorem 3.1, the proof of Theorem 4.1 is complete. □

Theorem 4.2. *Let $2<p<N^*$ and $(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$. Assume that $J(u_0,v_0)=d$, $I(u_0,v_0)<0$. Then the weak solution (u,v) of problem (1.3) blows up in finite time.*

Proof. We prove it by contradiction. Assume that (u,v) is the global weak solution of problem (1.3), we take the function $G(t)$ as in Theorem 3.2. Combining (3.28) and $J(u_0,v_0)=d$, $I(u_0,v_0)<0$, we can obtain

$$\begin{aligned}G(t)G''(t)-p\left[G'(t)\right]^2\geq{}&\left[a(p-1)C_B^{-1}G'(t)-4pd\right]G(t)\\&+\left[a(p-1)C_B^{-1}G(t)-2p\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\right]G'(t).\end{aligned}\tag{4.2}$$

Since $J(u,v)$ and $I(u,v)$ depend on the continuity of t, we know that there exists a sufficiently small t_0 such that $J(u(t_0),v(t_0))>0$ and $I(u,v)<0$ for all $t\in(0,t_0)$. According to $\left(\frac{u_t}{|x|^s},u\right)_2+\left(\frac{v_t}{|x|^s},v\right)_2=-I(u,v)$, we have

$$\left(\frac{u_t}{|x|^s},u\right)_2+\left(\frac{v_t}{|x|^s},v\right)_2>0,$$

and

$$\left\|\frac{u_t}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_t}{|x|^{\frac{s}{2}}}\right\|_2^2>0,\ \forall t\in[0,t_0].$$

From (2.3), we obtain

$$0<J(u(t_0),v(t_0))\leq d-\int_0^t\left(\left\|\frac{u_t}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_t}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau<d,$$

thus, if we take the initial time $t=t_0$, we can obtain $(u(t_0),v(t_0))\in \mathrm{V}$. From Lemma 2.8 (ii), we have $I(u,v)<0$ for all $t\in[t_0,+\infty)$. By Lemma 2.7 and (2.3), we can obtain

$$G''(t)=-2I(u,v)>4p(d-J(u,v))\geq 4p(d-J(u(t_0),v(t_0))):=M,\ \forall t\in[t_0,+\infty),$$

and

$$\begin{aligned}&G'(t)\geq M(t-t_0)+G'(t_0)=M(t-t_0),\ \forall t\in[t_0,+\infty),\\&G(t)\geq\frac{1}{2}Mt^2-Mt_0t+G(t_0),\forall t\in[t_0,+\infty).\end{aligned}$$

Moreover, for a sufficiently large t, it is possible to make

$$a(p-1)C_B^{-1}G'(t)>4pd,\tag{4.3}$$

$$a\left(p-1\right)C_B^{-1}G\left(t\right)>2p\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right). \tag{4.4}$$

Therefore, from (4.2)-(4.4), we have

$$G\left(t\right)G''\left(t\right)-p\big(G'\left(t\right)\big)^2>0,$$

for sufficiently large t. By the same processes as the proof of Theorem 3.2, there exists a sufficiently large $T>0$ such that

$$\lim_{t\to T^-}\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau=+\infty.$$

The proof of Theorem 4.2 is complete. □

5. High initial energy $J\left(u_0,v_0\right)>d$

In this section, we will continue to prove the finite-time blow-up of weak solutions in the case of arbitrarily high initial energy $J\left(u_0,v_0\right)>d$, which can be regarded as a generalization of the subcritical initial energy $J\left(u_0,v_0\right)<d$ and the critical initial energy $J\left(u_0,v_0\right)=d$. Since the initial energy in this case is no longer constrained by the depth d of the potential well, the potential well method is no longer applicable to this situation. Due to these characteristics, the invariant manifold will not hold. Therefore, we need to find three new lemmas to play the role of the invariant manifold.

Lemma 5.1. *Let $2<p<N^*$. Then $dist\left(0,\mathcal{N}\right)>0$ and $dist\left(0,\mathcal{N}_-\right)>0$.*

Proof. We first prove that $dist\left(0,\mathcal{N}\right)>0$. For $(u,v)\in\mathcal{N}$, based on the definition of d, it yields

$$\begin{aligned}
d\leq&\frac{a}{2}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{b}{4}\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)+\frac{1}{p^2}\left\|uv\right\|_p^p-\frac{1}{p}\int_\Omega u^pv^p\ln|uv|dx\\
=&\frac{a}{2}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{b}{4}\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)-\frac{1}{2p}\Big[a\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\\
&+b\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)\Big]+\frac{1}{p^2}\left\|uv\right\|_p^p\\
\leq&\frac{a\left(p-1\right)}{2p}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{b\left(p-2\right)}{4p}\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)+\frac{1}{2p^2}\left(\|u\|_{2p}^{2p}+\|v\|_{2p}^{2p}\right)\\
\leq&\frac{a\left(p-1\right)}{2p}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{b\left(p-2\right)}{4p}\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)+\frac{S_4^{2p}}{2p^2}\left(\|\nabla u\|_2^{2p}+\|\nabla v\|_2^{2p}\right),
\end{aligned}$$

where S_4 is the optimal constant of the embedding $H_0^1\left(\Omega\right)\hookrightarrow L^{2p}\left(\Omega\right)$. Therefore, according to the above inequality, there exists a constant $\tilde{C}>0$ such that

$$dist\left(0,\mathcal{N}\right)=\min_{(u,v)\in\mathcal{N}}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\geq\tilde{C}>0.$$

Next, we prove that $dist\,(0,\mathcal{N}_-)>0$. For $(u,v)\in\mathcal{N}_-$, we have $I\,(u,v)<0$ and $\|\nabla u\|_2\neq 0,\ \|\nabla v\|_2\neq 0$. Combining with (2.9), we obtain

$$\begin{aligned}
a\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right) &< a\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+b\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)\\
&< 2\int_\Omega u^pv^p\ln|uv|dx\\
&\le \frac{1}{\mathrm{e}\sigma}S^{2(p+\sigma)}\|\nabla u\|_2^{2(p+\sigma)}+\frac{1}{\mathrm{e}\sigma}S^{2(p+\sigma)}\|\nabla v\|_2^{2(p+\sigma)}\\
&\le \frac{1}{\mathrm{e}\sigma}S^{2(p+\sigma)}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)^{p+\sigma},
\end{aligned}$$

which implies

$$\|\nabla u\|_2^2+\|\nabla v\|_2^2\ge\left(\frac{a\mathrm{e}\sigma}{S^{2(p+\sigma)}}\right)^{\frac{1}{p+\sigma-1}}.$$

Furthermore, it follows from $S>0$ and the above inequality that

$$dist\,(0,\mathcal{N}_-)=\min_{(u,v)\in\mathcal{N}_-}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)>0.$$

The proof is complete. □

Lemma 5.2. *Let $2<p<N^*$, $(u,v)\in J^c\cap\mathcal{N}_+$ for any $c>d$. Then the weak solution (u,v) is uniformly bounded in $H_0^1(\Omega)\times H_0^1(\Omega)$.*

Proof. By $(u,v)\in J^c\cap\mathcal{N}_+$ for any $c>d$, we have $J\,(u,v)<c,\ I\,(u,v)>0$ and $\|\nabla u\|_2^2\neq 0,\ \|\nabla v\|_2^2\neq 0$. According to (2.4) and (2.5), we have

$$\begin{aligned}
J\,(u,v)&=\frac{a}{4}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{1}{p^2}\|uv\|_p^p+\frac{p-2}{2p}\int_\Omega u^pv^p\ln|uv|\,dx+\frac{1}{4}I\,(u,v)\\
&>\frac{a}{4}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{p-2}{2p}\int_\Omega u^pv^p\ln|uv|\,dx\\
&>\frac{a}{4}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)-\frac{|\Omega|}{2\mathrm{ep}},
\end{aligned}$$

which yields

$$\|\nabla u\|_2^2+\|\nabla v\|_2^2<\frac{4c}{a}+\frac{2\,|\Omega|}{aep}.$$

The proof is complete. □

Lemma 5.3. *Let $2<p<N^*$. Then $0<\lambda_c\le\Lambda_c<+\infty$ for any $c>d$.*

Proof. For any $c>d$, it follows from the Nirenberg inequality and the Cauchy-Schwarz that

$$\begin{aligned}
&\|\nabla u\|_2^2+\|\nabla v\|_2^2\\
<&\frac{2}{a}\int_\Omega u^pv^p\ln|uv|dx\\
\le&\frac{1}{ae\sigma}S^{2(p+\sigma)}\left(\|u\|_{2(p+\sigma)}^{2(p+\sigma)}+\|v\|_{2(p+\sigma)}^{2(p+\sigma)}\right)\\
\le&\frac{1}{ae\sigma}S^{2(p+\sigma)}C_N^{2(p+\sigma)}\left(\|\nabla u\|_2^{2\theta(p+\sigma)}\|u\|_2^{2(1-\theta)(p+\sigma)}+\|\nabla v\|_2^{2\theta(p+\sigma)}\|v\|_2^{2(1-\theta)(p+\sigma)}\right)
\end{aligned}$$

$$\leq \frac{1}{ae\sigma} S^{2(p+\sigma)} C_N^{2(p+\sigma)} \left[\left(\|\nabla u\|_2^{4\theta(p+\sigma)} + \|\nabla v\|_2^{4\theta(p+\sigma)} \right)^{\frac{1}{2}} \left(\|u\|_2^{4(1-\theta)(p+\sigma)} + \|v\|_2^{4(1-\theta)(p+\sigma)} \right)^{\frac{1}{2}} \right]$$
$$\leq \frac{1}{ae\sigma} S^{2(p+\sigma)} C_N^{2(p+\sigma)} \left[\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right)^{\theta(p+\sigma)} \left(\|u\|_2^2 + \|v\|_2^2 \right)^{(1-\theta)(p+\sigma)} \right]. \tag{5.1}$$

By Lemma 5.1, we derive $\|u\|_2^2 + \|v\|_2^2 > 0$. Combining with (5.1) and $|x| \leq L$, it yields

$$\frac{1}{2} \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) > \frac{1}{2L^s} \left(\|u\|_2^2 + \|v\|_2^2 \right) > 0,$$

which implies $\lambda_c > 0$. Furthermore, from the definition of $\mathcal{N}_c$ and $(u, v) \in \mathcal{N}_c$, we have

$$\frac{1}{2} \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) \leq \frac{C_B}{2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) < \frac{pcC_B}{a\,(p-1)},$$

which yields $\Lambda_c < +\infty$. Consequently, we conclude from above analysis that

$$0 < \lambda_c \leq \Lambda_c < +\infty.$$

The proof is complete. □

Theorem 5.1. *Let $2 < p < N^*$, for any $c > d$.*
(i) *If $(u_0, v_0) \in \Gamma_c$, then the weak solution (u, v) of problem (1.3) exists globally and $(u, v) \to (0, 0)$ as $t \to +\infty$.*
(ii) *If $(u_0, v_0) \in \Pi_c$, then the weak solution (u, v) of problem (1.3) blows up in finite time.*
Here

$$\Gamma_c = \mathcal{N}_+ \cap \left\{ (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \left| \frac{1}{2} \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) < \lambda_c, d < J(u, v) \leq c \right. \right\},$$
$$\Pi_c = \mathcal{N}_- \cap \left\{ (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \left| \frac{1}{2} \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) > \Lambda_c, \ d < J(u, v) \leq c \right. \right\}.$$

Proof. (i) If $(u_0, v_0) \in \Gamma_c$, then, based on the monotonicity of λ_c, we can know that $(u_0, v_0) \in \mathcal{N}_+$, $d < J(u_0, v_0) \leq c$ and

$$\frac{1}{2} \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_0}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) < \lambda_c \leq \lambda_{J(u_0, v_0)}. \tag{5.2}$$

We assert that $(u, v) \in \mathcal{N}_+$. If it is false, then there exists a $t_0 \in (0, T)$ such that $(u, v) \in \mathcal{N}_+$ for all $t \in [0, t_0)$ and $(u(t_0), v(t_0)) \in \mathcal{N}$. According to (3.19), we have

$$\frac{1}{2} \frac{d}{dt} \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) = -I(u, v). \tag{5.3}$$

From the definition of $\mathcal{N}_+$ and (5.3), we can conclude that $\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2$ is strictly monotonically decreasing in $[0,t_0)$. Furthermore, from (2.3), we can know that $J(u,v)$ is not increasing with respect to t. Therefore, $J(u(t_0),v(t_0)) \le J(u_0,v_0)$ for all $t\in[0,T)$. From $(u(t_0),v(t_0))\in\mathcal{N}$, we obtain $(u(t_0),v(t_0))\in\mathcal{N}_{J(u_0,v_0)}$. Review (5.2) leads us to conclude that

$$\frac{1}{2}\left(\left\|\frac{u(t_0)}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v(t_0)}{|x|^{\frac{s}{2}}}\right\|_2^2\right)<\frac{1}{2}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)<\lambda_{J(u_0,v_0)}. \tag{5.4}$$

However, according to the definition of $\lambda_{J(u_0,v_0)}$, we know that

$$\lambda_{J(u_0,v_0)}=\inf\left\{\frac{1}{2}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\middle|(u,v)\in\mathcal{N}_{J(u_0,v_0)}\right\}\le\frac{1}{2}\left(\left\|\frac{u(t_0)}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v(t_0)}{|x|^{\frac{s}{2}}}\right\|_2^2\right),$$

which contradicts with (5.4). Therefore, this assertion is proven, that is, $(u,v)\in\mathcal{N}_+$ and $(u,v)\in J^{J(u_0,v_0)}$ for all $t\in[0,T)$. It means that $(u,v)\in J^{J(u_0,v_0)}\cap\mathcal{N}_+$ for all $t\in[0,T)$. By Lemma 5.2, we know that (u,v) is uniformly bounded in $H_0^1(\Omega)\times H_0^1(\Omega)$. So, the maximal existence time $T=+\infty$. Thus, $\omega-$limit set is non-empty.

Next, for any $(\omega_1,\omega_2)\in\omega(u_0,v_0)=\bigcap_{t\ge0}\overline{\{(u(s),v(s)):s\ge t\}}$, by the above discussion, we obtain $J(\omega_1,\omega_2)\le J(u_0,v_0)$ and $\frac{1}{2}\left(\left\|\frac{\omega_1}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{\omega_2}{|x|^{\frac{s}{2}}}\right\|_2^2\right)<\lambda_{J(u_0,v_0)}$. Then, we can know $(\omega_1,\omega_2)\in J^{J(u_0,v_0)}$, $(\omega_1,\omega_2)\notin\mathcal{N}_{J(u_0,v_0)}$. Also, since $\mathcal{N}_{J(u_0,v_0)}=\mathcal{N}\cap J^{J(u_0,v_0)}$, we obtain $(\omega_1,\omega_2)\notin\mathcal{N}$. Therefore, $\omega(u_0,v_0)\cap\mathcal{N}=\emptyset$, and since $\mathcal{N}$ contains the non-trivial solutions of problem (1.3), it follows that $\omega(u_0,v_0)=(0,0)$, *i.e.* $(u,v)\to(0,0)$ as $t\to+\infty$.

(ii) If $(u_0,v_0)\in\Pi_c$, then from the definition of Λ_c. we can know $(u_0,v_0)\in\mathcal{N}_-$, and $d<J(u,v)\le c$. Considering the monotonicity of Λ_c, we can conclude that

$$\frac{1}{2}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)>\Lambda_c\ge\Lambda_{J(u_0,v_0)}. \tag{5.5}$$

We assert that $(u,v)\in\mathcal{N}_-$. If it is false, then there exists a $t_0\in(0,T)$ such that $(u,v)\in\mathcal{N}_-$ for all $t\in[0,t_0)$ and $(u(t_0),v(t_0))\in\mathcal{N}$. From the definition of $\mathcal{N}_-$ and (5.3), we obtain the property that $\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2$ is strictly increasing on $[0,t_0)$, and thus we get

$$\frac{1}{2}\left(\left\|\frac{u(t_0)}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v(t_0)}{|x|^{\frac{s}{2}}}\right\|_2^2\right)>\frac{1}{2}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)>\Lambda_{J(u_0,v_0)}, \tag{5.6}$$

$$J(u(t_0),v(t_0))\le J(u_0,v_0). \tag{5.7}$$

Based on (5.6) and (5.7), and $(u(t_0),v(t_0))\in\mathcal{N}$, it follows that $(u(t_0),v(t_0))\in\mathcal{N}_{J(u_0,v_0)}$. Consequently, by the definition of $\Lambda_{J(u_0,v_0)}$, we have

$$\Lambda_{J(u_0,v_0)}=\sup\left\{\frac{1}{2}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\middle|(u,v)\in N_{J(u_0,v_0)}\right\}\ge\frac{1}{2}\left(\left\|\frac{u(t_0)}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v(t_0)}{|x|^{\frac{s}{2}}}\right\|_2^2\right),$$

which contradicts with (5.6). Therefore, we can conclude $(u,v) \in \mathcal{N}_-$ and $(u,v) \in J^{J(u_0,v_0)}$ for all $t \in [0,T)$, which implies $(u,v) \in \mathcal{N}_- \cap J^{J(u_0,v_0)}$ for all $t \in [0,T)$.

Subsequently, since (u,v) exists globally, the maximal existence time $T = +\infty$. For every $(\omega_3,\omega_4) \in \omega(u_0,v_0)$, through the above analysis, we derive $J(\omega_3,\omega_4) \le J(u_0,v_0)$ and

$$\frac{1}{2}\left(\left\|\frac{\omega_3}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{\omega_4}{|x|^{\frac{s}{2}}}\right\|_2^2\right) > \Lambda_{J(u_0,v_0)}.$$

It follows that $(\omega_3,\omega_4) \in J^{J(u_0,v_0)}$ and $(\omega_3,\omega_4) \notin \mathcal{N}_{J(u_0,v_0)}$. Furthermore, when $\mathcal{N}_{J(u_0,v_0)} = \mathcal{N} \cap J^{J(u_0,v_0)}$, $(\omega_3,\omega_4) \notin \mathcal{N}$, we obtain $\omega(u_0,v_0) \cap \mathcal{N} = \emptyset$. However, according to Lemma 5.1, we know that $(0,0) \notin \omega(u_0,v_0)$. Consequently, $\omega(u_0,v_0) = \emptyset$, which contradicts with the global existence of (u,v). Hence, the maximal existence time $T < +\infty$. The proof is complete. □

Corollary 5.1. *Let $2 < p < N^*$ and $J(u_0,v_0) > d$. If*

$$J(u_0,v_0) + \frac{|\Omega|}{2\mathrm{ep}} \le \frac{a}{4C_B}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right),$$

then $(u_0,v_0) \in \mathcal{N}_-$ and the weak solution (u,v) of problem (1.3) blows up in finite time.

Proof. From (2.4), (2.5), (2.7) and Lemma 2.1, it follows that

$$\begin{aligned}
J(u_0,v_0) &= \frac{1}{4}I(u_0,v_0) + \frac{a}{4}\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) + \frac{1}{p^2}\|u_0v_0\|_p^p + \frac{p-2}{2p}\int_\Omega u^p v^p \ln|uv|\,dx \\
&> \frac{1}{4}I(u_0,v_0) + \frac{a}{4}\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) - \frac{|\Omega|}{2\mathrm{ep}} \\
&\ge \frac{1}{4}I(u_0,v_0) + \frac{a}{4C_B}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right) - \frac{|\Omega|}{2\mathrm{ep}} \\
&\ge \frac{1}{4}I(u_0,v_0) + J(u_0,v_0),
\end{aligned}$$

which gives $I(u_0,v_0) < 0$, i.e. $(u_0,v_0) \in \mathcal{N}_-$.

To prove that (u,v) blows up in finite time, it suffices to demonstrate

$$\frac{1}{2}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right) > \Lambda_{J(u_0,v_0)}.$$

Hence, for any $(u,v) \in \mathcal{N}_{J(u_0,v_0)}$, we have

$$\begin{aligned}
\frac{1}{2}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right) &\le \frac{C_B}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\
&< \frac{pC_B}{a(p-1)}J(u_0,v_0) \\
&\le \frac{p}{4(p-1)}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right) - \frac{C_B|\Omega|}{2\mathrm{ea}(\mathrm{p}-1)}
\end{aligned}$$

$$< \frac{1}{2}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right).$$

Taking supremum over $\mathcal{N}_{J(u_0,v_0)}$, we get

$$\Lambda_{J(u_0,v_0)} \leq \frac{pC_B}{a\,(p-1)} J\,(u_0,v_0) \leq \frac{1}{2}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right).$$

By Lemma 5.1 (ii), the proof is complete. □

Theorem 5.2. *For any $R > d$, there exists a weak solution $(u_R, v_R) \in \mathcal{N}_-$ of problem (1.3) such that $J\,(u_R, v_R) \geq R$ and (u_R, v_R) blows up in finite time.*

Proof. For any $R > d$, we select Ω_a and Ω_b be two arbitrary disjoint open subdomains of Ω. And we assume that $(\tilde{u}, \tilde{v}) \in H_0^1\,(\Omega_a) \times H_0^1\,(\Omega_a)$ is an arbitrary nontrivial function, then we choose $\theta > 0$ large sufficiently such that

$$\left\|\frac{\theta\tilde{u}}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{\theta\tilde{v}}{|x|^{\frac{s}{2}}}\right\|_2^2 > \frac{4C_B}{a}\left(R+\frac{|\Omega|}{2\mathrm{ep}}\right).$$

Fixing θ and choosing a function $(\hat{u}, \hat{v}) \in H_0^1\,(\Omega_b) \times H_0^1\,(\Omega_b)$ such that $J\,(\theta\tilde{u}, \theta\tilde{v}) + J\,(\hat{u}, \hat{v}) = R$. Next, we extend $(\tilde{u}, \tilde{v})$ and $(\hat{u}, \hat{v})$ to

$$(\tilde{u},\tilde{v}) = \begin{cases} (\tilde{u},\tilde{v})\,, x \in \Omega_a, \\ 0, \qquad x \in \Omega\backslash\Omega_a, \end{cases} \qquad (\hat{u},\hat{v}) = \begin{cases} (\hat{u},\hat{v})\,, x \in \Omega_b, \\ 0, \qquad x \in \Omega\backslash\Omega_b, \end{cases}$$

and let $(u_R, v_R) = (\theta\tilde{u}, \theta\tilde{v}) + (\hat{u}, \hat{v})$. Then, we have

$$J\,(u_R, v_R) = J\,(\theta\tilde{u}, \theta\tilde{v}) + J\,(\hat{u}, \hat{v}) = R > d,$$

and

$$\left\|\frac{u_R}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_R}{|x|^{\frac{s}{2}}}\right\|_2^2 \geq \left\|\frac{\theta\tilde{u}}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{\theta\tilde{v}}{|x|^{\frac{s}{2}}}\right\|_2^2 > \frac{4C_B}{a}\left(J\,(u_R,v_R)+\frac{|\Omega|}{2\mathrm{ep}}\right).$$

Therefore, by Corollary 5.1, the proof is complete. □

6. Blow-up time estimates independent of d

The blow-up results in previous studies are closely related to the depth d of the potential well, but the value of d is very small, making it difficult to calculate precisely. Thus, by the concavity method, in this section, we prove the blow-up properties of weak solutions to problem (1.3) under different energy conditions without relying on d, and estimate the upper bound of blow-up time.

Theorem 6.1. *Let $p > 2$ and $(u, v) \in H_0^1\,(\Omega) \times H_0^1\,(\Omega) \setminus \{0, 0\}$. If $J\,(u_0, v_0) < 0$, then the weak solution (u, v) of problem (1.3) blows up in finite time with the upper bound of blow-up time is*

$$T \leq \frac{\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2}{4p\,(1-p)\,J\,(u_0,v_0)},$$

and

$$\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\le\left[\frac{\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^p}{4p\left(1-p\right)J\left(u_0,v_0\right)}\right]^{\frac{1}{p-1}}\left(T-t\right)^{-\frac{1}{p-1}}. \tag{6.1}$$

Proof. We define two functions, respectively,

$$\psi\left(t\right):=-4pJ\left(u,v\right),\ \eta\left(t\right):=\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2.$$

By a direct calculation, we obtain

$$\psi'\left(t\right)=-4p\frac{d}{dt}J\left(u,v\right)=4p\left(\left\|\frac{u_t}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_t}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\ge 0$$

and

$$\eta'\left(t\right)=\frac{d}{dt}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)=2\left(\frac{u_t}{|x|^s},u\right)_2+2\left(\frac{v_t}{|x|^s},v\right)_2=-2I\left(u,v\right).$$

Thus, using the Hölder inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\eta\left(t\right)\psi'\left(t\right)&=4p\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\left(\left\|\frac{u_t}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_t}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\\&\ge 4p\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2\left\|\frac{u_t}{|x|^{\frac{s}{2}}}\right\|_2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2\left\|\frac{v_t}{|x|^{\frac{s}{2}}}\right\|_2\right)^2\\&\ge p\left[2\left(\frac{u_t}{|x|^s},u\right)_2+2\left(\frac{v_t}{|x|^s},v\right)_2\right]^2\\&=p\left[\eta'\left(t\right)\right]^2.\end{aligned} \tag{6.2}$$

Additionally, we observe that

$$\begin{aligned}\eta'\left(t\right)&=-2I\left(u,v\right)\\&=-4pJ\left(u,v\right)+2a\left(p-1\right)\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+b\left(p-2\right)\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)+\frac{4}{p}\|uv\|_p^p\\&\ge -4pJ\left(u,v\right)\\&=\psi\left(t\right).\end{aligned} \tag{6.3}$$

Therefore, combining (6.2) and (6.3), we derive $\eta\left(t\right)\psi'\left(t\right)\ge p\eta'\left(t\right)\psi\left(t\right)$, which implies

$$\frac{\psi'\left(t\right)}{\psi\left(t\right)}\ge p\frac{\eta'\left(t\right)}{\eta\left(t\right)}. \tag{6.4}$$

Integrating (6.4) over $(0,t)$, and combining with (6.3), we get

$$\frac{\eta'(t)}{\psi(0)} \geq \frac{\eta^p(t)}{\eta^p(0)},$$

which can be rewritten as

$$\frac{\eta'(t)}{\eta^p(t)} \geq \frac{\psi(0)}{\eta^p(0)}. \tag{6.5}$$

Then, integrating (6.5) over $(0,t)$, it yields

$$0 \leq \frac{1}{\eta^{p-1}(t)} \leq \frac{1}{\eta^{p-1}(0)} - (p-1)\frac{\psi(0)}{\eta^p(0)}t. \tag{6.6}$$

Thus, letting $t \to T$ in (6.6), we obtain

$$T \leq \frac{\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2}{4p(1-p)J(u_0,v_0)}$$

such that $\lim\limits_{t\to T^-} \eta(t) = +\infty$.

Moreover, integrating (6.5) over (t,T), we get (6.1). The proof is complete. □

Theorem 6.2. *Assume that $p > 2$ and $(u_0,v_0) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{0,0\}$, and let*

$$0 \leq J(u_0,v_0) < \frac{a}{4C_B}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right) - \frac{|\Omega|}{2\mathrm{ep}}. \tag{6.7}$$

Then, the weak solution (u,v) of problem (1.3) blows up in finite time and the upper bound blow-up time is

$$T \leq \frac{2\varepsilon\alpha_0}{(\beta_0-1)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)}, \tag{6.8}$$

where

$$\alpha_0 > \frac{1}{4\varepsilon^2}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^2,$$

$$0 < \varepsilon < \frac{\frac{a(p-2)}{pC_B}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right) - 4\beta_0 J(u_0,v_0)}{\beta_0\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)},$$

$$1 < \beta_0 < \frac{a(p-2)}{4pC_B J(u_0,v_0)}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right).$$

Proof. According to (2.4), (2.5) and (2.7), we have

$$\begin{aligned} I\left(u_0,v_0\right) &= 4J\left(u_0,v_0\right) - a\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) - \frac{4}{p^2}\|u_0v_0\|_p^p - \frac{2\left(p-2\right)}{p}\int_\Omega u_0^p v_0^p \ln|u_0v_0|\,dx \\ &\le 4J\left(u_0,v_0\right) - \frac{a}{C_B}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right) - \frac{2\left(p-2\right)}{p}\int_\Omega u_0^p v_0^p \ln|u_0v_0|\,dx \\ &\le 4J\left(u_0,v_0\right) - \frac{a}{C_B}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right) + \frac{2\left|\Omega\right|}{\mathrm{ep}} \\ &< 0. \end{aligned}$$

We assert that $I\left(u,v\right) < 0,\ t \in [0,T)$. If it is not true, then there exists a $t_0 \in (0,T)$ such that $I\left(u\left(t_0\right),v\left(t_0\right)\right) = 0$ and $I\left(u,v\right) < 0,\ t \in [0,t_0)$. Since

$$G''\left(t\right) = 2\left(\frac{u_t}{|x|^s},u\right)_2 + 2\left(\frac{v_t}{|x|^s},v\right)_2 = -2I\left(u,v\right) > 0,$$

we know that $\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2$ is strictly increasing on $[0,t_0]$. Then,

$$0 \le J\left(u_0,v_0\right) < \frac{a}{4C_B}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right) - \frac{|\Omega|}{2\mathrm{ep}}.$$

Moreover, from (2.3), we have

$$\begin{aligned} J\left(u_0,v_0\right) &\ge J\left(u\left(t_0\right),v\left(t_0\right)\right) \\ &\ge \frac{1}{4}I\left(u\left(t_0\right),v\left(t_0\right)\right) + \frac{a}{4C_H}\left(\left\|\frac{u\left(t_0\right)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v\left(t_0\right)}{|x|^{\frac{s}{2}}}\right\|_2^2\right) - \frac{|\Omega|}{2\mathrm{ep}} \\ &= \frac{a}{4C_H}\left(\left\|\frac{u\left(t_0\right)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v\left(t_0\right)}{|x|^{\frac{s}{2}}}\right\|_2^2\right) - \frac{|\Omega|}{2\mathrm{ep}}, \end{aligned}$$

thus, it contradicts with (6.7). It means that $I\left(u,v\right) < 0,\ \mathit{for\ any}\ t \in [0,T)$. Then,

$$\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2$$

is strictly increasing with $t \in [0,t_0)$.

Next, by contradiction, we assume that the weak solution (u,v) exists globally and $J\left(u,v\right) \ge 0$ for all $t \in [0,+\infty)$. Then we know that the maximal existence $T = +\infty$. Let us define an auxiliary function

$$\varphi\left(t\right) := G^2\left(t\right) + \frac{1}{\varepsilon}G'\left(0\right)G\left(t\right) + \alpha_0,\ \forall t \in (0,+\infty).$$

It follows from (2.3)-(2.5) that

$$G''\left(t\right) = -\,2I\left(u,v\right)$$

$$
\begin{aligned}
&=-8J(u,v)+2a\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{8}{p^2}\|uv\|_p^p+\frac{4(p-2)}{p}\int_\Omega u^pv^p\ln|uv|\,dx\\
&\geq -8J(u,v)+\frac{4(p-2)}{p}\int_\Omega u^pv^p\ln|uv|\,dx\\
&=8(\beta_0-1)J(u,v)-8\beta_0J(u,v)+\frac{4(p-2)}{p}\int_\Omega u^pv^p\ln|uv|\,dx\\
&\geq -8\beta_0J(u,v)+\frac{4(p-2)}{p}\int_\Omega u^pv^p\ln|uv|\,dx\\
&\geq -8\beta_0J(u_0,v_0)+8\beta_0\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\\
&\quad+\frac{4(p-2)}{p}\int_\Omega u^pv^p\ln|uv|\,dx. \qquad (6.9)
\end{aligned}
$$

Since $I(u,v)<0$, we conclude

$$
\begin{aligned}
\int_\Omega u^pv^p\ln|uv|\,dx &> \frac{a}{2}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{b}{2}\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)\\
&\geq \frac{a}{2C_B}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)\\
&\geq \frac{a}{2C_B}G'(t). \qquad (6.10)
\end{aligned}
$$

Consequently, from (6.9), (6.10), we derive

$$
G''(t)\geq \frac{2a(p-2)}{pC_B}G'(t)-8\beta_0J(u_0,v_0). \qquad (6.11)
$$

By integrating both sides of (6.11) on $(0,t)$, we obtain

$$
G'(t)\geq G'(0)e^{\frac{2a(p-2)}{pC_B}t}+\frac{4\beta_0pC_BJ(u_0,v_0)}{a(p-2)}\left[1-e^{\frac{2a(p-2)}{pC_B}t}\right]. \qquad (6.12)
$$

Substituting (6.12) into (6.11) and combining with (6.9) yields

$$
\begin{aligned}
G''(t)\geq&\frac{2a(p-2)}{pC_B}\left\{G'(0)e^{\frac{2a(p-2)}{pC_B}t}+\frac{4\beta_0pC_BJ(u_0,v_0)}{a(p-2)}\left[1-e^{\frac{2a(p-2)}{pC_B}t}\right]\right\}-8\beta_0J(u_0,v_0)\\
&+8\beta_0\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau\\
=&\left[\frac{2a(p-2)}{pC_B}G'(0)-8\beta_0J(u_0,v_0)\right]e^{\frac{2a(p-2)}{pC_B}t}+8\beta_0\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau.
\end{aligned}
$$

Then, we obtain

$$
G''(t)\geq 2\varepsilon\beta_0G'(0)+8\beta_0\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2\right)d\tau,\forall t\in(0,+\infty). \qquad (6.13)
$$

Additionally, differentiating $\varphi(t)$ yields

$$\begin{aligned}
\varphi'(t) &= 2G(t)\,G'(t) + \frac{1}{\varepsilon}G'(0)\,G'(t),\\
\varphi''(t) &= 2\big[G'(t)\big]^2 + 2G(t)\,G''(t) + \frac{1}{\varepsilon}G'(0)\,G''(t).
\end{aligned}$$

We select a $\alpha_0 > \frac{1}{4\varepsilon^2}\big[G'(0)\big]^2$ and define

$$\kappa := 4\alpha_0 - \frac{1}{\varepsilon^2}\big[G'(0)\big]^2 > 0.$$

Consequently, it follows from the definition of $\varphi(t)$ that

$$\begin{aligned}
4\big[G'(t)\big]^2\varphi(t) &= 4\big[G'(t)\big]^2\left[G^2(t) + \frac{1}{\varepsilon}G'(0)\,G(t) + \alpha_0\right]\\
&= 4G^2(t)\,\big[G'(t)\big]^2 + \frac{4}{\varepsilon}G'(0)\,G(t)\,\big[G'(t)\big]^2 + 4\alpha_0\big[G'(t)\big]^2\\
&= \left[2G(t) + \frac{1}{\varepsilon}G'(0)\right]^2\big[G'(t)\big]^2 - \frac{1}{\varepsilon^2}\big[G'(0)\big]^2\big[G'(t)\big]^2 + 4\alpha_0\big[G'(t)\big]^2\\
&= \big[\varphi'(t)\big]^2 + \left\{4\alpha_0 - \frac{1}{\varepsilon^2}\big[G'(0)\big]^2\right\}\big[G'(t)\big]^2\\
&= \big[\varphi'(t)\big]^2 + \kappa\big[G'(t)\big]^2,\ \forall t\in(0,+\infty).
\end{aligned}\tag{6.14}$$

From the derivative $\varphi''(t)$, we notice that

$$\begin{aligned}
\big[\varphi'(t)\big]^2 + \kappa\big[G'(t)\big]^2 &= 4\big[G'(t)\big]^2\varphi(t)\\
&= 4\left[\frac{1}{2}\varphi''(t) - G(t)\,G''(t) - \frac{1}{2\varepsilon}G'(0)\,G''(t)\right]\varphi(t)\\
&= 2\varphi(t)\,\varphi''(t) - 2\varphi(t)\,G''(t)\left[2G(t) + \frac{1}{\varepsilon}G'(0)\right],
\end{aligned}$$

which implies

$$2\varphi(t)\,\varphi''(t) = \big[\varphi'(t)\big]^2 + \kappa\big[G'(t)\big]^2 + 2\varphi(t)\,G''(t)\left[2G(t) + \frac{1}{\varepsilon}G'(0)\right].\tag{6.15}$$

Using the Hölder inequality, the Young inequality with ε and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\big[G'(t)\big]^2\\
=&\left\{G'(0) + 2\int_0^t\left[\left(\frac{u_\tau}{|x|^s},u\right)_2 + \left(\frac{v_\tau}{|x|^s},v\right)_2\right]d\tau\right\}^2\\
\le&\left[G'(0) + 2\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2\left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2\right)d\tau\right]^2\\
\le&\left[G'(0) + 2\int_0^t\sqrt{\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2}\sqrt{\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2}d\tau\right]^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[G'(0) + 2\sqrt{\int_0^t \left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 d\tau} \sqrt{\int_0^t \left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 d\tau} \right]^2 \\
&\leq \left[G'(0)\right]^2 + 4G'(0) \left[\int_0^t \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau \right]^{\frac{1}{2}} \left[\int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau \right]^{\frac{1}{2}} \\
&\quad + 4G(t) \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau \\
&\leq \left[G'(0)\right]^2 + 2\varepsilon G'(0) G(t) + \frac{2}{\varepsilon} G'(0) \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau \\
&\quad + 4G(t) \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau. \qquad (6.16)
\end{aligned}
$$

Then, from (6.13)-(6.16), we have

$$
\begin{aligned}
&2\varphi(t)\varphi''(t) - (\beta_0+1)\left[\varphi'(t)\right]^2 \\
&= 2\varphi(t) G''(t) \left[2G(t) + \frac{1}{\varepsilon} G'(0) \right] - 4\beta_0 \left[G'(t)\right]^2 \varphi(t) + \kappa(\beta_0+1)\left[G'(t)\right]^2 \\
&\geq 2\varphi(t) \left[2G(t) + \frac{1}{\varepsilon} G'(0) \right] \left\{ 2\beta_0 \varepsilon G'(0) + 8\beta_0 \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) \mathrm{d}\tau \right\} \\
&\quad - 4\beta_0 \varphi(t) \left\{ [G'(0)]^2 + 2\varepsilon G'(0) G(t) + \frac{2}{\varepsilon} G'(0) \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) \mathrm{d}\tau \right. \\
&\quad \left. + 4G(t) \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) \mathrm{d}\tau \right\} \\
&= 16\beta_0 \varphi(t) G(t) \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau + \frac{8}{\varepsilon} \beta_0 G'(0) \varphi(t) \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 \right) d\tau \\
&> 0.
\end{aligned}
$$

Considering the case $\varphi(0) = \alpha_0 > 0$ and $\varphi'(0) = \frac{1}{\varepsilon}[G'(0)]^2 > 0$, we derive from Lemma 2.2 that there exists a

$$
0 < \tilde{t} \leq \frac{2\varepsilon\alpha_0}{(\beta_0 - 1)\left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \left\| \frac{v_0}{|x|^{\frac{s}{2}}} \right\|_2^2 \right)^2}
$$

such that $\lim_{t\to\tilde{t}} \varphi(t) = +\infty$, which directly conflicts with $T = +\infty$. Consequently, the weak solution (u, v) blows up in finite time with the maximum blow-up time occurring at (6.8). The proof of Theorem 6.2 is complete. □

7. Extinction and non-extinction in finite time

In this section, we derive new threshold criteria for extinction and non-extinction phenomena of weak solutions to problem (1.3), explicitly incorporating polynomial expansion technique into the proof. Under some appropriate conditions, we obtain the critical threshold time of extinction phenomenon.

Theorem 7.1. *Let* $0<p<1$. *If* $\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2>\left(D_2D_1^{-1}\right)^{\frac{1}{1-p-\sigma}}$, *then the weak solution* (u,v) *of problem* (1.3) *satisfies*

$$\begin{cases}\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\le\left\{\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^{1-p-\sigma}\right.\\ \qquad\left.+\left[D_2-D_1\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^{1-p-\sigma}\right]t\right\}^{\frac{1}{1-p-\sigma}}, & 0<t<t_*,\\ \left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2=0, & t\ge t_*,\end{cases}$$

with the corresponding extinction time is estimated as

$$t_*=\frac{\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^{1-p-\sigma}}{D_1\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v_0}{|x|^{\frac{s}{2}}}\right\|_2^2\right)^{1-p-\sigma}-D_2},$$

where $D_1=\frac{a(1-p-\sigma)}{C_B}$, $D_2=\frac{1}{\mathrm{e}\sigma}\left(1-p-\sigma\right)L^{s(p+\sigma)}S_5^{2(p+\sigma)}$, *and* S_5 *is the best constant of the embedding* $L^2\left(\Omega\right)\hookrightarrow L^{2(p+\sigma)}\left(\Omega\right)$.

Proof. We define a function $K\left(t\right):=\frac{1}{2}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2\right)$. Then,

$$K'\left(t\right)=\left(\frac{u_t}{|x|^s},u\right)_2+\left(\frac{v_t}{|x|^s},v\right)_2=-I\left(u,v\right),$$

which leads to

$$K'\left(t\right)+a\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+b\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)=2\int_\Omega u^pv^p\ln|uv|\,dx.$$

Based on (2.9) and the polynomial expansion technique, we obtain

$$\begin{aligned}K'\left(t\right)+\frac{2a}{C_B}K\left(t\right)&\le K'\left(t\right)+a\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+b\left(\|\nabla u\|_2^4+\|\nabla v\|_2^4\right)\\&=2\int_\Omega u^pv^p\ln|uv|\,dx\\&\le\frac{1}{\mathrm{e}\sigma}\left(\|u\|_{2(p+\sigma)}^{2(p+\sigma)}+\|v\|_{2(p+\sigma)}^{2(p+\sigma)}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{e\sigma} S_5^{2(p+\sigma)} \left(\|u\|_2^{2(p+\sigma)} + \|v\|_2^{2(p+\sigma)}\right) \\
&\leq \frac{1}{e\sigma} S_5^{2(p+\sigma)} \left(\|u\|_2^2 + \|v\|_2^2\right)^{p+\sigma} \\
&\leq \frac{2^{p+\sigma}}{e\sigma} S_5^{2(p+\sigma)} L^{s(p+\sigma)} K^{p+\sigma}(t). \qquad (7.1)
\end{aligned}
$$

By setting $\xi(t) = K^{1-p-\sigma}(t)$ and combining with (7.1), a direct calculation yields

$$
\begin{aligned}
\xi'(t) &= (1-p-\sigma) K^{-p-\sigma}(t) K'(t) \\
&\leq (1-p-\sigma) K^{-p-\sigma}(t) \left[\frac{2^{p+\sigma} L^{s(p+\sigma)}}{e\sigma} S_5^{2(p+\sigma)} K^{p+\sigma}(t) - \frac{2a}{C_B} K(t)\right] \\
&= \frac{2^{p+\sigma}}{e\sigma} (1-p-\sigma) L^{s(p+\sigma)} S_5^{2(p+\sigma)} - \frac{2a(1-p-\sigma)}{C_B} \xi(t) \\
&= 2^{p+\sigma} D_2 - 2D_1 \xi(t) \\
&:= \rho(t). \qquad (7.2)
\end{aligned}
$$

Then, we can know from (7.2) that $\rho(0) < 0$. By the continuity of $\rho(t)$, there exists a sufficiently small $\tilde{t} > 0$ such that $\rho(t) < \frac{\rho(0)}{2} < 0$ for all $t \in \left(0, \tilde{t}\right]$, which implies $\xi'(t) \leq \frac{\rho(0)}{2}$. Integrating it on $(0, t)$ yields

$$
\begin{cases}
\xi(t) \leq \xi(0) + \dfrac{\rho(0)}{2} t, & 0 < t < t_*, \\
\xi(0) = 0, & t \geq t_*.
\end{cases}
$$

Consequently, the proof of Theorem 7.1 is complete. □

Theorem 7.2. *Assume that* $0 < p < 1$. *If* $J(u_0, v_0) < 0$, *then the weak solution* (u, v) *of problem* (1.3) *does not extinct in finite time.*

Proof. From the function $K(t)$ defined as in Theorem 7.1, according to (2.4), (2.5) and (2.9), we obtain

$$
\begin{aligned}
K'(t) &= -I(u, v) \\
&= -4J(u, v) + a\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{4}{p^2} \|uv\|_p^p + \frac{2(p-2)}{p} \int_\Omega u^p v^p \ln|uv|\, dx \\
&\geq -4J(u_0, v_0) - \frac{2(2-p)}{p} \int_\Omega u^p v^p \ln|uv|\, dx \\
&\geq -4J(u_0, v_0) - \frac{2^{p+\sigma}}{pe\sigma} (2-p) L^{s(p+\sigma)} S_5^{2(p+\sigma)} K^{p+\sigma}(t),
\end{aligned}
$$

which implies

$$
K'(t) + \frac{2^{p+\sigma}}{pe\sigma} (2-p) L^{s(p+\sigma)} S_5^{2(p+\sigma)} K^{p+\sigma}(t) \geq -4J(u_0, v_0).
$$

By $J(u_0, v_0) < 0$, $0 < p + \sigma < 1$ and Lemma 2.3, it yields

$$
K(t) \geq \min\left\{K(0), \left[\frac{-4J(u_0, v_0)\, pe\sigma}{2^{p+\sigma}(2-p) L^{s(p+\sigma)} S_5^{2(p+\sigma)}}\right]^{\frac{1}{p+\sigma}}\right\}, \ \forall t \in (0, +\infty).
$$

Since $K(0) > 0$, we have $K(t) > 0,\ \forall t \in (0, +\infty)$. Consequently, for any $g > 1$, we derive

$$\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2 \le \left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_g \left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_{\frac{g}{g-1}} + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_g \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_{\frac{g}{g-1}}$$
$$\le \left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_g^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_g^2\right)^{\frac{1}{2}} \left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_{\frac{g}{g-1}}^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_{\frac{g}{g-1}}^2\right)^{\frac{1}{2}}.$$

Furthermore, since $\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_2^2 > 0$, there does not exist t_* such that

$$\lim_{t \to t_*} \left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_g^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_g^2\right) = 0 \qquad \text{or} \qquad \lim_{t \to t_*} \left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_{\frac{g}{g-1}}^2 + \left\|\frac{v}{|x|^{\frac{s}{2}}}\right\|_{\frac{g}{g-1}}^2\right) = 0.$$

The proof of Theorem 7.2 is complete. □

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