Journal of Applied Analysis and Computation
Volume 16, Number 4, August 2026, 2019–2033

Website:http://www.jaac-online.com
DOI:10.11948/20250176

# EXISTENCE AND MULTIPLICITY RESULTS FOR A CLASS OF DISCRETE FRACTIONAL PROBLEMS WITH SUMMATION BOUNDARY CONDITIONS

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**Abstract** We study a discrete boundary value problem involving the Riemann-Liouville fractional operator. In particular, we introduce a discrete fractional summing boundary value problem that parallels the integral boundary value problem for fractional differential equations. By generalizing a commonly used fractional difference operator, we derive the corresponding Green's function and reformulate the problem as a fixed-point equation. The existence and multiplicity of positive solutions are established. Two illustrative examples are provided to demonstrate the applicability of the theoretical results, and numerical simulations are included for further validation.

**Keywords** Fractional difference equation, fixed point theory, summing boundary condition, numerical simulation, positive solution.

**MSC(2010)** 39A12, 39A70, 34B10.

## 1. Introduction

Along with the rapid growth of computational power, new applications of fractional calculus have attracted increasing interests from researchers [22]. These applications span a broad range of disciplines, including signal processing, physics and engineering, finance, control theory, economics and population dynamics [3,5,27,29]. Fractional techniques have also found applications in machine learning and artificial intelligence [3]. Meanwhile, fractional differential equations (FDEs), as powerful tools for modelling complex phenomena with memory-dependent or non-local behaviours, can be effectively solved using neural network based methods, often requiring comparatively less computation time [29].

As the discrete analogue of FDEs, fractional difference equations have direct connections with numerical analysis and computational modelling [17,24]. In practical computation, discrete operators replace integrals by weighted sums, the memory effects are captured through sums rather than integrals, avoid singular behavious and naturally align with standard numerical schemes. Classical difference operators are typically classified as forward operators that look ahead to compute the change, and backward operators that look behind to compute the change.

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The most commonly applied discrete fractional operators include the Riemann-Liouville [17, 28] and the Caputo [1, 26] types. Both definitions have well-developed theories and numerous applications. These definitions, however, exhibit distinct monotonicity and comparison properties. More recently, the Caputo-Fabrizio derivative has been introduced to describe the temporal and spatial variables of systems with memory effects [2, 8]. This definition applied an exponential kernel as an extension of the Atangana-Baleanu sum/difference that utilizes a generalized Mittag-Leffler function as the kernel. Owing to its non-local and non-singular kernel, the Caputo-Fabrizio derivative has some advantages in solving some physical problems with initial conditions [4, 21].

Differential equations with nonlocal boundary conditions have been extensively studied due to their applications in areas such as thermostat control, heat transfer [20], and neural computing [27]. Boundary value problems with integral boundary conditions [11] generalize multipoint boundary value problems and naturally arise in many modelling contexts. Although fixed point theory has been widely used in both continuous and discrete settings [9, 10, 30], fractional difference equations have essential distinctions from their differential counterparts [7]. For instance, unlike in the integer-order case, a function can increase even if its fractional difference is negative [16], reflecting their complex nonlocal structure. Moreover, different discretization techniques also have great impacts on the resulting discrete systems [6].

Summation-type boundary conditions form a class of nonlocal boundary value problems. Several authors have investigated the existence and multiplicity of solutions for fractional difference equations subject to summation or fractional-sum boundary conditions, treating both delta and nabla fractional operators [23]. For example, most recently, the existence of positive solutions for the following nabla fractional difference equations with parameter-dependent summation boundary conditions is studied in [12] using the Leray-Schauder nonlinear alternative:

$$\begin{cases} -(\nabla^{\eta}_{k-1}\omega)(\zeta) = f(\zeta, \omega(\zeta)), \quad 1 < \eta \leq 2,\ \zeta \in \mathbb{N}^{l}_{k+2}, \\ \omega(k) = 0,\ \ \omega(l) = \mu \displaystyle\sum_{s=k+1}^{d} \omega(s),\ \ d \in \mathbb{N}^{l-1}_{k+1},\ \ \mu > 0, \end{cases}$$

where $f : \mathbb{N}^{l}_{k+2} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In [13], existence, uniqueness, and stability of sign-changing solutions for a nabla difference equation with summation boundary conditions are established. Motivated by these developments, the present work extends the theory to discrete fractional problems with generalized fractional-summing boundary conditions, providing new existence results and illustrative examples.

In particular, we are interested in the following fractional difference summing boundary value problem (SBVP)

$$\begin{cases} \triangle^{\alpha}_{\alpha-3} x(t) = -f(t+\alpha-2, x(t+\alpha-2)), \quad t \in \mathbb{N}^{b+1}_{0}, \\ x(\alpha-3) = \triangle x(\alpha-3) = 0, \\ x(b+\alpha) = \lambda \displaystyle\sum_{s=0}^{b+1} x(s+\alpha-2), \end{cases} \tag{1.1}$$

where $\alpha \in (2, 3]$, $b \geqslant 5$ is an integer, $0 < \lambda < \frac{\alpha}{b+1}$, $\triangle^{\alpha}_{\alpha-3}$ denotes the standard Riemann-Liouville fractional difference, and $f : \mathbb{N}^{b+\alpha}_{\alpha-3} \times [0, +\infty) \to [0, +\infty)$ is continuous. The condition $\alpha > 2$ ensures that the fractional difference operator $\triangle^{\alpha}_{\alpha-3}$ acts as a forward operator.

Problem (1.1) may be viewed as a discrete analogue of FDEs with integral boundary conditions. In continuous problems, multi-point boundary conditions can often be *reduced* to an equivalent three-point BVP, greatly simplifying the analysis [15]. However, such reduction techniques do not appear to transfer to the discrete fractional framework. We work directly with the multi-point boundary conditions, obtain an explicit representation of the associated Green's function and apply fixed point theorem to prove existence of multiple positive solutions.

Moreover, the difference equation of problem (1.1) is different from the common discrete BVPs employed of the form

$$\triangle_{\alpha-n}^{\alpha}x(t)=-f(t+\alpha-1,\ x(t+\alpha-1)),\ \alpha\in(n-1,n]. \tag{1.2}$$

This form does not yield an explicit Green's function when $n=3$. We introduce the following generalized form

$$\triangle_{\alpha-n}^{\alpha}x(t)=-f(t+\alpha-(n-1),\ x(t+\alpha-(n-1))),\ \alpha\in(n-1,n]. \tag{1.3}$$

Obviously, when $n=2$, equation (1.3) reduced to (1.2). When $n=3$, (1.3) leads to a symmetric structure that allows us to derive the Green's function. Consequently, our results provide a genuine extension and provide a new approach to nonlocal fractional BVPs. Our work also extends the nonlinear FBVP considered in [14] where existence of two positive solutions were obtained by iteration. To illustrate the results, we present two examples and include numerical simulations that virtually demonstrate the behavior of the solutions.

## 2. The Green's function

We first give two lemmas on the discrete fractional difference, and the fixed point theorem that will be used in the sequel [18].

For $a\in\mathbb{R}$, $\alpha>0$, $t\in\mathbb{N}_{a+\alpha}$, the $\alpha$-th fractional sum of a function $f$ is given by

$$\triangle_a^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}f(s),$$

where $t^{\underline{\alpha}}=\frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$. Let $n=\lceil\alpha\rceil$ (the ceiling function), then the $\alpha$-th fractional difference is defined by $\triangle_a^{\alpha}f(t)=\triangle^n\triangle_a^{-(n-\alpha)}f(t)$, $t\in\mathbb{N}_{a+n-\alpha}$.

**Lemma 2.1.** ( [19]). *Let $\alpha>0$, $n=\lceil\alpha\rceil$ and $f$ is a function defined on $\mathbb{N}_a\to\mathbb{R}$. Then*

$$\triangle_{a-\alpha+n}^{-\alpha}\triangle_a^{\alpha}f(t)=f(t)+c_1(t-a)^{\underline{\alpha-1}}+c_2(t-a)^{\underline{\alpha-2}}+\cdots+c_n(t-a)^{\underline{\alpha-n}},$$

*where $c_i\in\mathbb{R}(i=1,2,\cdots,n)$.*

**Lemma 2.2.** ( [19]). *Let $f:\mathbb{N}_{a+\alpha}\times\mathbb{N}_a\to\mathbb{R}$, then*

$$\triangle\sum_{s=a}^{t-\alpha}f(t,s)=\sum_{s=a}^{t-\alpha}\triangle_t f(t,s)+f(t+1,t+1-\alpha),\quad t\in\mathbb{N}_{a+\alpha}.$$

**Theorem 2.1.** ( [18]). *Let $E$ be a Banach space, $K\subseteq E$ a cone, and $\Omega_1$ and $\Omega_2$ two bounded open subsets of $E$ with $\theta\in\Omega_1\subset\overline{\Omega}_1\subseteq\Omega_2$. Suppose that $T:K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ be a completely continuous operator such that either* *(i)* $\|Tx\| \leqslant \|x\|$, $x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \geqslant \|x\|$, $x \in K \cap \partial\Omega_2$; *or*

*(ii)* $\|Tx\| \geqslant \|x\|$, $x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \leqslant \|x\|$, $x \in K \cap \partial\Omega_2$

*holds. Then* $T$ *has a fixed point in* $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 2.1 is the classical Guo-Krasnosel'skii fixed point theorem, also known as the cone expansion and cone compression theorem. Under condition (i), the operator $T$ is contractive on the inner boundary and expansive on the outer boundary. This guarantees that $T$ has a fixed point in the annulus. Condition (ii) implies opposite conditions and a fixed point exists in the region between these two spheres.

The following new lemmas are essential for the results of Section 3.

**Lemma 2.3.** *Let* $\alpha \in (2,3]$, $\alpha \neq \lambda(b+1)$, *and* $h : \mathbb{N}_{\alpha-3}^{b+\alpha} \to \mathbb{R}$. *A solution to the SBVP (2.1)-(2.3)*

$$\triangle_{\alpha-3}^{\alpha} x(t) = -h(t+\alpha-2),\ t \in \mathbb{N}_0^{b+1}, \tag{2.1}$$

$$x(\alpha-3) = \triangle x(\alpha-3) = 0, \tag{2.2}$$

$$x(b+\alpha) = \lambda \sum_{s=0}^{b+1} x(s+\alpha-2), \tag{2.3}$$

*can be written as*

$$x(t) = \sum_{s=0}^{b+1} G(t,s) h(s+\alpha-2),$$

*where*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{\alpha t^{\underline{\alpha-1}}\,(b+\alpha-s-1)^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}\,[\alpha-\lambda(b+1)]} - (t-s-1)^{\underline{\alpha-1}}, & 0 \leqslant s \leqslant t-\alpha+1 \leqslant b+1, \\ \dfrac{\alpha t^{\underline{\alpha-1}}\,(b+\alpha-s-1)^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}\,[\alpha-\lambda(b+1)]}, & 0 \leqslant t-\alpha+1 \leqslant s \leqslant b+1. \end{cases}$$

**Proof.** Applying $\triangle^{-\alpha}$ on both sides of equation (2.1), we have

$$\triangle^{-\alpha} \triangle_{\alpha-3}^{\alpha} x(t) = -\triangle^{-\alpha} h(t+\alpha-2).$$

From Lemma 2.1,

$$\triangle^{-\alpha} \triangle_{\alpha-3}^{\alpha} x(t) = x(t) - c_1 (t-a)^{\underline{\alpha-1}} - c_2 (t-a)^{\underline{\alpha-2}} - c_3 (t-a)^{\underline{\alpha-3}}.$$

By definition,

$$\triangle^{-\alpha} h(t+\alpha-2) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} h(s+\alpha-2).$$

From the above two equations, we obtain the following

$$x(t) = -\frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} h(s+\alpha-2) + c_1 t^{\underline{\alpha-1}} + c_2 t^{\underline{\alpha-2}} + c_3 t^{\underline{\alpha-3}}\ ,$$

where $c_1, c_2, c_3$ are constants. From Lemma 2.2, we get

$$\begin{aligned}\triangle x(t) = &-\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t+1-\alpha}(\alpha-1)(t-s-1)^{\underline{\alpha-2}}h(s+\alpha-2)+c_1(\alpha-1)t^{\underline{\alpha-2}}\\ &+c_2(\alpha-2)t^{\underline{\alpha-3}}+c_3(\alpha-3)t^{\underline{\alpha-4}}.\end{aligned}$$

Applying the boundary condition (2.2), $c_2 = c_3 = 0$, and

$$x(t) = -\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}h(s+\alpha-2)+c_1 t^{\underline{\alpha-1}}.$$

Applying the boundary condition (2.3), we get

$$c_1 = \frac{1}{(b+\alpha)^{\underline{\alpha-1}}}\Bigg[\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{b+1}(b+\alpha-s-1)^{\underline{\alpha-1}}h(s+\alpha-2)+\lambda\sum_{s=0}^{b+1}x(s+\alpha-2)\Bigg].$$

Hence

$$\begin{aligned}x(t) = &-\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}h(s+\alpha-2)\\ &+\frac{t^{\underline{\alpha-1}}}{\Gamma(\alpha)(b+\alpha)^{\underline{\alpha-1}}}\sum_{s=0}^{b+1}(b+\alpha-s-1)^{\underline{\alpha-1}}h(s+\alpha-2)\\ &+\frac{\lambda t^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}}\sum_{s=0}^{b+1}x(s+\alpha-2).\end{aligned} \tag{2.4}$$

Let $c = \sum_{s=0}^{b+1}x(s+\alpha-2)$, equality (2.4) implies that

$$\begin{aligned}c = &\sum_{s=0}^{b+1}\frac{(s+\alpha-2)^{\underline{\alpha-1}}}{\Gamma(\alpha)(b+\alpha)^{\underline{\alpha-1}}}\sum_{\tau=0}^{b+1}(b+\alpha-\tau-1)^{\underline{\alpha-1}}h(\tau+\alpha-2)\\ &+\sum_{s=0}^{b+1}\frac{\lambda(s+\alpha-2)^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}}\sum_{s=0}^{b+1}x(s+\alpha-2).\end{aligned}$$

Then

$$\begin{aligned}&c\Bigg(1-\sum_{s=0}^{b+1}\frac{\lambda(s+\alpha-2)^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}}\Bigg)\\ &=\sum_{s=0}^{b+1}\sum_{\tau=0}^{b+1}\frac{(s+\alpha-2)^{\underline{\alpha-1}}}{\Gamma(\alpha)(b+\alpha)^{\underline{\alpha-1}}}(b+\alpha-\tau-1)^{\underline{\alpha-1}}h(\tau+\alpha-2)\\ &=\frac{1}{\Gamma(\alpha)}\sum_{\tau=0}^{b+1}\sum_{s=0}^{b+1}\frac{(s+\alpha-2)^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}}(b+\alpha-\tau-1)^{\underline{\alpha-1}}h(\tau+\alpha-2).\end{aligned}$$

Since

$$\sum_{s=0}^{b+1}(s+\alpha-2)^{\underline{\alpha-1}} = \frac{(b+\alpha)^{\underline{\alpha}}}{\alpha},$$

then

$$c=\frac{1}{\Gamma(\alpha)}\cdot\frac{b+1}{\alpha-\lambda(b+1)}\sum_{s=0}^{b+1}(b+\alpha-s-1)^{\underline{\alpha-1}}h(s+\alpha-2).$$

Replacing $c$ in (2.4), then the solution of SBVP (2.1)-(2.3) can be obtained as

$$\begin{aligned}x(t)=&-\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}h(s+\alpha-2)\\&+\frac{\alpha t^{\underline{\alpha-1}}}{\Gamma(\alpha)(b+\alpha)^{\underline{\alpha-1}}[\alpha-\lambda(b+1)]}\sum_{s=0}^{b+1}(b+\alpha-s-1)^{\underline{\alpha-1}}h(s+\alpha-2)\\=&\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\left[\frac{\alpha t^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}[\alpha-\lambda(b+1)]}(b+\alpha-s-1)^{\underline{\alpha-1}}-(t-s-1)^{\underline{\alpha-1}}\right]h(s+\alpha-2)\\&+\frac{1}{\Gamma(\alpha)}\sum_{s=t-\alpha+1}^{b+1}\frac{\alpha t^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}[\alpha-\lambda(b+1)]}(b+\alpha-s-1)^{\underline{\alpha-1}}h(s+\alpha-2)\\=&\sum_{s=0}^{b+1}G(t,s)h(s+\alpha-2).\end{aligned}$$

The proof is completed. □

**Remark 2.1.** It is notable that

$$G(\alpha-3,s)=G(\alpha-2,s)=G(t,b+1)=0.$$

So in the sequel, we only need to discuss the case $G(t,s),(t,s)\in\mathbb{N}_{\alpha-2}^{b+\alpha}\times\mathbb{N}_0^b$.

It is not difficult to prove the following Lemma.

**Lemma 2.4.** *The function $G(t,s)$ satisfies*

*(i)* $G(b+\alpha,s)=0$, $s\in\mathbb{N}_0^b$ *if and only if* $\lambda=0$.

*(ii)* $[\alpha-\lambda(b+1)]G(b+\alpha,s)>0$, $s\in\mathbb{N}_0^b$ *if and only if* $\alpha\neq\lambda(b+1)$.

*(iii)* $G(t,s)\leqslant\frac{(b+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha-1)[\alpha-\lambda(b+1)]}$, $(t,s)\in\mathbb{N}_{\alpha-2}^{b+\alpha}\times\mathbb{N}_0^b$ *if and only if* $\lambda\in[0,\alpha/(b+1))$.

*(iv) If* $\alpha\neq\lambda(b+1)$, *then* $G(t,s):\mathbb{N}_{\alpha-2}^{b+\alpha}\times\mathbb{N}_0^b\to\mathbb{R}$ *is continuous.*

**Lemma 2.5.** *Let* $2<\alpha\leqslant 3$ *and* $0<\lambda<\frac{\alpha}{b+1}$. *Then* $G(t,s)$ *satisfies the following inequalities*

$$\frac{\Gamma(\alpha)}{(b+\alpha)^{\underline{\alpha-1}}}G(b+\alpha,s)\leqslant G(t,s)\leqslant\frac{\alpha}{\lambda(b+1)}G(b+\alpha,s),\quad(t,s)\in\mathbb{N}_{\alpha-1}^{b+\alpha}\times\mathbb{N}_0^b.\tag{2.5}$$

**Proof.** The expression of $G(t,s)$ implies

$$\begin{aligned}&\frac{G(t,s)}{G(b+\alpha,s)}\\=&\begin{cases}\dfrac{\alpha t^{\underline{\alpha-1}}(b+\alpha-s-1)^{\underline{\alpha-1}}-(t-s-1)^{\underline{\alpha-1}}(b+\alpha)^{\underline{\alpha-1}}[\alpha-\lambda(b+1)]}{\lambda(b+1)(b+\alpha-s-1)^{\underline{\alpha-1}}(b+\alpha)^{\underline{\alpha-1}}}, & 0\leqslant s\leqslant t-\alpha+1\leqslant b,\\ \dfrac{\alpha t^{\underline{\alpha-1}}}{\lambda(b+1)(b+\alpha)^{\underline{\alpha-1}}}, & 0\leqslant t-\alpha+1\leqslant s\leqslant b.\end{cases}\end{aligned}$$

For the case $0 \leqslant t-\alpha+1 \leqslant s \leqslant b$, it is clear that

$$\frac{G(t,s)}{G(b+\alpha,s)} \leqslant \frac{\alpha(b+\alpha)^{\underline{\alpha-1}}}{\lambda(b+1)(b+\alpha)^{\underline{\alpha-1}}} = \frac{\alpha}{\lambda(b+1)},$$

and

$$\frac{G(t,s)}{G(b+\alpha,s)} \geqslant \frac{\lambda(b+1)t^{\underline{\alpha-1}}}{\lambda(b+1)(b+\alpha)^{\underline{\alpha-1}}} = \frac{t^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}} \geqslant \frac{\Gamma(\alpha)}{(b+\alpha)^{\underline{\alpha-1}}}.$$

For the other case $0 \leqslant s \leqslant t-\alpha+1 \leqslant b$, we know that $\frac{G(t,s)}{G(b+\alpha,s)} \leqslant \frac{\alpha}{\lambda(b+1)}$. In fact, for $0 \leqslant s < t-\alpha+1 \leqslant b$, the function $h(t) = \frac{t^{\underline{\alpha-1}}}{(t-s)^{\underline{\alpha-1}}}$ is decreasing with respect to $t$. So

$$\frac{t^{\underline{\alpha-1}}}{(t-s-1)^{\underline{\alpha-1}}} \geqslant \frac{(b+\alpha)^{\underline{\alpha-1}}}{(b+\alpha-s-1)^{\underline{\alpha-1}}},$$

namely

$$-(t-s-1)^{\underline{\alpha-1}} \geqslant -\frac{t^{\underline{\alpha-1}}(b+\alpha-s-1)^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}}.$$

Thus

$$\frac{t^{\underline{\alpha-1}}}{(b+\alpha)^{\underline{\alpha-1}}} G(b+\alpha,s) \leqslant G(t,s), \quad 0 \leqslant s < t-\alpha+1 \leqslant b.$$

As to $t = b+\alpha$, the conclusion is obvious. Therefore, inequalities (2.5) hold. □

**Corollary 2.1.** *If* $\alpha - \lambda(b+1) > 0$, *then* $G(t,s) > 0,\ (t,s) \in \mathbb{N}_{\alpha-1}^{b+\alpha} \times \mathbb{N}_0^b$.

From now on, we set

$$E = \{x : \mathbb{N}_{\alpha-3}^{b+\alpha} \to \mathbb{R} \ : \ x(\alpha-2) = x(\alpha-3) = \triangle x(\alpha-3) = 0,\ x(b+\alpha) = \lambda \sum_{s=0}^{b+1} x(s+\alpha-2)\}.$$

Obviously, $E$ is a Banach space with the maximum norm $\|x\| = \max_{t \in \mathbb{N}_{\alpha-3}^{b+\alpha}} |x(t)|$. Define the cone

$$K = \{x \in E \ : \ x(t) \geqslant \frac{\lambda(b+1)\Gamma(\alpha-1)}{(b+\alpha)^{\underline{\alpha-1}}}\|x\|,\ t \in \mathbb{N}_{\alpha-1}^{b+\alpha}\}$$

and the operator $T : E \to E$ by

$$(Tx)(t) = \sum_{s=0}^{b+1} G(t,s) f(s+\alpha-2, x(s+\alpha-2)), \quad t \in \mathbb{N}_{\alpha-2}^{b+\alpha}. \tag{2.6}$$

Since $G(t,b+1) = 0$, then we may define

$$(Tx)(t) = \sum_{s=0}^{b} G(t,s) f(s+\alpha-2, x(s+\alpha-2)), \quad t \in \mathbb{N}_{\alpha-2}^{b+\alpha}.$$

In order to keep the symmetry of domain, the form of (2.6) is needed.

It is known that a positive solution of SBVP (1.1) is equivalent to a fixed points of $T$ in $K$. Moreover, for each $x \in K$, if $t \in \mathbb{N}_{\alpha-1}^{b+\alpha}$, by (2.5) and (2.6), we have

$$(Tx)(t) \leqslant \frac{\alpha}{\lambda(b+1)} \sum_{s=0}^{b+1} G(b+\alpha,s) f(s+\alpha-2, x(s+\alpha-2)).$$

Notice that $(Tx)(\alpha-2)=(Tx)(\alpha-3)=0$, then

$$\|Tx\| \leqslant \frac{\alpha}{\lambda(b+1)} \sum_{s=0}^{b+1} G(b+\alpha,s) f(s+\alpha-2, x(s+\alpha-2)).$$

Using (2.5) again, we obtain

$$(Tx)(t) \geqslant \frac{\Gamma(\alpha)}{(b+\alpha)^{\underline{\alpha-1}}} \sum_{s=0}^{b+1} G(b+\alpha,s) f(s+\alpha-2, x(s+\alpha-2)).$$

Therefore $(Tx)(t) \geqslant \frac{\lambda(b+1)\Gamma(\alpha-1)}{(b+\alpha)^{\underline{\alpha-1}}}\|Tx\|$, $t \in \mathbb{N}_{\alpha-1}^{b+\alpha}$. Thus, $TK \subset K$. Since $T$ is a bounded operator defined on a finite space, $T$ is completely continuous.

## 3. Main results

We define the notations:

$$\underline{f}_0 = \liminf_{x\to 0^+} \min_{t\in\mathbb{N}_{\alpha-2}^{b+\alpha}} (f(t,x)/x),\ \overline{f}_0 = \limsup_{x\to 0^+} \max_{t\in\mathbb{N}_{\alpha-2}^{b+\alpha}} (f(t,x)/x),$$

$$\underline{f}_\infty = \liminf_{x\to +\infty} \min_{t\in\mathbb{N}_{\alpha-2}^{b+\alpha}} (f(t,x)/x),\ \overline{f}_\infty = \limsup_{x\to +\infty} \max_{t\in\mathbb{N}_{\alpha-2}^{b+\alpha}} (f(t,x)/x),$$

$$C_1 = \frac{\lambda(b+1)}{\alpha}\left(\sum_{s=0}^{b} G(b+\alpha,s)\right)^{-1},\ C_2 = \frac{(b+\alpha)^{\underline{\alpha-1}}}{\lambda\Gamma(\alpha-1)(b+1)}\left(\sum_{s=1}^{b} G(b+\alpha,s)\right)^{-1},$$

$$C_3 = \frac{(b+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)}\left(\sum_{s=1}^{b} G(\alpha+b,s)\right)^{-1}.$$

The following conditions are used to prove our main conclusions.

$(A_1)$ There exists $r>0$ such that $f(t,x) < rC_1$, $\alpha-2 \leqslant t \leqslant b+\alpha$, $0 \leqslant x \leqslant r$.

$(A_2)$ There exists $R>0$ such that $f(t,x) > RC_3$, $\alpha-1 \leqslant t \leqslant b+\alpha$,

$$\frac{\lambda(b+1)\Gamma(\alpha-1)}{(b+\alpha)^{\underline{\alpha-1}}} a \leqslant x \leqslant R.$$

$(H_1)$ $\underline{f}_0 > C_2$, $f_\infty > C_2$.

$(H_2)$ $\overline{f}_0 < C_1$, $\overline{f}_\infty < C_1$.

$(H_3)$ $\underline{f}_0 > C_2$, $\overline{f}_\infty < C_1$.

$(H_4)$ $\overline{f}_0 < C_1$, $\underline{f}_\infty > C_2$.

$(H_1^*)$ $\underline{f}_0 = +\infty$, $\underline{f}_\infty = +\infty$.

$(H_2^*)$ $\overline{f}_0 = 0$, $\overline{f}_\infty = 0$.

In the above definitions, $C_1$, $C_2$ and $C_3$ depend on the Green's function and parameters of the system. Conditions $A_1$ and $A_2$ connect the constants $C_1$, $C_2$, $C_3$ and the function $f$ to construct the region for the fixed point theorem. Conditions $(H_i)$ and $(H_i^\star)$ correspond to conditions (i) and (ii) of Theorem 2.1 respectively.

In the sequel, we assume that $\alpha \in (2,3], b \geq 5$ and $0 < \lambda < \frac{\alpha}{b+1}$.

**Theorem 3.1.** *Assume that $f$ satisfies $(A_1)$, $(A_2)$ and $r \neq R$. Then SBVP (1.1) has at least one positive solution $x_0 \in K$ with* $\min\{r, R\} < \|x_0\| < \max\{r, R\}$.

**Proof.** We may suppose $r < R$. For $x \in \partial\Omega_r = \{x \in K : \|x\| = r\}$, then $0 \leqslant x(t) \leqslant r$, $\alpha - 2 \leqslant t \leqslant b + \alpha$. From $(A_1)$, we see that

$$f(t, x(t)) < C_1 r, \quad t \in \mathbb{N}_{\alpha-2}^{b+\alpha}.$$

By (2.5), we get

$$(Tx)(t) \leqslant \frac{\alpha}{\lambda(b+1)} \sum_{s=0}^{b} G(b+\alpha, s) f(s+\alpha-2, x(s+\alpha-2)) < \frac{\alpha C_1 r}{\lambda(b+1)} \sum_{s=0}^{b} G(b+\alpha, s) = r.$$

So, $\|Tx\| < \|x\|$, $x \in \partial\Omega_r$.

If $x \in \partial\Omega_R$, then $\frac{\lambda(b+1)\Gamma(\alpha-1)}{(b+\alpha)^{\underline{\alpha-1}}} R \leqslant x(t) \leqslant R$, $t \in \mathbb{N}_{\alpha-1}^{b+\alpha}$. From $(A_2)$, we get $f(t, x(t)) > C_3 R$, $t \in \mathbb{N}_{\alpha-1}^{b+\alpha}$. Using (2.5) again, we have

$$\begin{aligned}(Tx)(\alpha) &= \sum_{s=0}^{b} G(\alpha, s) f(s+\alpha-2, x(s+\alpha-2)) \\ &\geqslant \frac{\Gamma(\alpha)}{(b+\alpha)^{\underline{\alpha-1}}} \sum_{s=1}^{b} G(b+\alpha, s) f(s+\alpha-2, x(s+\alpha-2)) \\ &> \frac{\Gamma(\alpha)}{(b+\alpha)^{\underline{\alpha-1}}} C_3 R \sum_{s=1}^{b} G(\alpha+b, s) \\ &= R.\end{aligned}$$

So, $\|Tx\| > \|x\|$, $x \in \partial\Omega_R$.

Therefore, by Theorem 2.1, operator $T$ has at least one fixed point $x_0 \in \overline{\Omega}_R \backslash \Omega_r$ with $r < \|x_0\| < R$, and $x_0$ is a positive solution of SBVP (1.1). □

**Theorem 3.2.** *Assume that $(A_1)$ and $(H_1)$ hold. Then SBVP (1.1) has at least two positive solutions $x_1$ and $x_2$ with* $0 < \|x_1\| < r < \|x_2\|$.

**Proof.** By $(H_1)$, on the one hand, since $f_0 > C_2$, one can find $\varepsilon > 0$ and $0 < r_0 < r$ satisfy

$$f(t, x) \geqslant (C_2 + \varepsilon) x, \quad t \in \mathbb{N}_{\alpha-2}^{b+\alpha},\ 0 \leqslant x \leqslant r_0.$$

Let $r_1 \in (0, r_0)$, for $x \in \partial\Omega_{r_1}$, we have $\frac{\lambda(b+1)\Gamma(\alpha-1)}{(b+\alpha)^{\underline{\alpha-1}}} r_1 \leqslant x(t) \leqslant r_1 < r_0$, $t \in \mathbb{N}_{\alpha-1}^{b+\alpha}$. Then

$$\begin{aligned}(Tx)(\alpha) &\geqslant \frac{\Gamma(\alpha)}{(b+\alpha)^{\underline{\alpha-1}}} \sum_{s=1}^{b} G(b+\alpha, s) f(s+\alpha-2, x(s+\alpha-2)) \\ &\geqslant \frac{\Gamma(\alpha)(C_2+\varepsilon)}{(b+\alpha)^{\underline{\alpha-1}}} \sum_{s=1}^{b} G(b+\alpha, s) x(s+\alpha-2) \\ &> \frac{\lambda(b+1)\Gamma(\alpha-1)}{(b+\alpha)^{\underline{\alpha-1}}} \cdot \frac{\Gamma(\alpha) C_2 r_1}{(b+\alpha)^{\underline{\alpha-1}}} \sum_{s=1}^{b} G(b+\alpha, s) \\ &= r_1.\end{aligned}$$

So, $\|Tx\| > \|x\|$, $x \in \partial\Omega_{r_1}$.

On the other hand, since $f_\infty > C_2$, one can find $\eta > 0$ and $R_0 > 0$ satisfy

$$f(t,x) \geqslant (C_2+\eta)x, \quad t \in \mathbb{N}_{\alpha-2}^{b+\alpha},\ x \geqslant R_0.$$

Choosing

$$R_1 > \max\left\{ \frac{(b+\alpha)^{\underline{\alpha-1}}}{\lambda(b+1)\Gamma(\alpha-1)} R_0,\ r \right\},$$

for $x \in \partial\Omega_{R_1}$, there exists

$$x(t) \geqslant \frac{\lambda(b+1)\Gamma(\alpha-1)}{(b+\alpha)^{\underline{\alpha-1}}}\|x\| > R_0, \quad \alpha-1 \leqslant t \leqslant b+\alpha.$$

Then

$$\begin{aligned}(Tx)(\alpha) &\geqslant \frac{\Gamma(\alpha)(C_2+\eta)}{(b+\alpha)^{\underline{\alpha-1}}}\sum_{s=1}^{b} G(b+\alpha,s)x(s+\alpha-2)\\ &> \frac{\lambda(b+1)\Gamma(\alpha-1)}{(b+\alpha)^{\underline{\alpha-1}}}\cdot\frac{\Gamma(\alpha)C_2\|y\|}{(b+\alpha)^{\underline{\alpha-1}}}\sum_{s=1}^{b} G(b+\alpha,s)\\ &= R_1.\end{aligned}$$

As to $x \in \partial\Omega_r$, from $(A_1)$, we see that $f(t,x) < C_1 r$, $t \in \mathbb{N}_{\alpha-2}^{b+\alpha}$. This implies

$$\begin{aligned}(Tx)(t) &< \frac{\alpha C_1 r}{\lambda(b+1)}\sum_{s=0}^{b} G(b+\alpha,s)\\ &= r.\end{aligned}$$

Thus, $\|Tx\| < \|x\|$, $x \in \partial\Omega_r$. Theorem 2.1 implies the conclusions of this theorem. □

**Corollary 3.1.** *The conclusion of Theorem 3.2 is valid if* $(H_1)$ *is replaced by* $(H_1^*)$.

**Theorem 3.3.** *Assume that* $(A_2)$ *and* $(H_2)$ *hold. Then the SBVP (1.1) has at least two positive solutions* $x_1$ *and* $x_2$ *with* $0 < \|x_1\| < R < \|x_2\|$.

**Proof.** By $(H_2)$, since $\overline{f}_0 < C_1$, we may choose $\varepsilon > 0$ $(\varepsilon < C_1)$ and $0 < r_0 < R$ such that

$$f(t,x) \leqslant (C_1-\varepsilon)x, \quad t \in \mathbb{N}_{\alpha-2}^{b+\alpha},\ 0 \leqslant y \leqslant r_0.$$

Let $r_2 \in (0, r_0)$, for $x \in \partial\Omega_{r_2}$, by (2.5),

$$\begin{aligned}(Tx)(t) &\leqslant \frac{\alpha}{\lambda(b+1)}\sum_{s=0}^{b} G(b+\alpha,s)(C_1-\varepsilon)r_2\\ &< \frac{C_1 r_2 \alpha}{\lambda(b+1)}\sum_{s=0}^{b} G(b+\alpha,s)\\ &= r_2.\end{aligned}$$

So, $\|Tx\| < \|x\|$, $x \in \partial\Omega_{r_2}$.

For the case of $\overline{f}_\infty < C_1$, there exist $\sigma \in (0, C_1)$ and $R_0 > 0$ such that

$$f(t,x) \leqslant \sigma x, \quad t \in \mathbb{N}_{\alpha-2}^{b+\alpha},\ x \geqslant R_0.$$

Choosing $M = \max_{(t,x)\in\mathbb{N}_{\alpha-2}^{b+\alpha}\times[0,R_0]} f(t,x)$, there is $0 \leqslant f(t,x) \leqslant \sigma x + M$, $0 \leqslant x < +\infty$. Let $R_2 > \max\{R, \frac{M}{A-\sigma}\}$, then for $x \in \partial\Omega_{R_2}$, we have

$$\begin{aligned}\|Tx\| &\leqslant \frac{\alpha}{\lambda(b+1)}\sum_{s=0}^{b} G(b+\alpha,s)f(s+\alpha-2,x(s+\alpha-2))\\ &\leqslant \frac{\alpha}{\lambda(b+1)}(\sigma\|x\|+M)\sum_{s=0}^{b} G(b+\alpha,s)\\ &< R_2.\end{aligned}$$

So, $\|Tx\| < \|x\|$, $x \in \partial\Omega_{R_2}$.

Finally, if $x \in \partial\Omega_R$, then $x(t) \geqslant \frac{\lambda(b+1)\Gamma(\alpha-1)}{(\alpha+b)^{\underline{\alpha-1}}}\|x\|$, $t \in \mathbb{N}_{\alpha-1}^{b+\alpha}$. By $(A_2)$, we have

$$\begin{aligned}(Tx)(\alpha) &> \frac{\Gamma(\alpha)}{(b+\alpha)^{\underline{\alpha-1}}}RC_3\sum_{s=1}^{b} G(\alpha+b,s)\\ &= R.\end{aligned}$$

So, $\|Tx\| > \|x\|$, $x \in \partial\Omega_R$. Theorem 2.1 ensures the conclusions of this theorem. □

**Corollary 3.2.** *If $(A_2)$ holds, $(H_2)$ is replaced by $(H_2^*)$. Then the conclusion of Theorem 3.3 also holds.*

**Theorem 3.4.** *If one condition $(H_3)$ or $(H_4)$ holds. Then the SBVP (1.1) has at least one positive solution.*

**Proof.** Condition $(H_4)$ implies $(A_1)$ and $(A_2)$. By Theorem 3.1, we therefore obtain the stated conclusion. Moreover, condition $(H_3)$ implies that the operator $T$ stretches at zero and compresses at infinity. Hence the conclusion follows by the argument used in the proof of Theorem 3.1. □

**Corollary 3.3.** *If one of the two following conditions holds*

*(i)* $\underline{f}_0 = +\infty$ *and* $\overline{f}_\infty = 0$;

*(ii)* $\underline{f}_0 = 0$ *and* $\underline{f}_\infty = \infty$.

*Then the SBVP (1.1) has at least one positive solution.*

## 4. Examples

To explain the results obtained in Section 3, we construct two concrete examples. Let $\alpha = \frac{35}{16}$, $b = 20$ and $\lambda = \frac{1}{64}$. We can calculate the estimated values of $C_1 \approx 0.00248$ and $C_2 \approx 9.61983$.

**Example 4.1.** Consider the following SBVP

$$\begin{cases}\triangle_{-\frac{13}{16}}^{\frac{35}{16}} x(t) = -\dfrac{1}{140}e^{t-21}\left(\dfrac{1}{3}x^{\frac{1}{3}}(t+\dfrac{3}{16}) + \dfrac{1}{16}x^{\frac{5}{3}}(t+\dfrac{3}{16})\right),\\ x(-\dfrac{13}{16}) = \triangle x(-\dfrac{13}{16}) = 0,\\ x(\dfrac{355}{16}) = \dfrac{1}{64}\displaystyle\sum_{s=0}^{21} x(s+\dfrac{3}{16}),\end{cases} \tag{4.1}$$

where

$$f(t,x)=\frac{1}{140}e^{t-\frac{355}{16}}\left(\frac{1}{3}x^{\frac{1}{3}}+\frac{1}{16}x^{\frac{5}{3}}\right).$$

By simple calculation, we have $\underline{f}_0=\underline{f}_\infty=+\infty$. Choosing $r=8$, then $f(t,x)\leqslant 0.01905<8C_1$, $t\in\mathbb{N}_{\alpha-2}^{b+\alpha}$, and $0\leqslant x\leqslant 8$. Thus all conditions of Corollary 3.1 are satisfied. Therefore, the SBVP (4.1) has at least two positive solutions $x_1$ and $x_2$ such that $0<\|x_1\|<8<\|x_2\|$. Figure 1 illustrates numerical results of the two solutions for Example 4.1 by Matlab. The values of solutions are also presented in the table Figure 2. The numerical solutions align with the theoretical results.

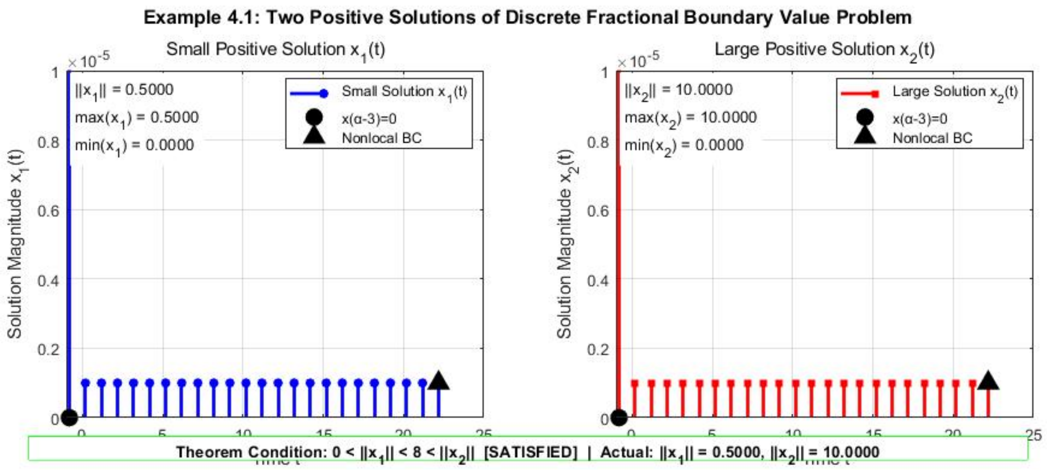
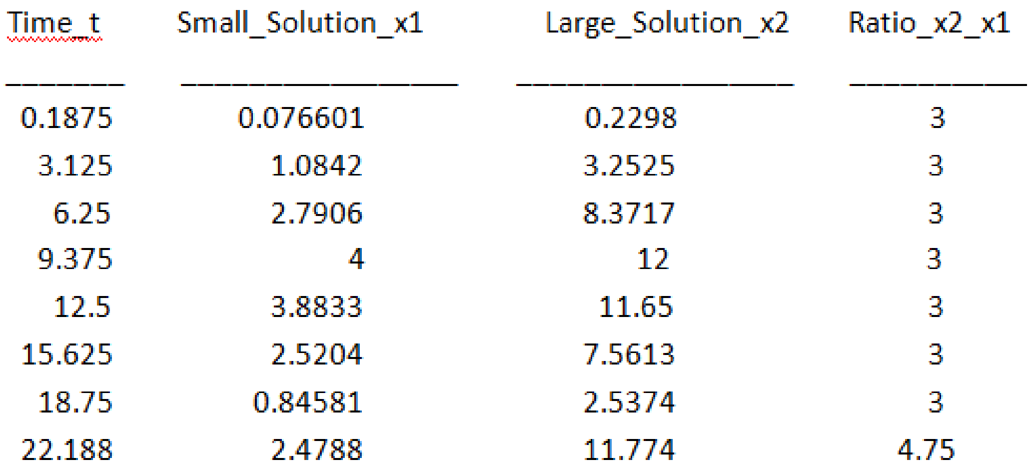


**Figure 1.** Example 4.1 simulation.



| Time_t | Small_Solution_x1 | Large_Solution_x2 | Ratio_x2_x1 |
|---|---|---|---|
| 0.1875 | 0.076601 | 0.2298 | 3 |
| 3.125 | 1.0842 | 3.2525 | 3 |
| 6.25 | 2.7906 | 8.3717 | 3 |
| 9.375 | 4 | 12 | 3 |
| 12.5 | 3.8833 | 11.65 | 3 |
| 15.625 | 2.5204 | 7.5613 | 3 |
| 18.75 | 0.84581 | 2.5374 | 3 |
| 22.188 | 2.4788 | 11.774 | 4.75 |

**Figure 2.** Example 4.1 numerical values.

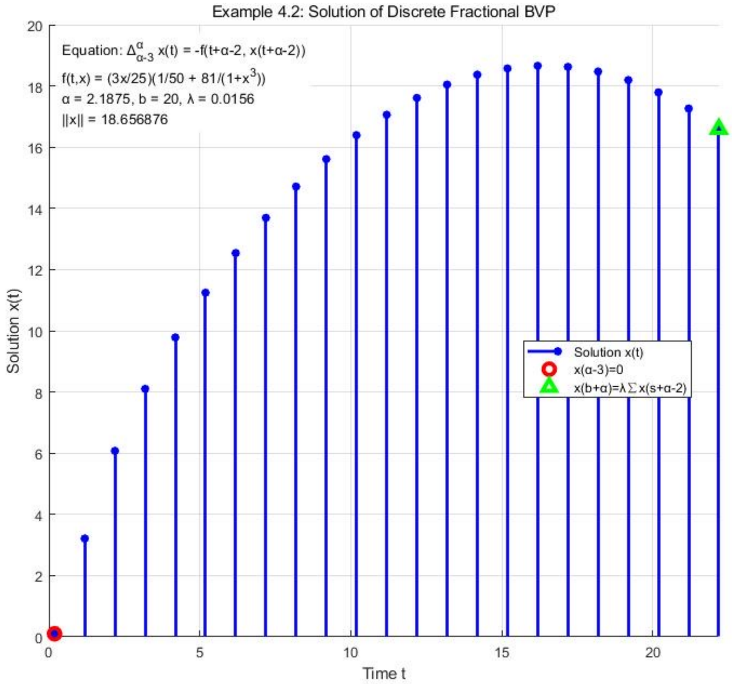


**Figure 3.** Example 4.2 simulation.

**Example 4.2.** Consider the following SBVP

$$\begin{cases} \triangle^{\frac{35}{16}}_{-\frac{13}{16}} x(t) = -\dfrac{3}{25} x(t+\dfrac{3}{16}) \left( \dfrac{1}{50} + \dfrac{81}{1+x^3(t+\frac{3}{16})} \right), \\ x(-\dfrac{13}{16}) = \triangle x(-\dfrac{13}{16}) = 0, \\ x(\dfrac{355}{16}) = \dfrac{1}{64} \displaystyle\sum_{s=0}^{21} x(s+\dfrac{3}{16}), \end{cases} \tag{4.2}$$

where $f(t,x) = \frac{3x}{25}\left(\frac{1}{50} + \frac{81}{1+x^3}\right)$. By calculation, we can obtain that $\overline{f}_{\infty} = 0.0024 < C_1$ and $\underline{f}_0 = 9.7224 > C_2$, which yields to the condition $(H_3)$. With Theorem 3.4, SBVP (4.2) has at least one positive solution. Numerical simulation for this example is presented in Figure 3.

## Acknowledgements

The research was supported by the Natural Science Foundation of Shanxi Province (202303021221168), Shanxi Province and Shanxi Datong University Science and Technology Innovation Team Project (2020CXTDZ1), and the Natural Sciences and Engineering Research Council of Canada (NSERC). The authors thank the anonymous reviewers for their detailed and constructive comments.

## References

[1] T. Abdeljawad, *On delta and nabla Caputo fractional differences and dual identities*, Discrete Dyn. Nat. Soc., 2013, 406910.

[2] T. Abdeljawad and D. Baleanu, *On fractional derivatives with exponential kernel and their discrete versions*, Rep. Math. Phys., 2017, 80, 11–27.

[3] H. Antil, R. Khatri, R. Lohner and D. Verma, *Fractional deep neural network via constrained optimization*, Mach. Learn. Sci. Technol., 2019, 2, 015003.

[4] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model*, Therm. Sci., 2016, 20, 763–769.

[5] F. M. Atici and S. Sengül, *Modeling with fractional difference equations*, J. Math. Anal. Appl., 2010, 369, 1–9.

[6] S. Bhalekar, P. M. Gade and D. Joshi, *Stability and dynamics of complex order fractional difference equations*, Chaos, Solitons and Fractals, 2022, 158, 112063.

[7] A. Cabada, N. D. Dimitrov and J. M. Jonnalagadda, *Green's functions for fractional difference equations with Dirichlet boundary conditions*, Chaos, Solitons and Fractals, 2021, 153, 111455.

[8] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Prog. Fract. Differ. Appl., 2015, 1, 73–85.

[9] H. Chen, Z. Jin and S. Kang, *Existence of positive solutions for Caputo fractional difference equation*, Adv. Differ. Equ., 2015, 44, 1–12.

[10] Y. Cui, *Uniqueness of solution for boundary value problems for fractional differential equations*, Appl. Math. Lett., 2016, 51, 48–54.

[11] Y. Cui, S. Kang and H. Chen, *Uniqueness of solutions for an integral boundary value problem with fractional q-differences*, J. Appl. Anal. Comput., 2018, 8, 525–532.

[12] N. D. Dimitrov and J. M. Jonnalagadda, *Existence of positive solutions for a class of nabla fractional difference equations with parameter-dependent summation boundary conditions*, Fractal Fract., 2025, 9(8), 513.

[13] N. D. Dimitrov and J. M. Jonnalagadda, *Existence, uniqueness and stability of solutions of a nabla fractional difference equations*, Fractal Fract., 2024, 8, 591.

[14] W. Feng, *Topological methods on solvability, multiplicity and eigenvalues of a nonlinear fractional boundary value problem*, Electronic Journal of Qualitative Theory of Differential Equations, 2015, 70, 1–16.

[15] W. Feng and J. R. L. Webb, *Solvability of $m$-point boundary value problems with nonlinear growth*, J. Math. Anal. Appl., 1997, 212(2), 467–480.

[16] C. S. Goodrich, B. Lyons and M. T. Velcsov, *Analytical and numerical monotonicity results for discrete fractional sequential differences with negative lower bound*, Commun. Pur. Appl. Anal., 2021, 20, 339–358.

[17] C. S. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Springer International Publishing, 2015.

[18] D. J. Guo, *Nonlinear Functional Analysis*, Shandong Science and Technology Press, Shandong, 1985.

[19] M. Holm, *Sum and difference compositions in discrete fractional calculus*, Cubo., 2011, 13, 153–184.

[20] S. M. AI-Issa, A. M. A. El-Sayed and H. H. G. Hashem, *An outlook on hybrid fractional modelling of a heat controller with multi-valued feedback control*, Fractal Fract., 2023, 7(10).

[21] P. O. Mohammed and T. Abdeljawad, *Discrete generalized fractional operators defined using h-discrete Mittag-Leffler kernels and applications to AB fractional difference systems*, Math. Methods Appl. Sci., 2023, 46, 7688–7713.

[22] K. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[23] J. Reunsumrit and T. Sitthiwirattham, *Existence results of fractional delta–nabla difference equations via mixed boundary conditions*, Advances in Difference Equations, 2020, 18, 370.

[24] A. Sharma, S. N. Mishra and A. Shukla, *Stability analysis of nonlinear fractional discrete dynamical systems*, Circuits Syst. Signal Processing, 2025.

[25] B. Shiri, G. Wu and D. Baleanu, *Collocation methods for terminal value problems of tempered fractional differential equations*, Appl. Numer. Math., 2020, 156, 385–395.

[26] H. M. Srivastava, P. O. Mohammed, C. S. Ryoo and Y. S. Hamed, *Existence and uniqueness of a class of uncertain Liouville-Caputo fractional difference equations*, J. King Saud Univ. Sci., 2021, 33, 101497.

[27] M. Vellappandi and S. Lee, *Physics-informed neural fractional differential equations*, Applied Mathematical Modelling, 2025, 145, 116–127.

[28] G. Wu, D. Baleanub and W. Luo, *Lyapunov functions for Riemann Liouville like fractional difference equations*, Appl. Math. Comput., 2017, 314, 228–236.

[29] M. Wu, J. Zhang, Z. Huang, X. Li and Y. Dong, *Numerical solutions of wavelet neural networks for fractional differential equations*, Math. Methods Appl. Sci., 2021.

[30] Y. Zhou, *Attractivity for fractional differential equations in Banach space*, Appl. Math. Lett., 2018, 75, 1–6.

Received June 2025; Accepted February 2026; Available online March 2026.