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EXISTENCE RESULTS FOR SYSTEM OF GENERALIZED HYBRID PANTOGRAPH EQUATIONS OF SEQUENTIAL TYPE

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Abstract The aim of this work is to prove the existence and uniqueness of solutions for a coupled system that generalizes hybrid pantograph equations incorporating some fractional operators of Caputo and Riemann-Liouville types. This is achieved throughout Banach and Leray-Schauder fixed point theorems. In addition, we investigate the stability by using the Ulam-Hyer technique, and an illustrative example is provided to show the validity of our findings.

Keywords Hybrid, equations of pantograph, stability, fractional operators.

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1. Introduction

Fractional differential equations have attracted considerable attention over the past decades, due to their ability to model memory and nonlocal effects inherent in many physical and engineering systems. These equations have been successfully applied in diverse fields such as biophysics, bioengineering, signal and image processing, viscoelasticity, and optical communication. For further details, the reader is referred to [12, 14, 18, 21]. Recent developments in the broader field of fractional differential equations have further increased interest in this topic, leading to a substantial expansion of the literature; see, for example, [2, 20, 22, 24]. These results motivate further investigation of generalized fractional pantograph equations and provide a solid

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theoretical foundation for our analysis.

Pantograph-type equations constitute an important subclass of functional differential equations in which the delay depends proportionally on the independent variable. Owing to this distinctive feature, such equations have attracted sustained interest from researchers in both theoretical and applied mathematics. They have been successfully employed in a variety of scientific fields, including stochastic processes, nonlinear dynamical analysis, electromagnetic theory, and quantum physics. The extension of these models to the fractional-order setting has significantly enhanced their descriptive power, making them particularly suitable for capturing systems in which memory effects coexist with delayed responses. For studies on fuzzy fractional pantograph equations, the reader is referred to [1,7,11,15,17,25], while investigations addressing multi-pantograph fractional models can be found in [5,6,8,9].

The authors in [16] examined the following generalized fractional hybrid pantograph equation:

$$\begin{cases} D_{0+}^{\delta}\left[\dfrac{\sigma(v)}{\omega(v,\sigma(v),\sigma(\tau(v)))}\right]=\mu(v,\sigma(v),\sigma(\rho(v))), \\ \sigma(0)=0, \quad v\in[0,1], \end{cases}$$

where D_{0+}^{δ} represents the (RL) Riemann-Liouville fractional derivative of order δ within the interval $[0,1]$, and the functions τ and ρ are defined on $[0,1]$.

In [10], the authors established the existence and uniqueness of a fractional pantograph equation given by the form:

$$\begin{cases} {}^{RL}D^{\vartheta}\left(\dfrac{\varpi(t)}{\zeta(t,\varpi(t),\varpi(\kappa t))}\right)=\phi(s,\varpi(t),\varpi(\iota t)),\ t\in[0,1], \\ \varpi(0)=0,\ 0<\vartheta\leq 1,\ 0<\kappa,\iota<1, \end{cases} \tag{1.1}$$

where ${}^{RL}D^{\vartheta}$ denote the Riemann-Liouville fractional derivative, $\zeta:[0,1]\times\mathbb{R}^2\longrightarrow\mathbb{R}\backslash\{0\}$ and $\phi:[0,1]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are two continuous functions.

Recently, the authors in [13], have considered the following problem:

$$\begin{cases} {}^{RL}D^{\gamma}\left({}^{C}D^{\delta}\left(\dfrac{w(x)}{p(x,w(x),w(\theta x))}\right)\right)=q(x,w(x),w(\lambda x)),\ 0\leq x\leq 1, \\ w(0)=0, \quad w(1)=0,\ 0<\theta,\lambda<1,\ 0<\gamma,\delta\leq 1, \end{cases} \tag{1.2}$$

where ${}^{RL}D^{\gamma}$ and ${}^{C}D^{\delta}$ denote the fractional derivatives of Riemann-Liouville and Caputo, respectively. The functions $p:[0,1]\times\mathbb{R}^2\to\mathbb{R}\backslash\{0\}$ and $q:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are given continuous functions.

In this paper, we analyze the following hybrid pantograph system:

$$\begin{cases} \left({}^{C}D_{0+}^{\vartheta}\,{}^{RL}D_{0+}^{\rho}\right)\left(\dfrac{\hat{\mathfrak{U}}(\mathfrak{r})}{\phi_1(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))}\right)=\mathcal{H}_1(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\zeta_2(\mathfrak{r}))), \\ \left({}^{RL}D_{0+}^{\upsilon}\,{}^{C}D_{0+}^{\nu}\right)\left(\dfrac{\hat{\mathfrak{V}}(\mathfrak{r})}{\phi_2(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})))}\right)=\mathcal{H}_2(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{r}))),\ 0<\vartheta,\rho,\upsilon,\nu\leq 1, \\ \hat{\mathfrak{U}}(0)=0,\ \phi_1(\varsigma_1,\hat{\mathfrak{U}}(\varsigma_1),\hat{\mathfrak{V}}(\zeta_1(\varsigma_1)))\hat{\mathfrak{U}}(1)=\phi_1(1,\hat{\mathfrak{U}}(1),\hat{\mathfrak{V}}(\zeta_1(1)))\hat{\mathfrak{U}}(\varsigma_1),\varsigma_1\in\,]0,1[\,, \\ \hat{\mathfrak{V}}(0)=0,\ \phi_2(\varsigma_2,\hat{\mathfrak{U}}(\varsigma_2),\hat{\mathfrak{V}}(\varepsilon_1(\varsigma_2)))\hat{\mathfrak{U}}(1)=\phi_2(1,\hat{\mathfrak{U}}(1),\hat{\mathfrak{U}}(\varepsilon_1(1)))\hat{\mathfrak{V}}(\varsigma_2),\varsigma_2\in\,]0,1[\,, \end{cases} \tag{1.3}$$

where ${}^{C}D^{\vartheta}$, ${}^{C}D^{\nu}$, ${}^{RL}D^{\rho}$, and ${}^{RL}D^{\upsilon}$ are the Caputo and RL fractional derivatives, respectively. The functions $\phi_1, \phi_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \setminus \{0\}$ and $\mathcal{H}_1, \mathcal{H}_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions, and ζ_i, ε_i (for $i = 1, 2$) are continuous functions mapping $[0,1]$ into itself.

2. Basic concepts and results

The following auxiliary results can be found in [19, 23].

Definition 2.1. Let $\mathfrak{y} > 0$. The Riemann-Liouville integral of order $\mathfrak{y}$ for any $\psi \in L^1[0,1]$ is:

$$I^{\mathfrak{y}}\psi(\mathfrak{v}) = \frac{1}{\Gamma(\mathfrak{y})} \int_0^{\mathfrak{v}} (\mathfrak{v} - \varepsilon)^{\mathfrak{y}-1} \psi(\varepsilon) d\varepsilon, \quad \text{for } \mathfrak{v} \in [0,1]. \tag{2.1}$$

Definition 2.2. We consider $\psi \in C^m[0,1]$. The Caputo fractional derivatives of ψ is given by:

$${}^{C}D^{\rho}\psi(\mu) = \frac{1}{\Gamma(m - \sigma)} \int_0^{\mu} \frac{\psi^{(m)}(\xi)}{(\mu - \xi)^{\sigma+1-m}} d\xi, \quad \mu > 0.$$

Definition 2.3. Given $\sigma > 0$, then the Riemann-Liouville fractional derivatives for $\psi : (0, \infty) \to \mathbb{R}$ is defined as:

$${}^{RL}D^{\rho}\psi(\mu) = \frac{1}{\Gamma(m - \sigma)} \left(\frac{d}{d\mu}\right)^m \int_0^{\mu} \frac{\psi(\xi)}{(\mu - \xi)^{\sigma+1-m}} d\xi, \quad \mu > 0.$$

For the reader the following lemmas are needed.

Lemma 2.1. *Let $\sigma > 0$. If $y \in L^1(0,1) \cap C(0,1)$, and ${}^{RL}D^{\sigma}y \in C(0,1) \cap L^1(0,1)$, then we have:*

$$I^{\sigma}\left({}^{RL}D^{\sigma}y(\xi)\right) = \sum_{i=1}^{\mathfrak{p}} b_i \xi^{\sigma-i} + y(\xi),$$

for some $b_i \in \mathbb{R}, i = \overline{1, \mathfrak{p}}$, $\mathfrak{p} = [\sigma] + 1$.

Lemma 2.2. *Given $z \in C^m[0,1], m \in \mathbb{N}$. For $\gamma \in]m-1, m]$, then we have*

$$I^{\gamma}({}^{C}D^{\gamma}z(\xi)) = z(\xi) + \sum_{i=0}^{m-1} a_i \xi^i,$$

in which $a_i \in \mathbb{R}, i = 0, 1, 2, \ldots, m-1, m = [\gamma] + 1$.

Theorem 2.1. [4] *Let $\mathcal{B}$ be a convex subset of a Banach space $\mathcal{E}$, and assume $0 \in \mathcal{B}$. Let $\mathcal{T} : \mathcal{B} \longrightarrow \mathcal{B}$ be a completely continuous operator, and let*

$$\mathcal{U} = \{x \in \mathcal{B} : x = \lambda \mathcal{T}(x) \text{ for some } 0 < \lambda < 1\}.$$

Then, either the set $\mathcal{U}$ is unbounded or $\mathcal{T}$ has at least one fixed point.

We start by proving the following key result which will be useful in the sequel:

Lemma 2.3. *Let* $0 < \vartheta, \rho, \upsilon, \nu \leq 1$. *If* ϕ_1 *and* ϕ_2 *belong to* $C([0,1] \times \mathbb{R}^2, \mathbb{R} - \{0\})$, *and* $\widehat{w}_1, \widehat{w}_2 \in C([0,1], \mathbb{R})$*), then the solution of the problem*

$$\begin{cases} \left({}^{C}D^{\vartheta}_{0^+}\, {}^{RL}D^{\rho}_{0+} \right) \left(\dfrac{\hat{\mathfrak{U}}(\mathfrak{r})}{\phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))} \right) = \widehat{w}_1(\mathfrak{r}), \\ \left({}^{RL}D^{\upsilon}_{0^+}\, {}^{C}D^{\nu}_{0+} \right) \left(\dfrac{\hat{\mathfrak{V}}(\mathfrak{r})}{\phi_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})))} \right) = \widehat{w}_2(\mathfrak{r}), \\ \hat{\mathfrak{U}}(0) = 0,\ \phi_1(\varsigma_1, \hat{\mathfrak{U}}(\varsigma_1), \hat{\mathfrak{V}}(\zeta_1(\varsigma_1)))\hat{\mathfrak{U}}(1) = \phi_1(1, \hat{\mathfrak{U}}(1), \hat{\mathfrak{V}}(\zeta_1(1)))\hat{\mathfrak{U}}(\varsigma_1), \varsigma_1 \in \,]0,1[\,, \\ \hat{\mathfrak{V}}(0) = 0,\ \phi_2(\varsigma_2, \hat{\mathfrak{U}}(\varsigma_2), \hat{\mathfrak{V}}(\varepsilon_1(\varsigma_2)))\hat{\mathfrak{U}}(1) = \phi_2(1, \hat{\mathfrak{U}}(1), \hat{\mathfrak{U}}(\varepsilon_1(1)))\hat{\mathfrak{V}}(\varsigma_2), \varsigma_2 \in \,]0,1[\,, \end{cases} \tag{2.2}$$

is given by

$$\hat{\mathfrak{U}}(\mathfrak{r}) = \phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r}))) \left(I^{\vartheta+\rho}\widehat{w}_1(\mathfrak{r}) + \left(I^{\vartheta+\rho}\widehat{w}_1(\varsigma_1) - I^{\vartheta+\rho}\widehat{w}_1(1) \right) \frac{\mathfrak{r}^{\rho}}{1 - \varsigma_1^{\rho}} \right),$$

and

$$\hat{\mathfrak{V}}(\mathfrak{r}) = \phi_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r}))) \left(I^{\upsilon+\nu}\widehat{w}_2(\mathfrak{r}) + \left(I^{\upsilon+\nu}\widehat{w}_1(\varsigma_2) - I^{\upsilon+\nu}\widehat{w}_1(1) \right) \frac{\mathfrak{r}^{\upsilon+\nu-1}}{1 - \varsigma_2^{\upsilon+\nu-1}} \right).$$

Proof. By applying the operator I^{ϑ} to the first equation in (2.2) and utilizing Lemma (2.2), we get

$${}^{RL}D^{\rho}_{0+} \left(\frac{\hat{\mathfrak{U}}(\mathfrak{r})}{\phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))} \right) = I^{\vartheta}\widehat{w}_1(\mathfrak{r}) + a_0,\ a_0 \in \mathbb{R}. \tag{2.3}$$

Taking I^{ρ} for (2.3) and by Lemma (2.1), we get

$$\frac{\hat{\mathfrak{U}}(\mathfrak{r})}{\phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))} = I^{\vartheta+\rho}\widehat{w}_1(\mathfrak{r}) + a_0 \frac{\mathfrak{r}^{\rho}}{\Gamma(\rho+1)} + b_0 \mathfrak{r}^{\rho-1},\ b_0 \in \mathbb{R}. \tag{2.4}$$

By using $\hat{\mathfrak{U}}(0) = 0$, we obtain $b_0 = 0$. Equation (2.4) becomes

$$\frac{\hat{\mathfrak{U}}(\mathfrak{r})}{\phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))} = I^{\vartheta+\rho}\widehat{w}_1(\mathfrak{r}) + a_0 \frac{\mathfrak{r}^{\rho}}{\Gamma(\rho+1)}, \tag{2.5}$$

and the condition $\phi_1(\varsigma_1, \hat{\mathfrak{U}}(\varsigma_1), \hat{\mathfrak{V}}(\zeta_1(\varsigma_1)))\hat{\mathfrak{U}}(1) = \phi_1(1, \hat{\mathfrak{U}}(1), \hat{\mathfrak{V}}(\zeta_1(1)))\hat{\mathfrak{U}}(\varsigma_1)$ yields

$$a_0 = \frac{\Gamma(\rho+1)}{1 - \varsigma_1^{\rho}} \left(I^{\vartheta+\rho}\widehat{w}_1(\varsigma_1) - I^{\vartheta+\rho}\widehat{w}_1(1) \right).$$

Inserting a_0 in equation (2.5) gives

$$\frac{\hat{\mathfrak{U}}(\mathfrak{r})}{\phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))} = I^{\vartheta+\rho}\widehat{w}_1(\mathfrak{r}) + \left(I^{\vartheta+\rho}\widehat{w}_1(\varsigma_1) - I^{\vartheta+\rho}\widehat{w}_1(1) \right) \frac{\mathfrak{r}^{\rho}}{1 - \varsigma_1^{\rho}}.$$

Consequently, we have

$$\hat{\mathfrak{U}}(\mathfrak{r}) = \phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r}))) \left(I^{\vartheta+\rho}\widehat{w}_1(\mathfrak{r}) + \left(I^{\vartheta+\rho}\widehat{w}_1(\varsigma_1) - I^{\vartheta+\rho}\widehat{w}_1(1) \right) \frac{\mathfrak{r}^{\rho}}{1 - \varsigma_1^{\rho}} \right).$$

For the second equation of (2.2), we utilize the RL fractional integral with order γ to get

$$ {}^{C}D_{0^+}^{\nu}\left(\frac{\hat{\mathfrak{V}}(\mathfrak{r})}{\phi_2(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})))}\right)=I^{\upsilon}\widehat{w}_2(\mathfrak{r})+c_0\mathfrak{r}^{\upsilon-1},\ c_0\in\mathbb{R}. \tag{2.6}$$

Again, By applying the Riemann–Liouville integral, we obtain

$$\frac{\hat{\mathfrak{V}}(\mathfrak{r})}{\phi_2(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})))}=I^{\upsilon+\nu}\widehat{w}_2(\mathfrak{r})+c_0\frac{\Gamma(\upsilon)}{\Gamma(\upsilon+\nu)}\mathfrak{r}^{\upsilon+\nu-1}+d_0,\ d_0\in\mathbb{R}. \tag{2.7}$$

Applying the condition $\hat{\mathfrak{V}}(0)=0$, gives us $d_0=0$..

According to the condition $\phi_2(\varsigma_2,\hat{\mathfrak{U}}(\varsigma_2),\hat{\mathfrak{V}}(\varepsilon_1(\varsigma_2)))\hat{\mathfrak{U}}(1)=\phi_2(1,\hat{\mathfrak{U}}(1),\hat{\mathfrak{U}}(\varepsilon_1(1)))\hat{\mathfrak{V}}(\varsigma_2)$ we get

$$I^{\upsilon+\nu}\widehat{w}_2(1)+c_0\frac{\Gamma(\nu)}{\Gamma(\upsilon+\nu)}=I^{\upsilon+\nu}\widehat{w}_2(\varsigma_2)+c_0\frac{\Gamma(\nu)}{\Gamma(\upsilon+\nu)}\varsigma_2^{\upsilon+\nu-1}.$$

Hence, from the above equation, we obtain

$$c_0=\frac{\Gamma(\upsilon+\nu)}{(1-\varsigma_2^{\upsilon+\nu-1})\Gamma(\nu)}\left(I^{\upsilon+\nu}\widehat{w}_2(\varsigma_2)-I^{\upsilon+\nu}\widehat{w}_2(1)\right).$$

Substituting the value of c_0 in (2.7), we obtain

$$\hat{\mathfrak{V}}(\mathfrak{r})=\phi_2(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})))\left(I^{\upsilon+\nu}\widehat{w}_2(\mathfrak{r})+\left(I^{\upsilon+\nu}\widehat{w}_2(\varsigma_2)-I^{\upsilon+\nu}\widehat{w}_2(1)\right)\frac{\mathfrak{r}^{\upsilon+\nu-1}}{(1-\varsigma_2^{\upsilon+\nu-1})}\right).$$

Thus the proof is established. □

3. Main results

We denote by $\mathcal{F}=\mathfrak{C}([0,1],\mathbb{R})$ the Banach space of continuous functional from $[0,1]$ to $\mathbb{R}$, with the norm $\|\hat{\mathfrak{U}}\|=\sup\{|\hat{\mathfrak{U}}(\mathfrak{r})|:\mathfrak{r}\in[0,1]\}$. Considering $\mathcal{F}=\mathfrak{C}\times\mathfrak{C}$, $(\mathcal{F},\|(\cdot,\cdot)\|)$ is a Banach space with:

$$\|(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\|=\|\hat{\mathfrak{U}}\|+\|\hat{\mathfrak{V}}\|.$$

Using Lemma (2.3), we can transform (1.3) as a fixed-point problem in the following manner:

$$\begin{aligned}&\mathfrak{K}:\mathcal{G}\longrightarrow\mathcal{G},\\&\mathfrak{K}(\hat{\mathfrak{U}},\hat{\mathfrak{V}})(\mathfrak{r})=\Big(\mathfrak{K}_1(\hat{\mathfrak{U}},\hat{\mathfrak{V}})(\mathfrak{r}),\mathfrak{K}_2(\hat{\mathfrak{U}},\hat{\mathfrak{V}})(\mathfrak{r})\Big),\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\mathfrak{K}_1(\hat{\mathfrak{U}},\hat{\mathfrak{V}})(\mathfrak{r})=\,&\phi_1(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))\times\left(\frac{1}{\Gamma(\vartheta+\rho)}\int_0^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{k})^{\vartheta+\rho-1}\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}(\mathfrak{k}),\hat{\mathfrak{V}}(\zeta_2(\mathfrak{k})))\,d\mathfrak{k}\right.\\&+\frac{\mathfrak{r}^{\rho}}{(1-\varsigma_1^{\rho})\,\Gamma(\vartheta+\rho)}\times\left[\int_0^{\varsigma_1}(\varsigma_1-\mathfrak{k})^{\vartheta+\rho-1}\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}(\mathfrak{k}),\hat{\mathfrak{V}}(\zeta_2(\mathfrak{k})))\,d\mathfrak{k}\right.\\&\left.\left.-\int_0^{1}(1-\mathfrak{k})^{\vartheta+\rho-1}\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}(\mathfrak{k}),\hat{\mathfrak{V}}(\zeta_2(\mathfrak{k})))\,d\mathfrak{k}\right]\right),\end{aligned}$$

and

$$\begin{aligned}\mathfrak{K}_2(\hat{\mathfrak{U}},\hat{\mathfrak{V}})(\mathfrak{r}) = \phi_2(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r}))) \times \Bigg(&\frac{1}{\Gamma(\upsilon+\nu)}\int_0^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{k})^{\upsilon+\nu-1}\mathcal{H}_2(\mathfrak{k},\hat{\mathfrak{U}}(\mathfrak{k}),\hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{k})))\,d\mathfrak{k}\\ &+\frac{\mathfrak{r}^{\upsilon+\nu-1}}{\left(1-\varsigma_2^{\upsilon+\nu-1}\right)\Gamma(\upsilon+\nu)}\times\left[\int_0^{\varsigma_2}(\varsigma_2-\mathfrak{k})^{\upsilon+\nu-1}\mathcal{H}_2(\mathfrak{k},\hat{\mathfrak{U}}(\mathfrak{k}),\hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{k})))\,d\mathfrak{k}\right.\\ &\left.-\int_0^1(1-\mathfrak{k})^{\upsilon+\nu-1}\mathcal{H}_2(\mathfrak{k},\hat{\mathfrak{U}}(\mathfrak{k}),\hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{k})))\,d\mathfrak{k}\right]\Bigg).\end{aligned}$$

We start by introducing the following assumptions:

(A1): We suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ continuous, and there are positive constants $L_{\mathcal{H}_1}$ and $L_{\mathcal{H}_2}$ such that for all $\mathfrak{r}\in[0,1]$, $\hat{\mathfrak{U}}_i,\hat{\mathfrak{V}}_i\in\mathbb{R},(i=1,2)$, we have

$$\left|\mathcal{H}_1\left(\mathfrak{r},\hat{\mathfrak{U}}_1,\hat{\mathfrak{V}}_1\right)-\mathcal{H}_1\left(\mathfrak{r},\hat{\mathfrak{U}}_2,\hat{\mathfrak{V}}_2\right)\right|\le L_{\mathcal{H}_1}\left(\left|\hat{\mathfrak{U}}_1-\hat{\mathfrak{U}}_2\right|+\left|\hat{\mathfrak{V}}_1-\hat{\mathfrak{V}}_2\right|\right),$$

and

$$\left|\mathcal{H}_2\left(\mathfrak{r},\hat{\mathfrak{U}}_1,\hat{\mathfrak{V}}_1\right)-\mathcal{H}_2\left(\mathfrak{r},\hat{\mathfrak{U}}_2,\hat{\mathfrak{V}}_2\right)\right|\le L_{\mathcal{H}_2}\left(\left|\hat{\mathfrak{U}}_1-\hat{\mathfrak{U}}_2\right|+\left|\hat{\mathfrak{V}}_1-\hat{\mathfrak{V}}_2\right|\right).$$

(A2): We suppose that ϕ and $\widehat{\phi}$ are continuous and bounded, that is, there exist $\varpi_1,\varpi_2>0$ such that

$$\left|\phi_1(\mathfrak{r},\hat{\mathfrak{U}},\hat{\mathfrak{V}})\right|\le\varpi_1,\quad\left|\phi_2(\mathfrak{r},\hat{\mathfrak{U}},\hat{\mathfrak{V}})\right|\le\varpi_2,\quad\forall\mathfrak{r}\in[0,1],(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\in\mathbb{R}^2.$$

(A3): Suppose that $\mathcal{S}$ is a bounded subset of $\mathcal{G}$. Then there exist $\widehat{\psi}_1,\widehat{\psi}_2>0$ such that

$$\left|\mathcal{H}_1(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(t\mathfrak{r}))\right|\le\widehat{\psi}_1,\quad\left|\mathcal{H}_2(\mathfrak{r},\hat{\mathfrak{U}}(\mathfrak{r}),\hat{\mathfrak{V}}(\mathfrak{r}))\right|\le\widehat{\psi}_2,\quad\forall(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\in\mathcal{S}.$$

(A4): Let the functions ϕ_1,ϕ_2 are assumed to be continuous, and $\exists L_{\phi_1},L_{\phi_2}>0$ such that

$$\left|\phi_1\left(\mathfrak{r},\hat{\mathfrak{U}}_1,\hat{\mathfrak{V}}_1\right)-\phi_1\left(\mathfrak{r},\hat{\mathfrak{U}}_2,\hat{\mathfrak{V}}_2\right)\right|\le L_{\phi_1}\left(\left|\hat{\mathfrak{U}}_1-\hat{\mathfrak{U}}_2\right|+\left|\hat{\mathfrak{V}}_1-\hat{\mathfrak{V}}_2\right|\right),$$

and

$$\left|\phi_2\left(\mathfrak{r},\hat{\mathfrak{U}}_1,\hat{\mathfrak{V}}_1\right)-\phi_2\left(\mathfrak{r},\hat{\mathfrak{U}}_2,\hat{\mathfrak{V}}_2\right)\right|\le L_{\phi_2}\left(\left|\hat{\mathfrak{U}}_1-\hat{\mathfrak{U}}_2\right|+\left|\hat{\mathfrak{V}}_1-\hat{\mathfrak{V}}_2\right|\right),$$

for all $\mathfrak{r}\in[0,1]$, $\hat{\mathfrak{U}}_i,\hat{\mathfrak{V}}_i\in\mathbb{R},(i=1,2)$.

Also, we introduce the following notations to simplify formulas of our estimates:

$$\begin{aligned}\mathcal{K}_1&:=\frac{1}{\Gamma(\vartheta+\rho+1)}+\frac{2}{|1-\varsigma_1^{\rho}|\,\Gamma(\vartheta+\rho+1)},\\ \mathcal{K}_2&:=\frac{1}{\Gamma(\upsilon+\nu+1)}+\frac{2}{\left|1-\varsigma_2^{\upsilon+\nu-1}\right|\Gamma(\upsilon+\nu+1)},\\ \mathcal{R}_1&:=\sup_{0\le\mathfrak{r}\le1}\left\{\frac{1}{\Gamma(\vartheta+\rho)}\int_0^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{k})^{\vartheta+\rho-1}\,d\mathfrak{k}+\frac{\mathfrak{r}^{\rho}}{|1-\varsigma_1^{\rho}|\,\Gamma(\vartheta+\rho)}\right.\\ &\qquad\left.\times\left[\int_0^{\varsigma_1}(\varsigma_1-\mathfrak{k})^{\vartheta+\rho-1}\,d\mathfrak{k}-\int_0^1(1-\mathfrak{k})^{\vartheta+\rho-1}\,d\mathfrak{k}\right]\right\}\\ &\le\frac{1}{\Gamma(\vartheta+\rho+1)}\left[1+\frac{\left|\varsigma_1^{\vartheta+\rho}-1\right|}{|1-\varsigma_1^{\rho}|}\right],\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_2 :=& \sup_{0\le\mathfrak{r}\le 1}\left\{\frac{1}{\Gamma(\upsilon+\nu)}\int_0^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{k})^{\upsilon+\nu-1}\,d\mathfrak{k}+\frac{\mathfrak{r}^{\upsilon+\nu-1}}{\left|1-\varsigma_2^{\upsilon+\nu-1}\right|\Gamma(\upsilon+\nu)}\right.\\ &\left.\times\left[\int_0^{\varsigma_2}(\varsigma_2-\mathfrak{k})^{\upsilon+\nu-1}\,d\mathfrak{k}-\int_0^1(1-\mathfrak{k})^{\upsilon+\nu-1}\,d\mathfrak{k}\right]\right\}\\ \le&\frac{1}{\Gamma(\upsilon+\nu+1)}\left[1+\frac{\left|\varsigma_2^{\upsilon+\nu}-1\right|}{\left|1-\varsigma_2^{\upsilon+\nu-1}\right|}\right].\end{aligned}$$

Now we give the proof of our first main theorem.

Theorem 3.1. *Suppose that the assumptions* $(\mathbf{A_i}), (i=\overline{1,4})$ *hold and that*

$$\sum_{i=1}^{2}\mathcal{K}_i\left(L_{\mathcal{H}_i}\varpi_i+L_{\phi_i}\widehat{\psi}_i\right)<1, \tag{3.2}$$

then (1.3) *has a unique solution.*

Proof. We show that $\mathfrak{K}$ is a contraction. Let $(\hat{\mathfrak{U}}_1,\hat{\mathfrak{V}}_1),(\hat{\mathfrak{U}}_2,\hat{\mathfrak{V}}_2)\in\mathcal{G}$ and $\mathfrak{r}\in[0,1]$. Then

$$\begin{aligned}&\left|\mathfrak{K}_1\left(\hat{\mathfrak{U}}_1,\hat{\mathfrak{V}}_1\right)(\mathfrak{r})-\mathfrak{K}_1\left(\hat{\mathfrak{U}}_2,\hat{\mathfrak{V}}_2\right)(\mathfrak{r})\right|\\ \le&\left|\phi_1(\mathfrak{r},\hat{\mathfrak{U}}_1(\mathfrak{r}),\hat{\mathfrak{V}}_1(\zeta_1(\mathfrak{r})))\right|\times\left(\frac{1}{\Gamma(\vartheta+\rho)}\int_0^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{k})^{\vartheta+\rho-1}\right.\\ &\times\left|\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}_1(\mathfrak{k}),\hat{\mathfrak{V}}_1(\zeta_2(\mathfrak{k})))-\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}_2(\mathfrak{k}),\hat{\mathfrak{V}}_2(\zeta_2(\mathfrak{k})))\right|d\mathfrak{k}+\frac{\mathfrak{r}^{\rho}}{(1-\varsigma_1^{\rho})\,\Gamma(\vartheta+\rho)}\\ &\times\left[\int_0^{\varsigma_1}(\varsigma_1-\mathfrak{k})^{\vartheta+\rho-1}\times\left|\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}_1(\mathfrak{k}),\hat{\mathfrak{V}}_1(\zeta_2(\mathfrak{k})))-\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}_2(\mathfrak{k}),\hat{\mathfrak{V}}_2(\zeta_2(\mathfrak{k})))\right|d\mathfrak{k}\right.\\ &\left.\left.+\int_0^1(1-\mathfrak{k})^{\vartheta+\rho-1}\left|\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}_1(\mathfrak{k}),\hat{\mathfrak{V}}_1(\zeta_2(\mathfrak{k})))-\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}_2(\mathfrak{k}),\hat{\mathfrak{V}}_2(\zeta_2(\mathfrak{k})))\right|d\mathfrak{k}\right]\right)\\ &+\left|\phi_1(\mathfrak{r},\hat{\mathfrak{U}}_1(\mathfrak{r}),\hat{\mathfrak{V}}_1(\zeta_1(\mathfrak{r})))-\phi_1(\mathfrak{r},\hat{\mathfrak{U}}_2(\mathfrak{r}),\hat{\mathfrak{V}}_2(\zeta_1(\mathfrak{r})))\right|\\ &\times\left(\frac{1}{\Gamma(\vartheta+\rho)}\int_0^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{k})^{\vartheta+\rho-1}\left|\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}_2(\mathfrak{k}),\hat{\mathfrak{V}}_2(\zeta_2(\mathfrak{k})))\right|d\mathfrak{k}+\frac{\mathfrak{r}^{\rho}}{(1-\varsigma_1^{\rho})\,\Gamma(\vartheta+\rho)}\right.\\ &\times\left[\int_0^{\varsigma_1}(\varsigma_1-\mathfrak{k})^{\vartheta+\rho-1}\times\left|\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}_2(\mathfrak{k}),\hat{\mathfrak{V}}_2(\zeta_2(\mathfrak{k})))\right|d\mathfrak{k}+\int_0^1(1-\mathfrak{k})^{\vartheta+\rho-1}\right.\\ &\left.\left.\times\left|\mathcal{H}_1(\mathfrak{k},\hat{\mathfrak{U}}_2(\mathfrak{k}),\hat{\mathfrak{V}}_2(\zeta_2(\mathfrak{k})))\right|d\mathfrak{k}\right]\right).\end{aligned}$$

Consequently

$$\begin{aligned}&\left|\mathfrak{K}_1\left(\hat{\mathfrak{U}}_1,\hat{\mathfrak{V}}_1\right)(\mathfrak{r})-\mathfrak{K}_1\left(\hat{\mathfrak{U}}_2,\hat{\mathfrak{V}}_2\right)(\mathfrak{r})\right|\\ \le&\varpi_1\left(\frac{L_{\mathcal{H}_1}}{\Gamma(\vartheta+\rho+1)}\left(\left|\hat{\mathfrak{U}}_1-\hat{\mathfrak{U}}_2\right|+\left|\hat{\mathfrak{V}}_1-\hat{\mathfrak{V}}_2\right|\right)+\frac{2L_{\mathcal{H}_1}}{|1-\varsigma_1^{\rho}|\,\Gamma(\vartheta+\rho+1)}\right.\end{aligned}$$

$$\times \left(\left|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right| + \left|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right|\right)\Bigg)$$

$$+ L_{\phi_1}\left(\left|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right| + \left|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right|\right)\left(\frac{\widehat{\psi}_1}{\Gamma(\vartheta+\rho+1)} + \frac{2\widehat{\psi}_1}{\left|1-\varsigma_1^{\rho}\right|\Gamma(\vartheta+\rho+1)}\right).$$

Hence

$$\left|\mathfrak{K}_1\left(\hat{\mathfrak{U}}_1, \hat{\mathfrak{V}}_1\right)(\mathfrak{r}) - \mathfrak{K}_1\left(\hat{\mathfrak{U}}_2, \hat{\mathfrak{V}}_2\right)(\mathfrak{r})\right| \leq \mathcal{K}_1\left(L_{\mathcal{H}_1}\varpi_1 + L_{\phi_1}\widehat{\psi}_1\right)\left(\left|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right| + \left|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right|\right). \tag{3.3}$$

It follows that

$$\left\|\mathfrak{K}_1\left(\hat{\mathfrak{U}}_1, \hat{\mathfrak{V}}_1\right) - \mathfrak{K}_1\left(\hat{\mathfrak{U}}_2, \hat{\mathfrak{V}}_2\right)\right\| \leq \mathcal{K}_1\left(L_{\mathcal{H}_1}\varpi_1 + L_{\phi_1}\widehat{\psi}_1\right)\left(\left\|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right\| + \left\|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right\|\right). \tag{3.4}$$

By an analogous reasoning, we find

$$\left\|\mathfrak{K}_2\left(\hat{\mathfrak{U}}_1, \hat{\mathfrak{V}}_1\right) - \mathfrak{K}_2\left(\hat{\mathfrak{U}}_2, \hat{\mathfrak{V}}_2\right)\right\| \leq \mathcal{K}_2\left(L_{\mathcal{H}_2}\varpi_2 + L_{\phi_2}\widehat{\psi}_2\right)\left(\left\|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right\| + \left\|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right\|\right). \tag{3.5}$$

The inequalities in (3.4) and (3.5) lead to

$$\left\|\mathfrak{K}\left(\hat{\mathfrak{U}}_1, \hat{\mathfrak{V}}_1\right) - \mathfrak{K}\left(\hat{\mathfrak{U}}_2, \hat{\mathfrak{V}}_2\right)\right\| \leq \sum_{i=1}^{2}\left(\mathcal{K}_i\left(L_{\mathcal{H}_i}\varpi_i + L_{\phi_i}\widehat{\psi}_i\right)\right)\left(\left\|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right\| + \left\|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right\|\right).$$

Under (3.2), it follows that $\mathcal{T}$ is a contraction operator. Therefore, $\mathcal{T}$ possesses a unique fixed point that represents the solution to (1.3). □

Theorem 3.2. *Provided that conditions* (**A2**) *and* (**A3**) *are fulfilled, the system* (1.3) *admits at least one solution.*

Proof. To begin, we establish the complete continuity of $\mathfrak{K}$. This property comes directly from the continuity assumptions of the functions ϕ, $\widehat{\phi}$, $\mathcal{H}_1$, and $\mathcal{H}_2$. The proof of this result can be completed by proceeding through the following steps.

Step 1. In this step, we show that the operator $\mathfrak{K}$ is uniformly bounded. For all $(\hat{\mathfrak{U}}, \hat{\mathfrak{V}}) \in \mathcal{S}$, we obtain

$$\begin{aligned}\left|\mathfrak{K}_1\left(\hat{\mathfrak{U}}, \hat{\mathfrak{V}}\right)(\mathfrak{r})\right| \leq\ & \varpi_1 \sup_{0\leq\mathfrak{r}\leq1}\Bigg\{\frac{1}{\Gamma(\vartheta+\rho)}\int_0^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{k})^{\vartheta+\rho-1}\left|\mathcal{H}_1(\mathfrak{k}, \hat{\mathfrak{U}}(\mathfrak{k}), \hat{\mathfrak{V}}(\zeta_2(\mathfrak{k})))\right| d\mathfrak{k} \\ & + \frac{\mathfrak{r}^{\rho}}{(1-\varsigma_1^{\rho})\Gamma(\vartheta+\rho)}\left[\int_0^{\varsigma_1}(\varsigma_1-\mathfrak{k})^{\vartheta+\rho-1}\left|\mathcal{H}_1(\mathfrak{k}, \hat{\mathfrak{U}}(\mathfrak{k}), \hat{\mathfrak{V}}(\zeta_2(\mathfrak{k})))\right| d\mathfrak{k}\right. \\ & \left. - \int_0^1(1-\mathfrak{k})^{\vartheta+\rho-1}\left|\mathcal{H}_1(\mathfrak{k}, \hat{\mathfrak{U}}(\mathfrak{k}), \hat{\mathfrak{V}}(\zeta_2(\mathfrak{k})))\right| d\mathfrak{k}\right]\Bigg\} \\ \leq\ & \varpi_1\widehat{\psi}_1\mathcal{R}_1.\end{aligned} \tag{3.6}$$

By following the same method, we obtain

$$\begin{aligned}\left|\mathfrak{K}_2\left(\hat{\mathfrak{U}}, \hat{\mathfrak{V}}\right)(\mathfrak{r})\right| \leq\ & \varpi_2 \sup_{0\leq\mathfrak{r}\leq1}\Bigg\{\frac{1}{\Gamma(\upsilon+\nu)}\int_0^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{k})^{\upsilon+\nu-1}\left|\mathcal{H}_2(\mathfrak{k}, \hat{\mathfrak{U}}(\mathfrak{k}), \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{k})))\right| d\mathfrak{k} \\ & + \frac{\mathfrak{r}^{\upsilon+\nu-1}}{\left(1-\varsigma_2^{\upsilon+\nu-1}\right)\Gamma(\upsilon+\nu)}\left[\int_0^{\varsigma_2}(\varsigma_2-\mathfrak{k})^{\upsilon+\nu-1}\left|\mathcal{H}_2(\mathfrak{k}, \hat{\mathfrak{U}}(\mathfrak{k}), \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{k})))\right| d\mathfrak{k}\right. \\ & \left. - \int_0^1(1-\mathfrak{k})^{\upsilon+\nu-1}\left|\mathcal{H}_2(\mathfrak{k}, \hat{\mathfrak{U}}(\mathfrak{k}), \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{k})))\right| d\mathfrak{k}\right]\Bigg\} \\ \leq\ & \varpi_2\widehat{\psi}_2\mathcal{R}_2.\end{aligned} \tag{3.7}$$

In view of the inequalities (3.6) and (3.7)it follows that

$$\left\|\mathfrak{K}\left(\hat{\mathfrak{U}},\hat{\mathfrak{V}}\right)\right\| \leq \sum_{i=1}^{2} \varpi_i \widehat{\psi}_i \mathcal{R}_i,$$

which means that $\mathfrak{K}$ is uniformly bounded.

Step 2. We demonstrate that the operator $\mathfrak{K}$ is equicontinuous. Let $\mathfrak{r}_1, \mathfrak{r}_2 \in [0,1]$ such that $\mathfrak{r}_1 < \mathfrak{r}_2$. Then, we obtain

$$\begin{aligned}\left|\mathfrak{K}_1(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\,(\mathfrak{r}_2) - \mathfrak{K}_1(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\,(\mathfrak{r}_1)\right| \leq & \varpi_1 \sup_{0\leq\mathfrak{r}\leq1}\left\{\frac{1}{\Gamma(\vartheta+\rho)}\int_0^{\mathfrak{r}_1}\left((\mathfrak{r}_1-\mathfrak{t})^{\vartheta+\rho+1} - (\mathfrak{r}_2-\mathfrak{t})^{\vartheta+\rho+1}\right)\right.\\ & \times\left|\mathcal{H}_1(\mathfrak{t},\hat{\mathfrak{U}}(\mathfrak{t}),\hat{\mathfrak{V}}(\zeta_2(\mathfrak{t})))\right|d\mathfrak{t} - \frac{1}{\Gamma(\vartheta+\rho)}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}(\mathfrak{r}_2-\mathfrak{t})^{\vartheta+\rho+1}\\ & \times\left|\mathcal{H}_1(\mathfrak{t},\hat{\mathfrak{U}}(\mathfrak{t}),\hat{\mathfrak{V}}(\zeta_2(\mathfrak{t})))\right|d\mathfrak{t} + \frac{\left|\mathfrak{r}_2^{\rho+1}-\mathfrak{r}_1^{\rho+1}\right|}{|1-\mathfrak{t}^{\rho+1}|\,\Gamma(\vartheta+\rho)}\\ & \times\left[\int_0^{\varsigma}(\varsigma-\mathfrak{t})^{\vartheta+\rho-1}\left|\mathcal{H}_1(\mathfrak{t},\hat{\mathfrak{U}}(\mathfrak{t}),\hat{\mathfrak{V}}(\zeta_2(\mathfrak{t})))\right|d\mathfrak{t}\right.\\ & \left.\left. - \int_0^1(1-\mathfrak{t})^{\vartheta+\rho-1}\left|\mathcal{H}_1(\mathfrak{t},\hat{\mathfrak{U}}(\mathfrak{t}),\hat{\mathfrak{V}}(\zeta_2(\mathfrak{t})))\right|d\mathfrak{t}\right]\right\},\end{aligned} \tag{3.8}$$

we therefore obtain

$$\begin{aligned}&\left|\mathfrak{K}_1(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\,(\mathfrak{r}_2) - \mathfrak{K}_1(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\,(\mathfrak{r}_1)\right|\\ \leq & \varpi_1\widehat{\psi}_1\left|\frac{1}{\Gamma(\vartheta+\rho)}\int_0^{\mathfrak{r}_1}\left((\mathfrak{r}_1-\mathfrak{t})^{\vartheta+\rho+1} - (\mathfrak{r}_2-\mathfrak{t})^{\vartheta+\rho+1}\right)d\mathfrak{t}\right.\\ & \left. - \frac{1}{\Gamma(\vartheta+\rho)}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}(\mathfrak{r}_2-\mathfrak{t})^{\vartheta+\rho+1}\,d\mathfrak{t}\right|\\ & + \frac{\left|\mathfrak{r}_2^{\rho+1}-\mathfrak{r}_1^{\rho+1}\right|}{|1-\mathfrak{t}^{\rho+1}|\,\Gamma(\vartheta+\rho)} \times\left[\int_0^{\varsigma}(\varsigma-\mathfrak{t})^{\vartheta+\rho-1}d\mathfrak{t} - \int_0^1(1-\mathfrak{t})^{\vartheta+\rho-1}d\mathfrak{t}\right].\end{aligned} \tag{3.9}$$

Thus

$$\left|\mathfrak{K}_1(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\,(\mathfrak{r}_2) - \mathfrak{K}_1(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\,(\mathfrak{r}_1)\right| \to 0, \quad \text{as} \quad \mathfrak{r}_1 \to \mathfrak{r}_2. \tag{3.10}$$

Similarly, we find the following inequality:

$$\left|\mathfrak{K}_2(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\,(\mathfrak{r}_2) - \mathfrak{K}_2(\hat{\mathfrak{U}},\hat{\mathfrak{V}})\,(\mathfrak{r}_1)\right| \to 0, \quad \text{as} \quad \mathfrak{r}_1 \to \mathfrak{r}_2. \tag{3.11}$$

Therefore, the set $\mathfrak{K}\mathcal{S}$ is equicontinuous. Utilizing the Arzela-Ascoli theorem, we deduce that $\mathfrak{K}\mathcal{S}$ is relatively compact, which means that the operator $\mathfrak{K}$ is completely continuous.

Step 3. We show that the set defined below

$$\Omega = \{(\hat{\mathfrak{U}},\hat{\mathfrak{V}}) \in \mathcal{F} : (\hat{\mathfrak{U}},\hat{\mathfrak{V}}) = \widetilde{\mu}\mathfrak{K}(\hat{\mathfrak{U}},\hat{\mathfrak{V}}), \widetilde{\mu} \in [0,1]\},$$

is bounded. The equation $(\hat{\mathfrak{U}},\hat{\mathfrak{V}}) = \widetilde{\mu}\mathfrak{K}(\hat{\mathfrak{U}},\hat{\mathfrak{V}})$ yields

$$\begin{aligned}\hat{\mathfrak{U}}(\mathfrak{r}) &= \widetilde{\mu}\mathfrak{K}_1(\hat{\mathfrak{U}},\hat{\mathfrak{V}})(\mathfrak{r}),\\ \hat{\mathfrak{V}}(\mathfrak{r}) &= \widetilde{\mu}\mathfrak{K}_2(\hat{\mathfrak{U}},\hat{\mathfrak{V}})(\mathfrak{r}).\end{aligned}$$

Then, we have

$$\|\hat{\mathfrak{U}}\| \leq \widehat{\psi}_1 \varpi_1 \mathcal{R}_1,$$
$$\|\hat{\mathfrak{V}}\| \leq \widehat{\psi}_2 \varpi_2 \mathcal{R}_2.$$

Consequently, we obtain

$$\|\hat{\mathfrak{U}}\| + \|\hat{\mathfrak{V}}\| \leq \widehat{\psi}_1 \varpi_1 \mathcal{R}_1 + \widehat{\psi}_2 \varpi_2 \mathcal{R}_2. \tag{3.12}$$

From inequality (3.12), it follows that Ω is bounded. Using Theorem 2.1, we infer that the system (1.3) possesses at least one solution over the interval $[0, 1]$. □

4. Stability

Here, we focus on the stability of the generalized hybrid pantograph system (1.3) in the sense of Ulam-Hyers.

Definition 4.1. The generalized hybrid pantograph system of Equations (1.3) is considered Ulam-Hyers stable provided there exists $m = \max(m_1, m_2) > 0$ so that, for any $\lambda = \max(\lambda_1, \lambda_2) > 0$ and for any $(\bar{\hat{\mathfrak{U}}}, \bar{\hat{\mathfrak{V}}}) \in \mathcal{G}$ satisfying

$$\begin{cases} \left| \left({}^{C}D^{\vartheta}_{0^+}\, {}^{RL}D^{\rho}_{0+}\right) \left(\dfrac{\hat{\mathfrak{U}}(\mathfrak{r})}{\phi(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))} \right) - \mathcal{H}_1(\mathfrak{r}, \bar{\hat{\mathfrak{U}}}(\mathfrak{r}), \bar{\hat{\mathfrak{V}}}(\zeta_2(\mathfrak{r}))) \right| < \lambda_1, \quad 0 \leq \mathfrak{r} \leq 1, \\ \left| \left({}^{RL}D^{\upsilon}_{0^+}\, {}^{C}D^{\nu}_{0+}\right) \left(\dfrac{\hat{\mathfrak{V}}(\mathfrak{r})}{\widehat{\phi}(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})))} \right) - \mathcal{H}_2(\mathfrak{r}, \bar{\hat{\mathfrak{U}}}(\mathfrak{r}), \bar{\hat{\mathfrak{V}}}(\varepsilon_2(\mathfrak{r}))) \right| < \lambda_2, \quad 0 \leq \mathfrak{r} \leq 1. \end{cases} \tag{4.1}$$

Then one can find a unique solution $(\hat{\mathfrak{U}}, \hat{\mathfrak{V}}) \in \mathcal{G}$ of the system (1.3) with

$$\left| \bar{\hat{\mathfrak{U}}}(\mathfrak{r}) - \hat{\mathfrak{U}}(\mathfrak{r}), \bar{\hat{\mathfrak{V}}}(\mathfrak{r}) - \hat{\mathfrak{V}}(\mathfrak{r}) \right| \leq m\lambda, \quad \mathfrak{r} \in [0, 1].$$

Theorem 4.1. *Assuming that the assumptions* (**A1**) *and* (**A2**) *are fulfilled, if*

$$\frac{L_{\mathcal{H}_1}}{\Gamma\left(\vartheta + \rho + 1\right)} < \frac{1}{\varpi_1} \text{ and } \frac{L_{\mathcal{H}_2}}{\Gamma\left(\upsilon + \nu + 1\right)} < \frac{1}{\varpi_2}, \tag{4.2}$$

thus the system (1.3) *is stable in the sense of Ulam-Hyers.*

Proof. Consider $(\bar{\hat{\mathfrak{U}}}, \bar{\hat{\mathfrak{V}}}) \in \mathcal{G}$ that satisfies (4.1) also, let's take $(\hat{\mathfrak{U}}, \hat{\mathfrak{V}}) \in \mathcal{G}$ representing the unique solution of the system

$$\begin{cases} \left({}^{C}D^{\vartheta}_{0^+}\, {}^{RL}D^{\rho}_{0+}\right) \left(\dfrac{\hat{\mathfrak{U}}(\mathfrak{r})}{\phi(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))} \right) = \mathcal{H}_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_2(\mathfrak{r}))),\ \mathfrak{r} \in [0, 1], \\ \left({}^{RL}D^{\upsilon}_{0^+}\, {}^{C}D^{\nu}_{0+}\right) \left(\dfrac{\hat{\mathfrak{V}}(\mathfrak{r})}{\widehat{\phi}(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})))} \right) = \mathcal{H}_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{r}))),\ 0 < \vartheta, \rho, \upsilon, \nu \leq 1, \\ \hat{\mathfrak{U}}(0) = \bar{\hat{\mathfrak{U}}}(0),\ \phi(\varsigma_1, \hat{\mathfrak{U}}(\varsigma_1), \hat{\mathfrak{V}}(\zeta_1(\varsigma_1)))\hat{\mathfrak{U}}(1) = \phi(\varsigma_1, \bar{\hat{\mathfrak{U}}}(\varsigma_1), \bar{\hat{\mathfrak{V}}}(\zeta_1(\varsigma_1)))\bar{\hat{\mathfrak{U}}}(1), \\ \phi(1, \hat{\mathfrak{U}}(1), \hat{\mathfrak{V}}(\zeta_1(1)))\hat{\mathfrak{U}}(\varsigma_1) = \phi(1, \bar{\hat{\mathfrak{U}}}(1), \bar{\hat{\mathfrak{V}}}(\zeta_1(1)))\bar{\hat{\mathfrak{U}}}(\varsigma_1),\ \varsigma_1 \in \left]0, 1\right[, \\ \hat{\mathfrak{V}}(0) = \bar{\hat{\mathfrak{V}}}(0),\ \widehat{\phi}(\varsigma_2, \hat{\mathfrak{U}}(\varsigma_2), \hat{\mathfrak{V}}(\varepsilon_1(\varsigma_2)))\hat{\mathfrak{U}}(1) = \widehat{\phi}(\varsigma_2, \bar{\hat{\mathfrak{U}}}(\varsigma_2), \bar{\hat{\mathfrak{V}}}(\varepsilon_1(\varsigma_2)))\bar{\hat{\mathfrak{U}}}(1), \\ \widehat{\phi}(1, \hat{\mathfrak{U}}(1), \hat{\mathfrak{V}}(\varepsilon_1(1)))\hat{\mathfrak{V}}(\varsigma_2) = \widehat{\phi}(1, \bar{\hat{\mathfrak{U}}}(1), \bar{\hat{\mathfrak{V}}}(\varepsilon_1(1)))\bar{\hat{\mathfrak{V}}}(\varsigma_2),\ \varsigma_2 \in \left]0, 1\right[. \end{cases}$$

By Lemma 2.3, we have:

$$\bar{\hat{\mathfrak{U}}}(\mathfrak{r}) = \phi(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varsigma_1(\mathfrak{r}))) \left[I^{\vartheta+\rho} \omega_1^{\hat{\mathfrak{U}}}(\mathfrak{r}) + a_0 \frac{\mathfrak{r}^{\rho}}{\Gamma(\rho+1)} \right],$$

and

$$\hat{\mathfrak{V}}(\mathfrak{r}) = \widehat{\phi}(\mathfrak{r}, \hat{\mathfrak{V}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r}))) \left[I^{\upsilon+\nu} \omega_2^{\hat{\mathfrak{V}}}(\mathfrak{r}) + a_0 \frac{\Gamma(\nu)}{\Gamma(\upsilon+\nu)} \mathfrak{r}^{\upsilon+\nu-1} + b_0 \right],$$

such that

$$\omega_1^{\hat{\mathfrak{U}}}(\mathfrak{r}) = \mathcal{H}_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varsigma_2(\mathfrak{r}))) \text{ and } \omega_2^{\hat{\mathfrak{V}}}(\mathfrak{r}) = \mathcal{H}_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{r}))).$$

Integrating (4.1), we obtain

$$\begin{aligned}
&\left| \bar{\hat{\mathfrak{U}}}(\mathfrak{r}) - \phi(\mathfrak{r}, \bar{\hat{\mathfrak{U}}}(\mathfrak{r}), \bar{\hat{\mathfrak{V}}}(\varsigma_1(\mathfrak{r}))) \left[I^{\vartheta+\rho} \omega_1^{\hat{\mathfrak{U}}}(\mathfrak{r}) + a_0 \frac{\mathfrak{r}^{\rho}}{\Gamma(\rho+1)} \right] \right| \\
&\leq \frac{\lambda_1 \mathfrak{r}^{\vartheta+\rho}}{\Gamma(\vartheta+\rho+1)} \\
&\leq \frac{\lambda_1}{\Gamma(\vartheta+\rho+1)},
\end{aligned}$$

and

$$\begin{aligned}
&\left| \bar{\hat{\mathfrak{V}}}(\mathfrak{r}) - \widehat{\phi}(\mathfrak{r}, \bar{\hat{\mathfrak{U}}}(\mathfrak{r}), \bar{\hat{\mathfrak{V}}}(\varepsilon_1(\mathfrak{r}))) \left[I^{\upsilon+\nu} \omega_2^{\hat{\mathfrak{V}}}(\mathfrak{r}) + c_0 \frac{\Gamma(\nu)}{\Gamma(\upsilon+\nu)} \mathfrak{r}^{\upsilon+\nu-1} + d_0 \right] \right| \\
&\leq \frac{\upsilon_2 \mathfrak{r}^{\upsilon+\nu}}{\Gamma(\upsilon+\nu+1)} \\
&\leq \frac{\lambda_2}{\Gamma(\upsilon+\nu+1)}.
\end{aligned}$$

From (A_q), for $q = 1, 2$, we obtain

$$\begin{aligned}
&\left| \bar{\hat{\mathfrak{U}}}(\mathfrak{r}) - \hat{\mathfrak{U}}(\mathfrak{r}) \right| \\
&\leq \left| \bar{\hat{\mathfrak{U}}}(\mathfrak{r}) - \phi(\mathfrak{r}, \bar{\hat{\mathfrak{U}}}(\mathfrak{r}), \bar{\hat{\mathfrak{V}}}(\varsigma_1(\mathfrak{r}))) \left[I^{\vartheta+\rho} \omega_1^{\hat{\mathfrak{U}}}(\mathfrak{r}) + a_0 \frac{\mathfrak{r}^{\rho}}{\Gamma(\rho+1)} \right] \right| \\
&\quad + \left| \phi(\mathfrak{r}, \bar{\hat{\mathfrak{U}}}(\mathfrak{r}), \bar{\hat{\mathfrak{V}}}(\varsigma_1(\mathfrak{r}))) \right| I^{\vartheta+\rho} \left| \omega_1^{\bar{\hat{\mathfrak{U}}}}(\mathfrak{r}) - \omega_1^{\hat{\mathfrak{U}}}(\mathfrak{r}) \right| \\
&\leq \frac{\lambda_1}{\Gamma(\vartheta+\rho+1)} + \varpi_1 I^{\vartheta+\rho} \left| \omega_1^{\bar{\hat{\mathfrak{U}}}}(\mathfrak{r}) - \omega_1^{\hat{\mathfrak{U}}}(\mathfrak{r}) \right|,
\end{aligned}$$

this implies that

$$\begin{aligned}
&\left| \bar{\hat{\mathfrak{U}}}(\mathfrak{r}) - \hat{\mathfrak{U}}(\mathfrak{r}) \right| \\
&\leq \frac{\lambda_1}{\Gamma(\vartheta+\rho+1)} + \frac{\varpi_1 L_{\mathcal{H}_1}}{\Gamma(\vartheta+\rho+1)} \left(\left| \bar{\hat{\mathfrak{U}}}(\mathfrak{r}) - \hat{\mathfrak{U}}(\mathfrak{r}) \right| + \left| \bar{\hat{\mathfrak{V}}}(\mathfrak{r}) - \hat{\mathfrak{V}}(\mathfrak{r}) \right| \right).
\end{aligned}$$

Also, we have

$$\left| \bar{\hat{\mathfrak{V}}}(\mathfrak{r}) - \hat{\mathfrak{V}}(\mathfrak{r}) \right|$$

$$\leq \frac{\lambda_2}{\Gamma(\upsilon+\nu+1)} + \frac{\varpi_2 L_{\mathcal{H}_2}}{\Gamma(\upsilon+\nu+1)} \left(\left| \bar{\hat{\mathfrak{U}}}(\mathfrak{r}) - \hat{\mathfrak{U}}(\mathfrak{r}) \right| + \left| \bar{\hat{\mathfrak{V}}}(\mathfrak{r}) - \hat{\mathfrak{V}}(\mathfrak{r}) \right| \right).$$

Thus

$$\begin{aligned} & \left| \bar{\hat{\mathfrak{U}}}(\mathfrak{r}) - \hat{\mathfrak{U}}(\mathfrak{r}), \bar{\hat{\mathfrak{V}}}(\mathfrak{r}) - \hat{\mathfrak{V}}(\mathfrak{r}) \right| \\ \leq & \frac{\frac{1}{\Gamma(\vartheta+\rho+1)} + \frac{1}{\Gamma(\upsilon+\nu+1)}}{\min\left(1 - \frac{\varpi_1 L_{\mathcal{H}_1}}{\Gamma(\vartheta+\rho+1)}, 1 - \frac{\varpi_2 L_{\mathcal{H}_2}}{\Gamma(\upsilon+\nu+1)}\right)} \lambda \\ := & \, m\lambda. \end{aligned}$$

Hence, the system (1.3) is Ulam-Hyers stable. □

5. Examples

Here we present some practical examples to support the theoretical results obtained.

Suppose that we have the following system:

$$\begin{cases} \left({}^{C}D_{0^+}^{\frac{1}{2}} \, {}^{RL}D_{0^+}^{\frac{3}{4}} \right) \left(\dfrac{\hat{\mathfrak{U}}(\mathfrak{r})}{\phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))} \right) = \mathcal{H}_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_2(\mathfrak{r}))), 0 \leq \mathfrak{r} \leq 1, \\ \left({}^{RL}D_{0^+}^{\frac{7}{9}} \, {}^{C}D_{0^+}^{\frac{5}{6}} \right) \left(\dfrac{\hat{\mathfrak{V}}(\mathfrak{r})}{\phi_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})))} \right) = \mathcal{H}_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{r}))), \\ \hat{\mathfrak{U}}(0) = 0, \ \phi_1(\frac{1}{4}, \hat{\mathfrak{U}}(\frac{1}{4}), \hat{\mathfrak{V}}(\zeta_1(\frac{1}{4})))\hat{\mathfrak{U}}(1) = \phi_1(1, \hat{\mathfrak{U}}(1), \hat{\mathfrak{V}}(\zeta_1(1)))\hat{\mathfrak{U}}(\frac{1}{4}), \\ \hat{\mathfrak{V}}(0) = 0, \ \phi_2(\frac{1}{5}, \hat{\mathfrak{U}}(\frac{1}{5}), \hat{\mathfrak{V}}(\varepsilon_1(\frac{1}{5})))\hat{\mathfrak{V}}(1) = \phi_2(1, \hat{\mathfrak{U}}(1), \hat{\mathfrak{V}}(\varepsilon_1(1)))\hat{\mathfrak{V}}(\frac{1}{5}), \end{cases} \tag{5.1}$$

where : $\vartheta = \frac{1}{2}, \ \rho = \frac{3}{4}, \ \upsilon = \frac{7}{9}, \ \nu = \frac{5}{6}, \ \varsigma_1 = \frac{1}{4}, \ \varsigma_2 = \frac{1}{5}$,

$$\begin{aligned} \phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r}))) &= \frac{1}{10(6+\mathfrak{r})} \left(\frac{\left|\hat{\mathfrak{U}}(\mathfrak{r})\right|}{1 + \left|\hat{\mathfrak{U}}(\mathfrak{r})\right|} + \cos \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})) \right), \\ \phi_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r}))) &= \frac{1}{80(4+\mathfrak{r})} \left(\sin \hat{\mathfrak{U}}(\mathfrak{r}) + \cos \hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})) \right). \end{aligned}$$

Example 5.1. To guarantee the existence and uniqueness of the solution for the considered fractional hybrid pantograph system, we consider two nonlinear functions: $\mathcal{H}_1, \mathcal{H}_2 : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by:

$$\begin{aligned} \mathcal{H}_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_2(\mathfrak{r}))) &= \frac{1}{30} \left(\sin\left(\hat{\mathfrak{U}}(\mathfrak{r}) + \hat{\mathfrak{V}}(\zeta_2(\mathfrak{r})) \right) \right) \\ &\quad + \frac{e^{-\mathfrak{r}}}{(\mathfrak{r}+2)^3} \left(\hat{\mathfrak{U}}(\mathfrak{r}) + \hat{\mathfrak{V}}(\zeta_2(\mathfrak{r})) \right), \\ \mathcal{H}_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{r}))) &= \frac{\cos \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{r}))}{200 + \mathfrak{r}} + \frac{1}{2} \frac{\tan^{-1} \hat{\mathfrak{U}}(\mathfrak{r})}{(\mathfrak{r}^2 + 50)}. \end{aligned}$$

For any $\hat{\mathfrak{U}}_1, \hat{\mathfrak{U}}_2, \hat{\mathfrak{V}}_1, \hat{\mathfrak{V}}_2 \in \mathbb{R}$ and $\mathfrak{r} \in [0,1]$, we have

$$\begin{aligned}
\left|\mathcal{H}_1\left(\mathfrak{r}, \hat{\mathfrak{U}}_1, \hat{\mathfrak{V}}_1\right) - \mathcal{H}_1\left(\mathfrak{r}, \hat{\mathfrak{U}}_2, \hat{\mathfrak{V}}_2\right)\right| &\leq \frac{38}{240}\left(\left|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right| + \left|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right|\right), \\
\left|\mathcal{H}_2\left(\mathfrak{r}, \hat{\mathfrak{U}}_1, \hat{\mathfrak{V}}_1\right) - \mathcal{H}_2\left(\mathfrak{r}, \hat{\mathfrak{U}}_2, \hat{\mathfrak{V}}_2\right)\right| &\leq \frac{1}{200}\left(\left|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right| + \left|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right|\right), \\
\left|\phi_1\left(\mathfrak{r}, \hat{\mathfrak{U}}_1, \hat{\mathfrak{V}}_1\right) - \phi_1\left(\mathfrak{r}, \hat{\mathfrak{U}}_2, \hat{\mathfrak{V}}_2\right)\right| &\leq \frac{1}{60}\left(\left|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right| + \left|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right|\right), \\
\left|\phi_2\left(\mathfrak{r}, \hat{\mathfrak{U}}_1, \hat{\mathfrak{V}}_1\right) - \phi_2\left(\mathfrak{r}, \hat{\mathfrak{U}}_2, \hat{\mathfrak{V}}_2\right)\right| &\leq \frac{1}{320}\left(\left|\hat{\mathfrak{U}}_1 - \hat{\mathfrak{U}}_2\right| + \left|\hat{\mathfrak{V}}_1 - \hat{\mathfrak{V}}_2\right|\right).
\end{aligned}$$

From the derived Lipschitz-type inequalities, it is clear that each nonlinear term satisfies a global Lipschitz condition with respect to the unknown functions. More precisely, the constants

$$L_{\mathcal{H}_1}, \quad L_{\mathcal{H}_2}, \quad L_{\phi_1} \text{ and } L_{\phi_2},$$

are explicitly computed and shown to be sufficiently small.

Based on the previous inequalities, we deduce that the functions ϕ_1 and ϕ_2 satisfy the assumption (**A2**), as

$$\left|\phi_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_1(\mathfrak{r})))\right| \leq 0.033333 = \varpi_1,$$

and

$$\left|\phi_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_1(\mathfrak{r})))\right| \leq 0.000625 = \varpi_2.$$

After performing a series of calculations, we obtain the following results

$$\begin{aligned}
\mathcal{K}_1\left(L_{\mathcal{H}_1}\varpi_1 + L_{\phi_1}\widehat{\psi}_1\right) &= 0.25 \times \left(\frac{1}{200} \times 0.03333 + \frac{1}{60} \times 0.11481\right) \\
&= 0.00051, \\
\mathcal{K}_2\left(L_{\mathcal{H}_2}\varpi_2 + L_{\phi_2}\widehat{\psi}_2\right) &= 0.33333 \times \left(\frac{1}{200} \times 0.00062 + \frac{1}{320} \times 0.01481\right) \\
&= 0.00014,
\end{aligned}$$

and we find that:

$$\sum_{i=1}^{2} \mathcal{K}_i\left(L_{\mathcal{H}_i}\varpi_i + L_{\phi_i}\widehat{\psi}_i\right) < 1.$$

Using Theorem 3.1, we can assert that the problem given by (5.1) has a unique solution defined over the interval $[0,1]$.

Example 5.2. To illustrate the theoretical results concerning the existence of at least one solution to the problem under study, we consider the functions $\mathcal{H}1$ and $\mathcal{H}2 : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as follows:

$$\begin{aligned}
\mathcal{H}_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_2(\mathfrak{r}))) &= \frac{e^{-\mathfrak{r}}}{16 + \mathfrak{r}^2} + \frac{\hat{\mathfrak{U}}^2(\mathfrak{r})}{1 + \hat{\mathfrak{U}}^2(\mathfrak{r})} + \frac{\hat{\mathfrak{V}}(\zeta_2(\mathfrak{r}))}{1 + \hat{\mathfrak{V}}(\zeta_2(\mathfrak{r}))}, \\
\mathcal{H}_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{r}))) &= \frac{1}{1 + \mathfrak{r}^2} + \frac{7 \sin \hat{\mathfrak{U}}(\mathfrak{r})}{\mathfrak{r} + 2} + \frac{e^{-\mathfrak{r}^2}}{(2\mathfrak{r} + 1)} \tan^{-1} \hat{\mathfrak{U}}(\varepsilon_2(\mathfrak{r})).
\end{aligned}$$

We can easily verify that the functions $\mathcal{H}_1$ and $\mathcal{H}_2$ are bounded, with

$$\left|\mathcal{H}_1(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\zeta_2(\mathfrak{r})))\right| \leq \frac{33}{16} \quad \text{and} \quad \left|\mathcal{H}_2(\mathfrak{r}, \hat{\mathfrak{U}}(\mathfrak{r}), \hat{\mathfrak{V}}(\varepsilon_2(\mathfrak{r})))\right| \leq \frac{11}{2}.$$

According to Theorem 3.2, the system (5.1) possesses at least one solution defined over $[0, 1]$.

Example 5.3. Using Theorem 4.1, we deduce that the system (5.1) exhibits Ulam–Hyers stability. This means that, for any approximate solution $(\bar{\hat{\mathfrak{U}}}, \bar{\hat{\mathfrak{V}}})$ satisfying the system under a small perturbation parameter $\lambda > 0$, there exists an exact solution $(\hat{\mathfrak{U}}, \hat{\mathfrak{V}})$ such that the deviations between the approximate and exact solutions are uniformly bounded.

More precisely, we have

$$\left|\bar{\hat{\mathfrak{U}}}(\mathfrak{r}) - \hat{\mathfrak{U}}(\mathfrak{r})\right|, \left|\bar{\hat{\mathfrak{V}}}(\mathfrak{r}) - \hat{\mathfrak{V}}(\mathfrak{r})\right| \leq 1.5836\lambda, \quad \mathfrak{r} \in [0, 1].$$

This inequality demonstrates that the system is not only theoretically solvable but also Ulam–Hyers stable.

6. Conclusion

We have shown the existence, uniqueness, and stability in the sense of Ulam-Hyers of solutions for a system of sequential generalized hybrid pantograph equations. This was accomplished through the application of Banach's contraction principle and Leray-Schauder's alternative fixed-point theorem. Additionally, we provided illustrative examples to enhance the clarity and comprehension of our findings.

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