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# ANTI-PERIODIC SYNCHRONIZATION FOR NABLA QUATERNION-VALUED COHEN-GROSSBERG NEURAL NETWORKS WITH TIME-VARYING DELAYS AND IMPULSIVE EFFECTS ON TIME SCALES*

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**Abstract** This paper aims to consider a class of Nabla quaternion-valued Cohen-Grossberg neural networks with time-varying delays and impulsive effects on time scales. By employing a continuation theorem of coincidence degree and calculus theory on time scales, we first establish a novel analytical framework for anti-periodic solutions to such networks. Secondly, by constructing appropriate Lyapunov functions and designing state feedback and impulsive controllers, some sufficient conditions are derived to ensure the global exponential synchronization of the response system and the driving system. The proposed results are extensions and supplements of existing findings in the field. Finally, a numerical example is given to verify the feasibility of the main results.

**Keywords** Quaternion-valued Cohen-Grossberg neural networks, anti-periodic solution, global exponential synchronization, impulse, time scale.

**MSC(2010)** 34K14, 34K45, 92B20.

## 1. Introduction

The Cohen-Grossberg neural network (CGNNs), introduced by Cohen and Grossberg [8], serves as a more generalized framework that can be readily adapted to different types of neural networks, including Hopfield neural networks, cellular neural networks, and recurrent neural networks. Over the past few years, CGNNs have found broad application in various real-world engineering scenarios, such as signal processing, pattern recognition, optimization tasks, and associative memory systems. Given that these applications heavily rely on the dynamic properties of the networks, the dynamical behaviors of CGNNs have attracted extensive research attention, leading to a wealth of [16,19,26,35,38]. For example, in [11], authors investigated the positivity and exponential stability a BAM-Cohen-Grossberg neural networks model with time-

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varying delays and nonlinear self-excitation rates via differential-integral inequalities. In [40], authors considered the problem of stabilization for neutral-type inertial Cohen-Grossberg neural networks with time-varying delays and proportional time delays by constructing two distinct Lyapunov functions.

In parallel, quaternions were firstly invented by Hamilton [33] in 1843. Quaternion multiplication does not meet the commutative law, so the investigation on quaternion is much harder than that on plurality. As is known that, quaternion-valued neural networks can deal with multi-level information, and require only half the connection weight parameters of complex-valued neural networks [18, 28]. Recently, the study of quaternion-valued Cohen-Grossberg neural networks (QVCGNNs) has received much attention of many scholars and some results about dynamical behaviors of QVCGNNs have been obtained. For example, in [21], authors investigate the exponential stabilization problem for the QVCGNNs by using the homeomorphism mapping theory and lexicographical order method. In [22], by starting from basic quaternion algebra properties and algorithms, several new sufficient conditions are derived to ensure existence and global asymptotic stability and global exponential stability of the equilibrium point for QVCGNNs. Moreover, synchronization is a widespread phenomenon in practical systems, which means that two or more systems are mutually regulated to achieve common dynamic behavior. Through synchronization, we can understand the unknown system through a known system. At present, many good results have been obtained about the synchronization of QVCGNNs [5–7, 13, 17].

Despite the above results of synchronization for QVCGNNs, there are still at least three problems which are essential to be discussed.

The first is to study the synchronicity of QVCGNNs on time scales. The synchronization discussed above pertains to continuous QVCGNNs, nevertheless, discrete systems exhibit distinct advantages in terms of computational efficiency and numerical simulation capabilities. In order to unify continuous analysis and discrete analysis, Stefan Hilger proposed the time scale theory in his 1988 doctoral dissertation [14]. This theory has attracted much attention due to its ability to cover both the conventional continuous time scale (real numbers) and the discrete time scale (integers), providing a universal tool for studying dynamic equations on time scales. Therefore, the research on artificial neural network models on time scales is both necessary and meaningful, such as [30, 32, 37]. Recently, in [20], authors investigated exponential lag synchronization results for the Cohen-Grossberg neural networks with discrete and distributed delays on an arbitrary time domain by applying feedback control. In [31], stability and synchronization of octonion-valued neural networks with leakage and mixed delays on time scales were investigated based on Halanay-type inequalities. However, these conclusions are invalid on the QVCGNNs on time scales.

The second is to involve impulsive effects into the QVCGNNs on time scales. Neural networks may undergo abrupt changes at certain moments due to instantaneous perturbations, which leads to impulsive effects and the existence of impulses is often a source of instability, bifurcation and chaos for neural networks, therefore, it is necessary to consider neural networks on time scales with impulsive effects [1, 9, 24, 34, 36]. Especially recently, the incorporation of impulsive effects into CGNNs has demonstrated many positive outcomes, such as [12, 38]. However, these studies have primarily focused on real-valued CGNNs of impulsive effects on the set of real numbers. Therefore, it is both necessary and meaningful to investigate the dynamical behavior of quaternion-valued CGNNs of impulsive effects on time scales.

The third is to explore the existence of anti-periodic solutions of the QVCGNNs on time scales by employing a continuation theorem of coincidence degree theory. In [23, 39, 41], applying the Mawhin continuation theorem, the authors respectively explored the existences of periodic solution and anti-periodic solution for delayed Cohen-Grossberg BAM neural networks with impulse on time scales, anti-periodic solutions for impulsive fuzzy Cohen-Grossberge neural networks on time scales, and periodic solutions for delayed Cohen-Grossberg BAM neural networks with impulses on time scales. In order to pursue a more straightforward computational methodology, many authors have explored the anti-periodic existence of CGNNs by using extension theorems different from the Marvian extension theorem [10,25]. However, due to the computational complexity of time scales and impulsive terms, no relevant results have been found yet on the exploration of QVCGNNs with impulsive effects on time scales using the new extension theorem.

Based on the content provided, the highlights and major contributions of this paper are as follows:

- **(i)** This study extends the previous research results on the synchronicity of continuous quaternion-valued Cohen-Grossberg neural networks to the time-scale theoretical framework. The conclusions obtained have broader applicability and are more general compared to the results in [20,30,32,37].
- **(ii)** We incorporate impulsive effects into the quaternion-valued Cohen-Grossberg neural networks on time scales to investigate their dynamic behavior, thereby more comprehensively and accurately depicting the dynamic evolution laws of this model.
- **(iii)** In addition, the continuation theorem of coincidence degree theory used to establish the existence of anti-periodic solutions of (2.1) in this paper is different from that used in [23,39,41] and its calculation method is more straightforward.
- **(iv)** In [1,8,12,20,23,24,30–32,34,35,37–39,41], the authors discuss the dynamic behavior of delta dynamical systems on time scales. It should be pointed out that the neural network (2.1) of this paper is described by nabla dynamic equations on time scales. Compared with the Delta dynamical system, it has more advantages in terms of practical modeling convenience, result reliability, and application scope adaptation.

Our main aim of this paper is to study the existence of anti-periodic solutions and the global exponential synchronization of (2.1) by using the methods of coincidence degree theory and Lyapunov functions. Our results of this paper are completely new and our proposed methods of this paper can be applied to study other types of QVNNs.

The rest of the paper is organized as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, we establish the existence of anti-periodic solutions of (2.1) by using the method of coincidence degree theory. The sufficient conditions for the exponential synchronization of system (2.1) are derived in Section 4. In Section 5, a example is given to verify the theoretical results. A brief conclusion is drawn in Section 6.

## 2. System description and preliminaries

### 2.1. System description

In this paper, we consider the following QVCGNNs with time-varying delays and impulsive effects on time scales:

$$\begin{cases} x_p^{\nabla}(t) = -a_p(x_p(t))\Big[\alpha_p(x_p(t)) - \displaystyle\sum_{q=1}^{n} b_{pq}(t) f_q(x_q(t-\tau_{pq}(t))) + I_p(t)\Big], \\ \qquad\qquad t \in \mathbb{T}, \quad t > 0, \quad t \neq t_r, \\ \Delta x_p(t_r) = \Phi_{pr}(x_p(t_r)), \quad r \in \mathbb{Z}^+, \end{cases} \tag{2.1}$$

where $p \in \{1, 2, \ldots, n\} := \Gamma$, $\mathbb{T}$ is an $\omega$-periodic time scale; $x_p(t) \in \mathbb{Q}$ is the state vector, $a_p : \mathbb{Q} \to \mathbb{Q}$ represents the amplification function, $\alpha_p : \mathbb{Q} \to \mathbb{Q}$ represents the behaved function; the time delay $\tau_{pq}(t) > 0$ corresponds to the transmission delay at time $t$, which denotes to the finite speed of the axonal signal transmission; $b_{pq} : \mathbb{T} \to \mathbb{R}$ is the connection weight; $I_p : \mathbb{T} \to \mathbb{Q}$ denotes the external input; $f_q : \mathbb{Q} \to \mathbb{Q}$ denotes the activation function of the $q$th neuron at time $t$; $\Delta x_p(t_r) = x_p(t_r^+) - x_p(t_r^-)$, $x_p(t_r^+), x_p(t_r^-)$ $(p \in \Gamma)$ represent the right and left limit of $x_p(t_r)$ in the sense of time scales, respectively, and $x_p(t_r^-) = x_p(t_r)$; $\{t_r\} \subset \mathbb{T}$ is a sequence of real numbers such that $0 < t_1 < t_2 < \ldots < t_r \to \infty$ as $r \to \infty$ and there is a nature number $\rho$ such that $t_{r+\rho} = t_r + \frac{\omega}{2}$, $\Phi_{pr+\rho} = -\Phi_{pr}$, $r \in \mathbb{Z}^+$, $p \in \Gamma$. Without loss of generality, we also assume that $[0, \frac{\omega}{2})_{\mathbb{T}} \cap \{t_r : r \in \mathbb{Z}^+\} = \{t_1, t_2, \ldots, t_\rho\}$.

Let $x_p = x_p^R + ix_p^I + jx_p^J + kx_p^K \in \mathbb{Q}$, $x_p^l \in \mathbb{R}$, $l \in \{R, I, J, K\} := \Lambda$, $p \in \Gamma$, for every function $h_q : \mathbb{Q} \to \mathbb{Q}$, we assume

$$\begin{aligned} h_p(x_p) = {} & h_p^R(x_p^R, x_p^I, x_p^J, x_p^K) + ih_p^I(x_p^R, x_p^I, x_p^J, x_p^K) \\ & + jh_p^J(x_p^R, x_p^I, x_p^J, x_p^K) + kh_p^K(x_p^R, x_p^I, x_p^J, x_p^K), \end{aligned}$$

where $h_p^l : \mathbb{R}^4 \to \mathbb{R}$, $l \in \Lambda$, $p \in \Gamma$.

Throughout this paper, we assume that:

$(A_1)$ $a_p^l \in C(\mathbb{R}^4, \mathbb{R}^+)$, $a_p^l(-x_p^R, -x_p^I, -x_p^J, -x_p^K) = a_p^l(x_p^R, x_p^I, x_p^J, x_p^K)$, and there are positive constants $(a_p^l)^m$, $(a_p^l)^M$ and $A_p$ such that

$$(a_p^l)^m \le a_p^l(t) \le (a_p^l)^M, \ t \in \mathbb{R}$$

and

$$\begin{aligned} & |a_p^l(x_p^R, x_p^I, x_p^J, x_p^K) - a_p^l(y_p^R, y_p^I, y_p^J, y_p^K)| \\ \le {} & A_p(|x_p^R - y_p^R| + |x_p^I - y_p^I| + |x_p^J - y_p^J| + |x_p^K - y_p^K|), \end{aligned}$$

where $p \in \Gamma$, $l \in \Lambda$;

$(A_2)$ $\alpha_p^l \in C(\mathbb{R}^4, \mathbb{R})$ satisfies

$$\alpha_p^l(-x_p^R, -x_p^I, -x_p^J, -x_p^K) = -\alpha_p^l(x_p^R, x_p^I, x_p^J, x_p^K),$$

$\alpha_p^l(0, 0, 0, 0) = 0$, and there are positive constants $\delta_p$ and $\tilde{\delta}_p$ such that

$$\tilde{\delta}_p(x_p^l - y_p^l) \le \alpha_p^l(x_p^R, x_p^I, x_p^J, x_p^K) - \alpha_p^l(y_p^R, y_p^I, y_p^J, y_p^K) \le \delta_p(x_p^l - y_p^l),$$

for $p \in \Gamma, l \in \Lambda$;

$(A_3)$ $f_q^l \in C(\mathbb{R}^4, \mathbb{R})$ satisfies

$$f_q^l(-x_q^R, -x_q^I, -x_q^J, -x_q^K) = f_q^l(x_q^R, x_q^I, x_q^J, x_q^K),$$

and there exists a positive constant $F$ such that

$$\begin{aligned} & |f_q^l(x_q^R, x_q^I, x_q^J, x_q^K) - f_q^l(y_q^R, y_q^I, y_q^J, y_q^K)| \\ \le {} & F(|x_q^R - y_q^R| + |x_q^I - y_q^I| + |x_q^J - y_q^J| + |x_q^K - y_q^K|), \end{aligned}$$

and $f_q^l(0, 0, 0, 0) = 0$, where $q \in \Gamma$, $l \in \Lambda$;

$(A_4)$ $I_p^l, b_{pq} \in C(\mathbb{T}, \mathbb{R})$ are $\frac{\omega}{2}$-anti-periodic functions, and $\tau_{pq} \in C^1(\mathbb{T}, \mathbb{R}^+)$ is a $\omega$-periodic function, $\dot{\tau}_{pq}(t) < 1$, $\Phi_{pr}^l \in C(\mathbb{R}^4, \mathbb{R})$, and there exists a positive number $\Phi_p^M$ such that $|\Phi_p^l(\cdot)| \leq \Phi_p^M$, $p \in \Gamma$, $l \in \Lambda$.

The system (2.1) is supplemented with initial values:

$$x_p(s) = \varphi_p(s), \;\; s \in [\tau, 0]_{\mathbb{T}}, \;\; \tau = \max_{1 \leq p,q \leq n} \sup_{t \in [0,\omega]_{\mathbb{T}}} \tau_{pq}(t), \;\; p \in \Gamma,$$

where $\varphi_p = \varphi_p^R + i\varphi_p^I + j\varphi_p^J + k\varphi_p^K$, $\varphi_p^l \in C([-\tau, 0]_{\mathbb{T}}, \mathbb{R})$, $l \in \Lambda$.

For convenience, we introduce the following notations:

$$\begin{aligned}
&a_p^m = \min_{l \in \Lambda}\{(a_p^l)^m\}, a_p^M = \max_{l \in \Lambda}\{(a_p^l)^M\}, \delta_p^M = \max\{\tilde{\delta}_p, \delta_p\},\\
&\tau_{pq} = \sup_{t \in [0,\omega]_{\mathbb{T}}} \tau_{pq}(t), \dot{\tau}_{pq} = \sup_{t \in [0,\omega]_{\mathbb{T}}} \dot{\tau}_{pq}(t), b^M = \max_{1 \leq p,q \leq n} \sup_{t \in [0,\omega]_{\mathbb{T}}} \{|b_{pq}(t)|\},\\
&z^M = \max\{(z^R)^M, (z^I)^M, (z^J)^M, (z^K)^M\}, z^m = \min\{(z^R)^m, (z^I)^m, (z^J)^m, (z^K)^m\},\\
&(z^l)^M = \sup_{t \in [0,\omega]_{\mathbb{T}}}\{|z^l(t)|\}, (z^l)^m = \inf_{t \in [0,\omega]_{\mathbb{T}}}\{|z^l(t)|\}, \quad ||z||_2 = \left(\int_0^\omega |z(t)|^2 \nabla t\right)^{\frac{1}{2}},
\end{aligned}$$

where $z(t) \in \mathbb{Q}$ is an $\omega$-periodic function, $l \in \Lambda$.

It follows from Hamilton rules that (2.1) can be decomposed to the following real-valued neural network:

$$\begin{aligned}
(x_p^R(t))^\nabla = &-\Big(a_p^R[t, x_p]\alpha_p^R[t, x_p] - a_p^I[t, x_p]\alpha_p^I[t, x_p] - a_p^J[t, x_p]\alpha_p^J[t, x_p]\\
&- a_p^K[t, x_p]\alpha_p^K[t, x_p]\Big) + \sum_{q=1}^n b_{pq}(t)\Big(a_p^R[t, x_p]f_q^R[t, x_q]\\
&- a_p^I[t, x_q]f_q^I[t, x_q] - a_p^J[t, x_p]f_q^J[t, x_q] - a_p^K[t, x_p]f_q^K[t, x_p]\Big)\\
&- \Big(a_p^R[t, x_p]I_p^R(t) - a_p^I[t, x_p]I_p^I(t) - a_p^J[t, x_p]I_p^J(t)\\
&- a_p^K[t, x_p]I_p^K(t)\Big)\\
:= &\, N_p^R(t, x(t)), \quad t > 0, \quad t \neq t_r,\\
\Delta x_p^R(t_r) = &\, \Phi_{pr}^R[t_r, x_p], \quad p \in \Gamma, \quad r \in \mathbb{Z}^+,\\
(x_p^I(t))^\nabla = &-\Big(a_p^R[t, x_p]\alpha_p^I[t, x_p] + a_p^I[t, x_p]\alpha_p^R[t, x_p] + a_p^J[t, x_p]\alpha_p^K[t, x_p]\\
&- a_p^K[t, x_p]\alpha_p^J[t, x_p]\Big) + \sum_{q=1}^n b_{pq}(t)\Big(a_p^R[t, x_p]f_q^I[t, x_q]\\
&+ a_p^I[t, x_p]f_q^R[t, x_q] + a_p^J[t, x_p]f_q^K[t, x_q] - a_p^K[t, x_p]f_q^J[t, x_q]\Big)\\
&- \Big(a_p^R[t, x_p]I_p^I(t) + a_p^I[t, x_p]I_p^R(t) + a_p^J[t, x_p]I_p^K(t)\\
&- a_p^K[t, x_p]I_p^J(t)\Big)\\
:= &\, N_p^I(t, x(t)), \quad t > 0, \quad t \neq t_r,\\
\Delta x_p^I(t_r) = &\, \Phi_{pr}^I[t_r, x_p], \quad p \in \Gamma, \quad r \in \mathbb{Z}^+,
\end{aligned}$$

$$
\begin{aligned}
(x_p^J(t))^{\nabla} = &-\Big(a_p^R[t,x_p]\alpha_p^J[t,x_p] - a_p^I[t,x_p]\alpha_p^K[t,x_p] + a_p^J[t,x_p]\alpha_p^R[t,x_p] \\
&+ a_p^K[t,x_p]\alpha_p^I[t,x_p]\Big) + \sum_{q=1}^{n} b_{pq}(t)\Big(a_p^R[t,x_p]f_q^J[t,x_q] \\
&- a_p^I[t,x_p]f_q^K[t,x_q] + a_p^J[t,x_p]f_q^R[t,x_q] + a_p^K[t,x_p]f_q^I[t,x_q]\Big) \\
&- \Big(a_p^R[t,x_p]I_p^J(t) - a_p^I[t,x_p]I_p^K(t) + a_p^J[t,x_p]I_p^R(t) + a_p^K[t,x_p]I_p^I(t)\Big) \\
:= &\, N_p^J(t,x(t)), \quad t>0, \quad t \neq t_r, \\
\Delta x_p^J(t_r) = &\, \Phi_{pr}^J[t_r,x_p], \quad p \in \Gamma, \quad r \in \mathbb{Z}^+
\end{aligned}
$$

and

$$
\begin{aligned}
(x_p^K(t))^{\nabla} = &-\Big(a_p^R[t,x_p]\alpha_p^K[t,x_p] + a_p^I[t,x_p]\alpha_p^J[t,x_p] - a_p^J[t,x_p]\alpha_p^I[t,x_p] \\
&+ a_p^K[t,x_p]\alpha_p^R[t,x_p]\Big) + \sum_{q=1}^{n} b_{pq}(t)\Big(a_p^R[t,x_p]f_q^K[t,x_q] \\
&+ a_p^I[t,x_p]f_q^J[t,x_q] - a_p^J[t,x_p]f_q^I[t,x_q] + a_p^K[t,x_p]f_q^R[t,x_q]\Big) \\
&- \Big(a_p^R[t,x_p]I_p^K(t) + a_p^I[t,x_p]I_p^J(t) - a_p^J[t,x_p]I_p^I(t) + a_p^K[t,x_p]I_p^R(t)\Big) \\
:= &\, N_p^K(t,x(t)), \quad t>0, \quad t \neq t_r, \\
\Delta x_p^K(t_r) = &\, \Phi_{pr}^K[t_r,x_p], \quad p \in \Gamma, \quad r \in \mathbb{Z}^+,
\end{aligned}
$$

where

$$
\begin{aligned}
&a_p^l[t,x_p] := a_p^l(x_p^R(t), x_p^I(t), x_p^J(t), x_p^K(t)), \\
&\alpha_p^l[t,x_p] := \alpha_p^l(x_p^R(t), x_p^I(t), x_p^J(t), x_p^K(t)), \\
&f_q^l[t,x_q] := f_q^l(x_q^R(t-\tau_{pq}(t)), x_q^I(t-\tau_{pq}(t)), x_q^J(t-\tau_{pq}(t)), x_q^K(t-\tau_{pq}(t))), \\
&\Phi_{pr}^l[t_r,x_p] := \Phi_{pr}^l(x_p^R(t_r), x_p^I(t_r), x_p^J(t_r), x_p^K(t_r)).
\end{aligned}
$$

That is system (2.1) can be transformed into

$$
\begin{cases}
(x_p^l(t))^{\nabla} = N_p^l(t,x(t)), \quad t>0, \quad t \neq t_r, \\
\Delta x_p^l(t_r) = \Phi_{pr}^l[t_r,x_p], \quad l \in \Lambda, \quad p \in \Gamma, \quad r \in \mathbb{Z}^+.
\end{cases}
\tag{2.2}
$$

The initial values of system (2.2) are

$$x_p^l(s) = \varphi_p^l(s), \quad s \in [-\tau, 0]_{\mathbb{T}},$$

where $\varphi_p^l \in C([-\tau,0]_{\mathbb{T}}, \mathbb{R})$, $p \in \Gamma$, $l \in \Lambda$.

## 2.2. Basic definitions and lemmas

Let $\mathbf{T}$ be a time scale. We denote by $\sigma$ and $\rho$ the forward jump and the backward jump operators, respectively. A point $t \in \mathbf{T}$ is called left-dense if $t > \inf \mathbf{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbf{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbf{T}$ has a left-scattered maximum $m$, then $\mathbf{T}^k = \mathbf{T} \setminus \{m\}$; otherwise $\mathbf{T}^k = \mathbf{T}$. If $\mathbf{T}$ has a right-scattered minimum $m$, then $\mathbf{T}_k = \mathbf{T} \setminus \{m\}$; otherwise $\mathbf{T}_k = \mathbf{T}$. Function $\nu : \mathbf{T}_k \to [0, \infty)$ defined by $\nu(t) = t - \rho(t)$ is called the backwards graininess function.

A function $f : \mathbf{T} \to \mathbb{R}$ is called ld-continuous if it is continuous at left-dense point in $\mathbf{T}$ and its right-side limits exist at right-dense points in $\mathbf{T}$.

**Definition 2.1** ( [4]). Let $f : \mathbf{T} \to \mathbb{R}$ be a function and $t \in \mathbf{T}_k$. Define $f^{\nabla}(t)$ to be the number (provided its exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e, $U = (t - \delta, t + \delta) \cap \mathbf{T}$ for some $\delta > 0$) such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \varepsilon |\rho(t) - s|$$

for all $s \in U$, we call $f^{\nabla}(t)$ the nabla derivative of $f$ at the point $t$.

**Lemma 2.1** ( [4]). *Let $f, g : \mathbf{T} \to \mathbb{R}$ be functions and let $t \in \mathbf{T}_k$. Then we have the following.*

*(i) If $f$ is nabla differentiable at $t$, then $f$ is continuous at $t$.*

*(ii) If $f$ is continuous at $t$ and $t$ is left-scattered, then $f$ is nabla differentiable at $t$ with*

$$f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

*(iii) If $t$ is left-dense, then $f$ is nabla differentiable at $t$ iff $\lim\limits_{s \to t} \frac{f(t) - f(s)}{t - s}$ exists as a finite number. In this case,*

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

**Lemma 2.2** ( [4]). *If $f, g$ are nabla differentiable functions on $\mathbf{T}$, then*

*(i) $(v_1 f + v_2 g)^{\nabla} = v_1 f^{\nabla} + v_2 g^{\nabla}$, for any constants $v_1, v_2$;*

*(ii) $(fg)^{\nabla}(t) = f^{\nabla}(t) g(t) + f(\rho(t)) g^{\nabla}(t) = f(t) g^{\nabla}(t) + f^{\nabla}(t) g(\rho(t))$;*

*(iii) If $f$ and $f^{\nabla}$ are continuous, then $(\int_a^t f(t, s) \nabla s)^{\nabla} = f(\rho(t), t) + \int_a^t f(t, s) \nabla s$.*

**Remark 2.1.** Let $f : \mathbf{T} \to \mathbb{R}$ be a function, for any $t \in \mathcal{T}$, if $\mathbf{T} = \mathbb{R}$, then $f^{\nabla}(t) = f'(t)$; If $\mathbf{T} = \mathbb{Z}$, then $f^{\nabla}(t) = f(t) - f(t - 1)$.

Let $f : \mathbf{T} \to \mathbb{R}$ be ld-continuous. If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by $\int_a^b f(t) \nabla t = F(b) - F(a)$. For more knowledge about time scale theory, we refer to [4].

**Definition 2.2** ( [15]). For $A \in C_{rd}(\mathbf{T}, \mathbb{R}^n)$ and $V \in C_{rd}(\mathbf{T} \times \mathbb{R}^n, R^+)$, we call $D^- V^{\nabla}(t, A(t))$ the left upper derivative of the function $V$ at $(t, A(t))$, if

$$D^- V^{\nabla}(t, A(t)) = \begin{cases} \dfrac{V(t, A(t)) - V(\rho(t), A(\rho(t)))}{t - \rho(t)}, & t > \rho(t); \\ \lim\limits_{s \to t^-} \sup \dfrac{V(t, A(t)) + (s - t) F(t, A(t)) - V(t, A(t))}{s - t}, & t = \rho(t). \end{cases}$$

**Definition 2.3** ( [27]). A matrix $Q = (q_{ij})_{n \times n}$ is said to be a nonsingular M-matrix, if $q_{ii} > 0$, $q_{ij} \leq 0$, for $i \neq j$, and $Q^{-1} \geq 0$, $i, j = 1, 2, \ldots, n$.

Similar to the proof of Lemma 2.4 in [3], one can easily prove

**Lemma 2.3.** *For any* $t_1, t_2 \in [0,\omega]_{\mathbf{T}}$. *If* $x : \mathbf{T} \to \mathbb{R}$ *is* $\omega$*-periodic, then*

$$x(t) \le x(t_1) + \int_0^{\omega} |x^{\nabla}(s)|\nabla s, \quad x(t) \ge x(t_2) - \int_0^{\omega} |x^{\nabla}(s)|\nabla s.$$

**Lemma 2.4** ( [29]). *If* $a, b \in \mathbf{T}$ *and functions* $f, g : [a,b] \to \mathbb{R}$ *are rd-continuous, then*

$$\int_a^b |f(t)g(t)|\nabla s \le \left(\int_a^b |f(t)|^2 \nabla s\right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2 \nabla s\right)^{\frac{1}{2}}.$$

**Definition 2.4.** A piecewise continuous function $x = (x_1, x_2, \dots, x_n)^T : [-\tau, +\infty)_{\mathbb{T}} \to \mathbb{Q}^n$ is said to be a solution of system (2.1), if

- $(i)$ $x(s) = \varphi(s)$ for $s \in [-\tau, 0]_{\mathbb{T}}$, where $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T \in C([-\tau, 0], \mathbb{Q}^n)$;
- $(ii)$ $x(t)$ satisfies system (2.1) for $t \ge 0$;
- $(iii)$ $x(t)$ is continuous everywhere except for some $t_k$ and left continuous at $t = t_k$, and the right limit $x(t_k^+)$ exists for $k \in \mathbb{N}$.

**Definition 2.5.** A solution $x$ of system (2.1) is said to be $\frac{\omega}{2}$-anti-periodic solution of system (2.1), if

$$\begin{cases} x(t + \dfrac{\omega}{2}) = -x(t), & t \ne t_k, \\ x((t_k + \dfrac{\omega}{2})^+) = -x(t_k^+), & k \in \mathbb{N}. \end{cases}$$

**Lemma 2.5** ( [2]). *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be two Banach spaces, and let* $L : \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Y}$ *be linear,* $N : \mathbb{X} \to \mathbb{Y}$ *continuous. Assume that* $L$ *is one-to-one and* $T := L^{-1}N$ *is compact. Furthermore, assume there exists a bounded and open subset* $\Omega \subset \mathbb{X}$ *with* $0 \in \Omega$ *such that the equation* $Lx = \lambda Nx$ *has no solutions in* $\partial\Omega \cap \operatorname{Dom} L$ *for any* $\lambda \in (0,1)$. *Then equation* $Lx = Nx$ *has at least one solution in* $\overline{\Omega}$.

**Remark 2.2.** To avoid the problems brought about by the non-commutative law of quaternion algebra in the application of continuation theorem, the system (2.1) is decomposed into four real-valued systems (2.2).

**Remark 2.3.** If $x = (x_1^R, x_1^I, x_1^J, x_1^K, \dots, x_n^R, x_n^I, x_n^J, x_n^K)^T$ is a solution to system (2.2), then $(x_1, x_2, \dots, x_n)^T$ is a solution to (2.1), where $x_p = x_p^R + ix_p^I + jx_p^J + kx_p^K$, $p \in \Gamma$.

## 3. Existence of anti-periodic solutions

In this section, by using Lemma 2.5, we shall study the existence of at least one anti-periodic solution of (2.1).

**Theorem 3.1.** *If assumptions* $(A_1)$*-*$(A_4)$ *hold. Suppose further that*

$$1 - 2n\omega a^M L^{\alpha} > 0,$$

*then system (2.1) has at least one* $\frac{\omega}{2}$*-anti-periodic solution.*

**Proof.** Define $C^k[0,\omega;t_1,t_2,\dots,t_\rho,\dots,t_{2\rho}]_{\mathbb{T}}=\big\{x=(x_1,x_2,\dots,x_n)^T:[0,\omega]_{\mathbb{T}}\to\mathbb{Q}^n\big|x^{(k)}(t)$ is a piecewise continuous map with first-class discontinuity points in $[0,\omega]_{\mathbb{T}}\cap\{t_r\}$, and at each discontinuity point it is continuous on the left$\big\}$, where $k=0,1$. Take

$$\mathbb{X}=\Big\{x\in C[0,\omega;t_1,t_2,\dots,t_\rho,\dots,t_{2\rho}]_{\mathbb{T}}:x(t+\frac{\omega}{2})=-x(t),\forall t\in[0,\frac{\omega}{2}]_{\mathbb{T}}\Big\},$$
$$\mathbb{Y}=\mathbb{X}\times\mathbb{Q}^{n\times\rho},$$

the norm of $\mathbb{X}$ is defined by

$$||x||_{\mathbb{X}}=\sum_{p=1}^{n}||x_p||_0,$$

where $||x_p||_0=\sup\limits_{t\in[0,\omega]_{\mathbb{T}}}||x_p(t)||, l\in\Lambda$, then $\mathbb{X}$ is a Banach space.

Define a linear operator $L:\operatorname{Dom}L\cap\mathbb{X}\to\mathbb{Y}$ by setting

$$Lx=(x^{\nabla},\Delta x(t_1),\dots,\Delta x(t_\rho)),$$

where Dom $L=\big\{x\in C^1[0,\omega;t_1,t_2,\dots,t_\rho,\dots,t_{2\rho}]_{\mathbb{T}}:x(t+\frac{\omega}{2})=-x(t),\forall t\in[0,\frac{\omega}{2}]_{\mathbb{T}}\big\}$. Define a continuous operator $N:\mathbb{X}\to\mathbb{Y}$ by setting

$$Nx=\left(\begin{pmatrix}\Pi_1(x(t))\\ \Pi_2(x(t))\\ \vdots\\ \Pi_n(x(t))\end{pmatrix},\begin{pmatrix}\Phi_{11}(x_1(t_1))\\ \Phi_{21}(x_1(t_1))\\ \vdots\\ \Phi_{n1}^R(x_n(t_1))\end{pmatrix},\dots,\begin{pmatrix}\Phi_{1q}(x_1(t_q))\\ \Phi_{2q}(x_1(t_q))\\ \vdots\\ \Phi_{nq}(x_n(t_q))\end{pmatrix}\right),$$

where

$$\Pi_p(x(t))=-a_p(x_p(t))\Big[\alpha_p(x_p(t))-\sum_{q=1}^{n}b_{pq}(t)f_q(x_q(t-\tau_{pq}(t)))+I_p(t)\Big],\ p=1,2,\cdots,n.$$

It is easy to see that $\ker L=\{\mathbf{0}\}$ and $\operatorname{Im}L=\{z=(v,C_1,\dots,C_\rho)\in\mathbb{Y}:\int_0^\omega v(s)\nabla s=0\}=\mathbb{Y}$. Hence, L is reversible and the inverse $L^{-1}$ is given by

$$L^{-1}z=\int_0^t v(s)\nabla s+\sum_{t>t_r}C_k-\frac{1}{2}\int_0^{\frac{\omega}{2}}v(s)\nabla s-\sum_{k=1}^{\rho}C_k.$$

Using the Arzela-Ascoli theorem, we can easily see that $T=LN^{-1}$ is compact.

Corresponding to the operator equation $Lx=\lambda Nx$, $\lambda\in(0,1)$, we have

$$\begin{cases}x_p^{\nabla}(t)=\lambda N_p(x(t)), & t\neq t_r,\\ \Delta x_p(t_r)=\lambda\Phi_{pr}(x_p(t_r)), & p\in\Gamma,\quad t=t_r,\quad r\in\mathbb{Z}^+.\end{cases}\tag{3.1}$$

Suppose that $x=(x_1,x_2,\dots,x_n)^T\in\mathbb{X}$ is a solution of (3.1) for some $\lambda\in(0,1)$. Set $t_0=t_0^+$, $t_{2\rho+1}=\omega$, in view of (3.1) and $(A_1)$-$(A_4)$, we obtain

$$\int_0^\omega||x_p^{\nabla}(t)||\nabla t$$

$$
\begin{aligned}
&=\sum_{r=1}^{2\rho+1}\int_{t_{r-1}^{+}}^{t_r}||x_p^{\nabla}(t)||\nabla t+\sum_{r=1}^{2\rho}||\Delta x_p(t_r)||\\
&\leq\Big(\int_0^{\omega}||a_p(x_p(t))\alpha_p(x_p(t))||\nabla t+\sum_{q=1}^{n}\int_0^{\omega}||a_p(x_p(t))b_{pq}(t)f_q(x_q(t-\tau_{pq}(t)))||\nabla t\\
&\quad+\int_0^{\omega}||a_p(x_p(t))I_p(t)||\nabla t++\sum_{r=1}^{2\rho}||\Phi_{pr}(x_p(t_r))||\\
&\leq a^M\int_0^{\omega}||\alpha_p(x_p(t))-\alpha_p(\mathbf{0})||\nabla t+\sum_{q=1}^{n}a^Mb^M\int_0^{\omega}||f_q(x_q(t-\tau_{pq}(t)))||\nabla t\\
&\quad+a^M\int_0^{\omega}||I_p(t)||\nabla t+2\rho\Phi^M\\
&\leq a^ML^{\alpha}\int_0^{\omega}||x_p(t)||\nabla t+n\omega a^Mb^MF+\omega a^MI^M+2\rho\Phi^M\\
&\leq\omega a^ML^{\alpha}||x_p||_0+n\omega a^Mb^MF+\omega a^MI^M+2\rho\Phi^M. \qquad (3.2)
\end{aligned}
$$

From the definition of space $\mathbb{X}$, it is known that $x$ is $\omega$-periodic. Therefore, from Lemma 2.3, for any $\zeta_p^l,\eta_p^l$, $p\in\Gamma$, $l\in\{R,I,J,K\}$, we have

$$\int_0^{\omega}x_p^l(t)\nabla t\leq\int_0^{\omega}x_p^l(\zeta_p^l)\nabla t+\int_0^{\omega}\Big(\int_0^{\omega}|(x_p^l(t))^{\nabla}|\nabla t\Big)\nabla t, \qquad (3.3)$$

$$\int_0^{\omega}x_p^l(t)\nabla t\geq\int_0^{\omega}x_p^l(\eta_p^l)\nabla t-\int_0^{\omega}\Big(\int_0^{\omega}|(x_p^l(t))^{\nabla}|\nabla t\Big)\nabla t. \qquad (3.4)$$

Noting that $\int_0^{\omega}x_p^l(t)\nabla t=0$, in view of (3.3) and (3.4), we can obtain

$$x_p^l(\zeta_p^l)\geq-\int_0^{\omega}|(x_p^l(t))^{\nabla}|\nabla t,\quad x_p^l(\eta_p^l)\leq\int_0^{\omega}|(x_p^l(t))^{\nabla}|\nabla t.$$

By the arbitrariness of $\zeta_p^l,\eta_p^l$, it is easy to see that for any $t\in[0,\omega]_{\mathbb{T}}$, $l\in\{R,I,J,K\}$

$$|x_p^l(t)|\leq\int_0^{\omega}|(x_p^l(t))^{\nabla}|\nabla t.$$

Therefore, we obtain

$$
\begin{aligned}
||x_p||_0&=\sup_{t\in[0,\omega]_{\mathbb{T}}}\big(\sum_{l\in\Lambda}|x_p^l(t)|^2\big)^{\frac{1}{2}}\\
&\leq\big(\sum_{l\in\Lambda}\big(\int_0^{\omega}|(x_p^l(t))^{\nabla}|\nabla t\big)^2\big)^{\frac{1}{2}}\\
&\leq\int_0^{\omega}\sum_{l\in\Lambda}|(x_p^l(t))^{\nabla}|\nabla t\\
&\leq2\int_0^{\omega}||x_p^{\nabla}(t)||\nabla t,\quad p\in\Gamma. \qquad (3.5)
\end{aligned}
$$

Hence, from (3.2) and (3.5), one has

$$||x||_{\mathbb{X}}=\sum_{p=1}^{n}||x_p||_0$$

$$\begin{aligned}
&\leq 2\int_0^{\omega}\sum_{p=1}^{n}||x_p^{\nabla}(t)||\nabla t\\
&\leq 2\sum_{p=1}^{n}a^M L^{\alpha}\omega||x_p||_0 + n^2\omega a^M b^M F + 2(n\omega a^M I^M + 2n\rho\Phi^M)\\
&\leq 2n\omega a^M L^{\alpha}||x||_{\mathbb{X}} + n^2\omega a^M b^M F + 2n\omega a^M I^M + 4n\rho\Phi^M.
\end{aligned}$$

That is,

$$||x||_{\mathbb{X}} \leq \frac{n^2\omega a^M b^M F + 2n\omega a^M I^M + 4n\rho\Phi^M}{1 - 2n\omega a^M L^{\alpha}} \triangleq \Pi.$$

Take

$$\Omega = \big\{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < \Pi + 1\big\}.$$

It is clear that $\Omega$ satisfies all the requirements in Lemma 2.5. Hence by Lemma 2.5, system (2.1) has at least one $\frac{\omega}{2}$-anti-periodic solution. This completes the proof. $\square$

## 4. Global exponential synchronization

In this section, we consider system (2.1) as the drive system, and the corresponding response system is designed as

$$\begin{cases} y_p^{\nabla}(t) = -a_p(y_p(t))\Big[\alpha_p(y_p(t)) - \displaystyle\sum_{q=1}^{n} b_{pq}(t)f_q(y_q(t-\tau_{pq}(t))) + I_p(t)\Big] + U_p(t),\\ \qquad\qquad t \in \mathbb{T}, \quad t > 0, \quad t \neq t_r,\\ \Delta y_p(t_r) = \Phi_{pr}(y_p(t_r)), \quad p \in \Gamma, \quad r \in \mathbb{Z}^+, \end{cases} \tag{4.1}$$

where $y_p(t) = y_p^R(t) + iy_p(t) + y_p^I(t) + jy_p^J(t) + ky_p^K(t)$ represents the state of the response system, $U_p(t)$ is the state-feedback controller, the rest of the notations are the same as those in system (2.1), and the initial condition is in the form of

$$y_p(s) = \psi_p(s), \;\; s \in [\tau, 0]_{\mathbb{T}}, \;\; p \in \Gamma,$$

where $\psi_p = \psi_p^R + i\psi_p^I + j\psi_p^J + k\psi_p^K$, $\psi_p^l \in C([-\tau, 0]_{\mathbb{T}}, \mathbb{R})$, $p \in \Gamma$, $l \in \Lambda$.

Put $e_p(t) = e_p^R(t) + ie_p^I(t) + je_p^J(t) + ke_p^K(t)$, $e_p^l(t) = x_p^l(t) - y_p^l(t)$ $(p \in \Gamma, l \in \Lambda)$. By subtracting (2.1) from (4.1), we obtain the error system

$$\begin{cases} e_p^{\nabla}(t) = -\big[a_p(y_p(t))\alpha_p(y_p(t)) - a_p(x_p(t))\alpha_p(x_p(t))\big] + \displaystyle\sum_{q=1}^{n} b_{pq}(t)\\ \qquad\qquad \times\big[a_p(y_p(t))f_q(y_q(t-\tau_{pq}(t))) - a_p(x_p(t))f_q(x_q(t-\tau_{pq}(t)))\big]\\ \qquad\qquad -\big[a_p(y_p(t))I_p(t) - a_p(x_p(t))I_p(t)\big] + U_p(t),\\ \qquad\qquad t \in \mathbb{T}, \quad t > 0, \quad t \neq t_r,\\ \Delta e_p(t_r) = \Phi_{pr}(y_p(t_r)) - \Phi_{pr}(x_p(t_r)), \quad p \in \Gamma, \quad \in \mathbb{Z}^+. \end{cases} \tag{4.2}$$

**Remark 4.1.** It is well known that in scenarios such as unmanned aircraft formations, distributed energy storage in smart grids, and autonomous vehicle formations where multi-node exponential synchronization occurs, system time-varying delays are ubiquitous and arise from communication link transmission delays, directly undermining synchronization performance. Therefore, the controller must introduce time-varying delays consistent with the system. The core reason lies in ensuring synchronization stability and accuracy. The essence of synchronous control is to track the reference state. However, system time-varying delays cause the reference state at the receiving end to lag behind the real-time state. For instance, delayed grid charging and discharging instructions lead to voltage fluctuations, and delayed brake signals in autonomous driving increase the risk of collision. Eventually, this leads to an expansion of synchronization errors or even instability. When the controller's delay is less than the system's delay, it will result in insufficient compensation, thereby generating residual synchronization errors. When the controller's delay is greater than the system's delay, it will introduce additional dynamic lag, further deteriorating control performance. Only under the condition that the controller's delay is strictly equal to the system's delay, the lag feedback term in the control law can accurately reflect the actual reference state after transmission through the system's delay. Through the collaborative operation of the current state and the lagging state, the phase deviation and error accumulation effect caused by the delay can be effectively eliminated, thereby ensuring the synchronization stability of the formation system.

**Remark 4.2.** In addition, in order to adaptively suppress the nonlinear terms of the system itself and ensure the energy attenuation of the error system, a nonlinear function $g_q(e_q)$ related to the error is added to the controller.

Therefore, to achieve global exponential synchronization of the drive-response system with lower costs, we take the following state-feedback controller:

$$U_p(t) = -\xi_p(t)e_p(t) + \sum_{q=1}^{n} \gamma_{pq}(t) g_q(e_q(t-\tau_{pq}(t))),$$

where

$$\begin{aligned}
\xi_p(t) &= \xi_p^R(t) + i\xi_p^I(t) + j\xi_p^J(t) + k\xi_p^K(t), \quad \xi_p^l \in C(\mathbb{T}, \mathbb{R}^+),\\
\gamma_{pq}(t) &= \gamma_{pq}^R(t) + i\gamma_{pq}^I(t) + j\gamma_{pq}^J(t) + k\gamma_{pq}^K(t), \quad \gamma_{pq}^l \in C(\mathbb{T}, \mathbb{R}),\\
g_q(e_q) &= g_q^R(e_q^R, e_q^I, e_q^J, e_q^K) + ig_q^I(e_q^R, e_q^I, e_q^J, e_q^K) + jg_q^J(e_q^R, e_q^I, e_q^J, e_q^K)\\
&\quad + kg_q^K(e_q^R, e_q^I, e_q^J, e_q^K), \quad g_q^l \in C(\mathbb{R}^4, \mathbb{R}),
\end{aligned}$$

for $p \in \Gamma, \ l \in \Lambda$.

**Definition 4.1.** Let $x = (x_1^R, x_1^I, x_1^J, x_1^K, \cdots, x_n^R, x_n^I, x_n^J, x_n^K)^T$ be an $\frac{\omega}{2}$-anti-periodic solution of system (2.1) with initial value $\varphi = (\varphi_1^R, \varphi_1^I, \varphi_1^J, \varphi_1^K, \cdots, \varphi_n^R, \varphi_n^I, \varphi_n^J, \varphi_n^K)^T$ and $y = (y_1^R, y_1^I, y_1^J, y_1^K, \cdots, y_n^R, y_n^I, y_n^J, y_n^K)^T$ be an $\frac{\omega}{2}$-anti-periodic solution of system (4.1) with initial value $\psi = (\psi_1^R, \psi_1^I, \psi_1^J, \psi_1^K, \cdots, \psi_n^R, \psi_n^I, \psi_n^J, \varphi_n^K)^T$. If there exist positive constants $\lambda$ and $M$ such that

$$\|y(t) - x(t)\| \le M\hat{e}_{\ominus\lambda}(t, 0)\|\psi - \varphi\|_\tau, \ t \in [0, +\infty)_\mathbb{T},$$

where

$$\|y(t) - x(t)\| = \sum_{p=1}^{n} \Big( |y_p^R(t) - x_p^R(t)| + |y_p^I(t) - x_p^I(t)|$$

$$
\begin{aligned}
&+ |y_p^J(t) - x_p^J(t)| + |y_p^K(t) - x_p^K(t)|\Big),\\
\|\psi - \varphi\|_\tau = &\sum_{p=1}^{n} \sup_{s\in[-\tau,0]} \Big(|\psi_p^R(s) - \varphi^R(s)| + |\psi_p^I(s) - \varphi^I(s)|\\
&+ |\psi_p^J(s) - \varphi^J(s)| + |\psi_p^K(s) - \varphi_p^K(s)|\Big).
\end{aligned}
$$

Then the response system (2.1) and the drive system (4.1) achieve globally exponentially synchronized.

System (4.2) can be decomposed to the following real-valued system:

$$
\begin{aligned}
(e_p^R(t))^{\nabla} = &-\Big[\big(a_p^R[t,y_p]\alpha_p^R[t,y_p] - a_p^R[t,x_p]\alpha_p^R[t,x_p]\big) - \big(a_p^I[t,y_p]\alpha_p^I[t,y_p]\\
&-a_p^I[t,x_p]\alpha_p^I[t,x_p]\big) - \big(a_p^J[t,y_p]\alpha_p^J[t,y_p] - a_p^J[t,x_p]\alpha_p^J[t,x_p]\big)\\
&-\big(a_p^K[t,y_p]\alpha_p^K[t,y_p] - a_p^K[t,y_p]\alpha_p^K[t,y_p]\big)\Big] + \sum_{q=1}^{n} b_{pq}(t)\\
&\times\Big[\big(a_p^R[t,y_p]f_q^R[t,y_q] - a_p^R[t,x_p]f_q^R[t,x_q]\big) - \big(a_p^I[t,y_p]f_q^I[t,y_q]\\
&-a_p^I[t,x_p]f_q^I[t,x_q]\big) - \big(a_p^J[t,y_p]f_q^J[t,y_q] - a_p^J[t,x_p]f_q^J[t,x_q]\big)\\
&-\big(a_p^K[t,y_p]f_q^K[t,y_q] - a_p^K[t,x_p]f_q^K[t,x_q]\big)\Big] - \Big[\big(a_p^R[t,y_p]I_p^R(t)\\
&-a_p^R[t,x_p]I_p^R(t)\big) - \big(a_p^I[t,y_p]I_p^I(t) - a_p^I[t,x_p]I_p^I(t)\big)\\
&-\big(a_p^J[t,y_p]I_p^J(t) - a_p^J[t,x_p]I_p^J(t)\big) - \big(a_p^K[t,y_p]I_p^K(t)\\
&-a_p^K[t,x_p]I_p^K(t)\big)\Big] - \big(\xi_p^R(t)e_p^R(t) - \xi_p^I(t)e_p^I(t) - \xi_p^J(t)e_p^J(t)\\
&-\xi_p^K(t)e_p^K(t)\big) + \sum_{q=1}^{n}\big(\gamma_{pq}^R(t)g_q^R[t,e_q] - \gamma_{pq}^I(t)g_q^I[t,e_q]\\
&-\gamma_{pq}^J(t)g_q^J[t,e_q] - \gamma_{pq}^K(t)g_q^K[t,e_q]\big)\\
:= &\,\Upsilon_p^R(t,x(t)),\ t\in\mathbb{T},\ t>0,\ t\neq t_r,\\
\Delta e_p^R(t_r) = &\,\Phi_{pr}^R(y_p(t_r)) - \Phi_{pr}^R(x_p(t_r)),\quad p\in\Gamma,\quad r\in\mathbb{Z}^+,\\
(e_p^I(t))^{\nabla} = &-\Big[\big(a_p^R[t,y_p]\alpha_p^I[t,y_p] - a_p^R[t,x_p]\alpha_p^I[t,x_p]\big) + \big(a_p^I[t,y_p]\alpha_p^R[t,y_p]\\
&- a_p^I[t,x_p]\alpha_p^R[t,x_p]\big) + \big(a_p^J[t,y_p]\alpha_p^K[t,y_p] - a_p^J[t,x_p]\alpha_p^K[t,x_p]\big)\\
&- \big(a_p^K[t,y_p]\alpha_p^J[t,y_p] - a_p^K[t,x_p]\alpha_p^J[t,x_p]\big)\Big] + \sum_{q=1}^{n} b_{pq}(t)\\
&\times\Big[\big(a_p^R[t,y_p]f_q^I[t,y_q] - a_p^R[t,x_p]f_q^I[t,x_q]\big) + \big(a_p^I[t,y_p]f_q^R[t,y_q]\\
&- a_p^I[t,x_p]f_q^R[t,x_q]\big) + \big(a_p^J[t,y_p]f_q^K[t,y_q] - a_p^J[t,x_p]f_q^K[t,x_q]\big)\\
&- \big(a_p^K[t,y_p]f_q^J[t,y_q] - a_p^K[t,x_p]f_q^J[t,x_q]\big)\Big] - \Big[\big(a_p^R[t,y_p]I_p^I(t)\\
&- a_p^R[t,x_p]I_p^I(t)\big) + \big(a_p^I[t,y_p]I_p^R(t) - a_p^I[t,x_p]I_p^R(t)\big)\\
&+ \big(a_p^J[t,y_p]I_p^K(t) - a_p^J[t,x_p]I_p^K(t)\big) - \big(a_p^K[t,y_p]I_p^J(t)\\
&- a_p^K[t,x_p]I_p^J(t)\big)\Big] - \big(\xi_p^R(t)e_p^I(t) + \xi_p^I(t)e_p^R(t) + \xi_p^J(t)e_p^K(t)
\end{aligned}
$$

$$
\begin{aligned}
&- \xi_p^K(t)e_p^J(t)\big) + \sum_{q=1}^{n} \big(\gamma_{pq}^R(t)g_q^I[t,e_q] + \gamma_{pq}^I(t)g_q^R[t,e_q] \\
&+ \gamma_{pq}^J(t)g_q^K[t,e_q] - \gamma_{pq}^K(t)g_q^J[t,e_q]\big) \\
:=&\Upsilon_p^I(t,x(t)),\ t\in\mathbb{T},\ t>0,\quad t\neq t_r, \\
\Delta e_p^I(t_r) =&\Phi_{pr}^I(y_p(t_r)) - \Phi_{pr}^I(x_p(t_r)),\quad p\in\Gamma,\quad r\in\mathbb{Z}^+, \qquad (4.3) \\
(e_p^J(t))^{\nabla} =& -\Big[\big(a_p^R[t,y_p]\alpha_p^J[t,y_p] - a_p^R[t,x_p]\alpha_p^J[t,x_p]\big) - \big(a_p^I[t,y_p]\alpha_p^K[t,y_p] \\
&-a_p^I[t,x_p]\alpha_p^K[t,x_p]\big) + \big(a_p^J[t,y_p]\alpha_p^R[t,y_p] - a_p^J[t,x_p]\alpha_p^R[t,x_p]\big) \\
&+\big(a_p^K[t,y_p]\alpha_p^I[t,y_p] - a_p^K[t,x_p]\alpha_p^I[t,x_p]\big)\Big] + \sum_{q=1}^{n} b_{pq}(t) \\
&\times\Big[\big(a_p^R[t,y_p]f_q^J[t,y_q] - a_p^R[t,x_p]f_q^J[t,x_q]\big) - \big(a_p^I[t,y_p]f_q^K[t,y_q] \\
&-a_p^I[t,x_p]f_q^K[t,x_q]\big) + \big(a_p^J[t,y_p]f_q^R[t,y_q] - a_p^J[t,x_p]f_q^R[t,x_q]\big) \\
&+\big(a_p^K[t,y_p]f_q^I[t,y_q] - a_p^K[t,x_p]f_q^I[t,x_q]\big)\Big] - \Big[\big(a_p^R[t,y_p]I_p^J(t) \\
&-a_p^R[t,x_p]I_p^J(t)\big) - \big(a_p^I[t,y_p]I_p^K(t) - a_p^I[t,x_p]I_p^K(t)\big) \\
&+\big(a_p^J[t,y_p]I_p^R(t) - a_p^J[t,x_p]I_p^R(t)\big) + \big(a_p^K[t,y_p]I_p^I(t) \\
&-a_p^K[t,x_p]I_p^I(t)\big)\Big] - \bigg(\xi_p^R(t)e_p^J(t) - \xi_p^I(t)e_p^K(t) + \xi_p^J(t)e_p^R(t) \\
&+\xi_p^K(t)e_p^I(t)\bigg) + \sum_{q=1}^{n} \big(\gamma_{pq}^R(t)g_q^J[t,e_q] - \gamma_{pq}^I(t)g_q^K[t,e_q] \\
&+\gamma_{pq}^J(t)g_q^R[t,e_q] + \gamma_{pq}^K(t)g_q^I[t,e_q]\big) \\
:=&\ \Upsilon_p^J(t,x(t)),\ t\in\mathbb{T},\ t>0,\quad t\neq t_r, \\
\Delta e_p^J(t_r) =&\ \Phi_{pr}^J(y_p(t_r)) - \Phi_{pr}^J(x_p(t_r)),\quad p\in\Gamma,\quad r\in\mathbb{Z}^+
\end{aligned}
$$

and

$$
\begin{aligned}
(e_p^K(t))^{\nabla} =& -\Big[\big(a_p^R[t,y_p]\alpha_p^K[t,y_p] - a_p^R[t,x_p]\alpha_p^K[t,x_p]\big) + \big(a_p^I[t,y_p]\alpha_p^J[t,y_p] \\
&-a_p^I[t,x_p]\alpha_p^J[t,x_p]\big) - \big(a_p^J[t,y_p]\alpha_p^I[t,y_p] - a_p^J[t,x_p]\alpha_p^I[t,x_p]\big) \\
&-\big(a_p^K[t,y_p]\alpha_p^R[t,y_p] - a_p^K[t,x_p]\alpha_p^R[t,x_p]\big)\Big] + \sum_{q=1}^{n} b_{pq}(t) \\
&\times\Big[\big(a_p^R[t,y_p]f_q^K[t,y_q] - a_p^R[t,x_p]f_q^K[t,x_q]\big) + \big(a_p^I[t,y_p]f_q^J[t,y_q] \\
&-a_p^I[t,x_p]f_q^J[t,x_q]\big) - \big(a_p^J[t,y_p]f_q^I[t,y_q] - a_p^J[t,x_p]f_q^I[t,x_q]\big) \\
&+\big(a_p^K[t,y_p]f_q^R[t,y_q] - a_p^K[t,x_p]f_q^R[t,x_q]\big)\Big] - \Big[\big(a_p^R[t,y_p]I_p^K(t) \\
&-a_p^R[t,x_p]I_p^K(t)\big) + \big(a_p^I[t,y_p]I_p^J(t) - a_p^I[t,x_p]I_p^J(t)\big) \\
&-\big(a_p^J[t,y_p]I_p^K(t) - a_p^J[t,x_p]I_p^K(t)\big) + \big(a_p^K[t,y_p]I_p^R(t) \\
&-a_p^K[t,x_p]I_p^R(t)\big)\Big] - \big(\xi_p^R(t)e_p^K(t) + \xi_p^I(t)e_p^J(t) - \xi_p^J(t)e_p^I(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\xi_p^K(t)e_p^R(t)\big) + \sum_{q=1}^{n}\big(\gamma_{pq}^R(t)g_q^K[t,e_q] + \gamma_{pq}^I(t)g_q^J[t,e_q] \\
&-\gamma_{pq}^J(t)g_q^I[t,e_q] - \gamma_{pq}^K(t)g_q^R[t,e_q]\big) \\
:=\ & \Upsilon_p^K(t,x(t)), \quad t\in\mathbb{T},\ t>0, \quad t\neq t_r, \\
\Delta e_p^K(t_r) =\ & \Phi_{pr}^K(y_p(t_r)) - \Phi_{pr}^K(x_p(t_r)), \quad p\in\Gamma, \quad r\in\mathbb{Z}^+,
\end{aligned}
$$

where $g_q^l[t,e_q] := g^l\big(e_q^R(t-\tau_{pq}(t)), e_q^I(t-\tau_{pq}(t)), e_q^J(t-\tau_{pq}(t)), e_q^K(t-\tau_{pq}(t))\big)$, $q\in\Gamma$, $l\in\Lambda$.

That is, system (4.2) can be decomposed to the following real-valued system:

$$
\begin{cases}
(e_p^l(t))^{\nabla} = \Upsilon_p^l(t,x(t)), \quad t>0, \quad t\neq t_r, \\
\Delta e_p^l(t_r) = \Phi_{pr}^l[t_r,x_p], \quad l\in\Lambda, \quad p\in\Gamma, \quad r\in\mathbb{Z}^+.
\end{cases}
\tag{4.4}
$$

**Theorem 4.1.** *Under the hypotheses of Theorem 3.1. Suppose further that*

($A_5$) $\gamma_{pq}^l\in C(\mathbb{T},\mathbb{R})$ *is an* $\frac{\omega}{2}$*-anti-periodic function,* $\xi_p^l\in C(\mathbb{T},\mathbb{R}^+)$, $\xi_p^l(t+\frac{\omega}{2}) = \xi_p^l(t)$, *and there exist positive constants* $(\xi_p^l)^m$, $(\xi_p^l)^M$ *such that for* $p\in\Gamma$, $l\in\Lambda$, *we have*

$$(\xi_p^l)^m \le \xi_p^l(t) \le (\xi_p^l)^M,\ t\in\mathbb{R};$$

($A_6$) $g_q^l\in C(\mathbb{R}^4,\mathbb{R})$, $g_q^l(-e_q^R,-e_q^I,-e_q^J,-e_q^K) = g^l\big(e_q^R,e_q^I,e_q^J,e_q^K\big)$, $g_q^l(0,0,0,0)=0$, *and there exists a positive constant* $G$ *such that*

$$
\begin{aligned}
&|g_p^l(x_p^R,x_p^I,x_p^J,x_p^K) - g_p^l(I_p^R,I_p^I,I_p^J,I_p^K)| \\
&\le G(|x_p^R-I_p^R| + |x_p^I-I_p^I| + |x_p^J-I_p^J| + |x_p^K-I_p^K|),
\end{aligned}
$$

*for* $p\in\Gamma$, $l\in\Lambda$*;*

($A_7$) *for* $p\in\Gamma$, *there exists a positive constant* $\lambda$ *such that*

$$\sup_{t\in[0,\omega]_{\mathbb{T}}} W_p(\lambda,t) > 0,$$

*where*

$$
\begin{aligned}
W_p(\lambda,t) =\ & -\lambda + (1-\lambda\nu(t))\Big([(a_p^R)^m\tilde{\delta}_p + (\xi_p^R)^m] - [(a_p^I)^M + (a_p^J)^M \\
&+ (a_p^K)^M]\delta_p - [(\xi_p^I)^M + (\xi_p^J)^M + (\xi_p^K)^M] - 16I_p^M A_p\Big) \\
&- \sum_{q=1}^{n}\frac{16}{1-\dot{\tau}_{pq}}(b^M a_q^M F + \gamma^M G)(1-\lambda\nu(t+\tau_{pq}(t)))\hat{e}_\lambda(t+\tau_{pq}(t),t), \\
\gamma^M =\ & \max_{1\le p,q\le n}\sup_{t\in[0,\omega]_{\mathbb{T}}}\{|\gamma_{pq}(t)|\};
\end{aligned}
$$

($A_8$) *the impulsive operator* $\Phi_{pr}(x_p(t_r))$ *satisfies* $\Phi_{pr}^l[t_r,x_p] = -\beta_{pr}x_p^l(t_r)$, $\beta_{pr}\in(0,2)$, *where* $p\in\Gamma$, $r\in\mathbb{Z}^+$, $l\in\Lambda$.

*Then the response system (2.1) and the drive system (4.1) achieve globally exponentially synchronized.*

**Proof.** By (4.4), for $l = R, t \neq t_r$, $r \in \mathbb{Z}^+$, $p \in \Gamma$, we can obtain from $(A_5)$-$(A_7)$ that

$$\begin{aligned}
D^-\big|e_p^R(t)\big|^\nabla \leq\, & \operatorname{sgn}(e_p^R(t))\big(e_p^R(t)\big)^\nabla \\
\leq\, & -(a_p^R)^m\tilde{\delta}_p|e_p^R(t)| + (a_p^I)^M\delta_p|e_p^I(t)| + (a_p^J)^M\delta_p|e_p^J(t)| \\
& +(a_p^K)^M\delta_p|e_p^K(t)| + \sum_{q=1}^n 4b^M a_p^M F\big(|e_q^R(t-\tau_{pq}(t))| \\
& +|e_q^I(t-\tau_{pq}(t))| + |e_q^J(t-\tau_{pq}(t))| + |e_q^K(t-\tau_{pq}(t))|\big) \\
& +4I_p^M A_p\big(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|\big) \\
& -(\xi_p^R)^m|e_p^R(t)| + (\xi_p^I)^M|e_p^I(t)| + (\xi_p^J)^M|e_p^J(t)| + (\xi_p^K)^M|e_p^K(t)| \\
& +\sum_{q=1}^n 4\gamma^M G\big(e(t-\tau_{pq}(t))| + |e_q^I(t-\tau_{pq}(t))| \\
& +|e_q^J(t-\tau_{pq}(t))| + |e_q^K(t-\tau_{pq}(t))|\big) \\
\leq\, & -\big[(a_p^R)^m\tilde{\delta}_p + (\xi_p^R)^m\big]|e_p^R(t)| + \big[(a_p^I)^M\delta_p + (\xi_p^I)^M\big]|e_p^I(t)| \\
& +\big[(a_p^J)\delta_p + (\xi_p^J)^M\big]|e_p^J(t)| + [(a_p^K)^M\delta_p + (\xi_p^K)^M]|e_p^K(t)| \\
& +\sum_{q=1}^n 4(b^M a_p^M F + \gamma^M G)\big(|e_q^R(t-\tau_{pq}(t))| + |e_q^I(t-\tau_{pq}(t))| \\
& +|e_q^J(t-\tau_{pq}(t))| + |e_q^K(t-\tau_{pq}(t))|\big) + 4I_p^M A_p \\
& \times\big(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|\big).
\end{aligned}$$

Similarly, by (4.4), for $l = I, J, K$, we have

$$\begin{aligned}
D^-\big|e_p^I(t)\big|^\nabla \leq\, & -[(a_p^R)^m\tilde{\delta}_p + (\xi_p^R)^m]|e_p^I(t)| + [(a_p^I)^M\delta_p + (\xi_p^I)^M]|e_p^R(t)| \\
& +[(a_p^J)^M\delta_p + (\xi_p^J)^M]|e_p^K(t)| + [(a_p^K)^M\delta_p + (\xi_p^K)^M]|e_p^J(t)| \\
& +\sum_{q=1}^n 4(b^M a_p^M F + \gamma^M G)(|e_q^R(t-\tau_{pq}(t))| + |e_q^I(t-\tau_{pq}(t))| \\
& +|e_q^J(t-\tau_{pq}(t))| + |e_q^K(t-\tau_{pq}(t))|) + 4I_p^M A_p \\
& \times(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|), \\
D^-\big|e_p^J(t)\big|^\nabla \leq\, & -[(a_p^R)^m\tilde{\delta}_p + (\xi_p^R)^m]|e_p^J(t)| + [(a_p^I)^M\delta_p + (\xi_p^I)^M]|e_p^K(t)| \\
& +[(a_p^J)^M\delta_p + (\xi_p^J)^M]|e_p^R(t)| + [(a_p^K)^M\delta_p + (\xi_p^K)^M]|e_p^I(t)| \\
& +\sum_{q=1}^n 4(b^M a_p^M F + \gamma^M G)(|e_q^R(t-\tau_{pq}(t))| + |e_q^I(t-\tau_{pq}(t))| \\
& +|e_q^J(t-\tau_{pq}(t))| + |e_q^K(t-\tau_{pq}(t))|) + 4I_p^M A_p \\
& \times(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|), \\
D^-\big|e_p^K(t)\big|^\nabla \leq\, & -[(a_p^R)^m\tilde{\delta}_p + (\xi_p^R)^m]|e_p^K(t)| + [(a_p^I)^M\delta_p + (\xi_p^I)^M]|e_p^J(t)| \\
& +[(a_p^J)^M\delta_p + (\xi_p^J)^M]|e_p^I(t)| + [(a_p^K)^M\delta_p + (\xi_p^K)^M]|e_p^R(t)| \\
& +\sum_{q=1}^n 4(b^M a_p^M F + \gamma^M G)(|e_q^R(t-\tau_{pq}(t))| + |e_q^I(t-\tau_{pq}(t))|
\end{aligned}$$

$$
\begin{aligned}
&+|e_q^J(t-\tau_{pq}(t))|+|e_q^K(t-\tau_{pq}(t))|)+4I_p^M A_p\\
&\times(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|).
\end{aligned}
$$

From (4.4) and $(A_8)$, we have

$$
\begin{aligned}
|y_p^l(t_r^+)-x_p^l(t_r^+)| &= |1-\beta_{pr}||y_p^l(t_r)-x_p^l(t_r)|\\
&\le |y_p^l(t_r)-x_p^l(t_r)|, \quad p\in\Gamma, \quad l\in\Lambda, \quad r\in\mathbb{Z}^+.
\end{aligned} \tag{4.5}
$$

Hence, for $t\in[0,+\infty)_{\mathbb{T}}$, $t\neq t_r$, $r\in\mathbb{Z}^+$, $q\in\Lambda$, we have

$$
\begin{aligned}
&D^-(|e_p^R(t)|^\nabla+|e_p^R(t)|^\nabla+|e_p^J(t)|^\nabla+|e_p^K(t)|^\nabla)\\
&\le -\Big[[(a_p^R)^m\tilde{\delta}_p+(\xi_p^R)^m]-[(a_p^I)^M+(a_p^J)^M+(a_p^K)^M]\delta_p\\
&\quad -[(\xi_p^I)^M+(\xi_p^J)^M+(\xi_p^K)^M]-16I_p^M A_p\Big](|e_q^R(t)|+|e_q^I(t)|\\
&\quad +|e_q^J(t)|+|e_q^K(t)|)+\sum_{q=1}^{n}16(b^M a_p^M F+\gamma^M G)(|e_q^R(t-\tau_{pq}(t))|\\
&\quad +|e_q^I(t-\tau_{pq}(t))|+|e_q^J(t-\tau_{pq}(t))|+|e_q^K(t-\tau_{pq}(t))|).
\end{aligned} \tag{4.6}
$$

Construct the following Lyapunov function:

$$
V(t)=V_1(t)+V_2(t),
$$

where

$$
\begin{aligned}
V_1(t) &= \sum_{p=1}^{n}\hat{e}_\lambda(t,0)\big(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|\big),\\
V_2(t) &= \sum_{p=1}^{n}\sum_{q=1}^{n}\int_{t-\tau_{pq}(t)}^{t}\frac{16}{1-\dot{\tau}_{pq}}\big(b^M a_p^M F+\gamma^M G\big)\big(1-\lambda\nu(s+\tau_{pq}(t))\big)\\
&\quad \times\hat{e}_\lambda(s+\tau_{pq}(t),0)\big(|e_q^R(s)|+|e_q^I(s)|+|e_q^J(s)|+|e_p^K(s)|\big)\nabla s.
\end{aligned}
$$

For $t\in[0,+\infty)_{\mathbb{T}}$, $t\neq t_r$, $r\in\mathbb{Z}^+$, calculating the nabla derivative of $V(t)$ along the solutions of (4.4) and using (4.6), we can get

$$
\begin{aligned}
D^-V_1^\nabla(t) &\le \sum_{p=1}^{n}\lambda\hat{e}_\lambda(t,0)(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)\\
&\quad +\sum_{p=1}^{n}\hat{e}_\lambda(\rho(t),0)D^-(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)\\
&\le \sum_{p=1}^{n}\lambda\hat{e}_\lambda(t,0)(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)\\
&\quad -\sum_{p=1}^{n}(1-\lambda\nu(t))\hat{e}_\lambda(t,0)\Big[\big[(a_p^R)^m\tilde{\delta}_p+(\xi_p^R)^m\big]-\big[(a_p^I)^M\\
&\quad +(a_p^J)^M+(a_p^K)^M\big]\delta_p-\big[(\xi_p^I)^M+(\xi_p^J)^M+(\xi_p^K)^M\big]
\end{aligned}
$$

$$\begin{aligned}
&-16I_p^M A_p\Big](|e_p^R(t)| + |e_p^I(t) + |e_p^J(t)| + |e_p^K(t)|)\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n} 16(b^M a_p^M F + \gamma^M G)(1-\lambda\nu(t))\hat{e}_\lambda(t,0)\\
&\times(|e_q^R(t-\tau_{pq}(t))| + |e_q^I(t-\tau_{pq}(t))|\\
&+|e_q^J(t-\tau_{pq}(t))| + |e_p^K(t-\sigma_{pq})|)\\
\leq & -\sum_{p=1}^{n}\Big[-\lambda + (1-\lambda\nu(t))\Big(\big[(a_p^R)^m\tilde{\delta}_p + (\xi_p^R)^m\big] - \big[(a_p^I)^M\\
&+(a_p^J)^M + (a_p^K)^M\big]\delta_p - \big[(\xi_p^I)^M + (\xi_p^J)^M + (\xi_p^K)^M\big]\\
&-16I_p^M A_p\Big)\Big]\hat{e}_\lambda(t,0)\big(|e_p^R(t)| + |e_p^I(t) + |e_p^J(t)| + |e_p^K(t)|\big)\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n} 16(b^M a_p^M F + \gamma^M G)(1-\lambda\nu(t))\hat{e}_\lambda(t,0)\\
&\times(|e_q^R(t-\tau_{pq}(t))| + |e_q^I(t-\tau_{pq}(t))|\\
&+|e_q^J(t-\tau_{pq}(t))| + |e_p^K(t-\tau_{pq}(t))|),\\
D^- V_2^\nabla(t) = &\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{16}{1-\dot{\tau}_{pq}}(b^M a_p^M F + \gamma^M G)(1-\lambda\nu(t+\tau_{pq}(t)))\\
&\times\hat{e}_\lambda(t+\tau_{pq}(t),0)(|e_q^R(t)| + |e_q^I(t)| + |e_q^J(t)| + |e_p^K(t)|)\\
&-\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{16(1-\dot{\tau}_{pq}(t))}{1-\dot{\tau}_{pq}}(b^M a_p^M F + \gamma^M G)(1-\lambda\nu(t))\\
&\times\hat{e}_\lambda(t,0)(|e_q^R(t-\tau_{pq}(t))| + |e_q^I(t-\tau_{pq}(t))|\\
&+|e_q^J(t-\tau_{pq}(t))| + |e_p^K(t-\tau_{pq}(t))|).
\end{aligned}$$

By condition $(A_7)$, it concludes that

$$\begin{aligned}
D^- V^\nabla(t) = & D^- V_1^\nabla(t) + D^- V_2^\nabla(t)\\
\leq & -\sum_{p=1}^{n}\Big[-\lambda + (1-\lambda\nu(t))\Big([(a_p^R)^m\tilde{\delta}_p + (\xi_p^R)^m] - [(a_p^I)^M\\
&+(a_p^J)^M + (a_p^K)^M]\delta_p - [(\xi_p^I)^M + (\xi_p^J)^M + (\xi_p^K)^M]\\
&-16I_p^M A_p\Big)\Big]\hat{e}_\lambda(t,0)(|e_p^R(t)| + |e_p^I(t) + |e_p^J(t)| + |e_p^K(t)|)\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{16}{1-\dot{\tau}_{pq}}(b^M a_q^M F + \gamma^M G)(1-\lambda\nu(t+\tau_{pq}(t)))\\
&\times\hat{e}_\lambda(t+\tau_{pq}(t),0)(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|)\\
&+\sum_{p=1}^{n}\sum_{q=1}^{n}16\bigg(1-\frac{1-\dot{\tau}_{pq}(t)}{1-\dot{\tau}_{pq}}\bigg)(b^M a_p^M F + \gamma^M G)\\
&\times(1-\lambda\nu(t))\hat{e}_\lambda(t,0)(|e_q^R(t-\tau_{pq}(t))| + |e_q^I(t-\tau_{pq}(t))|\\
&+|e_q^J(t-\tau_{pq}(t))| + |e_p^K(t-\tau_{pq}(t))|)
\end{aligned}$$

$$
\begin{aligned}
&\leq -\sum_{p=1}^{n}\Big\{-\lambda+(1-\lambda\nu(t))\Big([(a_p^R)^m\tilde{\delta}_p+(\xi_p^R)^m]-[(a_p^I)^M\\
&\quad +(a_p^J)^M+(a_p^K)^M]\delta_p-[(\xi_p^I)^M+(\xi_p^J)^M+(\xi_p^K)^M]\\
&\quad -16I_p^M A_p\Big)-\sum_{q=1}^{n}\frac{16}{1-\dot{\tau}_{pq}}(b^M a_q^M F+\gamma^M G)(1-\lambda\nu(t+\tau_{pq}(t)))\\
&\quad \times\hat{e}_\lambda(t+\tau_{pq}(t),t)\Big\}\hat{e}_\lambda(t,0)(|e_p^R(t)|+|e_p^I(t)+|e_p^J(t)|+|e_p^K(t)|)\\
&\leq 0,\ \ t\in[0,+\infty)_{\mathbb{T}},\ \ t\neq t_r,\ \ r\in\mathbb{Z}.
\end{aligned}
$$

Also, by (4.5), we have

$$
\begin{aligned}
V(t_r^+)&=V_1(t_r^+)+V_2(t_r^+)\\
&=\sum_{p=1}^{n}\hat{e}_\lambda(t_r^+,0)\big(|e_p^R(t_r^+)|+|e_p^I(t_r^+)|+|e_p^J(t_r^+)|+|e_p^K(t_r^+)|\big)\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\int_{t_r^+-\tau_{pq}(t_r^+)}^{t_r^+}\frac{16}{1-\dot{\tau}_{pq}}(b^M a_p^M F+\gamma^M G)\\
&\quad\times(1-\lambda\nu(s+\tau_{pq}(t)))\hat{e}_\lambda(s+\tau_{pq}(t),0)\\
&\quad\times\big(|e_q^R(s)|+|e_q^I(s)|+|e_q^J(s)|+|e_p^K(s)|\big)\nabla s\\
&\leq\sum_{p=1}^{n}\hat{e}_\lambda(t,0)\big(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|\big)\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\int_{t_r-\tau_{pq}(t_r)}^{t_r}\frac{16}{1-\dot{\tau}_{pq}}(b^M a_p^M F+\gamma^M G)\\
&\quad\times\big(1-\lambda\nu(s+\tau_{pq}(t))\big)\hat{e}_\lambda(s+\tau_{pq}(t),0)\\
&\quad\times\big(|e_q^R(s)|+|e_q^I(s)|+|e_q^J(s)|+|e_p^K(s)|\big)\nabla s\\
&=V(t_r),\ \ r\in\mathbb{Z}^+.
\end{aligned}
$$

Therefore, $V(t)\leq V(0)$ for all $t\in[0,+\infty)_{\mathbb{T}}$.

On the other hand, we have

$$
\begin{aligned}
V(0)&=V_1(0)+V_2(0)\\
&=\sum_{p=1}^{n}\hat{e}_\lambda(0,0)\big(|e_p^R(0)|+|e_p^I(0)|+|e_p^J(0)|+|e_p^K(0)|\big)\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\int_{-\tau_{pq}(t)}^{0}\frac{16}{1-\dot{\tau}_{pq}}(b^M a_p^M F+\gamma^M G)\big(1-\lambda\nu(s+\tau_{pq}(t))\big)\\
&\quad\times\hat{e}_\lambda(s+\tau_{pq}(t),0)\big(|e_q^R(s)|+|e_q^I(s)|+|e_q^J(s)|+|e_p^K(s)|\big)\nabla s\\
&\leq\sum_{p=1}^{n}\Big\{1+\sum_{q=1}^{n}\int_{-\tau}^{0}\frac{16}{1-\dot{\tau}_{pq}}(b^M a_q^M F+\gamma^M G)\big(1-\lambda\nu(s+\tau_{pq}(t))\big)\\
&\quad\times\hat{e}_\lambda(s+\tau_{pq}(t),0)\nabla s\Big\}\sup_{s\in[-\tau,0]}\big(|\psi_p^R(s)-\varphi_p^R(s)|
\end{aligned}
$$

$$
\begin{aligned}
&+|\psi_p^I(s)-\varphi_p^I(s)|+|\psi_p^J(s)-\varphi_p^J(s)|+|\psi_p^K(s)-\varphi_p^K(s)|\big)\\
&\leq M||\psi-\varphi||,
\end{aligned}
$$

where

$$
\begin{aligned}
M=\max_{1\leq p\leq n}\Bigg\{&2+\sum_{q=1}^{n}\int_{-\tau}^{0}\frac{16}{1-\dot{\tau}_{pq}}(b^M a_q^M F+\gamma^M G)\big(1-\lambda\nu(s+\tau_{pq}(t))\big)\\
&\times\hat{e}_\lambda(s+\tau_{pq}(t),0)\nabla s\Bigg\}>0.
\end{aligned}
$$

It is obvious that

$$
\sum_{p=1}^{n}\hat{e}_\lambda(t,0)\big(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|\big)\leq V(t)\leq V(0)\leq M||\psi-\varphi||.
$$

So we can finally get

$$
||y(t)-x(t)||\leq M\hat{e}_{\ominus\lambda}(t,0)||\psi-\varphi||_\tau,\ t\in[0,+\infty)_{\mathbb{T}}.
$$

From Definition 4.1, the response system (2.1) and the drive system (4.1) achieve globally exponentially synchronized. □

## 5. A numerical example

**Example 5.1.** Consider the following quaternion-valued Cohen-Grossberg neural network system with impulses on time scales as the drive system:

$$
\begin{cases}
x_p^\nabla(t)=-a_p(x_p(t))\Big[\alpha_p(x_p(t))-\displaystyle\sum_{q=1}^{2}b_{pq}(t)f_q(x_q(t-\tau_{pq}(t)))+I_p(t)\Big],\\
\qquad\qquad t\in\mathbb{T},\ t>0,\ t\neq t_r, r\in\mathbb{Z}^+,\\
\Delta x_p(t_r)=\Phi_{pr}(x_p(t_r)),\ \ p=1,2,
\end{cases}
\tag{5.1}
$$

the corresponding response system is given by

$$
\begin{cases}
y_p^\nabla(t)=-a_p(y_p(t))\Big[\alpha_p(y_p(t))-\displaystyle\sum_{q=1}^{2}b_{pq}(t)f_q(y_q(t-\tau_{pq}(t)))+I_p(t)\Big]+U_p(t),\\
\qquad\qquad t\in\mathbb{T},\ t>0,\ t\neq t_r, r\in\mathbb{Z}^+,\\
\Delta y_p(t_r)=\Phi_{pr}(y_p(t_r)),\ \ p=1,2,
\end{cases}
\tag{5.2}
$$

and the controller is as follow:

$$
U_p(t)=-\xi_p(t)e_p(t)+\sum_{q=1}^{2}\gamma_{pq}(t)g_q(e_q(t-\tau_{pq}(t))),
$$

where $t\in\mathbb{T}$, $x_p(t)=x_p^R(t)+ix_p^I(t)+jx_p^J(t)+kx_p^K(t)\in\mathbb{Q}$, $\omega=2\pi$, $[0,\omega]_{\mathbb{T}}\bigcap\{t_r:r\in\mathbb{Z}\}=\{t_1,t_2\}$, for $q=1,2$,

$$
f_q(x_q)=\frac{1}{10}\cos(x_q^R+x_q^I)-i\frac{1}{10}\sin|x_q^k|+j\frac{1}{10}\cos^2(x_q^I+x_q^J)-k\frac{1}{10}\cos x_q^J,
$$

$$\begin{aligned}
g_q(e_q) &= \frac{1}{100}\sin(|e_q^J| + |e_q^K|) + i\frac{1}{100}\cos e_q^K - j\frac{1}{100}\cos(e_q^R + e_q^I) - k\frac{1}{100}\sin|e_q^R|,\\
a_1(x_1) &= \frac{1}{100\pi}(2+\sin|x_1^R|) + i\frac{1}{50\pi}(2+\sin|x_1^J|) + j\frac{1}{100\pi}(3+\cos x_1^K) + k\frac{1}{100\pi}(2+\cos x_1^J),\\
a_2(x_2) &= \frac{1}{50\pi}(2+\sin|x_2^R|) + i\frac{1}{100\pi}(2+\cos x_2^I) + j\frac{1}{50\pi}(3+\sin|x_2^J|) + k\frac{1}{100\pi}(3+\sin|x_2^K|),\\
\xi_1(t) &= \frac{1}{10}(1+|\sin t|) + i\frac{1}{200}(1+|\sin t|) + j\frac{1}{100}(1+|\cos t|) + k\frac{1}{200}(1+|\cos t|),\\
\xi_2(t) &= \frac{1}{10}(1+|\sin t|) + i\frac{1}{200}(1+|\sin t|) + j\frac{1}{300}(2+|\cos t|) + k\frac{1}{400}(3+|\sin t|),\\
\alpha_1(x_1) &= \frac{1}{100}x_1^R + i\frac{1}{50}x_1^I + j\frac{1}{50}x_1^J + k\frac{1}{10}x_1^K,\\
\alpha_2(x_2) &= \frac{1}{100}x_1^R + i\frac{3}{100}x_1^I + j\frac{1}{20}x_1^J + k\frac{7}{100}x_1^K,\\
I_1(t) &= \frac{1}{1200}\cos t + i\frac{1}{1500}\sin t - j\frac{3}{3500}\cos t + k\frac{1}{1000}\sin t,\\
I_2(t) &= \frac{1}{1000}\sin t - i\frac{1}{1500}\cos t + j\frac{1}{1500}\sin t + k\frac{1}{1000}\sin t,\\
\Psi_{1k}(x_1(t_r)) &= -\frac{1}{10}x_1^R(t_r) - \frac{1}{100}x_1^I(t_r) - \frac{1}{5}x_1^J(t_r) - \frac{2}{5}x_1^K(t_r),\\
\Psi_{2k}(x_2(t_r)) &= -\frac{3}{10}x_2^R(t_r) - \frac{1}{50}x_2^I(t_r) - \frac{3}{100}x_2^J(t_r) - \frac{1}{25}x_2^K(t_r),\\
\gamma_{11}(t) &= \gamma_{12}(t) = \frac{1}{100}\sin t - i\frac{7}{100}\cos t + j\frac{2}{100}\cos t - k\frac{5}{100}\sin t,\\
\gamma_{21}(t) &= \gamma_{22}(t) = \frac{11}{100}\cos t + i\frac{13}{100}\cos t - j\frac{1}{100}\sin t - k\frac{3}{100}\sin t,\\
b_{11}(t) &= \frac{1}{1000}\cos t, \quad b_{12}(t) = \frac{1}{1001}\sin t, \quad b_{21}(t) = \frac{1}{1010}\sin t, \quad b_{22}(t) = \frac{1}{1200}\cos t,\\
\tau_{11}(t) &= 1, \quad \tau_{12}(t) = \frac{1}{5\pi}|\sin(\pi t)|, \quad \tau_{21}(t) = \frac{1}{10\pi}|\sin(2\pi t)|, \quad \tau_{22}(t) = 1.
\end{aligned}$$

By calculating, we have

$$\begin{aligned}
&F = \frac{1}{10}, \quad G = \frac{1}{100}, \quad (a_1^R)^m = \frac{1}{100\pi}, \quad a_1^m = \frac{1}{100\pi}, \quad a_1^M = \frac{1}{25\pi},\\
&(a_1^I)^M = \frac{3}{50\pi}, \quad (a_1^J)^M = \frac{1}{25\pi}, \quad (a_1^K)^M = \frac{3}{100\pi}, \quad (a_2^R)^m = \frac{1}{50\pi},\\
&a_2^m = \frac{1}{100\pi}, \quad a_2^M = \frac{2}{25\pi}, \quad (a_2^I)^M = \frac{3}{100\pi}, \quad (a_2^J)^M = \frac{2}{25\pi},\\
&(a_2^K)^M = \frac{1}{25\pi}, \quad A_1 = \frac{1}{50\pi}, \quad A_2 = \frac{1}{50\pi}, \quad \tilde{\delta}_1 = \frac{1}{100}, \quad \delta_1 = \frac{1}{10},\\
&\tilde{\delta}_2 = \frac{1}{100}, \quad \delta_2 = \frac{7}{100}, \quad b^M = \frac{1}{1000}, \quad I_1^M = I_2^M = \frac{1}{1000},\\
&(\xi_1^I)^M = (\xi_1^K)^M = (\xi_2^I)^M = (\xi_2^J)^M = (\xi_2^K)^M = \frac{1}{100},\\
&(\xi_1^R)^m = \frac{1}{10}, \quad (\xi_1^J)^M = \frac{1}{50}, \quad (\xi_2^R)^m = \frac{1}{10}, \quad \gamma^M = \frac{11}{100}, \quad \dot{\tau}_{11} = \dot{\tau}_{22} = 0, \quad \dot{\tau}_{12} = \dot{\tau}_{21} = \frac{1}{5}.
\end{aligned}$$

It is obvious that $(A_1)$-$(A_6)$ and $(A_8)$ are satisfied. Furthermore, whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, we can easily calculate that

$$\omega a_1^m \tilde{\delta}_1 - \omega^2 a_1^m \tilde{\delta}_1 a_1^M \delta_1^M - 4b^M F(\omega a_1^M + \omega^2 a_1^m \tilde{\delta}_1) \approx 1491 \times 10^{-7},$$

$$\omega a_2^m \tilde{\delta}_2 - \omega^2 a_2^m \tilde{\delta}_2 a_2^M \delta_2^M - 4b^M F(\omega a_2^M + \omega^2 a_2^m \tilde{\delta}_2) \approx 14926 \times 10^{-8}$$

and

$$-(\omega a_p^M \delta_p^M + \omega^2 a_p^m \tilde{\delta}_p a_p^M \delta_p^M) < 0, -4b^M F(\omega a_p^M + \omega^2 a_p^m \tilde{\delta}) < 0, \quad p = 1, 2.$$

Hence, $E$ is a nonsingular $M$ matrix when $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. By Theorem 3.1, system (5.1) has a $\pi$-anti-periodic solution. Take $\lambda = 0.01$, when $\mathbb{T} = \mathbb{R}$, $\nu(t) = 0$, we have that

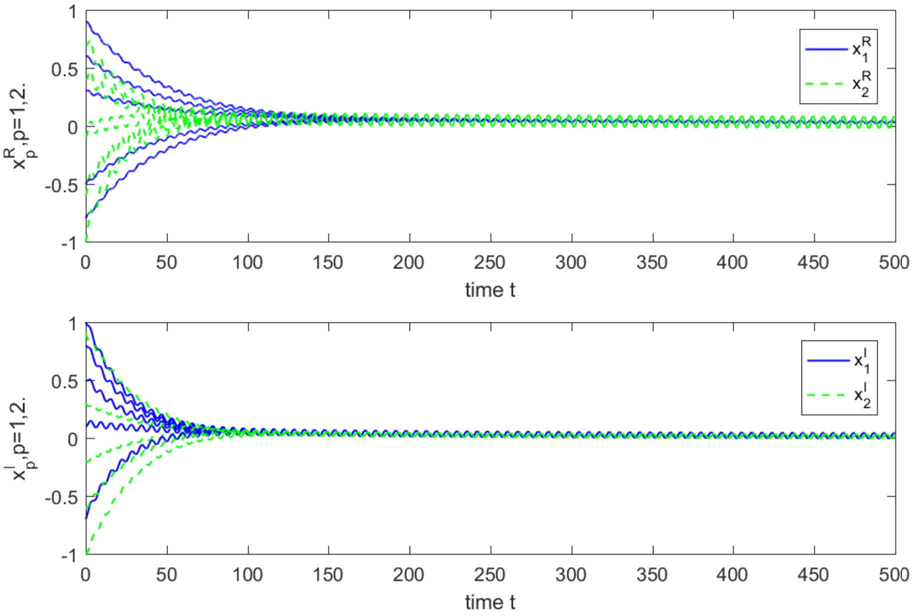

**Figure 1.** $\mathbb{T} = \mathbb{R}$. Curves of $x_p^R$ and $x_p^I$, p=1,2.

$$W_1(\lambda, t) \approx 0.906 > 0, \quad W_2(\lambda, t) \approx 0.946 > 0.$$

When $\mathbb{T} = \mathbb{Z}$, $\nu(t) = 1$. We obtain that

$$W_1(\lambda, t) \approx 0.069 > 0, \quad W_2(\lambda, t) \approx 0.073 > 0.$$

Hence, whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, $(A_7)$ holds. By Theorem 4.1, system (5.1) and (5.2) achieve global exponentially synchronization when $\mathbb{T} = \mathbb{R}$, see Figures 1-5, and when $\mathbb{T} = \mathbb{Z}$, see Figures 6-10.

**Remark 5.1.** From Figure 1–Figure 4 and Figure 6–Figure 9, it can be seen that all the components of all the state variables of the system (2.1) and (5.2) exhibit certain $\pi$-anti-periodic oscillations whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, indicating that both the drive system and response system have a $\pi$-anti-periodic oscillation state. At the same time, from Figure 5 and Figure 10, synchronization errors $e(t)$ are all approaching zero, which indicates that system (5.1) and (5.2) achieve achieve global exponential synchronization whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.

**Remark 5.2.** Since the quaternion-valued CGNNs considered in [5–7,13,17] is a special case of ours, the exponential synchronization they considered can be directly obtained from Theorem 2. Additionally, we extend our research to time scales. When $\mathbb{T} = \mathbb{R}$, our results are consistent with those on the real number set. Therefore, the synchronization results of this paper are more

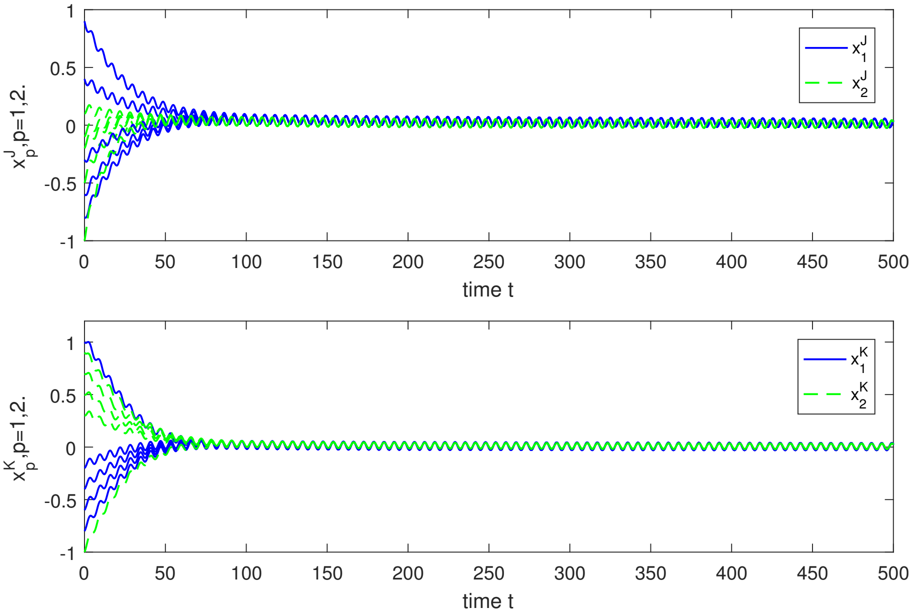

**Figure 2.** $\mathbb{T}=\mathbb{R}$. Curves of $x_p^J$ and $x_p^K$, p=1,2.

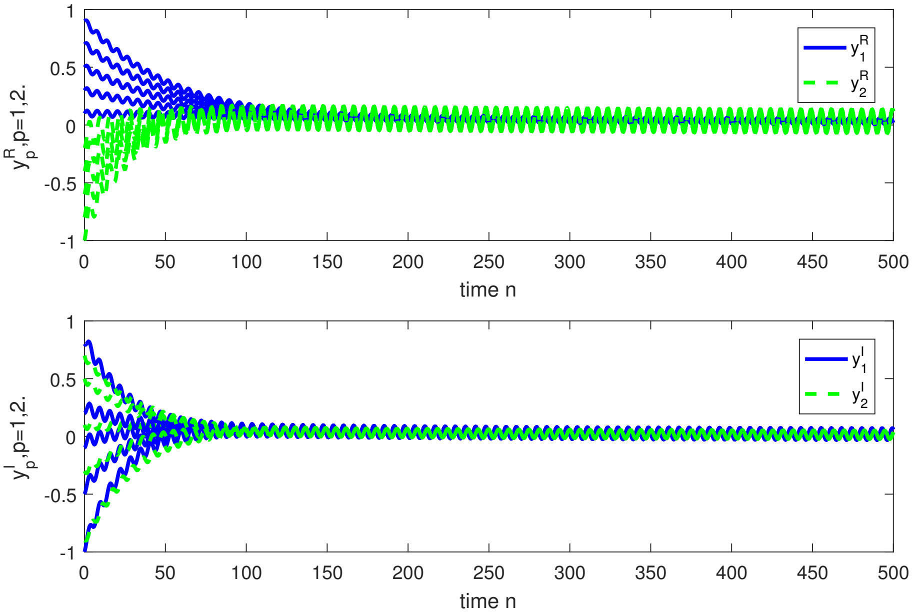

**Figure 3.** $\mathbb{T}=\mathbb{R}$. Curves of $y_p^R$ and $y_p^I$, p=1,2.

inclusive and extensive. Moreover, although [23, 39, 41] studied dynamic systems with impulsive effects on time scales, they were not Nabla dynamic systems with quaternion algebra. Clearly, it is evident that all results reported in [23, 39, 41] and the references therein are invalid for Example 5.1. This implies that the results established in this paper are essentially new.

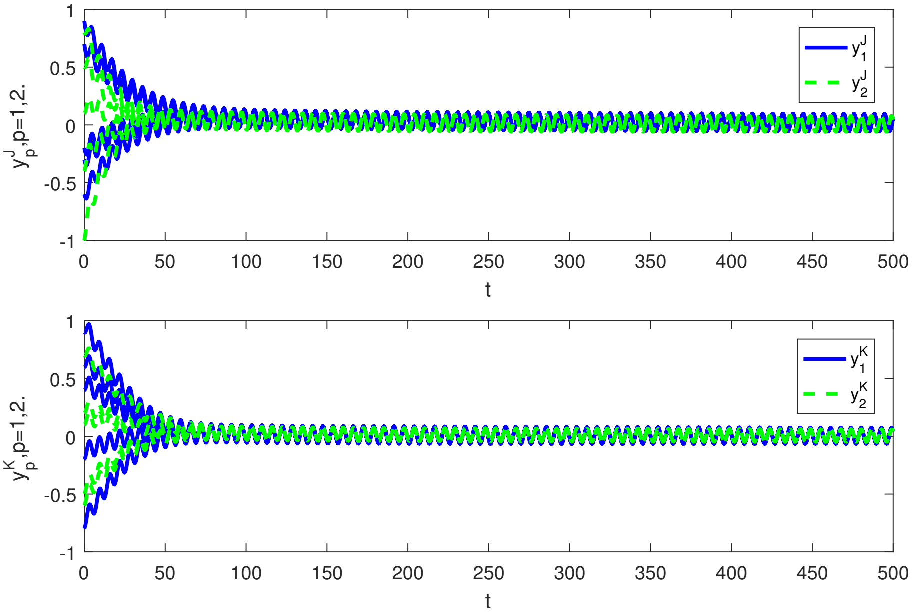

**Figure 4.** $\mathbb{T} = \mathbb{R}$. Curves of $y_p^J$ and $y_p^K$, p=1,2.

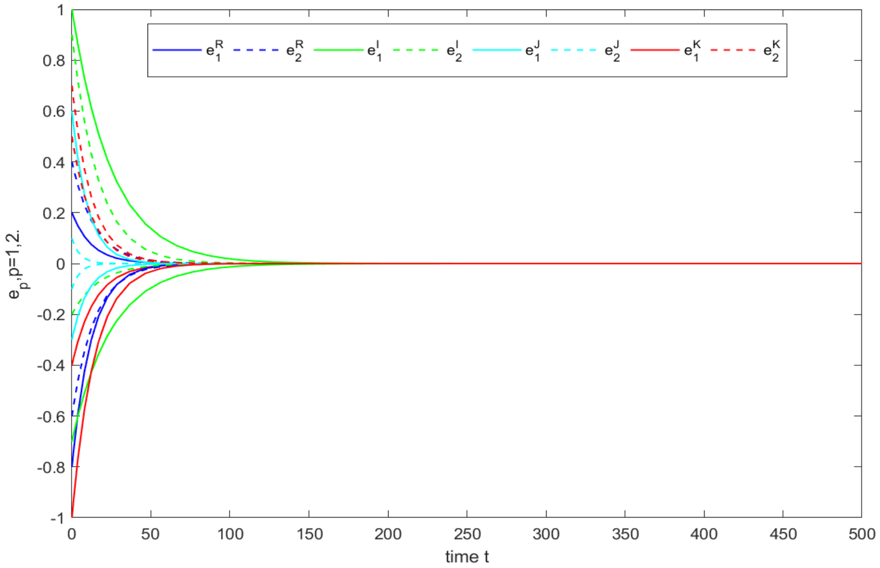

**Figure 5.** $\mathbb{T} = \mathbb{R}$. The drive system (5.1) and its response system (5.2) are synchronized.

## 6. Conclusion

In this paper, we proposed a class of quaternion-valued Cohen-Grossberg neural networks with impulses on time scales. By using a continuation theorem of coincidence degree theory and constructing a suitable Lyapunov function, we obtained sufficient conditions for the existence of anti-periodic solutions and the global exponential synchronization for quaternion-valued Cohen-

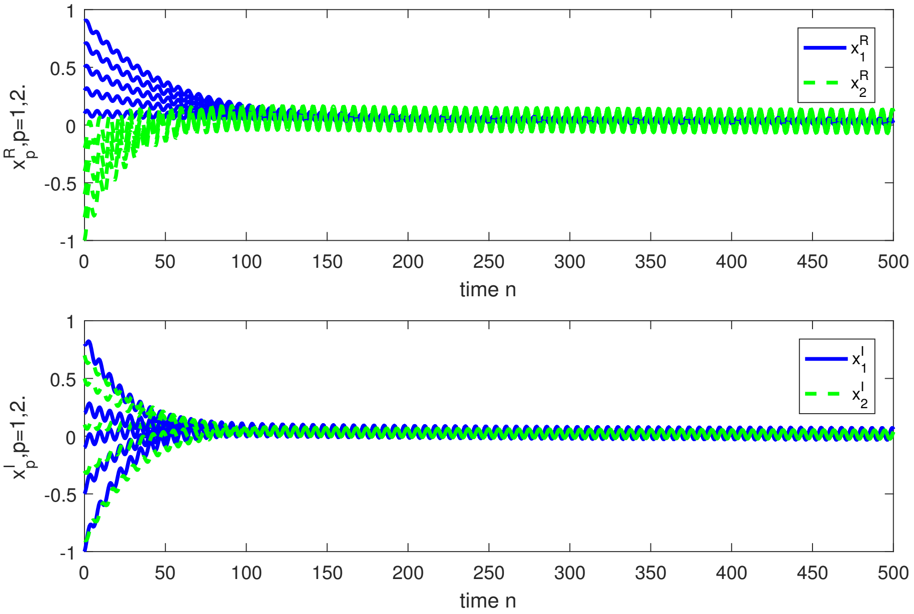

**Figure 6.** $\mathbb{T} = \mathbb{Z}$. Curves of $x_p^R$ and $x_p^I$, p=1,2.

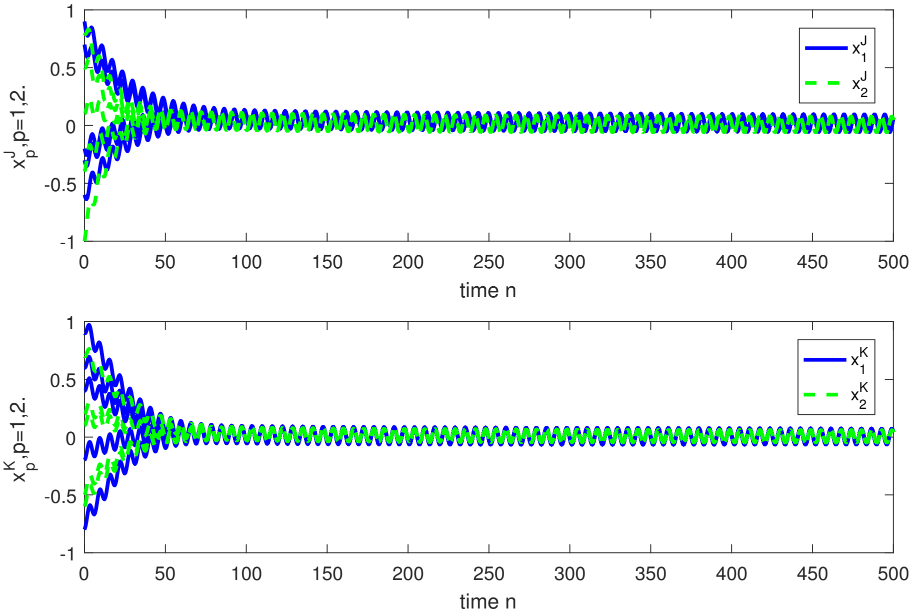

**Figure 7.** $\mathbb{T} = \mathbb{Z}$. Curves of $x_p^J$ and $x_p^K$, p=1,2.

Grossberg neural networks with impulses on time scales. This is the first time to study the existence of anti-periodic solutions and the global exponential synchronization of quaternion-valued Cohen-Grossberg neural networks with impulses on time scales. The method of this paper can be extended to study other types of QVNNs such as quaternion-valued BAM neural networks, high-order Hopfield neural networks, shunting inhibitory cellular neural networks and so on.

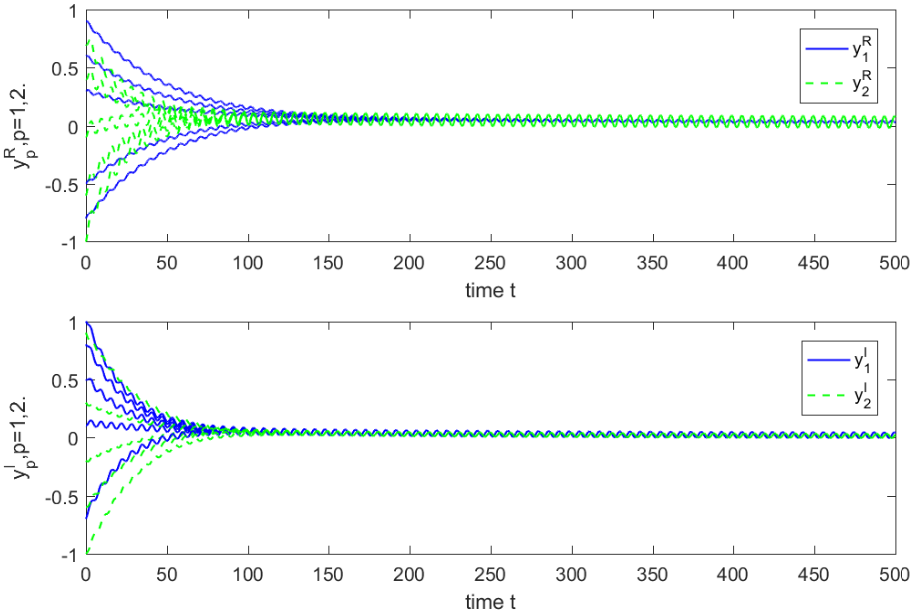

**Figure 8.** $\mathbb{T}=\mathbb{Z}$. Curves of $y_p^R$ and $y_p^I$, p=1,2.

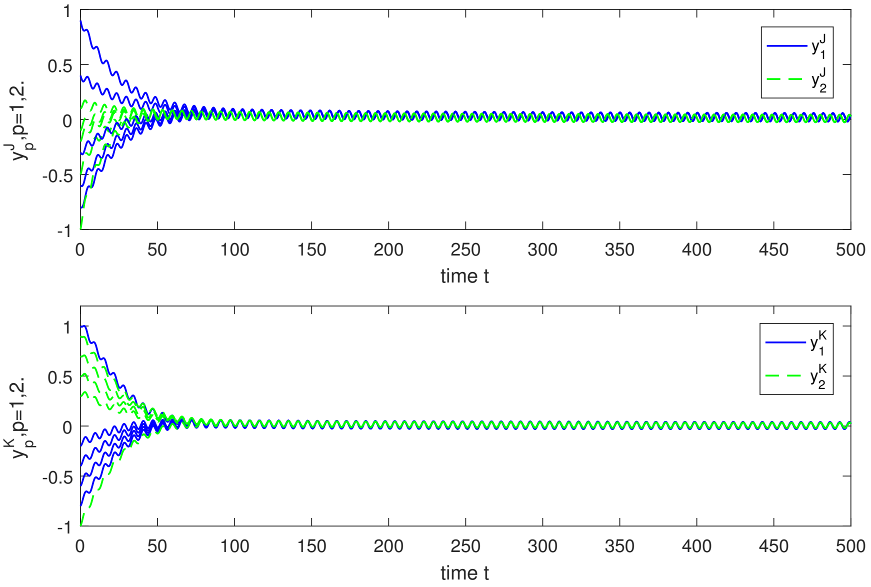

**Figure 9.** $\mathbb{T}=\mathbb{Z}$. Curves of $y_p^J$ and $y_p^K$, p=1,2.

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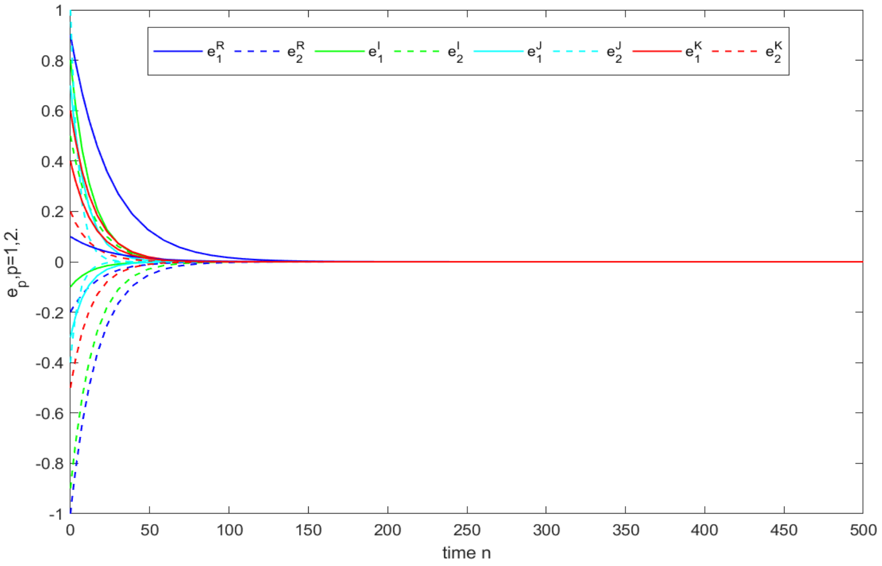

**Figure 10.** $\mathbb{T} = \mathbb{Z}$. The drive system (5.1) and its response system (5.2) are synchronized.

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