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INTERVAL-VALUED FRACTIONAL DIFFERENTIAL EQUATIONS WITH LENGTH CONSTRAINTS*

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Abstract This article investigates the existence of solutions for Caputo-type interval-valued fractional differential equations (IVFDEs) with length constraints. We begin by considering a class of IVFDEs that include impulsive effects. Then, we explore IVFDEs with impulses related to length constraints. Using fixed-point theory, we obtain several existence results. Importantly, we employ impulses to control the length of solutions to IVFDEs. Additionally, we present several examples to demonstrate our main results.

Keywords Interval-valued differential equation, Caputo fractional derivative, existence of solutions, length constraints, impulse control.

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1. Introduction

The study of fuzzy fractional integral and differential equations has attracted significant interest because of their ability to effectively characterise hereditary and memory properties in a diverse range of motion and materials processes [9, 21, 26, 32]. Furthermore, they can depict the incomplete and vague information present in uncertain phenomena across a range of disciplines, including physics [18], biology [1], chemistry [4, 7], engineering [38] and economics [8]. Consequently, fuzzy fractional calculus and fuzzy fractional differential equations have emerged as novel mathematical methods [10, 14].Notably, the theoretical framework of fractional calculus itself is still being expanded, with recent research efforts not only focusing on fractional differential equations and their boundary value problems [5, 6, 19, 33, 34] but also exploring its deep connections with zeta functions, such as the links between fractional calculus and Shannon entropy via zeta functions, the functional equations of Riemann zeta fractional derivatives, and the fractional calculus of Lerch zeta functions, which have further enriched the mathematical foundation of fractional calculus applications [15–17].

It is well known that interval analysis and fuzzy analysis are closely related [31]. Interval differential equation theory has also been proposed to handle interval uncertainties occurring in many different mathematical models. It has evolved in multiple theoretical avenues and found widespread applications in solving numerous practical problems [12, 20, 27–29, 35]. Therefore,

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the existence and uniqueness for solutions of interval-valued differential equations (IVDEs) naturally become important topics that have attracted considerable attention from experts. In [39], Wang et al. consider the existence for solutions of first-order IVDEs under generalized Hukuhara differentiability (referred to as gH-differentiability) and length constraints. By setting appropriate switching points, they provide certain sufficient conditions for the existence of solutions. However, interpreting the practical significance of these switching points in terms of dynamic systems as applied in biology or economics is challenging. In [41], Wang et al. introduce the concepts of forward and backward solutions of first-order IVDEs under gH-differentiability. According to the theory of stochastic processes outlined in [11], the stopping time is regarded as the termination point of the longest time interval for which a stochastic process can be reasonably forecasted from the initial conditions. For longer-term predictions, new data need to be collected before the stopping time. Thus, it is necessary to assess stopping time and collect new data at or before this point. Consequently, solutions with acceptable levels of uncertainty can be obtained.

On the other hand, the study of impulse differential equations has garnered significant interest for its broad practical applications in various real-life scenarios including epidemiology, optimal control [37, 43], population ecology [3], and mathematical modeling in mechanical and engineering domains [25]. In the field of IVDEs, Ho et al. study the existence and uniqueness of non-instantaneous impulses IVDEs under Caputo-Katugampola fractional derivatives in [22]. The authors also demonstrate some stability results regarding Ulam-Hyers-Mittag-Leffler. In [42], Wang et al. reconsidered first-order IVDEs with length constraints, controlling the length of solutions under gH-differentiability using impulses. They discuss the existence of forward and backward solutions. Compared to the approach in [39], the approach employed in [42] allows us to select the appropriate time for collecting new data based on the stopping time.

However, due to the complexity of IVFDEs, there are not many results concerning the existence of solutions for IVFDEs with length constraints under gH-differentiability. Therefore, in this paper, we study IVFDEs under length constraints in the form $len(\varphi(\zeta)) \le A$. By introducing some impulses at or before the stopping time with regard to $\{\varphi \in K_C | len(\varphi) > A\}$, we investigate the existence of solutions to the following IVFDEs.

To address this, in this paper, we study IVFDEs under length constraints in the form $len(\varphi(\zeta)) \le A$. By introducing some impulses at or before the stopping time with regard to $\{\varphi \in K_C | len(\varphi) > A\}$, we investigate the existence of solutions to the following IVFDEs

$$\begin{cases} {}^{C}_{\mathrm{g}H}\mathcal{D}^{\alpha}_{0+}\varphi(\zeta) = q(\zeta)\lambda(\varphi(\zeta)),\ \zeta \in [0,T], \\ \varphi(0) = \varphi_0, \end{cases} \tag{1.1}$$

where $\alpha \in (0,1)$, ${}^{C}_{\mathrm{g}H}\mathcal{D}^{\alpha}_{0+}\varphi$ is the Caputo generalized fractional derivative. $\varphi_0 \in K_C$, K_C is the set of all nonempty compact convex subsets of $\mathbb{R}$, $len(\varphi_0) > 0$, $\lambda : K_C \to K_C$ and $q : [0,T] \to \mathbb{R}$ are both continuous. Moreover, $q(\zeta) \neq 0$ holds almost everywhere on $[0,T]$. Compared with existing studies [22, 23, 39, 42], the core innovations and significance of this work are:

(i) Unlike [22, 23], this paper first integrates the $len(\varphi(\zeta)) \le A$ constraint into gH-differentiable IVFDEs, enabling coupled modeling of fractional memory and finite state amplitude to better align with practical system constraints.

(ii) Compared with first-order IVDEs in [42], this work addresses dual challenges: The non-locality of fractional operators and gH-differentiable endpoint asynchrony. An improved fixed-point theorem is proposed, providing a novel methodological basis for similar problems.

(iii) Unlike the fixed switching point control in [39], this paper adopts stopping time-triggered impulse control, automatically triggering impulses based on dynamic solution evolution. This adapts to fractional memory effects and allows flexible data collection and correction at stopping times, enhancing engineering applicability.

The paper is structured as follows: In Section 2, we introduce the key definitions, symbols, and lemmas necessary for the subsequent analysis. Section 3 explores a class of IVFDEs with impulses, presenting existence results for forward solutions. Section 4 focuses on IVFDEs with impulses subject to length constraints.

2. Preliminaries

K_C denotes the set of all nonempty compact convex subsets of $\mathbb{R}$. Let $\beta=[\underline{\beta},\overline{\beta}]$, $\gamma=[\underline{\gamma},\overline{\gamma}]$ belong to K_C, then the usual interval operations i.e. Minkowski addition and scalar multiplication, can be defined as follows:

$$\beta+\gamma=[\underline{\beta}+\underline{\gamma},\overline{\beta}+\overline{\gamma}],\ k\beta=\begin{cases}[k\underline{\beta},k\overline{\beta}] \text{ if } k>0,\\ 0 \qquad\quad\ \text{if } k=0,\\ [k\overline{\beta},k\underline{\beta}] \text{ if } k<0,\end{cases}$$

respectively. Let $k=-1$, we denote scalar multiplication $-\beta:=(-1)\beta=(-1)[\underline{\beta},\overline{\beta}]=[-\overline{\beta},-\underline{\beta}]$. In general, $\beta+(-\beta)\neq 0$; that is, the opposite β is not the inverse of β with respect to the Minkowski addition (unless β is a degenerate interval). The Minkowski difference is $\beta-\gamma=\beta+(-1)\gamma=[\underline{\beta}-\overline{\gamma},\overline{\beta}-\underline{\gamma}]$. Moreover, the length of β is defined by $len(\beta)=\overline{\beta}-\underline{\beta}$.

Let H be the Pomeiu-Hausdorff metric on K_C, that is, $H(\beta,\gamma)=\max\{|\underline{\beta}-\underline{\gamma}|,|\overline{\beta}-\overline{\gamma}|\}$ for $\beta,\gamma\in K_C$. Clearly, (K_C,H) is a complete and locally compact metric space.

Lemma 2.1 ([42]). *Let $\beta,\gamma,\eta,\mu\in K_C$. The following results hold:*

(i) $H(k\beta,k\gamma)=|k|\cdot H(\beta,\gamma), k\in\mathbb{R}$;

(ii) $H(\beta,\gamma)\leq H(\beta,\eta)+H(\gamma,\eta)$;

(iii) *if* $\gamma\subseteq\beta$, *then* $H(\gamma,0)\leq H(\beta,0)$;

(iv) $H(\beta+\gamma,\eta+\mu)\leq H(\beta,\eta)+H(\gamma,\mu)$, *especially,* $H(\beta+\gamma,\{0\})\leq H(\beta,\{0\})+H(\gamma+\{0\})$.

Lemma 2.2 ([40]). *Let $\beta,\gamma\in K_C$, $k\in\mathbb{R}$, then $len(\beta+\gamma)=len(\beta-\gamma)=len(\beta)+len(\gamma)$ and $len(k\beta)=|k|\cdot len(\beta)$ holds.*

Let $\beta,\gamma\in K_C$, Hukuhara difference of β and γ is defined as $\beta\ominus\gamma=[\underline{\beta}-\underline{\gamma},\overline{\beta}-\overline{\gamma}]$, see [24]. The g$H$-difference is defined as [36]:

$$\beta\ominus_{gH}\gamma=\begin{cases}(i)\beta\ominus\gamma, \text{if } \ \beta\ominus\gamma \ \text{ exists},\\ (ii)(-1)(\gamma\ominus\beta), \text{if } \ \gamma\ominus\beta \ \text{ exists}.\end{cases}$$

Definition 2.1 ([36]). Let $\ell:(a,b)\to K_C$ and $\zeta\in(a,b)$. The interval function ℓ is said to be generalized Hukuhara differentiable (gH-differentiable) at ζ, if there exists $\ell'(\zeta)\in K_C$ such that

$$\ell'(\zeta)=\lim_{h\to 0}\frac{\ell(\zeta+h)\ominus_{gH}\ell(\zeta)}{h}.$$

Let J is a bounded, closed interval, $\ell : J \subseteq \mathbb{R} \to K_C$ and $\ell(\zeta) = [\underline{\ell}(\zeta), \overline{\ell}(\zeta)]$, then

$$\int_J \ell(\zeta)dt = \left[\int_J \underline{\ell}(\zeta)dt, \int_J \overline{\ell}(\zeta)d\zeta\right]. \tag{2.1}$$

Suppose that $\int_J \ell(\zeta)d\zeta$ exist, $J_1 \cap J_2 = \Phi$ and $J_1 \cup J_2 = J$. According to the addition theorem for intervals of integration of real-valued functions,then

$$\begin{aligned}\int_J \ell(\zeta)d\zeta &= \left[\int_J \underline{\ell}(\zeta)d\zeta, \int_J \overline{\ell}(\zeta)d\zeta\right]\\ &= \left[\int_{J_1} \underline{\ell}(\zeta)d\zeta + \int_{J_2} \underline{\ell}(\zeta)d\zeta, \int_{J_1} \overline{\ell}(\zeta)d\zeta + \int_{J_2} \overline{\ell}(\zeta)d\zeta\right]\\ &= \left[\int_{J_1} \underline{\ell}(\zeta)d\zeta, \int_{J_1} \overline{\ell}(\zeta)d\zeta\right] + \left[\int_{J_2} \underline{\ell}(\zeta)dt, \int_{J_2} \overline{\ell}(\zeta)d\zeta\right]\\ &= \int_{J_1} \ell(\zeta)d\zeta + \int_{J_2} \ell(\zeta)d\zeta.\end{aligned} \tag{2.2}$$

Lemma 2.3 ([40]). *Assume that $\ell, \phi : J \subseteq \mathbb{R} \to K_C$ are continuous and $J \subset \mathbb{R}$ is a bounded closed interval. we have that,*

$$H\left(\int_J \ell(\zeta)dt, \int_J \phi(\zeta)d\zeta\right) \leq \int_J H\left(\ell(\zeta), \phi(\zeta)\right)d\zeta.$$

Definition 2.2 ([30]). Let $\alpha \in (0,1)$, then the Riemann-Liouville generalized fractional integral of order α of the interval function $\xi : [0,b] \to K_C$ are defined by:

$$\mathcal{I}^{\alpha}_{0+}\xi(\zeta) = \int_0^{\zeta} \frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}\xi(\vartheta)d\vartheta,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.3 ([30]). Let $\alpha \in (0,1)$, $\xi : [0,b] \to K_C$ is Hukuhara differentiable on $[0,b]$, then the Caputo generalized derivative of the interval function ξ are defined by:

$${}^{C}_{gH}\mathcal{D}^{\alpha}_{0+}\xi(\zeta) = \int_0^{\zeta} \frac{(\zeta-\vartheta)^{-\alpha}}{\Gamma(1-\alpha)}\xi^{'}(\vartheta)d\vartheta.$$

Definition 2.4 ([13]). Let $\omega : [a,b] \subseteq \mathbb{R} \to K_C$ and $\omega(\zeta) = [\underline{\omega}(\zeta), \overline{\omega}(\zeta)]$.

(i) ω is said to be (I)-gH-fractional differentiable at ζ_0, if $\underline{\omega}$ and $\overline{\omega}$ are both differentiable at ζ_0 and ${}^{C}_{gH}\mathcal{D}^{\alpha}\omega(\zeta_0) = [{}^{C}\mathcal{D}^{\alpha}\underline{\omega}(\zeta_0), {}^{C}\mathcal{D}^{\alpha}\overline{\omega}(\zeta_0)]$. ${}^{C}\mathcal{D}^{\alpha}$ denotes usual Caputo fractional derivative (see [30]).

(ii) ω is said to be (II)-gH-fractional differentiable at ζ_0, if $\underline{\omega}$ and $\overline{\omega}$ are both differentiable at ζ_0 and ${}^{C}_{gH}\mathcal{D}^{\alpha}\omega(\zeta_0) = [{}^{C}\mathcal{D}^{\alpha}\overline{\omega}(\zeta_0), {}^{C}\mathcal{D}^{\alpha}\underline{\omega}(\zeta_0)]$.

Lemma 2.4 ([2]). *Let $u_0 \in K_C$ and $h : \mathbb{R} \times K_C \to K_C$ is jointly continuous. Then, the IVFDEs*

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\alpha}_{\zeta_0^+}u(\zeta) = h(\zeta, u(\zeta)),\\ u(\zeta_0) = u_0 \in K_C, \end{cases} \tag{2.3}$$

is equivalent to the integral equation

$$u(\zeta) \ominus_{gH} u_0 = \tfrac{1}{\Gamma(\alpha)} \int_{\zeta_0^+}^{\zeta} (\zeta - \vartheta)^{\alpha-1} h(\vartheta, u(\vartheta)) d\vartheta, \tag{2.4}$$

on some interval $[\zeta_0, \zeta_0 + \delta]$*, where* $\delta > 0$*, if the length of the interval function* $\zeta \mapsto I^{\alpha}_{\zeta_0^+} h(\zeta, u)$ *is increasing and the length of the interval function* u *is either increasing or decreasing over the interval* $[\zeta_0, \zeta_0 + \delta]$*.*

By Lemma 2.4, the (I)-gH-fractional differentiable solution of (2.3) is also a solution of

$$\begin{cases} {}^C\mathcal{D}^{\alpha}_{\zeta_0^+} \underline{u}(\zeta) = \underline{h}(\zeta, u(\zeta)) = \min h(\zeta, u(\zeta)), \\ {}^C\mathcal{D}^{\alpha}_{\zeta_0^+} \overline{u}(\zeta) = \overline{h}(\zeta, u(\zeta)) = \max h(\zeta, u(\zeta)), \\ \underline{u}(\zeta_0) = \underline{u}_0, \overline{u}(\zeta_0) = \overline{u}_0, \end{cases} \tag{2.5}$$

where $len(u(\zeta))$ is monotonic increasing. The (II)-gH-fractional differentiable solution of (2.3) is also a solution of

$$\begin{cases} {}^C\mathcal{D}^{\alpha}_{\zeta_0^+} \underline{u}(\zeta) = \overline{h}(\zeta, u(\zeta)) = \max h(\zeta, u(\zeta)), \\ {}^C\mathcal{D}^{\alpha}_{\zeta_0^+} \overline{u}(\zeta) = \underline{h}(\zeta, u(\zeta)) = \min h(\zeta, u(\zeta)), \\ \underline{u}(\zeta_0) = \underline{u}_0, \overline{u}(\zeta_0) = \overline{u}_0, \end{cases} \tag{2.6}$$

where $len(u(\zeta))$ is monotonic decreasing.

Let $J = [a, b], a = \zeta_0 < \zeta_1 < \cdots < \zeta_n < \zeta_{n+1} = b$. Moreover, let $PC^*(J, K_C) = \{\varphi : J \to K_C |\ \varphi \text{ is continuous on } J \backslash \{\zeta_1, \zeta_2, \cdots, \zeta_n\}; \varphi(\zeta_k^+), \varphi(\zeta_k^-) \text{ exist}, \varphi(\zeta_k^+) = \varphi(\zeta_k) \text{ for } k = 1, 2, \cdots, n\}$ and

$$d(\varphi_1, \varphi_2) = \sup_{\zeta \in J} H(\varphi_1(\zeta), \varphi_2(\zeta))$$

for $\varphi_1, \varphi_2 \in PC^*(J, K_C)$.

Lemma 2.5 ([42]). $(PC^*(J, K_C), d)$ *is a complete metric space.*

3. Interval-valued fractional differential equation with impulses

This section investigates the existence of forward solutions impulsive IVFDEs. First of all, we consider the following IVFDEs with impulses:

$$\begin{cases} {}^C_{gH}\mathcal{D}^{\alpha}_{0^+} \varphi(\zeta) = q(\zeta)\lambda(\varphi(\zeta)),\ \zeta \in [0, T] \setminus \{\zeta_1, \zeta_2, \cdots, \zeta_n\}, \\ \varphi(0) = \varphi_0, \\ \varphi(\zeta_k^+) = I_k(\varphi(t_k)),\ k = 1, 2, \cdots, n, \end{cases} \tag{3.1}$$

where $\alpha \in (0, 1), \lambda : K_C \to K_C$ and $q : [0, T] \to \mathbb{R}$ are both continuous. $0 = \zeta_0 < \zeta_1 < \cdots < \zeta_n < \zeta_{n+1} = T, I_k : K_C \to K_C$ is also continuous. ${}^C_{gH}\mathcal{D}^{\alpha}_{0^+} \varphi(\zeta)$ denotes the Caputo fractional gH-derivative. Let $I_0 : K_C \to K_C$ be the identity map, meaning that, $I_0(\omega) = \omega$ for all $\omega \in K_C$.

In general, if we have no information other than the initial conditions, the uncertainty, as measured by the length of the solution, should increase as ζ moves away from the initial time, whether it increases or decreases. With this in mind, the solution $\varphi(\zeta)$ of (3.1) should satisfy

$$\varphi(\zeta)=\begin{cases}\varphi_0+\displaystyle\int_0^\zeta \frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\ \zeta\in[0,\zeta_1],\\ I_k(\varphi(\zeta_k))+\displaystyle\int_{\zeta_k}^\zeta \frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\ \zeta\in(\zeta_k,\zeta_{k+1}],\end{cases}\tag{3.2}$$

which is (I)-gH-fractional differentiable, $k=1,2,\cdots,n$.

Definition 3.1. $\varphi\in PC^*([0,T],K_C)$ is said to be a forward solution to (3.1), if φ satisfies (3.2) for $\zeta\in[0,T]$.

For every $\varphi\in PC^*([0,T],K_C)$, we define a mapping as follows:

$$F(\varphi(\zeta))=\begin{cases}\varphi_0+\displaystyle\int_0^\zeta \frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\ \zeta\in[0,\zeta_1],\\ I_k(\varphi(\zeta_k))+\displaystyle\int_{\zeta_k}^\zeta \frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\ \zeta\in(\zeta_k,\zeta_{k+1}],\end{cases}$$

where $k=1,2,\cdots,n$. Since λ and I_k are continuous, it follows that F is continuous on $PC^*([0,T],K_C)$.

Theorem 3.1. *Let $I_k(w)\subseteq w$ for every $w\in K_C$ and $k=1,2,\cdots,n$, then there is at least one forward solution to (3.1), if there exists $\varepsilon\in(0,1)$ such that $\max_{\zeta\in[0,T]}|q(\zeta)|\cdot\frac{T^\alpha}{\Gamma(\alpha+1)}<\frac{1}{\varepsilon}$ and*

$$\lim_{H(w,\{0\})\to+\infty}\frac{H(\lambda(w),\{0\})}{H(w,\{0\})}<\varepsilon\tag{3.3}$$

hold.

Proof. By (3.3), there exists $r_0>H(\varphi_0,\{0\})$ such that $H(\lambda(w),\{0\})\le\varepsilon\cdot H(w,\{0\}))$ for every $u\in\{w\in K_C|H(w,\{0\})>r_0\}$.

Let $r>r_0, B_{r_0}=\{w\in K_C\mid H(\omega,\{0\})<r_0\}, E_r=\{\varphi\in PC^*([0,T],K_C)\mid d(\varphi,\{0\})\le r\}$. For every $\varphi\in E_r$, let $W_\varphi=\{\zeta\in[0,T]|\varphi(\zeta)\in\overline{B}_{r_0}\}$. Based on Lemma 2.1 (iv), for every $\zeta\in[0,\zeta_1]$, we derive

$$\begin{aligned}H(F(\varphi(\zeta)),\{0\})&=H\left(\varphi_0+\int_0^\zeta\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right)\\&\le H(\varphi_0,\{0\})+H\left(\int_0^\zeta\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right).\end{aligned}\tag{3.4}$$

Suppose that $I_0(\varphi(\zeta_0))=\varphi_0$, by Lemma 2.1(iii) and $I_k(\omega)\subseteq\omega$, we can obtain $H\left(I_k(\varphi(\zeta_k)),0\right)\le H(\varphi(\zeta_k),0)$ For $k=1,2,\cdots,n$. Let $\zeta\in(\zeta_k,\zeta_{k+1}]$, by using Lemma 2.1(i) and (iv) to yield

$$\begin{aligned}&H(F(\varphi(\zeta)),\{0\})\\=&H\left(I_k(\varphi(\zeta_k))+\int_{\zeta_k}^\zeta\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq H\left(I_k(\varphi(\zeta_k)),\{0\}\right)+H\left(\int_{\zeta_k}^{\zeta}\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right)\\
&\leq H\left(\varphi(\zeta_k),\{0\}\right)+H\left(\int_{\zeta_k}^{\zeta}\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right)\\
&=H\left(I_{k-1}(\varphi(\zeta_{k-1}))+\int_{\zeta_{k-1}}^{\zeta_k}\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right)\\
&\quad+H\left(\int_{\zeta_k}^{\zeta}\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right)\\
&\leq H\left(I_{k-1}(\varphi(\zeta_{k-1})),\{0\}\right)+H\left(\int_{\zeta_{k-1}}^{\zeta_k}\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right)\\
&\quad+H\left(\int_{\zeta_k}^{\zeta}\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right).
\end{aligned}
$$

By repeating the previous steps, we obtain

$$
\begin{aligned}
H(F(\varphi(\zeta)),\{0\})\leq H\left(\varphi_0,\{0\}\right)+\sum_{i=0}^{k-1}H\left(\int_{\zeta_i}^{\zeta_{i+1}}\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right)\\
+H\left(\int_{\zeta_k}^{\zeta}\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right).
\end{aligned}
\tag{3.5}
$$

For every $\zeta\in[0,T]$, by Lemma 2.3 and (3.4), (3.5) to yield

$$
\begin{aligned}
H(F(\varphi(\zeta)),\{0\})&\leq H\left(\varphi_0,\{0\}\right)+\int_0^{\zeta}H\left(\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta,\{0\}\right)\\
&=H\left(\varphi_0,\{0\}\right)+\int_0^{\zeta}\left|\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\right|\cdot H\left(\lambda(\varphi(\vartheta)),\{0\}\right)d\vartheta\\
&\leq H\left(\varphi_0,\{0\}\right)+\max_{w\in\overline{B}_{r_0}}H(\lambda(w),\{0\})\cdot\int_{W_\varphi}\left|\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\right|d\vartheta\\
&\quad+\int_{[0,T]\setminus W_\varphi}\left|\frac{(\zeta-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\right|\cdot\varepsilon\cdot H(\varphi(\vartheta),\{0\})d\vartheta\\
&\leq H\left(\varphi_0,\{0\}\right)+\max_{w\in\overline{B}_{r_0}}H(\lambda(w),\{0\})\cdot\int_0^{T}\left|\frac{(T-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\right|d\vartheta\\
&\quad+r\cdot\varepsilon\int_0^{T}\left|\frac{(T-\vartheta)^{\alpha-1}}{\Gamma(\alpha)}q(\vartheta)\right|d\vartheta\\
&\leq H\left(\varphi_0,\{0\}\right)+\max_{\zeta\in[0,T]}|q(\zeta)|\cdot\frac{T^\alpha}{\Gamma(\alpha+1)}(\max_{w\in\overline{B}_{r_0}}H(\lambda(u),\{0\})+r\cdot\varepsilon).
\end{aligned}
\tag{3.6}
$$

If necessary, let r a larger positive number, for example,

$$
r\geq\frac{H(\varphi_0,\{0\})+\max_{w\in\overline{B}_{r_0}}H(\lambda(w),\{0\})\cdot\max_{\zeta\in[0,T]}|a(\zeta)|\cdot\frac{T^\alpha}{\Gamma(\alpha+1)}}{1-\max_{\zeta\in[0,T]}|q(\zeta)|\cdot\frac{T^\alpha\varepsilon}{\Gamma(\alpha+1)}},
$$

which means that $d(F(\varphi),\{0\}) \le r$. Thus, $F(E_r) \subseteq E_r$.

Moreover, for every $\varphi \in E_r$ and $\zeta_k \le \tilde{\zeta} \le \bar{\zeta} \le \zeta_{k+1}, k = 0,1,2,\cdots,n$, we have

$$
\begin{aligned}
&H(F(\varphi(\tilde{\zeta})), F(\varphi(\bar{\zeta}))\\
&\le H\left(\int_0^{\tilde{\zeta}} \frac{(\tilde{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta, \int_0^{\bar{\zeta}} \frac{(\bar{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta\right)\\
&= H\left(\int_0^{\tilde{\zeta}} \frac{(\tilde{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta, \int_0^{\tilde{\zeta}} \frac{(\bar{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta\right.\\
&\quad \left. + \int_{\tilde{\zeta}}^{\bar{\zeta}} \frac{(\bar{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta\right)\\
&\le H\left(\int_0^{\tilde{\zeta}} \frac{(\tilde{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta, \int_0^{\tilde{\zeta}} \frac{(\bar{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta\right)\\
&\quad + H\left(\int_{\tilde{\zeta}}^{\bar{\zeta}} \frac{(\bar{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta))d\vartheta, \{0\}\right)\\
&\le \int_0^{\tilde{\zeta}} H\left(\frac{(\tilde{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta)), \frac{(\bar{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta))\right)d\vartheta\\
&\quad + \int_{\tilde{\zeta}}^{\bar{\zeta}} H\left(\frac{(\bar{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\lambda(\varphi(\vartheta)), \{0\}\right)d\vartheta\\
&= \int_0^{\tilde{\zeta}} \left|\frac{[(\tilde{\zeta}-\vartheta)^{\alpha-1} - (\bar{\zeta}-\vartheta)^{\alpha-1}]q(\vartheta)}{\Gamma(\alpha)}\right| H(\lambda(\varphi(\vartheta)), \{0\})d\vartheta\\
&\quad + \int_{\tilde{\zeta}}^{\bar{\zeta}} \left|\frac{(\bar{\zeta}-\vartheta)^{\alpha-1}}{\Gamma(\alpha)} q(\vartheta)\right| \cdot H\left(\lambda(\varphi(\vartheta)), \{0\}\right)d\vartheta\\
&\le \max_{w\in \overline{B}_r} H(\lambda(w),\{0\}) \cdot \max_{\zeta\in[0,T]} |q(\zeta)| \cdot \left|\frac{(\bar{\zeta}-\tilde{\zeta})^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{\bar{\zeta}^{\alpha}-\tilde{\zeta}^{\alpha}}{\alpha\Gamma(\alpha)}\right|\\
&\quad + \max_{w\in \overline{B}_r} H(\lambda(w),\{0\}) \cdot \max_{\zeta\in[0,T]} |q(\zeta)| \cdot \left|\frac{(\bar{\zeta}-\tilde{\zeta})^{\alpha}}{\alpha\Gamma(\alpha)}\right|\\
&\le \frac{3}{\Gamma(\alpha+1)} \max_{w\in \overline{B}_r} H(\lambda(w),\{0\}) \cdot \max_{\zeta\in[0,T]} |q(\zeta)| \cdot |\bar{\zeta}-\tilde{\zeta}|^{\alpha}.
\end{aligned}
$$

Let $\underline{F}(\varphi(\zeta)) = \underline{F(\varphi(\zeta))}, \overline{F}(\varphi(\zeta)) = \overline{F(\varphi(\zeta))}$, then $\underline{F}(E_r)$ and $\overline{F}(E_r)$ are uniformly bounded and equicontinuous on $(\zeta_k, \zeta_{k+1}], k = 0,1,2,\cdots,n$. According to the Arzela-Ascoli theorem, $\underline{F}(E_r)$ and $\overline{F}(E_r)$ are both sequentially compact.

Let $\{\varphi_n\} \subset E_r$, then $\{\underline{F}(\varphi_n)\}$ is sequentially compact, i.e. there exists a convergent subsequence $\{\underline{F}(\varphi_{n_k})\} \subseteq \{\underline{F}(\varphi_n)\}$. And from the compactness of the sequence $\{\overline{F}(\varphi_{n_k})\}$ it follows that there exists at least one convergent subsequence $\{\overline{F}(\varphi_{n_{k_j}})\} \subseteq \{\overline{F}(\varphi_{n_k})\}$. Consequently, the sequence $\{F(\varphi_n)\}$ contains at least one convergent subsequence $\{F(\varphi_{n_{k_j}})\}$, indicating that $F(E_r)$ is sequentially compact as well.

The space $C^*[0,T] = \{\varphi : [0,T] \to \mathbb{R} \mid \varphi$ is continuous on $[0,T] \setminus \{\zeta_1, \zeta_2, \cdots, \zeta_n\}$; $\varphi(\zeta_k^+)$ and $\varphi(\zeta_k^-)$ exist and $\varphi(\zeta_k^+) = \varphi(\zeta_k)$ for $k = 1,2,\cdots,n\}$, forms a Banach space under the norm

$||(\varphi,\psi)|| = \max\{\max_{\zeta\in[0,T]}|\varphi(\zeta)|, \max_{\zeta\in[0,T]}|\psi(\zeta)|\}$. We define E_r as a subset of this space, which is nonempty, bounded, and convex. Additionally, From Lemma 2.5, it follows that E_r is closed. By applying Schauder's fixed point theorem, we conclude that, there is at least one fixed point of F within E_r, and this fixed point also serves as a forward solution to (3.1). □

Example 3.1. Consider impulsive IVFDEs

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\frac{1}{2}}_{0^+}\varphi(\zeta) = \sin\zeta\cdot([0,1]+\varphi(\zeta)),\ \zeta\in[0,\frac{5}{2}]\setminus\{1,2\}, \\ \varphi(0) = [-2,4], \\ \varphi(\zeta^+) = [\underline{\varphi}(\zeta), \frac{\underline{\varphi}(\zeta)+\overline{\varphi}(\zeta)}{5}],\ \zeta = 1,2, \end{cases} \tag{3.7}$$

where $q(\zeta) = \sin(\zeta)$, and $\lambda(\varphi) = [0,1]+\varphi$, by calculating $\max_{\zeta\in[0,\frac{5}{2}]}|\sin\zeta|\cdot\frac{(\frac{5}{2})^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \approx 1.7841$. In addition, Lemma 2.1 (ii) indicates that for every $\omega\in K_C$

$$H([0,1]+\omega,\{0\}) \le H([0,1],\{0\}) + H(\omega,\{0\}) = H(\omega,\{0\}) + 1.$$

Therefore, (3.3) is satisfied. According to Theorem 3.1, problem (3.7) has at least one forward solution. From (2.5), the forward solution of (3.7) satisfies

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{1}{2}}_{0^+}\underline{\varphi}(\zeta) = \sin\zeta\cdot\underline{\varphi}(\zeta), \\ {}^{C}\mathcal{D}^{\frac{1}{2}}_{0^+}\overline{\varphi}(\zeta) = \sin\zeta\cdot(1+\overline{\varphi}(\zeta)), \end{cases} \tag{3.8}$$

for $\zeta\in[0,\frac{5}{2}]\setminus\{1,2\}$, which implies that

$$\varphi(\zeta) = \begin{cases} [-2,4] + \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\zeta}(\zeta-\vartheta)^{-\frac{1}{2}}\sin\vartheta\cdot([0,1]+\varphi(\vartheta))d\vartheta,\ \zeta\in[0,1], \\ \left[\underline{\varphi}(1), \frac{\underline{\varphi}(1)+\overline{\varphi}(1)}{5}\right] + \frac{1}{\Gamma(\frac{1}{2})}\int_1^{\zeta}(\zeta-\vartheta)^{-\frac{1}{2}}\sin\vartheta\cdot([0,1]+\varphi(\vartheta))d\vartheta,\ \zeta\in(1,2], \\ \left[\underline{\varphi}(2), \frac{\underline{\varphi}(2)+\overline{\varphi}(2)}{5}\right] + \frac{1}{\Gamma(\frac{1}{2})}\int_2^{\zeta}(\zeta-\vartheta)^{-\frac{1}{2}}\sin\vartheta\cdot([0,1]+\varphi(\vartheta))d\vartheta,\ \zeta\in(2,\frac{5}{2}]. \end{cases} \tag{3.9}$$

We see the graph of the forward solution (3.9) is shown in Figure 1.

As shown in Figure 1, the X-axis represents time $\zeta\in[0,2.5]$ and the Y-axis represents $\varphi(\zeta)$. The interval-valued solution evolves continuously at non-impulse times and exhibits state jumps at $\zeta = 1,2$, which is consistent with the piecewise continuity of impulsive IVFDEs. Post-impulse adjustment of $\underline{\varphi}(\zeta)$ and $\overline{\varphi}(\zeta)$ further restrains unbounded growth, and the global existence of the solution over $[0,2.5]$ verifies the conclusion of Theorem 3.1.

Example 3.2. Consider impulsive IVFDEs

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\frac{1}{2}}_{0^+}\varphi(\zeta) = \frac{1}{8}\cdot([0,1]+\varphi(\zeta)),\ \zeta\in[0,4]\setminus\{1,2,3\}, \\ \varphi(0) = [1,8], \\ \varphi(\zeta^+) = [\frac{\underline{\varphi}(\zeta)+\overline{\varphi}(\zeta)}{2} - 1, \overline{\varphi}(\zeta)],\ \zeta = 1,2,3. \end{cases} \tag{3.10}$$

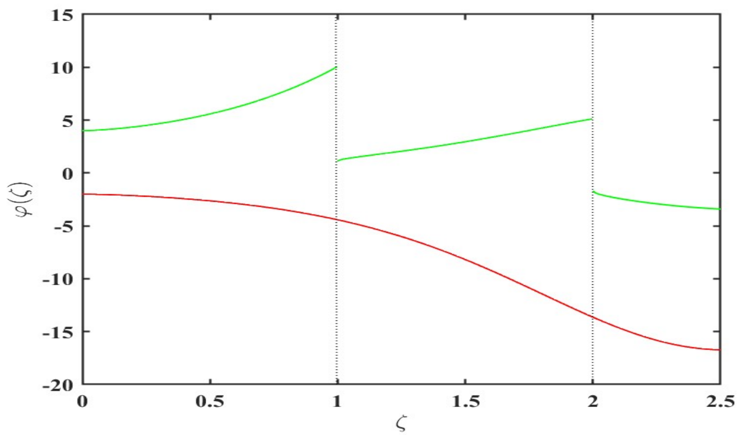

Figure 1. Graph of the forward solution (3.9) (the red curve represents $\underline{\varphi}(\zeta)$, and the green curve represents $\overline{\varphi}(\zeta)$).

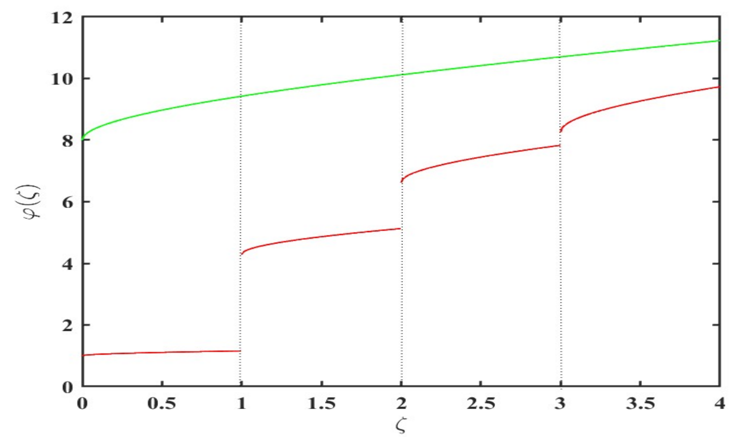

Figure 2. Graph of the forward solution (3.11) (the red curve represents $\underline{\varphi}(\zeta)$, and the green curve represents $\overline{\varphi}(\zeta)$).

Let $q(\zeta) = \frac{1}{4}$, and $\lambda(\varphi) = \frac{1}{2} \cdot ([0,1] + \varphi)$, we can check that $\frac{1}{4} \cdot \frac{4^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \approx 0.5642$ and (3.3) holds. By Theorem 3.1, problem (3.10) has at least one forward solution

$$\varphi(\zeta) = \begin{cases} [1,8] + \dfrac{1}{\Gamma(\frac{1}{2})} \displaystyle\int_0^\zeta \frac{1}{8}(\zeta - \vartheta)^{-\frac{1}{2}} \cdot ([0,1] + \varphi(\vartheta)) d\vartheta, \ \zeta \in [0,1], \\ \left[\dfrac{\underline{\varphi}(1) + \overline{\varphi}(1)}{2} - 1, \overline{\varphi}(1) \right] + \dfrac{1}{\Gamma(\frac{1}{2})} \displaystyle\int_1^\zeta \frac{1}{8}(\zeta - \vartheta)^{-\frac{1}{2}} \cdot ([0,1] + \varphi(\vartheta)) d\vartheta, \ \zeta \in (1,2], \\ \left[\dfrac{\underline{\varphi}(2) + \overline{\varphi}(2)}{2} - 1, \overline{\varphi}(2) \right] + \dfrac{1}{\Gamma(\frac{1}{2})} \displaystyle\int_2^\zeta \frac{1}{8}(\zeta - \vartheta)^{-\frac{1}{2}} \cdot ([0,1] + \varphi(\vartheta)) d\vartheta, \ \zeta \in (2,3], \\ \left[\dfrac{\underline{\varphi}(3) + \overline{\varphi}(3)}{2} - 1, \overline{\varphi}(3) \right] + \dfrac{1}{\Gamma(\frac{1}{2})} \displaystyle\int_3^\zeta \frac{1}{8}(\zeta - \vartheta)^{-\frac{1}{2}} \cdot ([0,1] + \varphi(\vartheta)) d\vartheta, \ \zeta \in (3,4]. \end{cases} \tag{3.11}$$

We see the graph of the forward solution (3.11) is shown in Figure 2.

As shown in Figure 2, adopting the same axis definition as Figure 1, the X-axis represents time $\zeta \in [0,4]$ and the Y-axis represents $\varphi(\zeta)$. It can be observed that the interval length is compressed after each impulse trigger at $\zeta = 1, 2, 3$, demonstrating the constraint effect of impulses on the solution. Meanwhile, the solution evolves in a smooth and gradual trend, which is consistent with the non-local memory property of fractional operators. Moreover, the solution exists throughout $[0, 4]$ and satisfies the impulse conditions, further verifying the universality of Theorem 3.1.

Remark 3.1. From the proof of Theorem 3.1, we see that the condition $I_k(w) \subseteq w$ is not a necessary condition. In fact, the results (3.4), (3.5), and (3.6) still hold under the weaker assumption that for every $w \in K_C$ and $k = 1, 2, \cdots, n$, the inequality $H(I_k(w), w) \leq \varepsilon$ satisfies for some $\varepsilon > 0$. Consequently, we state the following theorem without proof.

Theorem 3.2. *Suppose that there exists $\varepsilon > 0$ such that $H(I_k(w), w) \leq \varepsilon$ for every $w \in K_C, k = 1, 2, \cdots, n$, if there is an $\varepsilon \in (0, 1)$ such that*

$$\max_{\zeta \in [0,T]} |q(\zeta)| \cdot \frac{T^\alpha}{\Gamma(\alpha + 1)} < \frac{1}{\varepsilon}$$

and equation (3.3) is satisfied, then there exists at least one forward solution for Problem (3.1).

4. Interval-valued fractional differential equation with impulses relative to length constraints

Let us first study the characteristics the forward solution of the following IVFDEs

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\alpha}_{\zeta_0^+}\varphi(\zeta) = q(\zeta)\lambda(\varphi(\zeta)), \\ \varphi(\zeta_0) = \varphi_0, \end{cases} \tag{4.1}$$

where $\alpha \in (0, 1), \varphi_0 \in K_C$, $\lambda : K_C \to K_C$ and $q : \mathbb{R} \to \mathbb{R}$ are continuous. ${}^{C}_{gH}\mathcal{D}^{\alpha}_{\zeta_0^+}\varphi$ is the Caputo generalized fractional derivative. In addition,
(A1) there exists $L > 0$ such that $len(\lambda(w)) \leq L \cdot len(w)$ hold for every $w \in K_C$.

Lemma 4.1. *If (A1) is true, $0 < len(\varphi_0) < A, q(\zeta) \neq 0$ holds almost everywhere on $\mathbb{R}$,then the forward solution of problem (4.1) satisfies $len(\varphi(\zeta)) \leq A$ for $\zeta \in [\zeta_0, \alpha^+_{\zeta_0})$, where*

$$\alpha^+_{\zeta_0} = \begin{cases} P^{-1}_{\zeta_0}(\tilde{A}_{\varphi_0}), \ \textit{for}\ E_\alpha[\zeta^\alpha L \max\limits_{\tau \in [\zeta_0, \zeta]} |q(\tau)|] > \tilde{A}_{\varphi_0}, \\ +\infty, \ \textit{for}\ E_\alpha[\zeta^\alpha L \max\limits_{\tau \in [\zeta_0, \zeta]} |q(\tau)|] \leq \tilde{A}_{\varphi_0}, \end{cases}$$

E_α is the Mittag-Leffler function, $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$, $\tilde{A}_{\varphi_0} = \frac{A}{len(\varphi_0)}, P^{-1}_{\zeta_0}$ is the inverse function of $P_{\zeta_0} : [\zeta_0, +\infty) \to [0, +\infty)$, given by $P_{\zeta_0}(\zeta) = E_\alpha[\zeta^\alpha L \max\limits_{\tau \in [\zeta_0, \zeta]} |q(\tau)|]$.

Proof. Let φ is a forward solution of (4.1) on $[\zeta_0, +\infty)$.

Based on (2.5), we infer that ${}^{C}\mathcal{D}^{\alpha}_{\zeta_0^+}\overline{\varphi}(\zeta) -{}^{C}\mathcal{D}^{\alpha}_{\zeta_0^+}\underline{\varphi}(\zeta) = \overline{q(\zeta)\lambda(\varphi(\zeta))} - \underline{q(\zeta)\lambda(\varphi(\zeta))}$. Then, by (A1) and Lemma 2.2, we have

$${}^{C}_{gH}\mathcal{D}^{\alpha}_{\zeta_0^+}(len(\varphi(\zeta))) = len(q(\zeta)\lambda(\varphi(\zeta)))$$

$$= |q(\zeta)| \cdot len(\lambda(\varphi(\zeta)))$$
$$\leq |q(\zeta)| \cdot L \cdot len(\varphi(\zeta)).$$

Integrate the both sides from ζ_0 to ζ, and we have

$$len(\varphi(\zeta)) - len(\varphi_0) \leq \frac{L}{\Gamma(\alpha)} \int_{\zeta_0}^{\zeta} (\zeta - \vartheta)^{\alpha-1} |q(\vartheta)| len(\varphi(\vartheta)) d\vartheta,$$
$$len(\varphi(\zeta)) \leq len(\varphi_0) + \frac{L \max\limits_{\tau\in[\zeta_0,\zeta]} |q(\tau)|}{\Gamma(\alpha)} \int_{\zeta_0}^{\zeta} (\zeta - \vartheta)^{\alpha-1} len(\varphi(\vartheta)) d\vartheta.$$

In terms of the the Grönwal-Bellman inequality

$$len(\varphi(\zeta)) \leq len(\varphi_0) \cdot E_\alpha[\zeta^\alpha L \max_{\tau\in[\zeta_0,\zeta]} |q(\tau)|]. \tag{4.2}$$

Additionally, from the definition of $\alpha^+_{\zeta_0}$, we can derive

$$E_\alpha[(\alpha^+_{\zeta_0})^\alpha L \max_{\tau\in[\zeta_0,\alpha^+_{\zeta_0}]} |q(\tau)|] \leq \tilde{A}_{\varphi_0} = \frac{A}{len(\varphi_0)},$$

which implies that $len(\varphi_0) \cdot E_\alpha[(\alpha^+_{\zeta_0})^\alpha L \max\limits_{\tau\in[\zeta_0,\alpha^+_{\zeta_0}]} |q(\tau)|] \leq A$. From (4.2), to yield

$$\begin{aligned} len(\varphi(\zeta)) &\leq len(\varphi_0) \cdot E_\alpha[\zeta^\alpha L \max_{\tau\in[\zeta_0,\zeta]} |q(\tau)|] \\ &\leq len(\varphi_0) \cdot E_\alpha[(\alpha^+_{\zeta_0})^\alpha L \max_{\tau\in[\zeta_0,\alpha^+_{\zeta_0}]} |q(\tau)|] \\ &\leq A \end{aligned}$$

holds for every $\zeta \in [\zeta_0, \alpha^+_{\zeta_0})$. □

Remark 4.1. In practical scenarios, we can use Lemma 4.1 to estimate the time $\alpha^+_{\zeta_0}$, at which new data should be collected. This time corresponds to the impulse associated with the length constraint $len(\varphi(\zeta)) \leq A$. in problem (4.1). In [41], $\alpha^+_{\zeta_0}$ is referred to as the forward stopping time related to $len(\varphi(\zeta)) \leq A$.

Consider the existence of forward solution of problem (4.1) on $[0, T]$. Assume that (A1) is true, $0 < len(\varphi_0) < A, q(\zeta) \neq 0$ is holds almost everywhere on $[0, T]$. If $\zeta \in [0, T]$ satisfies $E_\alpha[T^\alpha L \max\limits_{\zeta\in[\zeta_0,T]} |q(\zeta)|] < \tilde{A}_{\varphi_0}$, Then by Lemma 4.1, it follows that if the forward solution of problem (4.1) exists, then it satisfies $len(\varphi(\zeta)) < A$ on $[0, T]$. If $E_\alpha[T^\alpha L \max\limits_{\zeta\in[\zeta_0,T]} |q(\zeta)|] > \tilde{A}_{\varphi_0}$, to ensure that the forward solution of problem (4.1) satisfies $len(\varphi(\zeta)) < A$ on $[0, T]$ if it exists, it is necessary to select a appropriate time $\zeta \in [0, T]$ for collecting new data.

Now, we consider the forward solution to problem (1.1) with $\varphi_0 \in K_C, 0 < len(\varphi_0) < A$. Moreover,the following two assumptions are introduced

(A2) $E_\alpha[T^\alpha L \max\limits_{\zeta\in[0,T]} |q(\zeta)|] > \tilde{A}_{\varphi_0}$.

(A3) $\lim_{H(w,\{0\})\to+\infty} \frac{H(\lambda(w),\{0\})}{H(w,\{0\})} < \frac{\Gamma(\alpha+1)}{\max_{\zeta\in[0,T]} |q(\zeta)| \cdot T^\alpha}$.

By Lemma 4.1, if (A1) holds and $E_\alpha[T^\alpha L \max_{\zeta\in[0,T]} |q(\zeta)|] \leq \tilde{A}_{\varphi_0}$, the forward solution of problem (1.1), if exists, satisfies $len(\varphi(\zeta)) \leq A$ on $[0,T]$.

Suppose that (A1) and (A2) hold, then if the forward solution of problem (1.1) exists on $[0,T]$, may have one or more impulses. The pulse conditions are defined as follows

$$\varphi(\zeta_k^+) = I(\varphi(\zeta_k)), k = 1, 2, \cdots, n, \tag{4.3}$$

where $I : K_C \to K_C$ is a continuous mapping which satisfies $len(I(u)) < A$ for every $u \in K_C, \zeta_0 = 0, \zeta_k = P^{-1}_{\zeta_{k-1}}(\tilde{M}_{I(\varphi(\zeta_{k-1}))})$ for $1 \leq k \leq n$ and $\zeta_{n+1} = T$. We present the following lemma concerning the value of n.

Lemma 4.2. *If (A1) and (A2) are satisfied, $len(I(w)) = len(\varphi_0)$ is true for every $w \in K_C$. Then, $n = \left[\frac{E_\alpha[T^\alpha L \max_{\zeta\in[0,T]} |q(\zeta)|]}{\tilde{A}_{\varphi_0}}\right]$ and $\zeta_k = P^{-1}_{\zeta_{k-1}}(\tilde{A}_{\varphi_0})$ for $1 \leq k \leq n$.*

Proof. From $len(I(w)) = len(\varphi_0)$, we have that $len(I(\varphi(\zeta_{k-1}))) = len(\varphi_0)$ for every $1 \leq k \leq n$. Hence,

$$\tilde{A}_{I(\varphi(\zeta_{k-1}))} = \frac{A}{len(I(\varphi(\zeta_{k-1})))} = \frac{A}{len(\varphi_0)} = \tilde{A}_{\varphi_0},\ 1 \leq k \leq n,$$

which provides that $E_\alpha[\zeta_k^\alpha L \max_{\zeta\in[\zeta_{k-1},\zeta_k]} |q(\zeta)|] = \tilde{A}_{\varphi_0}$ for $k = 1, 2, \cdots, n$ and $E_\alpha[\zeta_n^\alpha L \max_{\zeta\in[\zeta_n,T]} |q(\zeta)|] \leq \tilde{A}_{\varphi_0}$. Therefore, we derive that $\zeta_k = P^{-1}_{\zeta_{k-1}}(\tilde{A}_{\varphi_0})$ for $k = 1, 2, \cdots, n$, where $n = \left[\frac{E_\alpha[T^\alpha L \max_{\zeta\in[0,T]} |q(\zeta)|]}{\tilde{A}_{\varphi_0}}\right]$. □

According to the assumptions of lemma 4.2, n is a constant for the same M. Thus, regarding the existence for a forward solution of problem (1.1), the following theorem holds.

Theorem 4.1. *Assuming that (A1) and (A3) are both true, there exists at least one forward solution to problem (1.1) on the interval [0,T], for which $len(\varphi(\zeta)) \leq A$.*

Proof. Suppose that $E_\alpha[T^\alpha L \max_{\zeta\in[0,T]} |q(\zeta)|] \leq \tilde{A}_{\varphi_0}$ holds, Theorem 3.1 guarantees that problem (3.1) has at least one forward solution on $[0,T]$ with $n = 0$, therefore problem (1.1) has a forward solution.

If $E_\alpha[T^\alpha L \max_{\zeta\in[0,T]} |q(\zeta)|] \leq \tilde{A}_{\varphi_0}$ does not hold, implying that $E_\alpha[T^\alpha L \max_{\zeta\in[0,T]} |q(\zeta)|] > \tilde{A}_{\varphi_0}$. Let $I : K_C \to K_C$ be a continuous mapping. It satisfies $I(w) \subseteq w$ and $len(I(w)) = len(\varphi_0)$ for every $w \in K_C$. Consider problem (1.1)-(4.3). Lemma 4.24.2 guarantees that n_φ is fixed. Then, based on Theorem 3.1, there is at least one forward solution to problem (1.1)-(4.3). Finally, by Lemma 4.1 and the definition of ζ_k, the forward solution satisfies $len(\varphi(\zeta)) \leq A$ on the interval $[0,T]$. □

Example 4.1. We consider the following IVFDEs

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\frac{1}{2}}_{0^+}\varphi(\zeta) = \sin\zeta \cdot \varphi(\zeta),\ \zeta \in [0, \frac{5}{2}], \\ \varphi(0) = [1, 4]. \end{cases} \tag{4.4}$$

Let $q(\zeta) = \sin\zeta,\ \lambda(\varphi) = \varphi$. Since $len(\lambda(\varphi)) = len(\varphi)$, (A1) holds. Additionally, we find that $\max_{\zeta\in[0,\frac{5}{2}]} |\sin\zeta| \cdot \frac{(\frac{5}{2})^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \approx 1.7841$ and $\frac{H(\lambda(w),\{0\})}{H(w,\{0\})} = 1$ for $w \neq 0$ hence (A3) also is satisfied. According

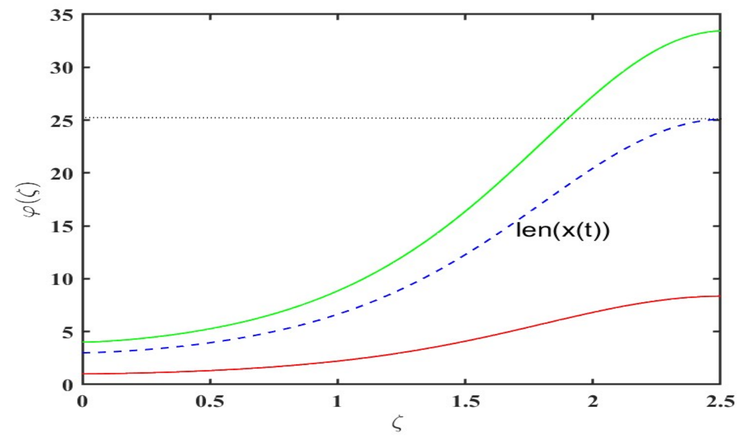

Figure 3. The forward solution to problem (4.4) with $A > 25$ (the red curve represents $\underline{\varphi}(\zeta)$, the green curve represents $\overline{\varphi}(\zeta)$, and the blue dashed line represents $len(\varphi(\zeta))$).

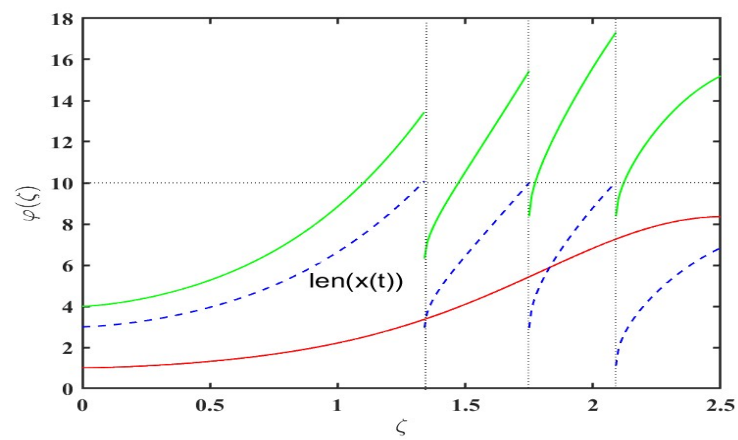

Figure 4. The forward solution to problem (4.4) with $A = 10$ (the red curve represents $\underline{\varphi}(\zeta)$, the green curve represents $\overline{\varphi}(\zeta)$, and the blue dashed line represents $len(\varphi(\zeta))$).

to Theorem 4.1, there exists at least one forward solution to problem (4.4) on $[0, \frac{5}{2}]$ that satisfies $len(\varphi(\zeta)) \leq A$.

If $A > 25$, then, according to the proof of Theorem 4.1, there is at least one forward solution to problem (4.4) that satisfies the condition $len(\varphi(\zeta)) \leq A$ on $[0, \frac{5}{2}]$ with no impulses. Refer to Figure 3.

Let $A = 10$, $\tilde{A}_{\varphi_0} = \frac{A}{len(\varphi_0)} = \frac{10}{3} \approx 3.33$, hence (A2) holds and $n_\varphi = 3$. By the proof of Theorem 4.1, there is at least one forward solution to problem (4.4) that satisfies the condition $len(\varphi(\zeta)) \leq A$ on the interval $[0, 2.5]$ and involves three impulses, $t_1 \approx 1.34, t_2 \approx 1.75$ and $t_3 \approx 2.09$. Refer to Figure 4, where $I(\varphi) = [\underline{\varphi}, \underline{\varphi} + len(\varphi_0)]$.

As shown in Figure 3, the X-axis matches Figure 1 time $\zeta \in [0, 2.5]$ with the length constraint threshold $A = 25$. For $A > 25$, $\text{len}(\varphi(\zeta)) < 25$ throughout $\zeta \in [0, 2.5]$ without impulses. The interval length increases gently, reflecting fractional-order systems'memory accumulation effect, with solutions evolving based on historical information unlike integer-order systems'abrupt changes. Solution continuity and constraint satisfaction verify Theorem 4.1 sufficiently large thresholds enable impulsive IVFDEs to have constraint-satisfying solutions without impulses.

Figure 4 adopts the same axis definition as Figure 3, where the X-axis represents dimensionless time, with impulse trigger times at $t_1 \approx 1.34$, $t_2 \approx 1.75$, and $t_3 \approx 2.09$. When $A = 10$, the interval length approaches the threshold during evolution. immediate impulses triggered when the length is on the verge of exceeding the limit quickly restore it to the constraint range. The trigger times depend on the stopping time at which the interval length would exceed the limit, and this flexible strategy can be adjusted according to the actual evolution of the solution. Throughout $\zeta \in [0, 2.5]$, the solution exists and satisfies $\text{len}(\varphi(\zeta)) \leq 10$, further verifying the core conclusion of Theorem 4.1 that impulsive IVFDEs under length constraints have solutions that meet all prescribed conditions.

Comparisons across different thresholds validate the rationality and universality of Theorem 4.1. These results confirm the existence of solutions for impulsive IVFDEs under varying length constraints, as well as the effectiveness of the stopping-time-based impulse control strategy, thereby providing a solid theoretical foundation for practical applications.

5. Conclusion

In this paper, we investigate interval-valued fractional-order dynamical systems with finite uncertainty (length constraints). In accordance with the concepts of forward solution and stopping time presented in [41], we examine the possibility of placing impulses at or before the stopping time for $\{\varphi \in K_C | len(\varphi) > A\}$. Thus, the problem can be transformed into a problem of the existence of solutions to IVFDEs with impulses, where the impulses are related to the length constraint $len(\varphi(\zeta)) \leq A$. By employing fixed-point theory, we prove some existence results. Importantly, under the gH-differentiability condition, we use impulses to control the length of the solution for the IVFDE. This enables more precise prediction of the entire process based on the initial conditions.

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Competing interests. The authors declare that they have no competing interest.

Authors' contributions. All authors read and approved the final manuscript.

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