Journal of Applied Analysis and Computation
Volume 16, Number 4, August 2026, 2097–2125
Website:http://www.jaac-online.com
DOI:10.11948/20250271

TURING INSTABILITY OF PERIODIC SOLUTIONS IN THE FITZHUGH–NAGUMO MODEL WITH CROSS-DIFFUSION*

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Abstract We investigate spatiotemporal pattern formation in a FitzHugh–Nagumo reaction-diffusion system with density-dependent cross-diffusion. In particular, we analyze how diffusion perturbs Hopf-bifurcating periodic orbits and derive conditions under which these oscillations lose stability via a Turing mechanism. Moreover, we obtain an explicit criterion, in terms of the diffusion coefficients, that predicts the onset of Turing instability for Hopf-bifurcating periodic solutions in the FitzHugh–Nagumo reaction-diffusion system with cross-diffusion. Numerical simulations validate the theoretical analysis and demonstrate the existence of spatiotemporal pattern formation.

Keywords FitzHugh–Nagumo, cross-diffusion, Turing instability, Hopf bifurcation, spatially homogeneous periodic solutions.

MSC(2010) 35K57, 35B36, 34C23, 92C20.

1. Introduction

Based on their seminal electrophysiological studies on squid giant axons, Hodgkin and Huxley [17] proposed, in the early 1950s, a four-variable system of nonlinear partial differential equations describing the conduction of nerve impulses along axons. Subsequently, FitzHugh [9] and Nagumo et al. [31] derived a simpler two-variable model that retains the essential features of the original Hodgkin–Huxley equations, given by:

$$\begin{aligned} \frac{du}{dt} &= u - \frac{u^3}{3} + w + I, \\ \frac{dw}{dt} &= \rho\big(a - u - bw\big), \end{aligned} \tag{1.1}$$

where u is the membrane potential, and w is a recovery variable representing processes such as potassium channel activation levels. The parameters I, a, b, ρ are all real numbers. Here, a and b relate to the number of channels in the cell membrane that are open to Na^+ and K^+ ions, ρ characterizes the time-scale separation between the fast membrane potential dynamics and slower recovery dynamics, and I represents the applied (injected) current.

System (1.1) is now widely known as the FitzHugh–Nagumo (FHN) model, although it was originally introduced under the name Bonhoeffer–van der Pol (BVP) model by FitzHugh. When $I = 0$, the steady state of system (1.1) is linearly stable but excitable. That is, an adequate

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[*]Y. Zhang is supported by the National Natural Science Foundation of China (NSFC) (Grant Nos. 12201151 and 12471181).
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initial displacement from the steady state drives the system on a large excursion through phase space before it eventually returns to the steady state [20]. For $I \neq 0$, there exist parameter ranges exhibiting regular repetitive firing, corresponding to limit cycle dynamics. For instance, Hsü et al. [19], Troy [37] and Hadeler et al. [13] carried out comprehensive bifurcation analyses, demonstrating the emergence of small-amplitude, orbitally stable limit cycles within certain intervals of the injected current I.

In physiological settings, the membrane potential propagates along the axon. Accordingly, the FitzHugh–Nagumo model is commonly extended with spatial diffusion terms, yielding the following reaction-diffusion system:

$$\begin{aligned} \frac{du}{dt} &= d_{11}\Delta u + u(\tilde{a} - u)(u - 1) - w + \tilde{I}, \\ \frac{dw}{dt} &= d_{22}\Delta w + \tilde{b}u - \tilde{\gamma}w, \end{aligned} \tag{1.2}$$

where x denotes the spatial coordinate vector, $u = u(x,t)$ denotes the membrane potential, $w = w(x,t)$ is the recovery variable, Δ denotes the Laplacian operator, and d_{11}, $d_{22} \geq 0$ are diffusion coefficients. As shown in [13,21], system (1.2) could be transformed from system (1.1) by a linear shift in u and w when $d_{11} = d_{22} = 0$.

The reaction-diffusion FHN model (1.2) has been widely used to study wave propagation and pattern formation. As early as 1952, Turing introduced the concept of diffusion-driven instability, showing that diffusion could destabilize a spatially homogeneous equilibrium and generate complex biological patterns [38]. Although the idea of Turing was later placed on a firm theoretical footing by Gierer and Meinhardt [12] by the model of Hydra regeneration, the first experimental demonstration of a Turing pattern was observed in 1990 by De Kepper et al. [5] in the chlorite-iodide-malonic acid-starch (CIMA) reaction in an open, non-stirred gel reactor. To date, numerous studies have shown that system (1.2) exhibits a rich variety of spatiotemporal patterns, including stationary Turing structures, traveling pulses/fronts, spiral waves and target patterns; see, for example, [6,8,14,15,22,25,29,30] and the references therein.

Cross-diffusion refers to the coupling in which a concentration gradient in one species drives the flux of another. Extensions of the FitzHugh–Nagumo framework that incorporate such effects have attracted growing attention in the pattern-formation literature. For instance, Berezovskaya et al. [1] showed that cross-diffusion admits a slow traveling wave in addition to the classical fast wave of the self-diffusive FHN equations; Biktashev and Tsyganov [2] demonstrated that replacing self-diffusion by linear cross-diffusion produces solitary waves with nonstandard profiles that undergo quasi-soliton interactions (penetration and reflection); Zemskov and coauthors analyzed traveling waves in cross-diffusion FHN systems, identifying front-type waves in the bistable regime [45] and oscillatory pulse solutions [44], and later provided an analytical description of oscillatory reaction-diffusion fronts in a piecewise-linear setting [46]; and Gambino et al. [10] showed that linear cross-diffusion can trigger long-range activation instabilities even outside the classical Turing parameter regime.

Most existing studies on cross-diffusion FitzHugh–Nagumo systems consider linear cross-diffusion with constant coefficients. In many settings, however, density-dependent cross-diffusion (i.e., Keller–Segel-type chemotaxis [18,23,24]), with cross fluxes of the form $\nabla(u\nabla w)$ or $\nabla(w\nabla u)$, is more realistic. Motivated by this gap, we investigate the emergence of Turing patterns in a FitzHugh–Nagumo-type model with density-dependent cross-diffusion. Specifically, we consider the system:

$$
\begin{cases}
u_t = d_{11}\Delta u + d_{12}\nabla\big(u\nabla w\big) + u - \dfrac{u^3}{3} + w + I, & x \in \Omega,\ t > 0,\\
w_t = d_{21}\nabla\big(w\nabla u\big) + d_{22}\Delta w + \rho\big(a - u - bw\big), & x \in \Omega,\ t > 0,\\
u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x \in \Omega,\\
\partial_\nu u = \partial_\nu w = 0, & x \in \partial\Omega,\ t > 0,
\end{cases}
\tag{1.3}
$$

where $\Omega \subset \mathbb{R}^n$ $(1 \le n \le 3)$ is an open bounded domain with smooth boundary $\partial\Omega$, and ν denotes the unit outward normal vector on $\partial\Omega$. In addition to the standard self-diffusion terms $d_{11}\Delta u$ and $d_{22}\Delta w$, the model incorporates cross-diffusion terms $d_{12}\nabla(u\nabla w)$ and $d_{21}\nabla(w\nabla u)$, thereby capturing not only spatial propagation of nerve signals but also their mutual interactions and richer spatiotemporal dynamics. We assume d_{11}, $d_{22} > 0$ and $d_{11}d_{22} - d_{12}d_{21} > 0$, so that the diffusion operator is (normally) elliptic. The parameters I, a, b, ρ are as in (1.1). The initial data u_0, w_0 specify the state at $t = 0$, and the homogeneous Neumann boundary conditions enforce no-flux across $\partial\Omega$.

Classical analyses of Turing instability have mainly concentrated on spatially homogeneous steady states. In contrast, the diffusion-driven destabilisation of periodic solutions has attracted comparatively little attention. Both Maginu [27] and Ruan [34] employed a regular perturbation approach to study how the minimal period of a spatially homogeneous limit cycle changes when diffusion is introduced, and they derived criteria for the diffusion-induced destabilisation of periodic solutions in reaction-diffusion systems posed on $\mathbb{R}^n$ for arbitrary spatial dimension. Extending Maginu's setting, Yi [42] investigated a general class of reaction-diffusion equations with linear cross-diffusion on a bounded spatial domain and proved that diffusion can likewise destabilise spatially homogeneous time-periodic solutions, thereby generating rich spatiotemporal patterns. Further results on the Turing instability of periodic solutions can be found in, for example, [26, 33, 35, 36, 41].

In addition to the continuum reaction-diffusion setting, heterogeneous patterning has also been explored in neuronal networks and in models with higher-order interactions. For example, Parastesh et al. [32] reviewed chimera states as a canonical example of partial synchronization. Majhi et al. [28] reviewed neuronal synchronization shaped by higher-order interactions and summarized master stability function based stability tools. Gao et al. [11] showed that higher-order connections on simplicial complexes can modify Turing instability conditions, with patterns in some regimes arising only when higher-order interactions are present.

In this paper, we study the reaction-diffusion FitzHugh–Nagumo system with cross-diffusion (1.3). Our aim is to quantify how diffusion, especially cross-diffusion, destabilizes Hopf-bifurcating periodic orbits, and to derive an explicit criterion predicting the onset of Turing patterns.

The remainder of the paper is organized as follows. Section 2 develops the stability and bifurcation analysis of the spatially uniform ODE system (1.1). Section 3 introduces two auxiliary perturbed ODEs, proves the existence of minimal-period periodic solutions, determines their minimal periods, and computes the first derivative of the period with respect to the perturbation parameter at zero. In Section 4, we derive general conditions on the diffusion coefficients and model parameters that trigger Turing instability of the Hopf-bifurcating periodic states of (1.3). Section 5 presents numerical simulations that corroborate the theoretical analysis.

2. Hopf bifurcation for the FitzHugh–Nagumo ODE system

In this section we establish the existence of spatially homogeneous time-periodic solutions of (1.3). To that end, we consider the FitzHugh–Nagumo ODE reduction of (1.3):

$$\begin{cases} \dfrac{du}{dt} = u - \dfrac{1}{3}u^3 + w + I, \\ \dfrac{dw}{dt} = \rho\big(a - u - bw\big). \end{cases} \tag{2.1}$$

This coincides with system (1.1). Throughout, we assume the parameters satisfy

$$0 < b < 1, \qquad 0 < \rho < 1, \qquad 1 - \frac{2b}{3} < a < 2 + \frac{2b}{3}, \qquad I \in \mathbb{R}.$$

We employ the classical Hopf bifurcation theorem (e.g., [16, 39]) to establish the existence of time-periodic solutions of (2.1). We then analyze the stability of the bifurcating orbits by deriving an explicit expression for the first Lyapunov coefficient following [16, 43]. While related analyses appear in [13,19,37], here we state the precise nondegeneracy and transversality conditions and identify the parameter intervals on which the bifurcating periodic orbits are asymptotically orbitally stable (or unstable). These results will be used subsequently to study the Turing instability of the spatially homogeneous time-periodic solutions in system (1.3).

The stationary state (u, w) of (2.1) satisfies

$$u - \frac{1}{3}u^3 + w + I = 0,$$
$$\rho\,(a - u - bw) = 0.$$

From the second equation, we have $w = (a - u)/b$. Substituting into the first yields

$$I = k(u), \qquad k(u) := \frac{u^3}{3} + \Big(\frac{1}{b} - 1\Big)u - \frac{a}{b}.$$

We note that $k(u) \to +\infty$ as $u \to +\infty$, $k(u) \to -\infty$ as $u \to -\infty$. Further,

$$k'(u) = u^2 + \left(\frac{1}{b} - 1\right) > 0,$$

where $0 < b < 1$ for all $u \in \mathbb{R}$, so $k : \mathbb{R} \to \mathbb{R}$ is strictly increasing and hence bijective. Therefore, for any $I \in \mathbb{R}$ there exists a unique $u_I \in \mathbb{R}$ with $k(u_I) = I$, and the corresponding equilibrium $(u_I,\, w_I)$ satisfies

$$\frac{u_I^3}{3} + \Big(\frac{1}{b} - 1\Big)u_I - \frac{a}{b} - I = 0, \qquad w_I = \frac{a - u_I}{b}. \tag{2.2}$$

In particular, $u_I = k^{-1}(I)$ depends monotonically on I, with

$$\frac{du_I}{dI} = \frac{1}{k'(u_I)} = \frac{1}{u_I^2 + \frac{1}{b} - 1} > 0.$$

Choosing I as the bifurcation parameter, we now perform a Hopf bifurcation analysis of (2.1). Translating the equilibrium of (2.1) to the origin via

$$x := u - u_I, \quad y := w - w_I,$$

yields

$$\begin{cases} \dfrac{dx}{dt} = (1-u_I^2)x + y - u_I x^2 - \dfrac{1}{3}x^3, \\ \dfrac{dy}{dt} = -\rho x - \rho b y, \end{cases} \tag{2.3}$$

for which (0, 0) is the unique equilibrium.

The Jacobian matrix of (2.3) at (0, 0) is

$$A(I) = \begin{pmatrix} 1-u_I^2 & 1 \\ -\rho & -\rho b \end{pmatrix}.$$

Let

$$T(I) = \operatorname{tr} A = 1 - \rho b - u_I^2, \qquad D(I) = \det A = \rho(b u_I^2 - b + 1). \tag{2.4}$$

Then, the eigenvalues of $A(I)$ satisfy

$$\lambda^2 - T(I)\lambda + D(I) = 0, \qquad \lambda_{1,2}(I) = \frac{T(I) \pm \sqrt{T^2(I) - 4D(I)}}{2}. \tag{2.5}$$

A pair of purely imaginary eigenvalues occurs iff

$$T(I) = 1 - \rho b - u_I^2 = 0 \quad \text{and} \quad D(I) = \rho(b u_I^2 - b + 1) > 0. \tag{2.6}$$

Since we recall that $0 < b < 1$ and $0 < \rho < 1$, the map $I \mapsto u_I$ is strictly increasing and u_I ranges over $\mathbb{R}$. Thus, there are exactly two values $I_1 < I_2$ for which $T(I) = 0$, corresponding to

$$u_{I_1} = -\sqrt{1-\rho b}, \qquad u_{I_2} = \sqrt{1-\rho b},$$

and

$$I_1 = k\left(-\sqrt{1-\rho b}\right), \qquad I_2 = k\left(\sqrt{1-\rho b}\right).$$

At $I = I_{1,2}$, we have

$$D(I_{1,2}) = \rho - \rho^2 b^2 > 0.$$

Consequently, $I = I_{1,2}$ are Hopf bifurcation values. The eigenvalues of $A(I)$ at $I = I_{1,2}$ are

$$\lambda_1 = -i\sqrt{\rho - \rho^2 b^2}, \quad \lambda_2 = i\sqrt{\rho - \rho^2 b^2}.$$

An eigenvector associated with λ_1 can be chosen as

$$\mu = \begin{pmatrix} 1 \\ -\rho b \end{pmatrix} + i \begin{pmatrix} 0 \\ -\sqrt{\rho - \rho^2 b^2} \end{pmatrix},$$

while the eigenvector for λ_2 is the complex conjugate of μ.

At the Hopf points $I = \bar{I} \in \{I_1, I_2\}$, system (2.3) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \rho b & 1 \\ -\rho & -\rho b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -u_{\bar{I}} x^2 - \dfrac{1}{3}x^3 \\ 0 \end{pmatrix}. \tag{2.7}$$

Setting $\omega := \sqrt{\rho - \rho^2 b^2}$, we introduce

$$A := \begin{pmatrix} \rho b & 1 \\ -\rho & -\rho b \end{pmatrix}, \qquad M := \begin{pmatrix} 1 & 0 \\ -\rho b & -\omega \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

A direct calculation shows that $AM = MB$, and

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ -\dfrac{\rho b}{\omega} & -\dfrac{1}{\omega} \end{pmatrix}.$$

Let $z = (z_1, z_2)^T = M^{-1}(x, y)^T$, then

$$(x, y)^T = M(z_1, z_2)^T = (z_1, -\rho b z_1 - \omega z_2)^T.$$

In these coordinates, (2.7) becomes

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = M^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} -u_{\bar{I}} z_1^2 - \dfrac{1}{3} z_1^3 \\ \dfrac{\rho b}{\omega} u_{\bar{I}} z_1^2 + \dfrac{\rho b}{3\omega} z_1^3 \end{pmatrix}.$$

Define

$$f^1(z_1, z_2, \bar{I}) := -u_{\bar{I}} z_1^2 - \frac{1}{3} z_1^3, \qquad f^2(z_1, z_2, \bar{I}) := \frac{\rho b}{\omega} u_{\bar{I}} z_1^2 + \frac{\rho b}{3\omega} z_1^3.$$

Following [16, 39], we now calculate a^* by

$$\begin{aligned} a^* =& \frac{1}{16} \left(f^1_{111} + f^1_{122} + f^2_{112} + f^2_{222} \right) \\ &+ \frac{1}{16\omega} \left(f^1_{12}(f^1_{11} + f^1_{22}) - f^2_{12}(f^2_{11} + f^2_{22}) - f^1_{11} f^2_{11} + f^1_{22} f^2_{22} \right), \end{aligned} \tag{2.8}$$

where subscripts denote partial derivatives with respect to z_1 and z_2, respectively.

At $(z_1, z_2, \bar{I}) = (0, 0, \bar{I})$, one has

$$f^1_{11} = -2u_{\bar{I}}, \qquad f^2_{11} = \frac{2\rho b u_{\bar{I}}}{\omega}, \qquad f^1_{111} = -2,$$

and

$$f^1_{12} = f^1_{22} = f^2_{12} = f^2_{22} = f^1_{122} = f^2_{112} = f^2_{222} = 0.$$

Recalling that $\omega = \sqrt{\rho - \rho^2 b^2}$ and $u_{\bar{I}}^2 = 1 - \rho b$, substitution into (2.8) yields that

$$a^* = -\frac{1 + \rho b^2 - 2b}{8(1 - \rho b^2)}.$$

Noting that u_I depends smoothly on I (by the implicit function theorem applied to $k(u_I) = I$), and that $T(\bar{I}) = 0$ and $D(\bar{I}) > 0$, it follows from (2.5) and (2.6) that, for I sufficiently close to $\bar{I}$, the eigenvalues $\lambda_{1,2}(I)$ form a complex conjugate pair and

$$\operatorname{Re} \lambda_{1,2}(I) \;=\; -\frac{1}{2} \left(u_I^2 - 1 + \rho b \right).$$

Hence

$$d := \frac{d}{dI}\operatorname{Re}\lambda_{1,2}(I)\Big|_{I=\bar{I}} = -u_{\bar{I}}\frac{du_I}{dI}\Big|_{I=\bar{I}}.$$

Recall that $u_{I_1} < 0$, $u_{I_2} > 0$, and $du_I/dI > 0$ for any $I \in \mathbb{R}$. Therefore d has the opposite sign of $u_{\bar{I}}$. In particular,

$$\bar{I} = I_1 \Rightarrow u_{I_1} < 0 \Rightarrow d > 0, \qquad \bar{I} = I_2 \Rightarrow u_{I_2} > 0 \Rightarrow d < 0.$$

Thus, the transversality condition holds at both Hopf points (cf. [4]).

According to [39, Theorem 20.2.3], we have the following results.

Theorem 2.1. *Suppose that $1 + \rho b^2 - 2b \neq 0$. Then, there exist positive constants τ_1 and τ_2, such that:*

(i) If $1 + \rho b^2 - 2b > 0$, then the FitzHugh–Nagumo system (2.1) *admits periodic solutions for every $I \in (I_1, I_1 + \tau_1) \cup (I_2 - \tau_2, I_2)$. These bifurcating periodic orbits are asymptotically orbitally stable, and the equilibrium (u_I, w_I) is unstable on those intervals.*

(ii) If $1 + \rho b^2 - 2b < 0$, then the FitzHugh–Nagumo system (2.1) *admits periodic solutions for every $I \in (I_1 - \tau_1, I_1) \cup (I_2, I_2 + \tau_2)$. These bifurcating periodic orbits are unstable, and the equilibrium (u_I, w_I) is locally asymptotically stable on those intervals.*

Remark 2.1. When $1 + \rho b^2 - 2b > 0$, we have $a^* < 0$, indicating that the bifurcation is supercritical. Conversely, when $1 + \rho b^2 - 2b < 0$, we have $a^* > 0$, corresponding to a subcritical bifurcation.

Remark 2.2. Near $I = \bar{I} \in \{I_1, I_2\}$, system (2.1) admits a family of periodic solutions bifurcating from $(u_{\bar{I}}, w_{\bar{I}})$. More precisely, there exist $r_p > 0$ and a smooth function $I_p(r)$ such that, for each $r \in (0, r_p)$, system (2.1) possesses a $P(r)$-periodic solution $(u_p(t,r),\, w_p(t,r))$ at $I = I_p(r)$, satisfying

$$\lim_{r\to 0} P(r) = \frac{2\pi}{\sqrt{\rho - \rho^2 b^2}}, \qquad \lim_{r\to 0}\big(I_p(r), u_p(t,r), w_p(t,r)\big) = \big(\bar{I}, u_{\bar{I}}, w_{\bar{I}}\big),$$

where $P(r)$ is the minimal period of the bifurcating periodic solution. For brevity, we write $(u_p(t), w_p(t))$ in place of $(u_p(t,r), w_p(t,r))$; these coincide with the spatially homogeneous time-periodic solutions of system (1.3).

3. Two auxiliary perturbed ODE systems

In this section, we consider the following auxiliary perturbed ordinary differential equation:

$$\left(E + \varepsilon\begin{pmatrix} d_{11} & d_{12}u_p(t) \\ d_{21}w_p(t) & d_{22}\end{pmatrix}\right)\begin{pmatrix} u' \\ w'\end{pmatrix} = \begin{pmatrix} u - \dfrac{u^3}{3} + w + I \\ \rho(a - u - bw)\end{pmatrix}, \tag{3.1}$$

where $(u_p(t), w_p(t)) = (u_p(t,r), w_p(t,r))$ denotes the bifurcating periodic solution of system (2.1) with the minimal period $P = P(r)$ as stated in Remark 2.2, the parameters are as in (2.1), ε is a small perturbation parameter, and $E \in \mathbb{R}^{2\times 2}$ denotes the identity matrix.

Taking I as the bifurcation parameter, we show that for $|\varepsilon|$ sufficiently small system (3.1) undergoes a Hopf bifurcation. We also determine the resulting minimal period and compute its derivative with respect to ε. To that end, we introduce the simpler auxiliary perturbed system:

$$\left(E+\varepsilon\begin{pmatrix} d_{11} & d_{12}u_I \\ d_{21}w_I & d_{22}\end{pmatrix}\right)\begin{pmatrix} u' \\ w'\end{pmatrix} = \begin{pmatrix} I+w+u-\frac{1}{3}u^3 \\ \rho(a-u-bw)\end{pmatrix}, \tag{3.2}$$

where (u_I, w_I) is the equilibrium of (2.1).

Remark 2.2 states that when I is close to $\bar{I}$, the bifurcating periodic solution $(u_p(t), w_p(t))$ is close to the equilibrium (u_I, w_I). Therefore, by [3, 7, 40], the dynamics of the perturbed system (3.1) are well approximated by those of (3.2). In particular, for $|\varepsilon|$ small, the periodic solution $(u_p^*(t,\varepsilon), w_p^*(t,\varepsilon))$ of (3.1) is accurately approximated by the periodic solution $(u_p(t,\varepsilon), w_p(t,\varepsilon))$ of (3.2). Hence, in what follows we analyze (3.2) and infer the corresponding conclusions for (3.1).

We rewrite the perturbed system (3.2) in the following form:

$$\begin{pmatrix} u' \\ w'\end{pmatrix} = \frac{1}{H(\varepsilon)}\begin{pmatrix} 1+d_{22}\varepsilon & -d_{12}u_I\varepsilon \\ -d_{21}w_I\varepsilon & 1+d_{11}\varepsilon\end{pmatrix}\begin{pmatrix} I+w+u-\frac{1}{3}u^3 \\ \rho(a-u-bw)\end{pmatrix}, \tag{3.3}$$

where

$$H(\varepsilon) := (d_{11}d_{22} - d_{12}d_{21}u_I w_I)\varepsilon^2 + (d_{11}+d_{22})\varepsilon + 1. \tag{3.4}$$

Since $H(\varepsilon)$ is continuous in ε and $H(0) = 1$, it follows that $H(\varepsilon) > 0$ for all $|\varepsilon|$ sufficiently small.

We now establish the existence of periodic solutions of the perturbed system (3.2).

Lemma 3.1. *Suppose that for I close to $\bar{I}$, system (2.1) admits a stable periodic solution $(u_p(t), w_p(t))$ bifurcating from $(u_{\bar{I}}, w_{\bar{I}})$, with minimal period P. Then there exists a constant $\varepsilon_1 > 0$ such that, for any $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, system (3.2) admits an ε-dependent periodic solution $(u_p(t,\varepsilon), w_p(t,\varepsilon))$ with minimal period $P(\varepsilon)$, satisfying*

$$\lim_{\varepsilon\to 0}(u_p(t,\varepsilon), w_p(t,\varepsilon)) = (u_p(t), w_p(t)) \qquad \text{and} \qquad \lim_{\varepsilon\to 0} P(\varepsilon) = P.$$

Proof. It is straightforward to verify that (u_I, w_I) is an equilibrium of system (3.2). The Jacobian matrix of system (3.2) at (u_I, w_I) is

$$\mathcal{A}(I,\varepsilon) = \begin{pmatrix} a_{11}(I,\varepsilon) & a_{12}(I,\varepsilon) \\ a_{21}(I,\varepsilon) & a_{22}(I,\varepsilon)\end{pmatrix}, \tag{3.5}$$

where

$$\begin{aligned} a_{11}(I,\varepsilon) &= \frac{1}{H(\varepsilon)}\left(1-u_I^2+\left((1-u_I^2)d_{22}+\rho d_{12}u_I\right)\varepsilon\right), \\ a_{12}(I,\varepsilon) &= \frac{1}{H(\varepsilon)}\left(1+\left(d_{22}+\rho b d_{12}u_I\right)\varepsilon\right), \\ a_{21}(I,\varepsilon) &= \frac{1}{H(\varepsilon)}\left(-\rho-\left((1-u_I^2)d_{21}w_I+\rho d_{11}\right)\varepsilon\right), \\ a_{22}(I,\varepsilon) &= \frac{1}{H(\varepsilon)}\left(-\rho b-\left(d_{21}w_I+\rho b d_{11}\right)\varepsilon\right) \end{aligned} \tag{3.6}$$

with $H(\varepsilon)$ as in (3.4).

The characteristic equation of $\mathcal{A}(I,\varepsilon)$ is given by

$$\lambda^2 - \mathcal{T}(I,\varepsilon)\lambda + \mathcal{D}(I,\varepsilon) = 0,$$

with

$$\begin{aligned}\mathcal{T}(I,\varepsilon) &= \frac{1}{H(\varepsilon)}\Big(T(I) + \Big((1-u_I^2)d_{22} + \rho d_{12}u_I - d_{21}w_I - \rho b d_{11}\Big)\varepsilon\Big),\\ \mathcal{D}(I,\varepsilon) &= \frac{D(I)}{H(\varepsilon)} = \frac{1}{H(\varepsilon)}\Big(\rho b u_I^2 - \rho b + \rho\Big),\end{aligned} \tag{3.7}$$

where $T(I) = 1 - u_I^2 - \rho b$ and $D(I) = \rho b u_I^2 - \rho b + \rho$ are as in (2.4).

It is clear that $\mathcal{T}(I,0) = T(I)$ and $\mathcal{D}(I,0) = D(I)$. In particular, we have

$$\mathcal{T}(\bar I,0) = T(\bar I) = 0, \quad \partial_I \mathcal{T}(I,0)|_{I=\bar I} = T'(\bar I) = -2\,u_{\bar I}\,\frac{du_I}{dI}\bigg|_{I=\bar I} \neq 0, \quad \mathcal{D}(\bar I,0) = D(\bar I) > 0.$$

According to the implicit function theorem, there exist a constant $h>0$ and a unique continuously differentiable function $\bar I_\varepsilon := I(\varepsilon)$ defined for $|\varepsilon| < h$, such that

$$\mathcal{T}(\bar I_\varepsilon, \varepsilon) = 0, \qquad I(0) = \bar I. \tag{3.8}$$

Since $\bar I_\varepsilon$ and $\mathcal{D}$ are continuous in ε and $\mathcal{D}(\bar I_0, 0) = D(\bar I) > 0$, we have

$$\mathcal{D}(\bar I_\varepsilon, \varepsilon) > 0$$

for all sufficiently small $|\varepsilon|$.

Thus, for any fixed ε with $|\varepsilon|$ sufficiently small, at $I = \bar I_\varepsilon$, the matrix $\mathcal{A}(I,\varepsilon)$ has a pair of purely imaginary eigenvalues

$$\lambda(\bar I_\varepsilon, \varepsilon) = \pm i\sqrt{\mathcal{D}(\bar I_\varepsilon,\varepsilon)}.$$

By continuity, there exists a neighborhood of $\bar I_\varepsilon$ in which $\mathcal{A}(I,\varepsilon)$ has a pair of complex conjugate eigenvalues denoted by

$$\hat\lambda_{1,2}(I,\varepsilon) = \hat l(I,\varepsilon) \pm i\,\hat k(I,\varepsilon),$$

where

$$\hat l(I,\varepsilon) = \frac{1}{2}\mathcal{T}(I,\varepsilon), \quad \hat k(I,\varepsilon) = \frac{1}{2}\sqrt{4\mathcal{D}(I,\varepsilon) - \mathcal{T}^2(I,\varepsilon)}. \tag{3.9}$$

By continuity and the sign-preserving property of $\partial_I \mathcal{T}(I,\varepsilon)$, for $|\varepsilon|$ sufficiently small, we have

$$\partial_I \hat l(\bar I_\varepsilon, \varepsilon) = \frac{1}{2}\partial_I \mathcal{T}(\bar I_\varepsilon,\varepsilon) \neq 0.$$

Thus, according to [39, Theorem 20.2.3], for all sufficiently small $|\varepsilon|$, system (3.2) undergoes a Hopf bifurcation at the equilibrium (u_I, w_I) when $I = \bar I_\varepsilon$. More precisely, for each such ε, there exists a family of periodic solutions $(u_p(t,\varepsilon), w_p(t,\varepsilon))$ bifurcating from the equilibrium (u_I, w_I) at $I = \bar I_\varepsilon$, with minimum period $P(\varepsilon)$. Furthermore, by Chapter 14 of [7], we have

$$\lim_{\varepsilon\to 0}(u_p(t,\varepsilon), w_p(t,\varepsilon)) = (u_p(t), w_p(t)) \quad \text{and} \quad \lim_{\varepsilon\to 0} P(\varepsilon) = P.$$

□

For the sake of simplicity, we explicitly define

$$\hat l'(\bar I_\varepsilon) = \frac{\partial \hat l(I,\varepsilon)}{\partial I}\bigg|_{I=\bar I_\varepsilon}, \qquad \hat k'(\bar I_\varepsilon) = \frac{\partial \hat k(I,\varepsilon)}{\partial I}\bigg|_{I=\bar I_\varepsilon}. \tag{3.10}$$

Lemma 3.2. *Suppose that for I close to $\bar{I}$, system* (2.1) *has a stable periodic solution* $(u_p(t), w_p(t))$ *bifurcating from* $(u_{\bar{I}}, w_{\bar{I}})$, *and that* $(u_p(t,\varepsilon), w_p(t,\varepsilon))$ *is the periodic solution of the perturbed system* (3.2) *bifurcating from* $(u_{\bar{I}_\varepsilon}, w_{\bar{I}_\varepsilon})$ *with minimal period* $P(\varepsilon)$, *where* $\bar{I}_\varepsilon$ *is given by* (3.8). *Then for I sufficiently close to* $\bar{I}_\varepsilon$,

$$P(\varepsilon) = \frac{2\pi}{\hat{k}(\bar{I}_\varepsilon,\varepsilon)}\left(1 + \left(\frac{\hat{l}'(\bar{I}_\varepsilon)\,\mathrm{Im}(c_1(\bar{I}_\varepsilon))}{\hat{k}(\bar{I}_\varepsilon,\varepsilon)\,\mathrm{Re}(c_1(\bar{I}_\varepsilon))} - \frac{\hat{k}'(\bar{I}_\varepsilon)}{\hat{k}(\bar{I}_\varepsilon,\varepsilon)}\right)(I - \bar{I}_\varepsilon)\right) + O\left((I-\bar{I}_\varepsilon)^2\right),$$

where $c_1(\bar{I}_\varepsilon)$ *is given by* (3.15), *and* $\hat{l}(I,\varepsilon)$, $\hat{k}(I,\varepsilon)$ *by* (3.9). *In particular, for I sufficiently close to* $\bar{I}$,

$$P'(0) = \frac{\pi}{\sqrt{D(\bar{I})D(\bar{I})}}(\delta_1 d_{11} + \delta_2 d_{22} + \delta_3 d_{12} u_{\bar{I}} + \delta_4 d_{21} w_{\bar{I}}) + O(I - \bar{I}),$$

where δ_1, δ_2, δ_3, δ_4 *are defined in* (3.28).

Proof. The proof proceeds in two steps.

Step 1. Evaluate the complex coefficient $c_1(\bar{I}_\varepsilon)$ at $\varepsilon = 0$ and determine its real and imaginary parts.

To simplify the analysis, we introduce the translation $\hat{u} := u - u_I$ and $\hat{w} := w - w_I$, which shifts the equilibrium (u_I, w_I) of (3.2) to the origin. For convenience, we henceforth write u and w in place of $\hat{u}$ and $\hat{w}$. In these coordinates, system (3.3) becomes

$$\begin{pmatrix} u' \\ w' \end{pmatrix} = \frac{1}{H(\varepsilon)}\begin{pmatrix} 1 + d_{22}\varepsilon & -d_{12}u_I\varepsilon \\ -d_{21}w_I\varepsilon & 1 + d_{11}\varepsilon \end{pmatrix}\begin{pmatrix} I + (w + w_I) + (u + u_I) - \dfrac{1}{3}(u + u_I)^3 \\ \rho\,(a - (u + u_I) - b(w + w_I)) \end{pmatrix}, \tag{3.11}$$

where $H(\varepsilon)$ is defined in (3.4).

Using the fact that (u_I, w_I) satisfies (2.2), system (3.11) can be rewritten as

$$\begin{aligned} \begin{pmatrix} u' \\ w' \end{pmatrix} &= \frac{1}{H(\varepsilon)}\begin{pmatrix} 1 + d_{22}\varepsilon & -d_{12}u_I\varepsilon \\ -d_{21}w_I\varepsilon & 1 + d_{11}\varepsilon \end{pmatrix}\begin{pmatrix} w + (1 - u_I^2)u - u_I u^2 - \dfrac{1}{3}u^3 \\ \rho(-u - bw) \end{pmatrix} \\ &= \mathcal{A}(I,\varepsilon)\begin{pmatrix} u \\ w \end{pmatrix} + \frac{1}{H(\varepsilon)}\begin{pmatrix} 1 + d_{22}\varepsilon & -d_{12}u_I\varepsilon \\ -d_{21}w_I\varepsilon & 1 + d_{11}\varepsilon \end{pmatrix}\begin{pmatrix} -u_I u^2 - \dfrac{1}{3}u^3 \\ 0 \end{pmatrix}, \end{aligned} \tag{3.12}$$

where $\mathcal{A}(I,\varepsilon)$ is given in (3.5).

By Lemma 3.1, the matrix $\mathcal{A}(I,\varepsilon)$ has a pair of complex conjugate eigenvalues $\hat{\lambda}_{1,2}(I,\varepsilon) = \hat{l}(I,\varepsilon) \pm i\,\hat{k}(I,\varepsilon)$ for any I sufficiently close to $\bar{I}_\varepsilon$. For brevity, write $a_{ij} = a_{ij}(I,\varepsilon)$ for the entries of $\mathcal{A}(I,\varepsilon)$, $i,j = 1,2$. By (3.6), $a_{12} \neq 0$ for any sufficiently small $|\varepsilon|$. Therefore, an eigenvector $\hat{\mu}$ corresponding to $\hat{\lambda}(I,\varepsilon) = \hat{l}(I,\varepsilon) - i\,\hat{k}(I,\varepsilon)$ can be taken as

$$\hat{\mu} = \begin{pmatrix} 1 \\ \dfrac{\hat{l}(I,\varepsilon) - a_{11}}{a_{12}} \end{pmatrix} + i\begin{pmatrix} 0 \\ -\dfrac{\hat{k}(I,\varepsilon)}{a_{12}} \end{pmatrix}.$$

Define

$$\mathcal{B}(I,\varepsilon) := \begin{pmatrix} \hat{l}(I,\varepsilon) & -\hat{k}(I,\varepsilon) \\ \hat{k}(I,\varepsilon) & \hat{l}(I,\varepsilon) \end{pmatrix}, \quad N(I,\varepsilon) := \begin{pmatrix} 1 & 0 \\ \dfrac{\hat{l}(I,\varepsilon)-a_{11}}{a_{12}} & -\dfrac{\hat{k}(I,\varepsilon)}{a_{12}} \end{pmatrix}.$$

Then, a direct calculation shows that $\mathcal{A}(I,\varepsilon)N(I,\varepsilon) = N(I,\varepsilon)\mathcal{B}(I,\varepsilon)$. Moreover, the inverse matrix of $N(I,\varepsilon)$ is given by

$$N^{-1}(I,\varepsilon) = \begin{pmatrix} 1 & 0 \\ \dfrac{\hat{l}(I,\varepsilon)-a_{11}}{\hat{k}(I,\varepsilon)} & -\dfrac{a_{12}}{\hat{k}(I,\varepsilon)} \end{pmatrix}.$$

By applying the coordinate transformation $(x,y)^T = N^{-1}(I,\varepsilon)(u,w)^T$, system (3.12) is transformed into

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \hat{l}(I,\varepsilon) & -\hat{k}(I,\varepsilon) \\ \hat{k}(I,\varepsilon) & \hat{l}(I,\varepsilon) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} L(x,y,I) \\ Q(x,y,I) \end{pmatrix}, \tag{3.13}$$

where

$$\begin{aligned} L(x,y,I) &= -\frac{1}{H(\varepsilon)}(1+d_{22}\varepsilon)\left(u_I x^2 + \frac{1}{3}x^3\right), \\ Q(x,y,I) &= -\frac{1}{H(\varepsilon)}\left(\frac{\hat{l}(I,\varepsilon)-a_{11}}{\hat{k}(I,\varepsilon)}(1+d_{22}\varepsilon) + \frac{a_{12}}{\hat{k}(I,\varepsilon)}d_{21}w_I\varepsilon\right)\left(u_I x^2 + \frac{1}{3}x^3\right). \end{aligned} \tag{3.14}$$

At $I = \bar{I}_\varepsilon$, (3.13) reduces to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -\hat{k}(\bar{I}_\varepsilon,\varepsilon) \\ \hat{k}(\bar{I}_\varepsilon,\varepsilon) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} L(x,y,\bar{I}_\varepsilon) \\ Q(x,y,\bar{I}_\varepsilon) \end{pmatrix},$$

and (3.14) becomes

$$\begin{aligned} L(x,y,\bar{I}_\varepsilon) &= -\frac{1}{H(\varepsilon)}(1+d_{22}\varepsilon)\left(u_{\bar{I}_\varepsilon} x^2 + \frac{1}{3}x^3\right), \\ Q(x,y,\bar{I}_\varepsilon) &= -\frac{1}{H(\varepsilon)}\left(\frac{\hat{l}(\bar{I}_\varepsilon,\varepsilon)-a_{11}(\bar{I}_\varepsilon,\varepsilon)}{\hat{k}(\bar{I}_\varepsilon,\varepsilon)}(1+d_{22}\varepsilon) + \frac{a_{12}(\bar{I}_\varepsilon,\varepsilon)}{\hat{k}(\bar{I}_\varepsilon,\varepsilon)}d_{21}w_{\bar{I}_\varepsilon}\varepsilon\right)\left(u_{\bar{I}_\varepsilon} x^2 + \frac{1}{3}x^3\right). \end{aligned}$$

In what follows, we compute the real and imaginary parts of the complex coefficient $c_1(\bar{I}_\varepsilon)$, as these are essential for determining the minimal period of the Hopf-bifurcating periodic orbit. Following [16, 43], we define

$$c_1(\bar{I}_\varepsilon) := \frac{i}{2\hat{k}(\bar{I}_\varepsilon,\varepsilon)}\left(s_{20}(\varepsilon)s_{11}(\varepsilon) - 2|s_{11}(\varepsilon)|^2 - \frac{1}{3}|s_{02}(\varepsilon)|^2\right) + \frac{s_{21}(\varepsilon)}{2}, \tag{3.15}$$

where

$$\begin{aligned} s_{11}(\varepsilon) &:= \frac{1}{4}(L_{xx}+L_{yy})+\frac{1}{4}(Q_{xx}+Q_{yy})\,i,\\ s_{02}(\varepsilon) &:= \frac{1}{4}(L_{xx}-L_{yy}-2Q_{xy})+\frac{1}{4}(Q_{xx}-Q_{yy}+2L_{xy})\,i,\\ s_{20}(\varepsilon) &:= \frac{1}{4}(L_{xx}-L_{yy}+2Q_{xy})+\frac{1}{4}(Q_{xx}-Q_{yy}-2L_{xy})\,i,\\ s_{21}(\varepsilon) &:= \frac{1}{8}(L_{xxx}+L_{xyy}+Q_{xxy}+Q_{yyy})+\frac{1}{8}(Q_{xxx}+Q_{xyy}-L_{xxy}-L_{yyy})\,i, \end{aligned} \tag{3.16}$$

with subscripts x and y denoting partial derivatives with respect to x and y, respectively. All derivatives are evaluated at $(0,\,0,\,\bar{I}_\varepsilon)$.

A direct calculation yields the following identities at $(0,\,0,\,\bar{I}_\varepsilon)$:

$$\begin{aligned} L_{xx} &= -\frac{2u_{\bar{I}_\varepsilon}}{H(\varepsilon)}(1+d_{22}\varepsilon),\\ L_{xxx} &= -\frac{2}{H(\varepsilon)}(1+d_{22}\varepsilon),\\ Q_{xx} &= -\frac{2u_{\bar{I}_\varepsilon}}{H(\varepsilon)}\left(\frac{\hat{l}(\bar{I}_\varepsilon,\varepsilon)-a_{11}(\bar{I}_\varepsilon,\varepsilon)}{\hat{k}(\bar{I}_\varepsilon,\varepsilon)}(1+d_{22}\varepsilon)+\frac{a_{12}(\bar{I}_\varepsilon,\varepsilon)}{\hat{k}(\bar{I}_\varepsilon,\varepsilon)}d_{21}w_{\bar{I}_\varepsilon}\varepsilon\right),\\ Q_{xxx} &= -\frac{2}{H(\varepsilon)}\left(\frac{\hat{l}(\bar{I}_\varepsilon,\varepsilon)-a_{11}(\bar{I}_\varepsilon,\varepsilon)}{\hat{k}(\bar{I}_\varepsilon,\varepsilon)}(1+d_{22}\varepsilon)+\frac{a_{12}(\bar{I}_\varepsilon,\varepsilon)}{\hat{k}(\bar{I}_\varepsilon,\varepsilon)}d_{21}w_{\bar{I}_\varepsilon}\varepsilon\right), \end{aligned} \tag{3.17}$$

and

$$L_{xy}=L_{yy}=L_{xxy}=L_{xyy}=L_{yyy}=Q_{xy}=Q_{yy}=Q_{xxy}=Q_{xyy}=Q_{yyy}=0. \tag{3.18}$$

Evaluating the first two relations in (3.6) at $(I,\,\varepsilon)=(\bar{I},\,0)$ gives

$$a_{11}(\bar{I},0)=1-u_{\bar{I}}^2,\qquad a_{12}(\bar{I},0)=1.$$

Since $\bar{I}_0=\bar{I}$ and $H(0)=1$, it follows from (3.7), (3.8), and (3.9) that

$$\hat{l}(\bar{I}_0,0)=\frac{1}{2}\mathcal{T}(\bar{I}_0,0)=\frac{1}{2}T(\bar{I})=0,\qquad \hat{k}(\bar{I},0)=\sqrt{\mathcal{D}(\bar{I},0)}=\sqrt{D(\bar{I})}=\sqrt{\rho-\rho^2b^2}.$$

Therefore, evaluating (3.17) at $(0,0,\bar{I}_0)$ yields

$$L_{xx}=-2u_{\bar{I}},\qquad L_{xxx}=-2,\qquad Q_{xx}=\frac{2u_{\bar{I}}\,(1-u_{\bar{I}}^2)}{\sqrt{\rho-\rho^2b^2}},\qquad Q_{xxx}=\frac{2(1-u_{\bar{I}}^2)}{\sqrt{\rho-\rho^2b^2}}. \tag{3.19}$$

Setting $\varepsilon=0$, and substituting (3.18) and (3.19) into (3.16), we have

$$\begin{aligned} s_{11}(0) &:= \frac{1}{2}u_{\bar{I}}\left(\frac{1-u_{\bar{I}}^2}{\sqrt{\rho-\rho^2b^2}}i-1\right),\qquad s_{02}(0):=\frac{1}{2}u_{\bar{I}}\left(\frac{1-u_{\bar{I}}^2}{\sqrt{\rho-\rho^2b^2}}i-1\right),\\ s_{20}(0) &:= \frac{1}{2}u_{\bar{I}}\left(\frac{1-u_{\bar{I}}^2}{\sqrt{\rho-\rho^2b^2}}i-1\right),\qquad s_{21}(0):=\frac{1}{4}\left(\frac{1-u_{\bar{I}}^2}{\sqrt{\rho-\rho^2b^2}}i-1\right). \end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.15) and employing $u_{\bar{I}}^2 = 1 - \rho b$, we obtain

$$c_1(\bar{I}_0) = -\frac{1 + \rho b^2 - 2b}{8(1 - \rho b^2)} + \frac{3\rho^2 b^3 - 6\rho b^2 + 7\rho b - 4}{24\sqrt{\rho - \rho^2 b^2}(1 - \rho b^2)} i.$$

Thus, we have explicitly,

$$\begin{aligned} \mathrm{Re}(c_1(\bar{I}_0)) &= -\frac{1 + \rho b^2 - 2b}{8(1 - \rho b^2)}, \\ \mathrm{Im}(c_1(\bar{I}_0)) &= \frac{3\rho^2 b^3 - 6\rho b^2 + 7\rho b - 4}{24\sqrt{\rho - \rho^2 b^2}(1 - \rho b^2)}. \end{aligned} \tag{3.21}$$

Step 2. Determine the minimal period $P(\varepsilon)$ and its derivative at $\varepsilon = 0$, namely $P'(0)$.

Following the analysis in [16, p90], for I sufficiently close to $\bar{I}_\varepsilon$, the minimal period $P(\varepsilon)$ of the periodic solution $(u_p(t, \varepsilon), w_p(t, \varepsilon))$ can be expressed as

$$P(\varepsilon) = \frac{2\pi}{\hat{k}(\bar{I}_\varepsilon, \varepsilon)} \left(1 + \left(\frac{\hat{l}'(\bar{I}_\varepsilon)\mathrm{Im}(c_1(\bar{I}_\varepsilon))}{\hat{k}(\bar{I}_\varepsilon, \varepsilon)\mathrm{Re}(c_1(\bar{I}_\varepsilon))} - \frac{\hat{k}'(\bar{I}_\varepsilon)}{\hat{k}(\bar{I}_\varepsilon, \varepsilon)}\right)(I - \bar{I}_\varepsilon)\right) + O\left((I - \bar{I}_\varepsilon)^2\right),$$

where $\hat{l}'(\bar{I}_\varepsilon) := \partial_I \hat{l}(I, \varepsilon)\big|_{I=\bar{I}_\varepsilon}$ and $\hat{k}'(\bar{I}_\varepsilon) := \partial_I \hat{k}(I, \varepsilon)\big|_{I=\bar{I}_\varepsilon}$, as defined in (3.10). Setting $K(\varepsilon) := \hat{k}(\bar{I}_\varepsilon, \varepsilon)$, differentiation of $P(\varepsilon)$ with respect to ε yields

$$P'(\varepsilon) = -\frac{2\pi}{\hat{k}^2(\bar{I}_\varepsilon, \varepsilon)} \cdot \frac{dK(\varepsilon)}{d\varepsilon} - \frac{2\pi}{\hat{k}(\bar{I}_\varepsilon, \varepsilon)} \left(\frac{\hat{l}'(\bar{I}_\varepsilon)\,\mathrm{Im}(c_1(\bar{I}_\varepsilon))}{\hat{k}(\bar{I}_\varepsilon, \varepsilon)\,\mathrm{Re}(c_1(\bar{I}_\varepsilon))} - \frac{\hat{k}'(\bar{I}_\varepsilon)}{\hat{k}(\bar{I}_\varepsilon, \varepsilon)}\right) \frac{d\bar{I}_\varepsilon}{d\varepsilon} + O\left(I - \bar{I}_\varepsilon\right). \tag{3.22}$$

We now compute $P'(0)$. Note that $\bar{I}_0 = \bar{I}$, and that for I sufficiently close to $\bar{I}$, the term $O(I - \bar{I})$ in (3.22) is negligible. Hence, at $\varepsilon = 0$, the sign of $P'(0)$ is determined by the first two terms on the right-hand side of (3.22).

First, we calculate the value of $d\bar{I}_\varepsilon/d\varepsilon|_{\varepsilon=0}$. At $I = \bar{I}_\varepsilon$, the condition $\mathcal{T}(\bar{I}_\varepsilon, \varepsilon) = 0$ together with (3.7) gives

$$T(\bar{I}_\varepsilon) + \varepsilon\left(-\rho b d_{11} + \rho d_{12} u_{\bar{I}_\varepsilon} - d_{21} w_{\bar{I}_\varepsilon} + (1 - u_{\bar{I}_\varepsilon}^2) d_{22}\right) = 0. \tag{3.23}$$

Differentiating (3.23) with respect to ε and then evaluating at $\varepsilon = 0$, we obtain

$$T'(\bar{I}) \left.\frac{d\bar{I}_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} + \left(-\rho b d_{11} + \rho d_{12} u_{\bar{I}} - d_{21} w_{\bar{I}} + (1 - u_{\bar{I}}^2) d_{22}\right) = 0,$$

so that

$$\left.\frac{d\bar{I}_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\rho b d_{11} - \rho d_{12} u_{\bar{I}} + d_{21} w_{\bar{I}} - (1 - u_{\bar{I}}^2) d_{22}}{T'(\bar{I})}. \tag{3.24}$$

Since $T'(\bar{I}) = -2u_{\bar{I}} u'_{\bar{I}} \neq 0$, the quotient in (3.24) is well defined.

Next, we compute the derivatives $\hat{l}'(\bar{I}_0) = \partial_I \hat{l}(I, \varepsilon)|_{I=\bar{I}_0}$ and $\hat{k}'(\bar{I}_0) = \partial_I \hat{k}(I, \varepsilon)|_{I=\bar{I}_0}$. Recall that

$$\mathcal{T}(I, 0) = T(I), \qquad \mathcal{T}(\bar{I}_0, 0) = 0, \qquad \mathcal{D}(\bar{I}_0, 0) = D(\bar{I}), \qquad H(0) = 1.$$

Differentiating the expressions in (3.9) with respect to I at $I = \bar{I}_0$, and using (3.10), we obtain

$$\hat{l}'(\bar{I}_0) = \frac{1}{2}\frac{\partial \mathcal{T}(I,0)}{\partial I}\bigg|_{I=\bar{I}_0} = \frac{1}{2}T'(\bar{I}), \tag{3.25}$$

and

$$\hat{k}'(\bar{I}_0) = \frac{2\,\partial_I \mathcal{D}(I,0)\big|_{I=\bar{I}_0} - \mathcal{T}(\bar{I}_0,0)\,\partial_I \mathcal{T}(I,0)\big|_{I=\bar{I}_0}}{2\sqrt{4\mathcal{D}(\bar{I}_0,0) - \mathcal{T}^2(\bar{I}_0,0)}} = \frac{D'(\bar{I})}{2\sqrt{D(\bar{I})}}. \tag{3.26}$$

We now compute

$$\frac{dK(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \left(\frac{\partial \hat{k}(\bar{I}_\varepsilon,\varepsilon)}{\partial \bar{I}_\varepsilon}\cdot\frac{d\bar{I}_\varepsilon}{d\varepsilon}\right)\bigg|_{\varepsilon=0} + \frac{\partial \hat{k}(\bar{I}_\varepsilon,\varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0}.$$

From (3.8) and (3.9), $\hat{k}(I,\varepsilon) = \sqrt{\mathcal{D}(I,\varepsilon)}$ with $\mathcal{D}(I,\varepsilon) = D(I)/H(\varepsilon)$, where $H(\varepsilon)$ is given by (3.4). It then follows from (3.26) that

$$\begin{aligned}
\frac{dK(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} &= \hat{k}'(\bar{I}_0)\frac{d\bar{I}_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0} + \frac{\partial\sqrt{\mathcal{D}(\bar{I}_\varepsilon,\varepsilon)}}{\partial\varepsilon}\bigg|_{\varepsilon=0}\\
&= \frac{D'(\bar{I})}{2\sqrt{D(\bar{I})}}\frac{d\bar{I}_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0} + \frac{1}{2\sqrt{D(\bar{I})}}\frac{d}{d\varepsilon}\left(\frac{D(\bar{I})}{H(\varepsilon)}\right)\bigg|_{\varepsilon=0}\\
&= \frac{D'(\bar{I})}{2\sqrt{D(\bar{I})}}\frac{d\bar{I}_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0} - \frac{(d_{11}+d_{22})\sqrt{D(\bar{I})}}{2}.
\end{aligned} \tag{3.27}$$

When $\varepsilon = 0$, substituting (3.21), (3.24), (3.25), (3.26) and (3.27) into (3.22), and using $\hat{k}(\bar{I},0) = \sqrt{D(\bar{I})}$, $D(\bar{I}) = \rho - \rho^2 b^2$ and $u_{\bar{I}}^2 = 1 - \rho b$, we have

$$\begin{aligned}
P'(0) = &-\frac{2\pi}{\hat{k}^2(\bar{I},0)}\cdot\frac{d(K(\varepsilon))}{d\varepsilon}\bigg|_{\varepsilon=0} - \frac{2\pi}{\hat{k}(\bar{I},0)}\left(\frac{\hat{l}'(\bar{I}_0)\mathrm{Im}(c_1(\bar{I}_0))}{\hat{k}(\bar{I},0)\mathrm{Re}(c_1(\bar{I}_0))} - \frac{\hat{k}'(\bar{I}_0)}{\hat{k}(\bar{I},0)}\right)\frac{d\bar{I}_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0} + O(I-\bar{I})\\
=&\frac{\pi(3\rho^2b^3 - 6\rho b^2 + 7\rho b - 4)}{\sqrt{D(\bar{I})}D(\bar{I})(3+3\rho b^2 - 6b)}\left(\rho b d_{11} - \rho d_{12}u_{\bar{I}} + d_{21}w_{\bar{I}} - (1-u_{\bar{I}}^2)d_{22}\right) + \frac{\pi(d_{11}+d_{22})}{\sqrt{D(\bar{I})}}\\
&+ O(I-\bar{I})\\
=&\frac{\pi}{\sqrt{D(\bar{I})}D(\bar{I})}(\delta_1 d_{11} + \delta_2 d_{22} + \delta_3 d_{12}u_{\bar{I}} + \delta_4 d_{21}w_{\bar{I}}) + O(I-\bar{I}),
\end{aligned}$$

where

$$\begin{aligned}
\delta_1 &= \frac{\rho(7\rho b^2 - 10b + 3)}{3+3\rho b^2 - 6b}, \quad \delta_2 = \frac{\rho(-6\rho^2b^4 + 12\rho b^3 - 7\rho b^2 - 2b + 3)}{3+3\rho b^2 - 6b},\\
\delta_3 &= -\frac{\rho(3\rho^2b^3 - 6\rho b^2 + 7\rho b - 4)}{3+3\rho b^2 - 6b}, \quad \delta_4 = \frac{3\rho^2b^3 - 6\rho b^2 + 7\rho b - 4}{3+3\rho b^2 - 6b}.
\end{aligned} \tag{3.28}$$

This completes the proof. □

The following result follows immediately from Lemmas 3.1 and 3.2.

Corollary 3.1. *Suppose that for I close to $\bar{I}$, system* (2.1) *has a stable periodic solution* $(u_p(t), w_p(t))$ *bifurcating from* $(u_{\bar{I}}, w_{\bar{I}})$, *with minimal period P. Then there exists an $\varepsilon_2 > 0$ such that, for any $\varepsilon \in (-\varepsilon_2, \varepsilon_2)$, system* (3.1) *has an ε-dependent periodic solution* $(u_p^*(t,\varepsilon), w_p^*(t,\varepsilon))$ *with minimal period $P_*(\varepsilon)$, and as $\varepsilon \to 0$,*

$$(u_p^*(t,\varepsilon), w_p^*(t,\varepsilon)) \to (u_p(t), w_p(t)) \quad \textit{and} \quad P_*(\varepsilon) \to P.$$

In particular, for I sufficiently close to $\bar{I}$,

$$P_*'(0) = P'(0) = \frac{\pi}{\sqrt{D(\bar{I})D(\bar{I})}}(\delta_1 d_{11} + \delta_2 d_{22} + \delta_3 d_{12} u_{\bar{I}} + \delta_4 d_{21} w_{\bar{I}}) + O(I - \bar{I}), \tag{3.29}$$

where δ_1, δ_2, δ_3, δ_4 are defined in (3.28).

4. Turing instability of the periodic solutions

In this section, we adopt the framework of [27, 42] to determine when system (1.3) exhibits Turing instability around periodic solutions. Let $(u_p(t), w_p(t))$ be an asymptotically orbitally stable periodic solution of (2.1) with minimal period P. Then, the linearization of system (1.3) around $(u_p(t), w_p(t))$ is given by

$$\begin{pmatrix} \tilde{u}_t \\ \tilde{w}_t \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} u_p(t) \\ d_{21} w_p(t) & d_{22} \end{pmatrix} \begin{pmatrix} \Delta \tilde{u} \\ \Delta \tilde{w} \end{pmatrix} + A_p(t) \begin{pmatrix} \tilde{u} \\ \tilde{w} \end{pmatrix}, \tag{4.1}$$

where the time-dependent coefficient matrix $A_p(t)$ is defined as

$$A_p(t) := \begin{pmatrix} 1 - u_p^2(t) & 1 \\ -\rho & -\rho b \end{pmatrix}. \tag{4.2}$$

We work in the L^2 framework. Let $-\Delta$ denote the Laplacian operator on a bounded domain $\Omega \subset \mathbb{R}^n$ with homogeneous Neumann boundary conditions. Its spectrum is a nonnegative discrete sequence of eigenvalues, counted with multiplicities. That is,

$$0 = \zeta_0 < \zeta_1 \le \zeta_2 \le \cdots \le \zeta_k \to \infty, \quad \text{as} \quad k \to \infty,$$

where $k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. Let $\{\phi_k\}_{k \ge 0}$ be an orthonormal basis of $L^2(\Omega)$ consisting of the corresponding eigenfunctions. More precisely, we have

$$-\Delta \phi_k(x) = \zeta_k \phi_k(x) \quad \text{for } x \in \Omega, \qquad \partial_\nu \phi_k(x) = 0 \quad \text{for } x \in \partial\Omega.$$

Suppose that the solution of (4.1) is expressed as

$$\begin{pmatrix} \tilde{u}(x,t) \\ \tilde{w}(x,t) \end{pmatrix} = \sum_{k=0}^{\infty} \phi_k(x) \begin{pmatrix} \alpha_k(t) \\ \beta_k(t) \end{pmatrix}.$$

Substituting this expansion into (4.1), we have

$$\begin{aligned} \alpha_k'(t) &= -\zeta_k d_{11} \alpha_k(t) - \zeta_k d_{12} u_p(t) \beta_k(t) + \left(1 - u_p^2(t)\right) \alpha_k(t) + \beta_k(t), \\ \beta_k'(t) &= -\zeta_k d_{21} w_p(t) \alpha_k(t) - \zeta_k d_{22} \beta_k(t) - \rho \alpha_k(t) - \rho b \beta_k(t) \end{aligned} \tag{4.3}$$

for each $k \in \mathbb{N}_0$.

We now establish a Turing-instability criterion for the time-periodic state $(u_p(t), w_p(t))$ of system (1.3).

Theorem 4.1. *Suppose that I is fixed sufficiently close to $\bar{I}$ so that the kinetic system* (2.1) *admits a stable periodic solution $(u_p(t), w_p(t))$ bifurcating from $(u_{\bar{I}}, w_{\bar{I}})$ at $I = \bar{I}$. If the spatial domain Ω is sufficiently large and*

$$\delta_1 d_{11} + \delta_2 d_{22} + \delta_3 d_{12} u_{\bar{I}} + \delta_4 d_{21} w_{\bar{I}} < 0, \tag{4.4}$$

then the time-periodic solution $(u_p(t), w_p(t))$ undergoes a Turing instability as in the cross-diffusion system (1.3). *Here δ_1, δ_2, δ_3, δ_4 are defined in* (3.28).

Proof. The proof is divided into four steps.

Step 1. We compute the Floquet multipliers of the variational system (4.3) and the corresponding eigenvectors of the monodromy matrix.

Suppress the mode index k and write α, β, ζ for α_k, β_k, ζ_k. With this convention, (4.3) can be written as

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \left(-\zeta \begin{pmatrix} d_{11} & d_{12} u_p(t) \\ d_{21} w_p(t) & d_{22} \end{pmatrix} + A_p(t) \right) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{4.5}$$

where $A_p(t)$ is given in (4.2).

At $\zeta = 0$, system (4.5) reduces to the variational equation

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = A_p(t) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{4.6}$$

which is precisely the linearization of (2.1) along the periodic orbit $(u_p(t), w_p(t))$. Let $\Theta(t)$ denote the fundamental solution matrix of (4.6) with $\Theta(0) = E$. Denote by κ_i $(i = 1, 2)$ the Floquet multipliers of (4.6), with corresponding eigenvectors $(r_i,\, q_i)^T$. That is,

$$\Theta(P) \begin{pmatrix} r_i \\ q_i \end{pmatrix} = \kappa_i \begin{pmatrix} r_i \\ q_i \end{pmatrix}, \qquad i = 1, 2. \tag{4.7}$$

Differentiating (2.1) with respect to t, we have

$$\begin{cases} u''(t) = w'(t) + (1 - u^2(t)) u'(t), \\ w''(t) = \rho \left(-u'(t) - b w'(t) \right). \end{cases} \tag{4.8}$$

Since $(u_p(t), w_p(t))$ is a periodic solution of system (2.1), it satisfies (4.8). Thus, substituting $(u_p(t), w_p(t))$ into (4.8) gives

$$\begin{pmatrix} u_p''(t) \\ w_p''(t) \end{pmatrix} = A_p(t) \begin{pmatrix} u_p'(t) \\ w_p'(t) \end{pmatrix}.$$

This fact indicates that $(u_p'(t), w_p'(t))$ solves the linearized system (4.6). Consequently, for the fundamental matrix $\Theta(t)$ of (4.6), we have

$$\begin{pmatrix} u_p'(t) \\ w_p'(t) \end{pmatrix} = \Theta(t) \begin{pmatrix} u_p'(0) \\ w_p'(0) \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} u_p'(P) \\ w_p'(P) \end{pmatrix} = \Theta(P) \begin{pmatrix} u_p'(0) \\ w_p'(0) \end{pmatrix} = \begin{pmatrix} u_p'(0) \\ w_p'(0) \end{pmatrix}. \tag{4.9}$$

Therefore, 1 is a Floquet multiplier of (4.6), with an eigenvector proportional to $\big(u_p'(0),\, w_p'(0)\big)^T$.

Without loss of generality, we assume that $\kappa_1 = 1$ and choose the associated eigenvector as

$$(r_1, q_1)^T = (u_p'(0), w_p'(0))^T. \tag{4.10}$$

Because the periodic orbit $(u_p(t), w_p(t))$ of system (2.1) is asymptotically stable, the second Floquet multiplier satisfies $|\kappa_2| < 1$.

Let $\Theta(t, \zeta)$ denote the fundamental solution matrix of system (4.5) satisfying the initial condition $\Theta(0, \zeta) = E$. Then, $\Theta(t, \zeta)$ satisfies the linear periodic equation

$$\partial_t \Theta(t, \zeta) = \left(-\zeta \begin{pmatrix} d_{11} & d_{12} u_p(t) \\ d_{21} w_p(t) & d_{22} \end{pmatrix} + A_p(t) \right) \Theta(t, \zeta).$$

Clearly, $\Theta(t, 0) = \Theta(t)$. Moreover, it is straightforward to see that, for sufficiently small $|\zeta|$, $\Theta(t, \zeta)$ is continuously differentiable with respect to both t and ζ for all $t \in [0, P]$.

We now compute the Floquet multipliers of system (4.5) for $\zeta \geq 0$. Let

$$F(\zeta, \sigma) := |\Theta(P, \zeta) - \sigma E| = \sigma^2 - \operatorname{tr}(\Theta(P, \zeta))\sigma + \det \Theta(P, \zeta), \tag{4.11}$$

where $\sigma \in \mathbb{C}$, $\operatorname{tr}(\Theta(P, \zeta))$ and $\det \Theta(P, \zeta)$ denote the trace and the value of the determinant of $\Theta(P, \zeta)$, respectively. In particular, $\operatorname{tr}(\Theta(P, 0)) = \operatorname{tr}(\Theta(P))$ and $\det \Theta(P, 0) = \det \Theta(P)$.

Consider

$$F(\zeta, \sigma) = 0,$$

for ζ in a neighborhood of 0. At $\zeta = 0$, we have $F(0, \kappa_i) = 0$ for $i = 1, 2$.

Differentiating (4.11) with respect to σ, we obtain

$$\frac{\partial F}{\partial \sigma}(\zeta, \sigma) = 2\sigma - \operatorname{tr}(\Theta(P, \zeta)).$$

Evaluating the derivative at $(\zeta, \sigma) = (0, \kappa_i)$ yields

$$\frac{\partial F}{\partial \sigma}(0, \kappa_i) = 2\kappa_i - \operatorname{tr} \Theta(P) = \kappa_i - \kappa_j \neq 0, \quad j \neq i,$$

since the eigenvalues of $\Theta(P)$ satisfy $\kappa_1 + \kappa_2 = \operatorname{tr} \Theta(P)$ and $\kappa_1 \neq \kappa_2$. Therefore, for each $i = 1, 2$, the implicit function theorem ensures the existence of a continuously differentiable function

$$\sigma_i : (-\tau, \tau) \to \mathbb{C}, \qquad \text{for some } \tau > 0,$$

such that

$$\sigma_i(0) = \kappa_i \quad \text{and} \quad F\big(\zeta, \sigma_i(\zeta)\big) = 0 \quad \text{for all } \zeta \in (-\tau, \tau).$$

Furthermore, for each $i = 1, 2$, there exist continuously differentiable functions $R_i(\zeta)$ and $Q_i(\zeta)$ defined on $(-\tau, \tau)$, such that

$$\Theta(P,\zeta)\begin{pmatrix}R_i(\zeta)\\ Q_i(\zeta)\end{pmatrix}=\sigma_i(\zeta)\begin{pmatrix}R_i(\zeta)\\ Q_i(\zeta)\end{pmatrix}, \quad \text{with} \quad \sigma_i(0)=\kappa_i, \quad \begin{pmatrix}R_i(0)\\ Q_i(0)\end{pmatrix}=\begin{pmatrix}r_i\\ q_i\end{pmatrix}. \tag{4.12}$$

Hence, $\sigma_1(\zeta)$ and $\sigma_2(\zeta)$ are the Floquet multipliers associated with system (4.5) for $|\zeta| < \tau$.

Step 2. We construct a solution to equation (4.6), denoted by $\widetilde{\Phi}(t)$.

Let $(\phi_1(t,\zeta), \psi_1(t,\zeta))^T$ be the solution of (4.5) with initial conditions

$$\begin{pmatrix}\phi_1(0,\zeta)\\ \psi_1(0,\zeta)\end{pmatrix}=\begin{pmatrix}R_1(\zeta)\\ Q_1(\zeta)\end{pmatrix}, \tag{4.13}$$

where $R_1(\zeta), Q_1(\zeta)$ are as in (4.12). Then, we have

$$\begin{pmatrix}\phi_1(t,\zeta)\\ \psi_1(t,\zeta)\end{pmatrix}=\Theta(t,\zeta)\begin{pmatrix}R_1(\zeta)\\ Q_1(\zeta)\end{pmatrix}. \tag{4.14}$$

When $\zeta = 0$, it follows from (4.9), (4.10), (4.12) and (4.14) that

$$\begin{pmatrix}\phi_1(t,0)\\ \psi_1(t,0)\end{pmatrix}=\Theta(t,0)\begin{pmatrix}R_1(0)\\ Q_1(0)\end{pmatrix}=\Theta(t)\begin{pmatrix}r_1\\ q_1\end{pmatrix}=\begin{pmatrix}u_p'(t)\\ w_p'(t)\end{pmatrix}. \tag{4.15}$$

Recall that, for sufficiently small $|\varepsilon|$, system (3.1) admits a periodic solution $(u_p^*(t,\varepsilon), w_p^*(t,\varepsilon))$ with period $P_*(\varepsilon)$. We now show that

$$\widetilde{\Phi}(t):=\begin{pmatrix}\partial_\zeta\phi_1(t,0)\\ \partial_\zeta\psi_1(t,0)\end{pmatrix}-\begin{pmatrix}\partial_\varepsilon u_p^*(t,0)\\ \partial_\varepsilon w_p^*(t,0)\end{pmatrix} \tag{4.16}$$

is a solution of equation (4.6).

Substituting $(\phi_1(t,\zeta), \psi_1(t,\zeta))^T$ into (4.5), we have

$$\begin{pmatrix}\partial_t\phi_1(t,\zeta)\\ \partial_t\psi_1(t,\zeta)\end{pmatrix}=-\zeta\begin{pmatrix}d_{11} & d_{12}u_p(t)\\ d_{21}w_p(t) & d_{22}\end{pmatrix}\begin{pmatrix}\phi_1(t,\zeta)\\ \psi_1(t,\zeta)\end{pmatrix}+A_p(t)\begin{pmatrix}\phi_1(t,\zeta)\\ \psi_1(t,\zeta)\end{pmatrix}. \tag{4.17}$$

Differentiating (4.17) with respect to ζ, evaluating the result at $\zeta = 0$, and employing (4.15), we obtain

$$\begin{pmatrix}\partial_t\partial_\zeta\phi_1(t,0)\\ \partial_t\partial_\zeta\psi_1(t,0)\end{pmatrix}=-\begin{pmatrix}d_{11} & d_{12}u_p(t)\\ d_{21}w_p(t) & d_{22}\end{pmatrix}\begin{pmatrix}u_p'(t)\\ w_p'(t)\end{pmatrix}+A_p(t)\begin{pmatrix}\partial_\zeta\phi_1(t,0)\\ \partial_\zeta\psi_1(t,0)\end{pmatrix}. \tag{4.18}$$

Since $(u_p^*(t,\varepsilon), w_p^*(t,\varepsilon))$ is the periodic solution of (3.1), it satisfies

$$\left(E+\varepsilon\begin{pmatrix} d_{11} & d_{12}u_p(t) \\ d_{21}w_p(t) & d_{22}\end{pmatrix}\right)\begin{pmatrix}\partial_t u_p^*(t,\varepsilon) \\ \partial_t w_p^*(t,\varepsilon)\end{pmatrix} = \begin{pmatrix} I + w_p^*(t,\varepsilon) + u_p^*(t,\varepsilon) - \frac{1}{3}\left(u_p^*(t,\varepsilon)\right)^3 \\ \rho(a - u_p^*(t,\varepsilon) - bw_p^*(t,\varepsilon))\end{pmatrix}. \tag{4.19}$$

Recall that $u_p^*(t,0) = u_p(t)$ and $w_p^*(t,0) = w_p(t)$. Consequently, $\partial_t u_p^*(t,0) = u_p'(t)$ and $\partial_t w_p^*(t,0) = w_p'(t)$. Differentiating (4.19) with respect to ε and setting $\varepsilon = 0$, we obtain

$$\begin{pmatrix}\partial_t\partial_\varepsilon u_p^*(t,0) \\ \partial_t\partial_\varepsilon w_p^*(t,0)\end{pmatrix} + \begin{pmatrix} d_{11} & d_{12}u_p(t) \\ d_{21}w_p(t) & d_{22}\end{pmatrix}\begin{pmatrix} u_p'(t) \\ w_p'(t)\end{pmatrix} = A_p(t)\begin{pmatrix}\partial_\varepsilon u_p^*(t,0) \\ \partial_\varepsilon w_p^*(t,0)\end{pmatrix}.$$

Hence,

$$\begin{pmatrix}\partial_t\partial_\varepsilon u_p^*(t,0) \\ \partial_t\partial_\varepsilon w_p^*(t,0)\end{pmatrix} = -\begin{pmatrix} d_{11} & d_{12}u_p(t) \\ d_{21}w_p(t) & d_{22}\end{pmatrix}\begin{pmatrix} u_p'(t) \\ w_p'(t)\end{pmatrix} + A_p(t)\begin{pmatrix}\partial_\varepsilon u_p^*(t,0) \\ \partial_\varepsilon w_p^*(t,0)\end{pmatrix}. \tag{4.20}$$

Subtracting (4.20) from (4.18), we have

$$\begin{aligned}\frac{d}{dt}\widetilde{\Phi}(t) &= \begin{pmatrix}\partial_t\partial_\zeta \phi_1(t,0) \\ \partial_t\partial_\zeta \psi_1(t,0)\end{pmatrix} - \begin{pmatrix}\partial_t\partial_\varepsilon u_p^*(t,0) \\ \partial_t\partial_\varepsilon w_p^*(t,0)\end{pmatrix} \\ &= A_p(t)\left(\begin{pmatrix}\partial_\zeta \phi_1(t,0) \\ \partial_\zeta \psi_1(t,0)\end{pmatrix} - \begin{pmatrix}\partial_\varepsilon u_p^*(t,0) \\ \partial_\varepsilon w_p^*(t,0)\end{pmatrix}\right) \\ &= A_p(t)\widetilde{\Phi}(t).\end{aligned}$$

Therefore, $\widetilde{\Phi}(t)$ solves the linear equation (4.6).

Step 3. We prove that

$$\sigma_1'(0) + P_*'(0) = 0.$$

By (4.12), (4.13) and (4.14), we obtain

$$\begin{pmatrix}\phi_1(P,\zeta) \\ \psi_1(P,\zeta)\end{pmatrix} = \Theta(P,\zeta)\begin{pmatrix} R_1(\zeta) \\ Q_1(\zeta)\end{pmatrix} = \sigma_1(\zeta)\begin{pmatrix}\phi_1(0,\zeta) \\ \psi_1(0,\zeta)\end{pmatrix}. \tag{4.21}$$

Differentiating (4.21) with respect to ζ yields that

$$\begin{pmatrix}\partial_\zeta\phi_1(P,\zeta) \\ \partial_\zeta\psi_1(P,\zeta)\end{pmatrix} = \sigma_1'(\zeta)\begin{pmatrix}\phi_1(0,\zeta) \\ \psi_1(0,\zeta)\end{pmatrix} + \sigma_1(\zeta)\begin{pmatrix}\partial_\zeta\phi_1(0,\zeta) \\ \partial_\zeta\psi_1(0,\zeta)\end{pmatrix}.$$

Setting $\zeta = 0$, and using (4.12), (4.13), we obtain

$$\begin{aligned}\begin{pmatrix}\partial_\zeta \phi_1(P,0)\\ \partial_\zeta \psi_1(P,0)\end{pmatrix} &= \sigma_1'(0)\begin{pmatrix}\phi_1(0,0)\\ \psi_1(0,0)\end{pmatrix} + \kappa_1 \begin{pmatrix}\partial_\zeta \phi_1(0,0)\\ \partial_\zeta \psi_1(0,0)\end{pmatrix}\\ &= \sigma_1'(0)\begin{pmatrix} r_1\\ q_1\end{pmatrix} + \begin{pmatrix}\partial_\zeta \phi_1(0,0)\\ \partial_\zeta \psi_1(0,0)\end{pmatrix},\end{aligned} \tag{4.22}$$

where we have used $\sigma_1(0) = \kappa_1 = 1$.

Since $(u_p^*(t,\varepsilon), w_p^*(t,\varepsilon))$ is periodic with minimal period $P_*(\varepsilon)$, it satisfies

$$\begin{pmatrix} u_p^*(t,\varepsilon)\\ w_p^*(t,\varepsilon)\end{pmatrix} = \begin{pmatrix} u_p^*(t+P_*(\varepsilon),\varepsilon)\\ w_p^*(t+P_*(\varepsilon),\varepsilon)\end{pmatrix}. \tag{4.23}$$

By the chain rule, differentiating (4.23) with respect to ε and evaluating at $(t,\, \varepsilon) = (0,\, 0)$ yields

$$\begin{aligned}\begin{pmatrix}\partial_\varepsilon u_p^*(t,\varepsilon)\\ \partial_\varepsilon w_p^*(t,\varepsilon)\end{pmatrix}\Bigg|_{(t,\,\varepsilon)=(0,\,0)} &= \begin{pmatrix}\partial_t u_p^*(t,\varepsilon)P_*'(\varepsilon) + \partial_\varepsilon u_p^*(t+P_*(\varepsilon),\varepsilon)\\ \partial_t w_p^*(t,\varepsilon)P_*'(\varepsilon) + \partial_\varepsilon w_p^*(t+P_*(\varepsilon),\varepsilon)\end{pmatrix}\Bigg|_{(t,\,\varepsilon)=(0,\,0)}\\ &= \begin{pmatrix}\partial_t u_p^*(0,0)P_*'(0) + \partial_\varepsilon u_p^*(P_*(0),0)\\ \partial_t w_p^*(0,0)P_*'(0) + \partial_\varepsilon w_p^*(P_*(0),0)\end{pmatrix}.\end{aligned} \tag{4.24}$$

Recall that $\partial_t u_p^*(t,0) = u_p'(t)$ and $\partial_t w_p^*(t,0) = w_p'(t)$. It then follows from (4.10) that

$$\partial_t u_p^*(0,0) = u_p'(0) = r_1, \qquad \partial_t w_p^*(0,0) = w_p'(0) = q_1.$$

Thus, using $P_*(0) = P$ and rearranging (4.24) gives

$$\begin{pmatrix}\partial_\varepsilon u_p^*(P,0)\\ \partial_\varepsilon w_p^*(P,0)\end{pmatrix} = -P_*'(0)\begin{pmatrix} r_1\\ q_1\end{pmatrix} + \begin{pmatrix}\partial_\varepsilon u_p^*(0,0)\\ \partial_\varepsilon w_p^*(0,0)\end{pmatrix}. \tag{4.25}$$

Using (4.16), and subtracting (4.25) from (4.22), we have

$$\begin{aligned}\widetilde{\Phi}(P) - \widetilde{\Phi}(0) &= \left(\begin{pmatrix}\partial_\zeta \phi_1(P,0)\\ \partial_\zeta \psi_1(P,0)\end{pmatrix} - \begin{pmatrix}\partial_\varepsilon u_p^*(P,0)\\ \partial_\varepsilon w_p^*(P,0)\end{pmatrix}\right) - \left(\begin{pmatrix}\partial_\zeta \phi_1(0,0)\\ \partial_\zeta \psi_1(0,0)\end{pmatrix} - \begin{pmatrix}\partial_\varepsilon u_p^*(0,0)\\ \partial_\varepsilon w_p^*(0,0)\end{pmatrix}\right)\\ &= (\sigma_1'(0) + P_*'(0))\begin{pmatrix} r_1\\ q_1\end{pmatrix}.\end{aligned} \tag{4.26}$$

Suppose that $\widetilde{\Phi}(0) = (\theta_1, \theta_2)^T \in \mathbb{R}^2$. Because $\Theta(t)$ is the principal fundamental solution matrix of (4.6) with $\Theta(0) = E$, we have

$$\widetilde{\Phi}(t) = \Theta(t) \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \tag{4.27}$$

Since $(r_1, q_1)^T$ and $(r_2, q_2)^T$ are linearly independent, there exist γ_1, $\gamma_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \gamma_1 \begin{pmatrix} r_1 \\ q_1 \end{pmatrix} + \gamma_2 \begin{pmatrix} r_2 \\ q_2 \end{pmatrix}. \tag{4.28}$$

Using (4.7), (4.26), (4.27), (4.28) and $\kappa_1 = 1$, we compute that

$$\begin{aligned} \widetilde{\Phi}(P) - \widetilde{\Phi}(0) &= \Theta(P) \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} - \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \\ &= \gamma_1 \left(\Theta(P) \begin{pmatrix} r_1 \\ q_1 \end{pmatrix} - \begin{pmatrix} r_1 \\ q_1 \end{pmatrix} \right) + \gamma_2 \left(\Theta(P) \begin{pmatrix} r_2 \\ q_2 \end{pmatrix} - \begin{pmatrix} r_2 \\ q_2 \end{pmatrix} \right) \\ &= \gamma_2(\kappa_2 - 1) \begin{pmatrix} r_2 \\ q_2 \end{pmatrix} \\ &= (\sigma_1'(0) + P_*'(0)) \begin{pmatrix} r_1 \\ q_1 \end{pmatrix}. \end{aligned}$$

Since $(r_1, q_1)^T$ and $(r_2, q_2)^T$ are linearly independent and $|\kappa_2| < 1$, it follows immediately that

$$\sigma_1'(0) + P_*'(0) = 0 \quad \text{and} \quad \gamma_2 = 0. \tag{4.29}$$

Step 4. We now examine the conditions for the onset of Turing instability in system (1.3).

If $P_*'(0) < 0$, it follows from (4.29) immediately that $\sigma_1'(0) > 0$. Suppose that the domain Ω is sufficiently large such that the smallest positive eigenvalue of $-\Delta$, denoted ζ_1, is arbitrarily small. Since $\sigma_1(0) = \kappa_1 = 1$ and $\sigma_1'(0) > 0$, continuity of $\sigma_1(\zeta)$ in ζ implies that $\sigma_1(\zeta) > 1$ for all sufficiently small $\zeta > 0$. In particular, there exists at least one ζ_n such that $\sigma_1(\zeta) = \sigma_1(\zeta_n) > 1$. Therefore, the spatially homogeneous time-periodic solution $(u_p(t), w_p(t))$ of (1.3) is destabilized by diffusion.

According to Theorem 2.1 and Remark 2.2, for I sufficiently close to $\bar{I}$, system (1.3) admits a family of stable periodic solutions, denoted by $(u_p(t),\ w_p(t))$, bifurcating from $(u_{\bar{I}},\ w_{\bar{I}})$ provided that $1 + \rho b^2 - 2b > 0$. Furthermore, Corollary 3.1 provides the explicit formula (3.29) for $P_*'(0)$. Consequently, for I sufficiently close to $\bar{I}$, the condition

$$\delta_1 d_{11} + \delta_2 d_{22} + \delta_3 d_{12} u_{\bar{I}} + \delta_4 d_{21} w_{\bar{I}} < 0$$

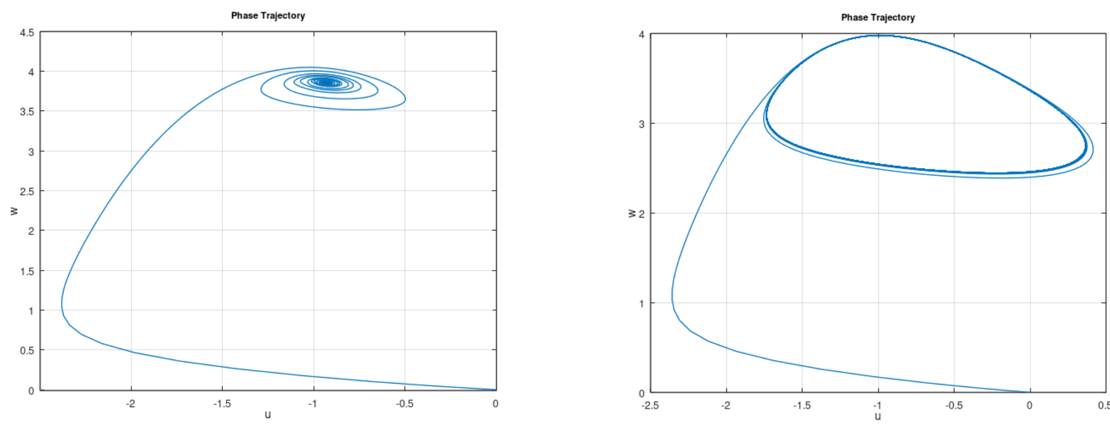
is sufficient to ensure $P_*'(0) < 0$. Thus, the spatially homogeneous periodic solutions $(u_p(t),$ $w_p(t))$ of (1.3) undergo a Turing instability. This completes the proof. □

5. Numerical simulations

In this section, we present numerical simulations to support the theoretical analysis. By Theorem 2.1, if $1 + \rho b^2 - 2b > 0$, then system (2.1) admits a stable periodic orbit for every $I \in (I_1, I_1 + \tau_1) \cup (I_2 - \tau_2, I_2)$, where $\tau_1, \tau_2 > 0$ are some constants. Accordingly, we set $\rho = 0.5$, $b = 0.4$, and $a = 1$, for which

$$1 + \rho b^2 - 2b = 1 + 0.5 \times 0.16 - 0.8 = 0.28 > 0.$$

Then, from (2.2) and (2.6), we obtain $I_1 \approx -3.1329$ and $I_2 \approx -0.8671$; consequently, the FitzHugh–Nagumo system (2.1) undergoes two Hopf bifurcations at the equilibria $(u_{I_1}, w_{I_1}) \approx (-0.8944,\ 3.7889)$ and $(u_{I_2}, w_{I_2}) \approx (0.8944,\ 0.2111)$. In the simulations below, we therefore take $I = -3.1$ and $I = -0.9$ as representative values, and we consider the one-dimensional spatial domain $\Omega = (0, \ell)$ with $\ell > 0$.

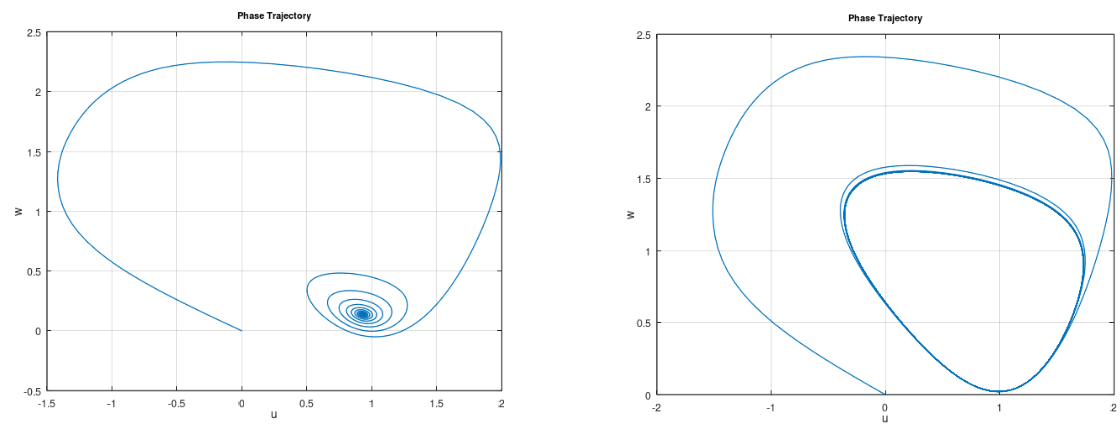


(a) $I = -3.2$ (stable equilibrium) (b) $I = -3.1$ (stable periodic orbit)

Figure 1. Trajectories in the (u, w) plane of the spatially homogeneous FitzHugh–Nagumo system (2.1) near the Hopf bifurcation. Parameters: $a = 1$, $b = 0.4$, $\rho = 0.5$. Numerical integration on $t \in [0, 500]$ with step size $\Delta t = 0.1$, starting from $u(0) = w(0) = 0$. (a) $I = -3.2$: Convergence to the asymptotically stable equilibrium. (b) $I = -3.1$: Convergence to the stable periodic orbit born at the Hopf bifurcation.

Figures 1 and 2 illustrate the existence of stable periodic solutions in the FitzHugh–Nagumo ODE system (2.1). We take $I = -3.2$ and $I = -3.1$ for Figure 1, and $I = -0.8$ and $I = -0.9$ for Figure 2. In Figure 1(a) ($I = -3.2$), the equilibrium of (2.1) is asymptotically stable. In Figure 1(b) ($I = -3.1$), a stable periodic solution $(u_p(t), w_p(t))$ bifurcates from the equilibrium $(u_{I_1}, w_{I_1}) \approx (-0.8944,\ 3.7889)$ and yields a limit cycle, in agreement with Theorem 2.1. Similarly, in Figure 2(a) ($I = -0.8$) the equilibrium is asymptotically stable, whereas in Figure 2(b) ($I = -0.9$) a stable periodic solution bifurcating from $(u_{I_2}, w_{I_2}) \approx (0.8944,\ 0.2111)$, again producing a limit cycle that is consistent with Theorem 2.1.

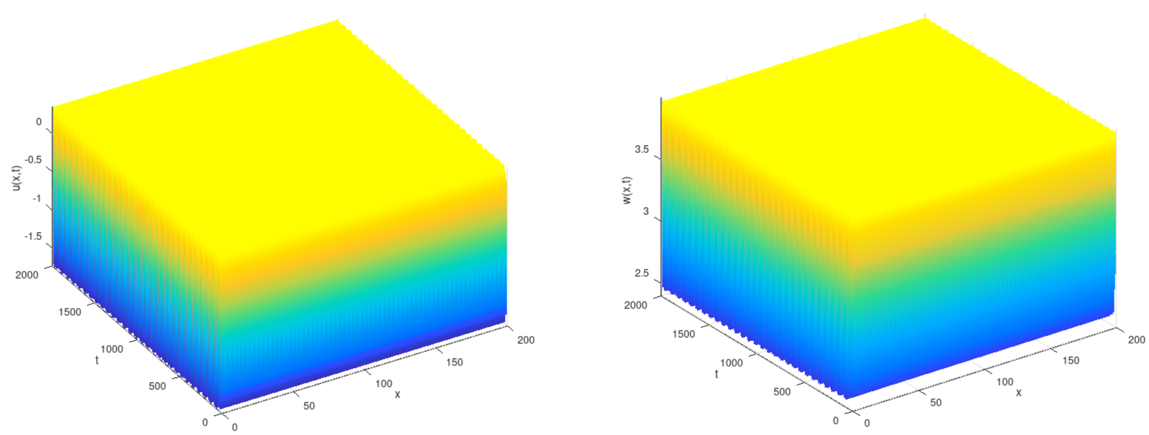
Figures 3 and 4 present the cases where diffusion-driven instability of periodic solutions does not occur. We set $d_{11} = d_{22} = 1$ and $d_{12} = d_{21} = 0$. In Figure 3, for $I = -3.1$, no Turing instability arises for the periodic solution $(u_p(t), w_p(t))$ bifurcating from the equilibrium $(u_{I_1}, w_{I_1}) \approx (-0.8944,\ 3.7889)$. Similarly, in Figure 4, for $I = -0.9$, the periodic solution $(u_p(t), w_p(t))$ bifurcating from $(u_{I_2}, w_{I_2}) \approx (0.8944,\ 0.2111)$ also exhibits no Turing instability. Thus, the time-periodic solution $(u_p(t), w_p(t))$ remains stable in the reaction-diffusion FitzHugh–Nagumo system with cross-diffusion (1.3).



(a) $I = -0.8$ (stable equilibrium)

(b) $I = -0.9$ (stable periodic orbit)

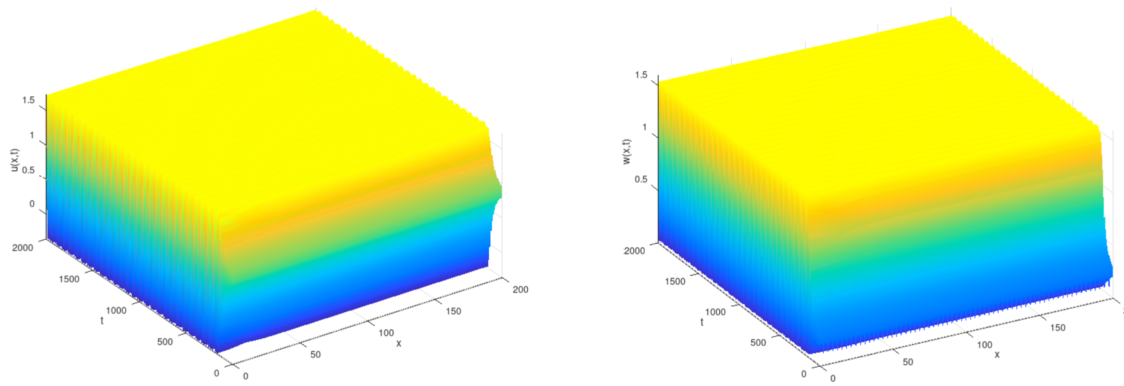
Figure 2. Trajectories in the (u, w) plane of the spatially homogeneous FitzHugh–Nagumo system (2.1) near the Hopf bifurcation. Parameters: $a = 1$, $b = 0.4$, $\rho = 0.5$. Numerical integration on $t \in [0, 500]$ with step size $\Delta t = 0.1$, starting from $u(0) = w(0) = 0$. (a) $I = -0.8$: Convergence to the asymptotically stable equilibrium. (b) $I = -0.9$: Convergence to the stable periodic orbit born at the Hopf bifurcation.



(a) Space-time plot of $u(x, t)$

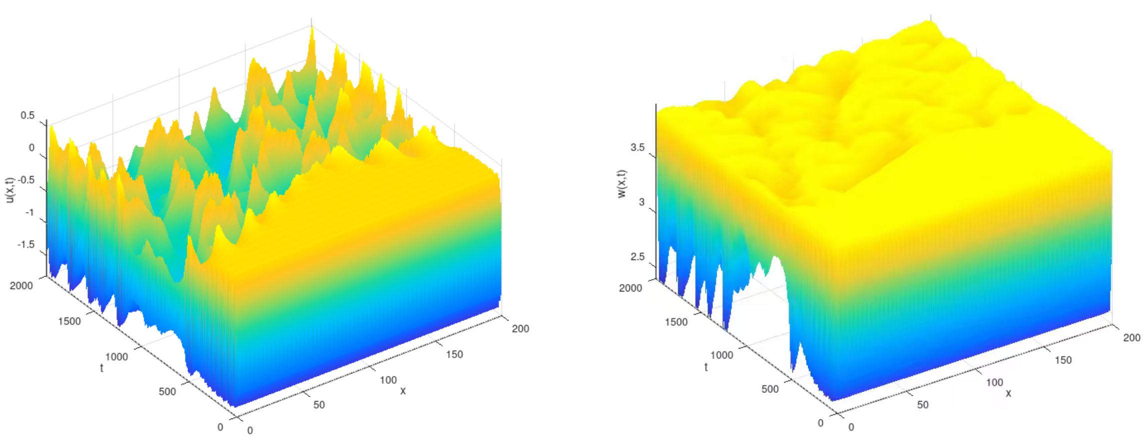
(b) Space-time plot of $w(x, t)$

Figure 3. Numerical solution of the FitzHugh–Nagumo system (1.3) on the one-dimensional domain $\Omega = (0, 200)$ with homogeneous Neumann boundary conditions $\partial_\nu u = \partial_\nu w = 0$. Parameters: $a = 1$, $b = 0.4$, $\rho = 0.5$, $I = -3.1$, and $(d_{11}, d_{12}, d_{21}, d_{22}) = (1, 0, 0, 1)$. Panels show space-time plots of (a) $u(x, t)$ and (b) $w(x, t)$ for $t \in [0, 2000]$. Discretization: $\Delta x = 0.5$ and $\Delta t = 1$. Initial data: $u(x, 0) = -0.9 + 0.1 \sin(2x)$ and $w(x, 0) = 3 + 0.1 \sin(2x)$. The spatially homogeneous periodic orbit bifurcating from $(u_{I_1}, w_{I_1}) \approx (-0.8944, 3.7889)$ is stable: The solution approaches a uniform oscillation and no Turing pattern develops.



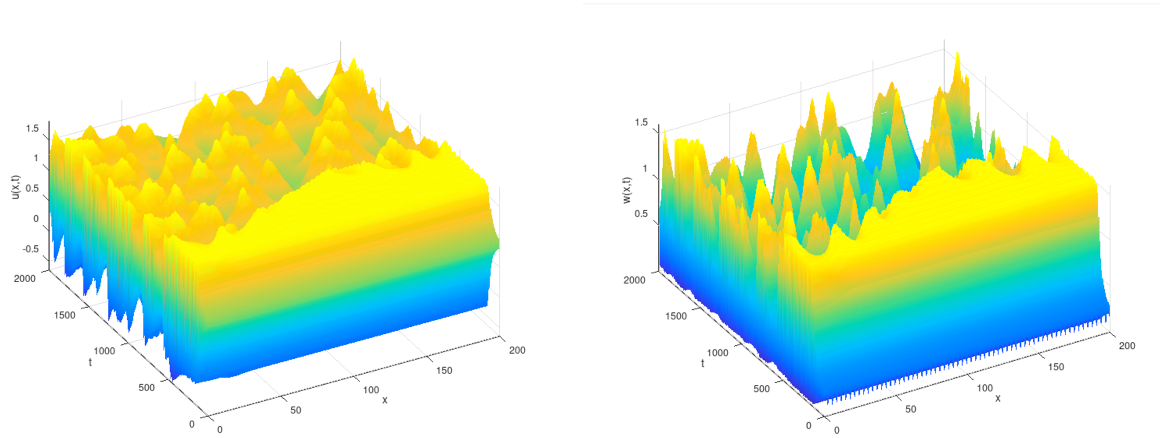
(a) Space-time plot of $u(x,t)$ (b) Space-time plot of $w(x,t)$

Figure 4. Numerical solution of the FitzHugh–Nagumo system (1.3) on the one-dimensional domain $\Omega = (0, 200)$ with homogeneous Neumann boundary conditions $\partial_\nu u = \partial_\nu w = 0$. Parameters: $a = 1$, $b = 0.4$, $\rho = 0.5$, $I = -0.9$, and $(d_{11}, d_{12}, d_{21}, d_{22}) = (1, 0, 0, 1)$. Panels show space-time plots of (a) $u(x,t)$ and (b) $w(x,t)$ for $t \in [0, 2000]$. Discretization: $\Delta x = 0.5$ and $\Delta t = 1$. Initial data: $u(x, 0) = 0.9 + 0.1 \sin(2x)$ and $w(x, 0) = 0.2 + 0.1 \sin(2x)$. The spatially homogeneous periodic orbit bifurcating from $(u_{I_2}, w_{I_2}) \approx (0.8944, 0.2111)$ is stable: The solution approaches a uniform oscillation and no Turing pattern develops.



(a) Space-time plot of $u(x,t)$ (b) Space-time plot of $w(x,t)$

Figure 5. Numerical solution of the FitzHugh–Nagumo system (1.3) on the one-dimensional domain $\Omega = (0, 200)$ with homogeneous Neumann boundary conditions $\partial_\nu u = \partial_\nu w = 0$. Parameters: $a = 1$, $b = 0.4$, $\rho = 0.5$, $I = -3.1$, and $(d_{11}, d_{12}, d_{21}, d_{22}) = (1, 0, 0, 0.001)$. Panels show space-time plots of (a) $u(x,t)$ and (b) $w(x,t)$ for $t \in [0, 2000]$. Discretization: $\Delta x = 0.5$ and $\Delta t = 1$. Initial data: $u(x, 0) = -0.9 + 0.1 \sin(2x)$ and $w(x, 0) = 3 + 0.1 \sin(2x)$. The spatially homogeneous periodic orbit bifurcating from $(u_{I_1}, w_{I_1}) \approx (-0.8944, 3.7889)$ becomes diffusion-driven unstable, leading to spatiotemporal patterns.



(a) Space-time plot of $u(x,t)$

(b) Space-time plot of $w(x,t)$

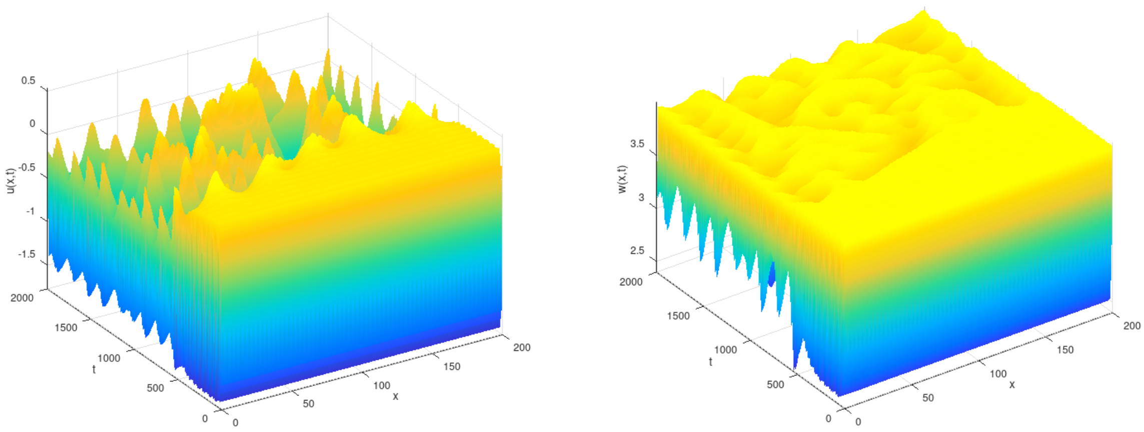
Figure 6. Numerical solution of the FitzHugh–Nagumo system (1.3) on the one-dimensional domain $\Omega = (0, 200)$ with homogeneous Neumann boundary conditions $\partial_\nu u = \partial_\nu w = 0$. Parameters: $a = 1$, $b = 0.4$, $\rho = 0.5$, $I = -0.9$, and $(d_{11}, d_{12}, d_{21}, d_{22}) = (1, 0, 0, 0.001)$. Panels show space-time plots of (a) $u(x,t)$ and (b) $w(x,t)$ for $t \in [0, 2000]$. Discretization: $\Delta x = 0.5$ and $\Delta t = 1$. Initial data: $u(x,0) = 0.9 + 0.1 \sin(2x)$ and $w(x,0) = 0.2 + 0.1 \sin(2x)$. The spatially homogeneous periodic orbit bifurcating from $(u_{I_2}, w_{I_2}) \approx (0.8944, 0.2111)$ becomes diffusion-driven unstable, leading to spatiotemporal pattern formation.

Figures 5 and 6 present the cases where diffusion-driven Turing instability of periodic solutions occurs. We fix $d_{11} = 1$, $d_{22} = 10^{-3}$, and $d_{12} = d_{21} = 0$. Clearly, the criterion (4.4) is satisfied. In Figure 5 ($I = -3.1$), the time-periodic solution $(u_p(t), w_p(t))$ bifurcating from $(u_{I_1}, w_{I_1}) \approx (-0.8944, 3.7889)$ is destabilized in the cross-diffusion system (1.3); a diffusion-driven (Turing) instability occurs and irregular spatiotemporal patterns emerge. In Figure 6 ($I = -0.9$), the time-periodic solution $(u_p(t), w_p(t))$ bifurcating from $(u_{I_2}, w_{I_2}) \approx (0.8944, 0.2111)$ likewise becomes unstable in (1.3), again leading to irregular spatiotemporal patterns. Moreover, increasing the diffusion contrast d_{11}/d_{22} tends to promote the onset of the Turing instability.

Figures 7 and 8 present cases in which cross-diffusion induces a Turing instability of periodic solutions. For the periodic solution $(u_p(t), w_p(t))$ bifurcating from $(u_{I_1}, w_{I_1}) \approx (-0.8944,\ 3.7889)$, we set $(d_{11},\ d_{22},\ d_{12},\ d_{21}) = (1,\ 1,\ -1,\ 0.2)$; for the periodic solution bifurcating from $(u_{I_2}, w_{I_2}) \approx (0.8944,\ 0.2111)$, we set $(d_{11},\ d_{22},\ d_{12},\ d_{21}) = (1,\ 1,\ 0.1,\ 2)$. Under these parameter sets, the criterion (4.4) is satisfied and $(u_p(t), w_p(t))$ is destabilized in the cross-diffusion system (1.3), leading to diffusion-driven (Turing) instability and irregular spatiotemporal patterns. This demonstrates that, although no diffusion-driven instability occurs with self-diffusion alone, the inclusion of cross-diffusion can trigger destabilization of the time-periodic orbit.

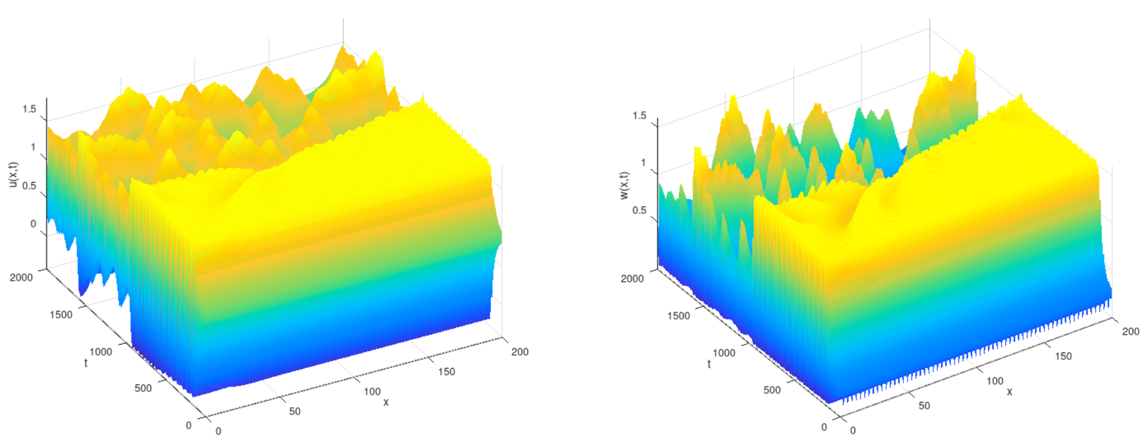
6. Discussion

In this work, we investigated diffusion-driven spatiotemporal pattern formation in a FitzHugh–Nagumo reaction-diffusion system with density-dependent cross-diffusion under homogeneous Neumann boundary conditions. In contrast to the classical Turing theory for equilibria, we focused on spatially homogeneous time-periodic oscillations bifurcating from a Hopf point and how diffusion, in particular cross-diffusion, can destabilize such uniform periodic states via a Turing-type mechanism.



(a) Space-time plot of $u(x,t)$ (b) Space-time plot of $w(x,t)$

Figure 7. Numerical solution of the cross-diffusion FitzHugh–Nagumo system (1.3) on the one-dimensional domain $\Omega = (0, 200)$ with homogeneous Neumann boundary conditions $\partial_\nu u = \partial_\nu w = 0$. Parameters: $a = 1$, $b = 0.4$, $\rho = 0.5$, $I = -3.1$, and $(d_{11}, d_{12}, d_{21}, d_{22}) = (1, -1, 0.2, 1)$ (nonzero cross-diffusion). Panels show space-time plots of (a) $u(x,t)$ and (b) $w(x,t)$ for $t \in [0, 2000]$. Discretization: $\Delta x = 0.5$ and $\Delta t = 1$. Initial data: $u(x,0) = -0.9 + 0.1\sin(2x)$ and $w(x,0) = 3 + 0.1\sin(2x)$. The spatially homogeneous periodic orbit bifurcating from $(u_{I_1}, w_{I_1}) \approx (-0.8944, 3.7889)$ is destabilized by cross-diffusion, and spatial patterns develop.



(a) Space-time plot of $u(x,t)$ (b) Space-time plot of $w(x,t)$

Figure 8. Numerical solution of the cross-diffusion FitzHugh–Nagumo system (1.3) on the one-dimensional domain $\Omega = (0, 200)$ with homogeneous Neumann boundary conditions $\partial_\nu u = \partial_\nu w = 0$. Parameters: $a = 1$, $b = 0.4$, $\rho = 0.5$, $I = -0.9$, and $(d_{11}, d_{12}, d_{21}, d_{22}) = (1, 0.1, 2, 1)$ (nonzero cross-diffusion). Panels show space-time plots of (a) $u(x,t)$ and (b) $w(x,t)$ for $t \in [0, 2000]$. Discretization: $\Delta x = 0.5$ and $\Delta t = 1$. Initial data: $u(x,0) = 0.9 + 0.1\sin(2x)$ and $w(x,0) = 0.2 + 0.1\sin(2x)$. The spatially homogeneous periodic orbit bifurcating from $(u_{I_2}, w_{I_2}) \approx (0.8944, 0.2111)$ is destabilized by cross-diffusion, and spatial patterns develop.

A key strength of our analysis is the explicit instability condition, formulated directly in terms of the diffusion coefficients, which characterizes the instability region in the diffusion parameter space. The criterion accounts for density-dependent cross-diffusion and thus goes beyond standard self-diffusion settings. Moreover, the analytical predictions are supported by numerical simulations, showing good agreement between theory and computation for the onset of diffusion-induced patterns.

Our analysis is performed in a neighborhood of the Hopf bifurcation and therefore mainly captures the destabilization of small-amplitude periodic orbits. The dynamics farther from the bifurcation point, including larger-amplitude oscillations, are not addressed here and will be considered in future work.

We expect that our results provide a useful theoretical basis for the study of diffusion-driven instabilities and pattern formation in excitable reaction-diffusion systems, and may motivate further studies of related cross-diffusion models.

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Received September 2025; Accepted February 2026; Available online March 2026.