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DOUBLE PHASE PROBLEMS IN MUSIELAK SPACES INVOLVING THE φ-HILFER FRACTIONAL DERIVATIVE

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Abstract In this paper, we introduce a functional framework that extends the φ-Hilfer space by incorporating the Laplacian operator within a double-phase structure in the setting of Musielak–Orlicz–Sobolev spaces. We define the φ-Riemann–Liouville fractional partial integral and derivative, as well as the Hilfer fractional derivative (HFD), within this generalized context. Furthermore, we construct a Musielak–Orlicz–Sobolev $\mathbb{H}$ space that integrates both the Laplacian operator and variable exponent Lebesgue spaces, which are central to the analysis of double-phase problems. The paper establishes the fundamental propositions, definitions, and theorems, formulates the necessary hypotheses, and demonstrates the existence of weak solutions for a suitable elliptic problem involving the HFD. To illustrate and support our theoretical results, several examples are also presented.

Keywords φ-Hilfer fractional derivative, double-phase functional spaces, Musielak-Orlicz-Sobolev spaces, variational methods in differential equations.

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1. Introduction

The study of functional spaces has long been a cornerstone of modern analysis, particularly in the formulation and resolution of differential equations that model a wide spectrum of physical and engineering phenomena. In recent years, significant attention has been directed toward generalizing classical function spaces to more flexible frameworks, capable of capturing the complex behavior associated with heterogeneous materials and anomalous diffusion processes. Among these advancements, double-phase functional spaces have proven especially effective in addressing problems involving non-standard growth conditions and spatial irregularities [10, 18, 46].

Simultaneously, the field of fractional calculus has witnessed rapid development due to its ability to model memory effects and non-local phenomena in diverse scientific contexts. Fractional differential equations now serve as fundamental tools in areas such as biology, physics, chemistry, mechanics, and engineering [6, 17, 33–35, 39]. Their inherent flexibility allows for more precise representations of real-world dynamics than classical integer-order models.

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A particular emphasis has been placed on the qualitative behavior and stability analysis of fractional differential equations. These equations, involving derivatives and integrals of non-integer order, find applications in biomechanics, electrical circuits, control systems, and medical technologies, including ultrasound imaging [16, 29, 31, 57]. While classical models often assume constant fractional orders, practical scenarios in signal processing, physics, and control theory frequently demand variable-order derivatives, motivating the development of generalized operators [2, 53].

Historically, the Riemann-Liouville fractional integral and differential operators laid the foundation of fractional calculus. Building on this, a broad spectrum of operators including the Caputo, Hilfer, Hilfer-Katugampola, Katugampola, and Hadamard derivatives has been introduced to provide flexible modeling capabilities and are now extensively studied in the literature [7, 9, 12–14, 19–23, 41, 62]. Of particular interest is the Hilfer fractional derivative, which interpolates between the Riemann-Liouville and Caputo operators [27] and plays a key role in applied fields such as polymer science, rheology, electrical engineering, and dielectric relaxation simulations.

The double-phase framework was initially motivated by advances in the calculus of variations and nonlinear potential theory. It involves energy functionals whose integrands exhibit two distinct growth behaviors modulated by a coefficient function [18, 46]. This formulation naturally generalizes classical Orlicz and variable exponent spaces, providing an effective tool for modeling materials undergoing phase transitions or exhibiting spatially heterogeneous properties [32, 52, 54]. The foundational work of Zhikov [61] significantly advanced this area by examining variational integrals with nonstandard growth conditions. Double-phase operators, generalizing the classical κ_2-Laplace operator, are particularly suited for capturing the behavior of systems with spatially varying properties and non-uniform growth. They have found applications in composite materials, nonlinear elasticity, heterogeneous media, and mathematical physics, including contexts involving strongly anisotropic structures, the Lavrentiev gap phenomenon, plasma dynamics, biophysical processes, and nonlinear chemical reactions. Recent studies have further advanced this field [1, 3, 18, 59, 60].

Parallel to these developments, fractional calculus has introduced a wide range of operators extending classical differentiation and integration, capturing memory and hereditary effects in physical systems. Among these, the φ-Hilfer fractional operator provides a unified framework encompassing numerous well-known fractional derivatives through an appropriate choice of the kernel function φ. Incorporating this operator within a double-phase structure leads to a novel class of functional spaces capable of modeling increasingly intricate phenomena.

Double-phase functional spaces are particularly important in problems where the energy functional alternates between two growth conditions typically of the form $|u|^{\kappa_1(x)}$ and $|u|^{\kappa_2(x)}$-governed by a modulating function $\eta(x)$. This framework effectively represents materials whose mechanical or physical properties, such as stiffness or conductivity, vary spatially, making it highly relevant in nonlinear elasticity, electrorheology, and material science. To rigorously analyze such structures, researchers employ Musielak-Orlicz-Sobolev spaces, which generalize classical Orlicz and variable exponent spaces by allowing the growth behavior to depend on the spatial variable. This flexibility makes them particularly suitable for handling the irregular and non-uniform phenomena inherent to double-phase problems.

In this context, a central focus of our study is the generalized formulation involving the φ-Hilfer fractional derivative (denoted φ-HFD) within the Musielak-Orlicz-Sobolev framework. This setting allows the analysis of fractional singular double-phase equations involving the φ-

Hilfer operator and non-standard growth structures. Specifically, we consider the operator

$$\mathbb{L}(u) := \mathbb{D}_{-}^{\alpha_1,\alpha_2;\varphi}\left(\left|\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_1(x)-2}\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u+\eta(x)\left|\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_2(x)-2}\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right), \tag{1.1}$$

where $\mathbb{D}_{-}^{\alpha_1,\alpha_2;\varphi}$ and $\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}$ denote the left- and right-sided φ-Hilfer fractional operators of order $0<\alpha_1<1$ and type $0\leq\alpha_2\leq 1$, and $\kappa_1(\cdot),\kappa_2(\cdot)\in C^+(\overline{\Theta})$ satisfy $\frac{1}{\kappa_2^-}<\alpha_1<\frac{1}{\kappa_1^+}<1$.

The primary goal of this work is to introduce and rigorously investigate a novel class of double-phase functional spaces that extend the φ-Hilfer fractional framework within Musielak-Orlicz-Sobolev spaces. We examine their structural and embedding properties and demonstrate their applicability to differential equations governed by double-phase-type operators. This framework supports the analysis of elliptic and parabolic problems with slope-dependent nonlinearities and non-homogeneous growth, frequently arising in fluid mechanics, image processing, and electrorheological fluids.

Motivated by the significant role of fractional differential equations with variable exponent and double-phase structures in modeling complex phenomena, this study addresses a novel combination of double-phase structures with variable exponent φ-Hilfer fractional operators a synthesis not previously explored in the literature. This approach allows for the rigorous development of fundamental properties of the space ${}^{H}W_0^1L_{\mathcal{A}}(\Theta)$, advancing both theoretical understanding and the originality of the contribution.

Recent works [4,5,11,58] have explored Musielak-Orlicz-Sobolev spaces and associated double-phase operators, establishing key results on the existence and uniqueness of solutions to elliptic equations with slope-dependent terms. While our work builds on these foundations, our assumptions, methodology, and results differ significantly. For further developments in elliptic and parabolic problems in Musielak-Orlicz-Sobolev spaces, we refer the reader to [24,32,54–56]. The depth, utility, and broad range of applications in these studies have provided strong motivation for the present research.

In recent developments on double-phase problems, Benslimane et al. [8] investigate a class of equations featuring both singularities and homogeneous Choquard-type nonlinearities, establishing the existence of positive nontrivial solutions via variational methods and the Nehari manifold approach within the framework of Musielak-Orlicz Sobolev spaces. Their study effectively combines the handling of double-phase growth with singular and nonlocal terms, employing tools such as the Hardy-Littlewood-Sobolev inequality to control the Choquard interaction. This work not only extends classical results on double-phase problems but also provides a methodological foundation that can inspire extensions to more general settings, including fractional operators and variable exponent Sobolev spaces. Motivated by these techniques, we consider an analogous class of double-phase problems with ϕ-Hilfer fractional derivatives, aiming to address the interplay of non-standard growth, singularities, and fractional nonlocal effects within variable exponent spaces.

The aim of this research is to show the existence of a weak solution for the following fractional differential equation:

$$\begin{cases}\displaystyle\sum_{i=1}^{N}{}^{H}\mathbb{D}_{x_i^-}^{\alpha_1,\alpha_2;\varphi}\left(\left|{}^{H}\mathbb{D}_{x_i^+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_1(x)-2}{}^{H}\mathbb{D}_{x_i^+}^{\alpha_1,\alpha_2;\varphi}u+\eta(x)\left|{}^{H}\mathbb{D}_{x_i^+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_2(x)-2}{}^{H}\mathbb{D}_{x_i^+}^{\alpha_1,\alpha_2;\varphi}u\right)\\ +|u|^{\kappa_1(x)-2}u+\eta(x)|u|^{\kappa_2(x)-2}u=g(x,u)\quad\text{for}\quad x\in\Theta,\\ I_{x_i^+}^{1-\alpha_2;\varphi}u(x)=I_{x_i^-}^{1-\alpha_1;\varphi}u(x)=0\quad\text{on}\quad\partial\Theta,\end{cases} \tag{1.2}$$

where $\Theta \subset \mathbb{R}^N$ $(N \geq 2)$ is a limited region with a Lipschitz continuous edge $\partial\Theta$.

Due to the complexity of high-order φ-Hilfer differential equations of fractional order, there has been comparatively limited examination regarding these equations in the scientific literature.

The transition from classical elliptic models to those incorporating the φ-Hilfer fractional derivative framework is motivated by the fundamental limitations of local operators in capturing the pervasive non-local and memory-dependent phenomena observed in real-world systems. Classical formulations, based on standard derivatives, are inadequate for processes where the current state depends on the entire history or spatial configuration, such as fluid flow in fractal reservoirs, drug diffusion in heterogeneous tumors, or stress relaxation in viscoelastic composites. The φ-Hilfer operator, defined before provides a mathematically rigorous and flexible alternative. Its parameters grant precise physical control: α quantifies the degree of non-locality (e.g., the fractal dimension in porous media), γ interpolates between initial condition types, and the kernel $\varphi(t)$ (e.g., t^ρ, $\log t$, $e^{\omega t}$) adapts the model to multi-scale temporal or spatial behaviors. This is exemplified by its successful application in modified elliptic equations like

$$-\nabla \cdot \left(K(\mathbf{x})\, {}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;t^\beta} \nabla p\right) = Q(\mathbf{x}),$$

which significantly improves production forecasting in petroleum engineering, and in biomedical models like

$$\frac{\partial C}{\partial t} = {}^H\, \mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi} \left(K(x)\, {}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi} C\right) - \lambda C,$$

which accurately predicts drug concentration gradients in tumor tissue. By offering enhanced predictive capability, direct physical interpretability of its parameters, and a clear path for experimental validation, the φ-Hilfer framework presents a transformative tool for addressing the complex, multi-scale challenges that elude classical analysis. For more details see [25].

In [45], Srivastava and Sousa investigated quasi-linear fractional differential equations with variable exponents, specifically those of the form:

$$\begin{cases} \mathbb{D}_T^{\alpha_1,\alpha_2;\varphi} \left(\left|\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_1(x)-2} \mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u \right) = p|u|^{\gamma(x)-2}u + \mathcal{A}(x,u) \quad \text{in } \Theta = [0,T]^3, \\ u = 0 \quad \text{on } \partial\Theta. \end{cases} \tag{1.3}$$

To establish the existence and multiplicity of solutions for problem (1.3), the authors employed the Genus Theory in combination with the Concentration-Compactness Principle and the Mountain Pass Theorem.

Furthermore, in [46], Sousa et al. applied the fibering method alongside the Nehari manifold approach to establish the presence of a minimum of two generalized answers to the following singular fractional double-phase problem:

$$\begin{cases} \mathbb{D}_T^{\alpha_1,\alpha_2;\varphi} \left(\left|\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u + \mu(x)\right|^{\kappa_1-2} \mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u \right) = qu^{-\sigma} + pu^{r-1} \quad \text{in } \Theta = [0,T]^2, \\ u = 0 \quad \text{on } \partial\Theta, \end{cases} \tag{1.4}$$

under the assumption that κ_1 is sufficiently small.

Additionally, in [48], Sousa et al. established existence and multiplicity results for the following curvature-type problem by employing the Nehari manifold technique:

$$\mathbb{D}_T^{\alpha_1,\alpha_2;\varphi} \left(\left(\left| \frac{1 + \left|\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_1(x)}}{\sqrt{1 + \left|\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{2\kappa_1(x)}}} \right| \mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u \right|^{\kappa_1(x)-2} \mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u \right) \tag{1.5}$$

$$= |u|^{\alpha_2(x)-2}u + p(x)\mathcal{A}_u(x,u), \quad \text{in } \Theta,$$
$$u = 0 \quad \text{on } \partial\Theta.$$

Other notable contributions relevant to the analysis of problem (1.3) can be found in [28, 44, 46, 47, 49, 51].

Although we have not established the uniqueness of the solution in our study, it is important to highlight several works in the literature where the uniqueness has been rigorously proven, thereby providing valuable insights for further investigation.

In [30], the authors investigated the existence and uniqueness of solutions to the initial value problem involving the (k,φ)-Hilfer fractional derivative, employing Banach's fixed point theorem. The problem is formulated as follows:

$$\begin{cases} {}^{k,H}\mathbb{D}_{a^+}^{\alpha_1,\mu;\varphi} z(t) = g(t, z(t)), & t \in (a,b], \quad 0 < \alpha_1 < k, \quad 0 \le \mu \le 1, \\ {}^{k}I_{a^+}^{k-\alpha_1-\mu(k-\alpha_1);\varphi} z(a) = z_a \in \mathbb{R}, \end{cases} \tag{1.6}$$

where ${}^{k,H}\mathbb{D}^{\alpha_1,\mu;\varphi}$ denotes the (k,φ)-Hilfer fractional derivative of order α_1, g is a suitably defined real-valued function, and $\mu \in [0,1]$ is a continuous parameter.

In [43], Sitho et al. examined a more generalized system:

$$\begin{cases} \left({}^{H}\mathbb{D}^{\alpha_1,\mu;\varphi} + k\,{}^{H}\mathbb{D}^{\alpha_1-1,\mu;\varphi}\right) z(t) = g(t, z(t)), \\ \left({}^{H}\mathbb{D}^{\alpha_1;\mu} + k\,{}^{H}\mathbb{D}^{\alpha_1-1,\mu;\varphi}\right) z(t) \in g(t, z(t)), \quad t \in [a,b], \\ z(a) = 0, \quad z(b) = \displaystyle\sum_{i=1}^{q} \xi_i \int_a^{\vartheta_i} \varphi'(s) z(s)\, ds + \sum_{j=1}^{p} \theta_j z(\zeta_j), \end{cases} \tag{1.7}$$

where $1 < \alpha_1 \le 2$, ${}^{H}D^{\alpha_1,\mu;\varphi}$ is the φ-Hilfer fractional derivative of order α_1 and parameter μ, $0 < \mu \le 1$, and g is an appropriate function with a continuous derivative.

In [26], the authors established sufficient conditions ensuring the uniqueness of solutions for a class of nonlocal boundary value problems, formulated as:

$$\begin{cases} {}^{H}\mathbb{D}^{\alpha_1,\mu_1;\varphi}\left({}^{H}\mathbb{D}^{\alpha_1,\mu_2;\varphi} z(t)\right) + \lambda z(t) = g(t, z(t)), & a \le t \le b, \\ z(a) = 0, \quad z(b) = \displaystyle\sum_{i=1}^{N} \xi_i \left(I^{\alpha_1} z\right)(\vartheta), & a < \vartheta < b, \end{cases}$$

where $0 < \alpha_1 < 1$, $i = 1, 2$, ${}^{H}\mathbb{D}^{\alpha_1,\mu_i;\varphi}$ denotes the φ-Hilfer fractional derivative of order α_1 and parameter μ_i with $0 \le \mu_i \le 1$, $1 < \alpha_1 + \alpha_2 \le 2$, $\lambda \in \mathbb{R}$, and I^{α_1} represents the Riemann-Liouville fractional integral of order $\alpha_1 > 0$. The constants $\xi_i \in \mathbb{R}$ and g is a suitable nonlinear function.

This paper is structured as follows. In Section 2, we introduce the φ-Riemann-Liouville fractional partial integral and derivative, and define the φ-Hilfer fractional derivative (HFD) within the framework of Musielak-Orlicz-Sobolev spaces. We also review essential concepts including the log-Hölder continuity condition, Musielak-Orlicz-Sobolev spaces, and variable exponent Lebesgue spaces, which play a fundamental role in analyzing double-phase problems. Additionally, we present key propositions and theorems that will be utilized throughout the paper (see Proposition 2.3 and Proposition 2.4). Section 3 outlines the main hypotheses and states the principal results. Section 4 is dedicated to proving a collection of lemmas that provide essential support for the demonstration of our main results. Finally, in Section 5, we apply the developed framework to investigate double-phase problems in Musielak-Orlicz-Sobolev spaces, as demonstrated by Theorem 3.1.

2. Mathematical background

Let $n-1<\alpha_1<n$ with $n\in\mathbb{N}$, and let $I=[a,b]$ be an interval such that $-\infty\le a<b\le\infty$. Let $u,\varphi\in C^n([a,b],\mathbb{R})$ be two functions such that φ is increasing and $\varphi'(x)\neq 0$ for all $x\in I$. The left-sided and right-sided φ-Hilfer fractional derivatives of order α_1 and type $0\le\alpha_2\le 1$, denoted respectively by

$$^{H}\mathbb{D}_{a^+}^{\alpha_1,\alpha_2;\varphi}u(x) \quad \text{and} \quad ^{H}\mathbb{D}_{b^-}^{\alpha_1,\alpha_2;\varphi}u(x),$$

are defined as:

$$^{H}\mathbb{D}_{a^+}^{\alpha_1,\alpha_2;\varphi}u(x)=I_{a^+}^{\alpha_2(n-\alpha_1);\varphi}\left(\frac{1}{\varphi'(x)}\frac{d}{dx}\right)^n I_{a^+}^{(1-\alpha_2)(n-\alpha_1);\varphi}u(x),$$

and

$$^{H}\mathbb{D}_{b^-}^{\alpha_1,\alpha_2;\varphi}u(x)=I_{b^-}^{\alpha_2(n-\alpha_1);\varphi}\left(-\frac{1}{\varphi'(x)}\frac{d}{dx}\right)^n I_{b^-}^{(1-\alpha_2)(n-\alpha_1);\varphi}u(x),$$

where $I_{a^+}^{\cdot;\varphi}$ and $I_{b^-}^{\cdot;\varphi}$ denote the left- and right-sided fractional integrals with respect to the function φ given by

$$I_{a^+}^{\alpha_1;\varphi}u(x)=\frac{1}{\Gamma(\alpha_1)}\int_a^x\varphi'(t)\left(\varphi(x)-\varphi(t)\right)^{\alpha_1-1}u(t)\,dt,$$

and

$$I_{b^-}^{\alpha_1;\varphi}u(x)=\frac{1}{\Gamma(\alpha_1)}\int_x^b\varphi'(t)\left(\varphi(t)-\varphi(x)\right)^{\alpha_1-1}u(t)\,dt.$$

Let $\Theta=\prod_{i=1}^N[a_i,b_i]\subset\mathbb{R}^N$ be a rectangular domain, and let $u:\Theta\to\mathbb{R}$ be a sufficiently regular function. For each $i\in\{1,\dots,N\}$, we define the left-sided and right-sided fractional integrals of order $\alpha_1>0$ with respect to a strictly increasing function $\varphi\in C^1([a_i,b_i])$, with $\varphi'(x_i)\neq 0$, as follows:

$$I_{x_i^+}^{\alpha_1;\varphi}u(x)=\frac{1}{\Gamma(\alpha_1)}\int_{a_i}^{x_i}\varphi'(t)\left(\varphi(x_i)-\varphi(t)\right)^{\alpha_1-1}u(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_N)\,dt,$$

and

$$I_{x_i^-}^{\alpha_1;\varphi}u(x)=\frac{1}{\Gamma(\alpha_1)}\int_{x_i}^{b_i}\varphi'(t)\left(\varphi(t)-\varphi(x_i)\right)^{\alpha_1-1}u(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_N)\,dt.$$

The left- and right-sided φ-Hilfer fractional derivatives of order α_1 and type α_2 with respect to the variable x_i are defined as follows:

$$^{H}\mathbb{D}_{x_i^+}^{\alpha_1,\alpha_2;\varphi}u(x)=I_{x_i^+}^{\alpha_2(n-\alpha_1);\varphi}\left(\frac{1}{\varphi'(x_i)}\frac{\partial}{\partial x_i}\right)^n I_{x_i^+}^{(1-\alpha_2)(n-\alpha_1);\varphi}u(x),$$

and

$$^{H}\mathbb{D}_{x_i^-}^{\alpha_1,\alpha_2;\varphi}u(x)=I_{x_i^-}^{\alpha_2(n-\alpha_1);\varphi}\left(-\frac{1}{\varphi'(x_i)}\frac{\partial}{\partial x_i}\right)^n I_{x_i^-}^{(1-\alpha_2)(n-\alpha_1);\varphi}u(x).$$

Example 2.1 (Fractional derivative of power functions)**.** Let $u(x)=[\varphi(x)]^m$ with $m>-1$. Then the φ-Hilfer derivative is

$$^{H}\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}[\varphi(x)]^m=\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha_1)}[\varphi(x)]^{m-\alpha_1}. \tag{2.1}$$

Power-law functions are frequently used in modeling anomalous diffusion, where the mean squared displacement of particles scales as t^m. The fractional derivative introduces memory effects into this scaling, allowing for sub-diffusive or super-diffusive behavior. Similarly, in viscoelastic materials, stress or strain often follows a power-law creep model; the φ-Hilfer derivative captures the history-dependent response of the material. In heat conduction in fractal or porous media, temperature profiles can follow power laws, and using a φ-Hilfer derivative allows modeling nonlocal temporal effects and variable time scales.

Example 2.2 (Constant function)**.** For $u(x) = 1$, the φ-Hilfer derivative is

$$^{H}\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u(x) = \frac{[\varphi(x)-\varphi(0)]^{-\alpha_1}}{\Gamma(1-\alpha_1)}. \tag{2.2}$$

For $\varphi(x) = x$, we recover

$$^{H}\mathbb{D}_{0^+}^{\alpha,\beta;x}u(x) = \frac{x^{-\alpha_1}}{\Gamma(1-\alpha_1)}.$$

The fractional derivative of a constant function models the initial rate of change in systems with memory. In fractional relaxation processes, the initial state (e.g., initial stress or voltage) decays not exponentially but with a history-dependent power-law kernel. In heat conduction with memory, the singularity at the origin reflects the strong influence of the initial state on future dynamics, which is crucial for modeling materials with long-term memory effects, such as polymers or gels.

Example 2.3 (Exponential-type function)**.** For $u(x) = e^{\lambda\varphi(x)}$, the derivative is

$$^{H}\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}e^{\lambda\varphi(x)} = \lambda^{\alpha_1}e^{\lambda\varphi(x)}. \tag{2.3}$$

Exponential functions describe decay or growth in many classical physical systems. Using the φ-Hilfer derivative generalizes this to systems with memory. For example, in fractional RC circuits, the voltage across a capacitor may decay according to a fractional exponential law. In chemical kinetics, the concentration of a reactant may decrease in a non-Markovian way. In heat conduction problems, an exponential source term combined with a fractional derivative models materials where the temperature changes with both local effects and a history-dependent contribution.

Example 2.4 (Trigonometric functions)**.** For $u(x) = \sin(\lambda\varphi(x))$ or $u(x) = \cos(\lambda\varphi(x))$, we have

$$^{H}\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}\sin(\lambda\varphi(x)) = \lambda^{\alpha_1}\sin\Big(\lambda\varphi(x) + \frac{\alpha_1\pi}{2}\Big), \tag{2.4}$$

$$^{H}\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}\cos(\lambda\varphi(x)) = \lambda^{\alpha_1}\cos\Big(\lambda\varphi(x) + \frac{\alpha_1\pi}{2}\Big). \tag{2.5}$$

Sinusoidal functions are fundamental in modeling oscillatory phenomena. Fractional derivatives of sine and cosine introduce phase shifts and amplitude scaling, reflecting memory effects and energy dissipation. In viscoelastic or mechanical systems, fractional oscillators exhibit damping and frequency modification not captured by classical derivatives. In electrical circuits with fractional elements (like constant phase elements), the current or voltage response is sinusoidal but modified by the fractional order. Similarly, in wave propagation through complex media, fractional derivatives account for dispersive and nonlocal effects that alter the wave profile over time.

Finally, we set

$$ {}^{H}\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u(x) = \left({}^{H}\mathbb{D}_{x_1^+}^{\alpha_1,\alpha_2;\varphi}u(x), \dots, {}^{H}\mathbb{D}_{x_N^+}^{\alpha_1,\alpha_2;\varphi}u(x)\right) $$

and,

$$ {}^{H}\mathbb{D}_{-}^{\alpha_1,\alpha_2;\varphi}u(x) = \left({}^{H}\mathbb{D}_{x_1^-}^{\alpha_1,\alpha_2;\varphi}u(x), \dots, {}^{H}\mathbb{D}_{x_N^-}^{\alpha_1,\alpha_2;\varphi}u(x)\right). $$

In this paper, we consider a bounded Lipschitz domain in $\Theta \subset \mathbb{R}^N$ $(N \geq 2)$. Regarding the explanations and terminology that we will introduce next, we employ [10, 37, 38] and the cited sources.

We introduce the collection $C^+(\overline{\Theta})$ as

$$ C^+(\overline{\Theta}) = \left\{s : s \in C(\overline{\Theta}), \quad \beta(x) > 1 \quad \text{for a.e.} \quad x \in \overline{\Theta}\right\}. $$

Consider $C^+_{\log}(\overline{\Theta})$ the set of functions $\beta(\cdot) \in C^+(\overline{\Theta})$ that satisfy the log-Hölder continuity condition

$$ \sup\left\{|\beta(x) - \beta(y)| \log \frac{1}{|x-y|} : x, y \in \overline{\Theta}, \quad 0 < |x-y| < \frac{1}{2}\right\} < \infty. $$

For any $\beta(\cdot) \in C^+(\overline{\Theta})$, we define $\beta^+ = \sup_{x\in\Theta} \beta(x)$, $\beta^- = \inf_{x\in\Theta} \beta(x)$.

For a function $\beta(\cdot) \in C_+(\overline{\Theta})$ we define the Sobolev space with variable exponent as follows:

$$ L^{\beta(\cdot)}(\Theta) = \left\{u : \overline{\Theta} \mapsto \mathbb{R}, \int_{\Theta} |u|^{\beta(x)}\, dx < \infty\right\}, $$

endowed with the norm

$$ \|u\|_{L^{\beta(\cdot)}(\Theta)} = \inf\left\{\varsigma > 0 : \int_{\Theta} \left(\frac{|u|}{\varsigma}\right)^{\beta(x)} dx \leq 1\right\}. $$

Now we introduce the seminormed space variable exponent $\beta(\cdot)$ as follows:

$$ L^{\beta(\cdot)}_{\eta(\cdot)}(\Theta) = \left\{u : \overline{\Theta} \mapsto \mathbb{R}, \int_{\Theta} \eta(x)\, |u|^{\beta(x)}\, dx < \infty\right\} $$

and endow it with the seminorm

$$ \|u\|_{L^{\beta(\cdot)}_{\eta(\cdot)}(\Theta)} = \inf\left\{\tau > 0 : \int_{\Theta} \eta(x) \left(\frac{|u|}{\tau}\right)^{\beta(x)} dx \leq 1\right\}. $$

For $\kappa_1(\cdot), \kappa_2(\cdot) \in C_+(\overline{\Theta})$ and a positive function $\eta(\cdot)$ which satisfies the following conditions:

$$ 1 < \kappa_1(x) < \kappa_2(x) < N \quad \text{for all} \quad x \in \overline{\Theta}, \quad \text{and} \quad \eta(\cdot) \in L^1(\Theta). \tag{2.6} $$

We define the following Musielak-function given by:

$$ \mathcal{A}(x,t) = |t|^{\kappa_1(x)} + \eta(x)|t|^{\kappa_2(x)}, $$

and the associated Musielak space:

$$L_{\mathcal{A}}(\Theta)=\left\{u:\overline{\Theta}\mapsto\mathbb{R},\int_{\Theta}\mathcal{A}\left(x,\frac{u}{\sigma}\right)\,dx<\infty\quad\text{for somme }\sigma>0\right\},$$

endowed with the norm

$$\|u\|_{L_{\mathcal{A}}(\Theta)}=\inf\left\{\sigma>0:\int_{\Theta}\mathcal{A}\left(x,\frac{u}{\sigma}\right)\,dx\leq 1\right\}.$$

Note that this space is a Banach, uniformly convex, separable and reflexive space (see [10, 15, 36, 50]).

Bellow, we denote by c_i a generic positive constant.

Proposition 2.1. *1. Given $\mathcal{B},\mathcal{A}$, we say that $\mathcal{B}$ is weaker than $\mathcal{A}$, denoted by $\mathcal{B}\prec\mathcal{A}$, if there exists a non negative function $h\in L^1(\Theta)$ such that*

$$\mathcal{B}(x,t)\leq c_1\mathcal{A}(x,c_2t)+h(x),$$

for almost all $x\in Q$ and for all $t\in[0,\infty)$.

2. Let $\mathcal{B},\mathcal{A}$ be locally integrable with $\mathcal{B}\prec\mathcal{A}$. Then

$$L_{\mathcal{A}}(\Theta)\hookrightarrow L_{\mathcal{B}}(\Theta).$$

The Musielak Sobolev space associate the function $\mathcal{A}$ is defined as

$${}^{H}W^1L_{\mathcal{A}}(\Theta)=\left\{u\in L_{\mathcal{A}}(\Theta):\left|{}^{H}\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right|\in L_{\mathcal{A}}(\Theta)\right\},$$

endowed with the norm

$$\|u\|_{{}^{H}W^1L_{\mathcal{A}}(\Theta)}=\|u\|_{L_{\mathcal{A}}(\Theta)}+\left\|\left|{}^{H}\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right|\right\|_{L_{\mathcal{A}}(\Theta)}.$$

We consider the following space

$$\mathbb{H}=\left\{u\in{}^{H}W^1L_{\mathcal{A}}(\Theta):I_{x_i^+}^{1-\alpha_2;\varphi}u(x)=I_{x_i^-}^{1-\alpha_2;\varphi}u(x)=0,\quad\text{almost every }x\in\partial\Theta\right\}.$$

We present the fundamental proprieties of the space $\mathbb{H}$ in the following proposition.

Proposition 2.2. *The space $\mathbb{H}$ equipped with the norm induced by ${}^{H}W^1L_{\mathcal{A}}(\Theta)$ is a uniformly convex, separable, reflexive, Banach space.*

Proof. Since $L_{\mathcal{A}}(\Theta)$ is a reflexive and separable space, it follows the product of this space $(L_{\mathcal{A}}(\Theta))^{N+1}$ with respect to the norm

$$\|u\|_{(L_{\mathcal{A}}(\Theta))^{N+1}}=\sum_{i=0}^{N}\|u_i\|_{L_{\mathcal{A}}(\Theta)},\tag{2.7}$$

where $u=(u_0,u_1,\ldots,u_N)\in(L_{\mathcal{A}}(\Theta))^{N+1}$ is also reflexive and separable space. However, on the other side, let us consider the space $\Upsilon=\left\{\left(u,{}^{H}\mathbb{D}_{x_1^+}^{\alpha_1,\alpha_2;\varphi}u,\ldots,{}^{H}\mathbb{D}_{x_N^+}^{\alpha_1,\alpha_2;\varphi}u\right):u\in\mathbb{H}\right\}$, which is a closed subset of $(L_{\mathcal{A}}(\Theta))^{N+1}$ as $\mathbb{H}$ is complete. Hence, Υ is also a reflexive, separable Banach space in relation to the norm (2.7) for $u=(u_0,\ldots,u_N)\in\Upsilon$.

Define the operator $\Xi : \mathbb{H} \longrightarrow \Upsilon$ given by

$$\Xi(u) =: \left(u, {}^H\mathbb{D}^{\alpha_1,\alpha_2;\varphi}_{x_1^+} u, \dots, {}^H\mathbb{D}^{\alpha_1,\alpha_2;\varphi}_{x_N^+} u\right), \quad u \in \mathbb{H}.$$

Therefore, it follows that $\|u\|_{\mathbb{H}} = \|\Xi(u)\|_{(L_{\mathcal{A}}(\Theta))^{N+1}}$, which means that the operator Ξ is a isometric isomorphic mapping, which implies that the space $\mathbb{H}$ is isometric to the space Υ. Consequently, $\mathbb{H}$ is a separable Banach space and reflexive, which concludes the proof. □

Proposition 2.3. *Let hypotheses* (2.6) *be satisfied and let us we set $\widehat{\rho}_{\mathcal{A}}$ be defined by as*

$$\widehat{\rho}_{\mathcal{A}}(u) = \int_{\Theta} \left[\mathcal{A}\left(x, u\right) + \mathcal{A}\left(x, \left|{}^H\mathbb{D}^{\alpha_1,\alpha_2;\varphi}_{+} u\right|\right)\right] \, dx.$$

Then, the following relations hold true:

1. *If $u \neq 0$, then $\|u\|_{\mathbb{H}} = \tau$ if and only if $\widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right) = 1$.*
2. *If $\|u\|_{\mathbb{H}} < 1(=, > 1)$ if and only if $\widehat{\rho}_{\mathcal{A}}(u) < 1(=, > 1)$.*
3. *If $\|u\|_{\mathbb{H}} > 1$, then $\|u\|_{\mathbb{H}}^{\kappa_1^-} \leq \widehat{\rho}_{\mathcal{A}}(u) \leq \|u\|_{\mathbb{H}}^{\kappa_2^+}$.*
4. *If $\|u\|_{\mathbb{H}} < 1$, then $\|u\|_{\mathbb{H}}^{\kappa_2^+} \leq \widehat{\rho}_{\mathcal{A}}(u) \leq \|u\|_{\mathbb{H}}^{\kappa_1^-}$.*
5. *$\|u_n - u\|_{\mathbb{H}} \longrightarrow 0$ if and only if $\widehat{\rho}_{\mathcal{A}}(u_n - u) \longrightarrow 0$.*

Proof.

1. Let $u \in \mathbb{H}$. It follows that the function $\widehat{\rho}_{\mathcal{A}}(\tau u)$ is convexity, non-negative of is continuous, and non-decreasing when $\tau \in [0, \infty)$ for all $u \in \mathbb{H}$. This, given by the definition of $\widehat{\rho}_{\mathcal{A}}$, we obtain that

$$\|u\|_{\mathbb{H}} = \tau \Longleftrightarrow \widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right).$$

2. Moreover, we know that $\widehat{\rho}_{\mathcal{A}}$ is continuous function

$$\begin{aligned} \widehat{\rho}_{\mathcal{A}}(\tau u) = \widehat{\rho}_{\mathcal{A}}(\tau u) \leq \tau \widehat{\rho}_{\mathcal{A}}(u) \quad \text{for all} \quad \tau \leq 1, \\ \widehat{\rho}_{\mathcal{A}}(\tau u) = \widehat{\rho}_{\mathcal{A}}(\tau u) \geq \tau \widehat{\rho}_{\mathcal{A}}(u) \quad \text{for all} \quad \tau \geq 1. \end{aligned} \tag{2.8}$$

If $\|u\|_{\mathbb{H}} \leq 1$, then $\widehat{\rho}_{\mathcal{A}}(u) \leq \|u\|_{\mathbb{H}}$. The claim is obvious for u = 0, so let us assume that $0 < \|u\|_{\mathbb{H}} \leq 1$, $\|u\|_{\mathbb{H}} < 1$, then there exists $\tau < 1$ with $\widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right) \leq 1$, and $\left\|\frac{u}{\|u\|_{\mathbb{H}}}\right\|_{\mathbb{H}} = 1$ it follows that $\widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\|u\|_{\mathbb{H}}}\right) \leq 1$. Since $\|u\|_{\mathbb{H}} \leq 1$. Hence by the first this equation (2.8) it follows that $\widehat{\rho}_{\mathcal{A}}(u) \leq \tau \widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right) \leq \tau < 1$ this given us this $\frac{\widehat{\rho}_{\mathcal{A}}(u)}{\|u\|_{\mathbb{H}}} \leq 1$.
If $1 < \|u\|_{\mathbb{H}}$, then $\|u\|_{\mathbb{H}} \leq \widehat{\rho}_{\mathcal{A}}(u)$. Assume that $\|u\|_{\mathbb{H}} > 1$, then $\widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right) > 1$ for $1 < \tau < \|u\|_{\mathbb{H}}$ and by the second of the equation (2.8) it follows that $1 < \widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right)$. Since τ was arbitrary, $\widehat{\rho}_{\mathcal{A}}(u) > \|u\|_{\mathbb{H}}$.

3. Let us know for $u \in \mathbb{H}$ we obtain the following inequalities

$$\begin{aligned} \lambda^{\kappa_1^-} \widehat{\rho}_{\mathcal{A}}(u) \leq \widehat{\rho}_{\mathcal{A}}(\lambda u) \leq \lambda^{\kappa_2^+} \widehat{\rho}_{\mathcal{A}}(u) \quad \text{if} \quad \lambda > 1, \\ \lambda^{\kappa_2^+} \widehat{\rho}_{\mathcal{A}}(u) \leq \widehat{\rho}_{\mathcal{A}}(\lambda u) \leq \lambda^{\kappa_1^-} \widehat{\rho}_{\mathcal{A}}(u) \quad \text{if} \quad 0 < \lambda < 1. \end{aligned} \tag{2.9}$$

Let $\|u\|_{\mathbb{H}} = \tau$ with $0 < \tau < 1$. Then, we have $\widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right) = 1$ from the first proposition. Since $\frac{1}{\tau} > 1$ we can apply the first inequality in (2.9) in order to obtain

$$\frac{\widehat{\rho}_{\mathcal{A}}(u)}{\tau^{\kappa_1^-}} \leq \widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right) = 1 \leq \frac{\widehat{\rho}_{\mathcal{A}}(u)}{\tau^{\kappa_2^+}}.$$

4. If $\|u\|_{\mathbb{H}} < 1$ with $0 < \tau < 1$, for the defination $\widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right) = 1$ for the $\|u\|_{\mathbb{H}} = \tau$, then we take the second equation for the inequality in (2.9) with $0 < \tau < 1$ this meaning by the following

$$\frac{\widehat{\rho}_{\mathcal{A}}(u)}{\tau^{\kappa_2^+}} \leq \widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\tau}\right) = 1 \leq \frac{\widehat{\rho}_{\mathcal{A}}(u)}{\tau^{\kappa_1^-}}.$$

5. For the ultimate proposition when $\|u_n - u\|_{\mathbb{H}} \longrightarrow 0$ this means $u_n \longrightarrow u \in \mathbb{H}$, are to gether non negative in $\mathbb{H}$, that follows $\widehat{\rho}_{\mathcal{A}}(u_n - u) \longrightarrow 0$, we know $\kappa_1(x) < \kappa_2(x)$ and the usual embedding $\|u_n - u\|_{\mathbb{H}} \longrightarrow 0$, $u_n \longrightarrow u$ a.e. using a subselection (designated by u_n). Conversely, as

$$\begin{aligned} &\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_1(x)} + \eta(x)\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_2(x)} + |u_n|^{\kappa_1(x)} + \eta(x)\,|u_n|^{\kappa_2(x)} \\ \leq & 2^{\kappa_2^+}\begin{pmatrix} \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n - {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_1(x)} + \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_1(x)} \\ +\eta(x)\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n - {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_2(x)} + \eta(x)\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_2(x)} \\ +\,|u_n - u|^{\kappa_1(x)} + |u|^{\kappa_1(x)} + \eta(x)\,|u_n - u|^{\kappa_2(x)} + \eta(x)\,|u|^{\kappa_2(x)} \end{pmatrix} \end{aligned}$$

and there holds $\widehat{\rho}_{\mathcal{A}}(u_n - u) \longrightarrow 0$, we know that

$$\left\{\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_1(x)} + \eta(x)\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_2(x)} + |u_n|^{\kappa_1(x)} + \eta(x)\,|u_n|^{\kappa_2(x)}\right\}_{n\in\mathbb{N}},$$

is a uniformly integrable sequence, which furthermore converges almost every where to

$$\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_1(x)} + \eta(x)\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_2(x)} + |u|^{\kappa_1(x)} + \eta(x)\,|u|^{\kappa_2(x)},$$

by the almost every convergence of $u_n \longrightarrow u$. By [37, Theorem 1.4] it follows that $\widehat{\rho}_{\mathcal{A}}(u_n) \longrightarrow \widehat{\rho}_{\mathcal{A}}(u)$ through this subsequence.

□

Proposition 2.4. *Let $r(\cdot) \in \mathcal{C}(\overline{\Theta})$ be a continuous function satisfying $1 < r(x) \leq \kappa_1(x) < N$ for all $x \in \overline{\Theta}$. We define the functions $\overline{\kappa_1}$ and $\underline{\kappa_1}$ pointwise on $\overline{\Theta}$ as follows:*

$$\overline{\kappa_1}(x) = \begin{cases} \dfrac{N\kappa_1(x)}{N - \kappa_1(x)}, & \text{if } \kappa_1(x) < N, \\ \infty, & \text{if } \kappa_1(x) \geq N, \end{cases} \quad \text{and} \quad \underline{\kappa_1}(x) = \begin{cases} \dfrac{(N-1)\kappa_1(x)}{N - \kappa_1(x)}, & \text{if } \kappa_1(x) < N, \\ \infty, & \text{if } \kappa_1(x) \geq N. \end{cases} \tag{2.10}$$

Then the following embedding results hold:

1. *The embedding $L_{\mathcal{A}}(\Theta) \hookrightarrow L^{r(\cdot)}(\Theta)$ is continuous. Similarly, the embedding $\mathbb{H} \hookrightarrow W_0^{1,r(\cdot)}(\Theta)$ is continuous for every $r(\cdot) \in \mathcal{C}(\overline{\Theta})$ with $1 \leq r(x) \leq \kappa_1(x)$ for all $x \in \overline{\Theta}$.*

2. *The embedding* $\mathbb{H} \hookrightarrow L^{r(\cdot)}(\Theta)$ *is compact under the same conditions on* $r(\cdot)$. *Moreover, there exists a constant* $c_3 > 0$ *such that*

$$\|u\|_{L^{r(\cdot)}(\Theta)} \le c_3 \|u\|_{\mathbb{H}} \quad \textit{for all } u \in \mathbb{H}.$$

3. *The embedding* $L_{\mathcal{A}}(\Theta) \hookrightarrow L^{\kappa_2(\cdot)}_{\eta(\cdot)}(\Theta)$ *is continuous.*

4. *Suppose that* $r(\cdot) : \overline{\Theta} \to (1, \infty)$ *satisfies* $1 \le r(x) \le \underline{\kappa_1}(x) < \overline{\kappa_1}(x) \quad$ *for all* $x \in \overline{\Theta}$., *Then the following embedding is compact:*

$$\mathbb{H} \hookrightarrow L_{\mathcal{A}}(\Theta). \tag{2.11}$$

Proof.

1. We take $\Psi_{\kappa_1(\cdot)}(x,t) = t^{\kappa_1(x)}$ for all $t \ge 0$ and for all $x \in \overline{\Theta}$. It is easy to see that $\Psi_{\kappa_1(\cdot)} \prec \mathcal{A}$, see Proposition 2.3 (i). Hence, from Proposition 2.3 (ii) we obtain that $L_{\mathcal{A}}(\Theta) \hookrightarrow L^{\kappa_1(\cdot)}(\Theta)$ continuously, and by definition it follows that $\mathbb{H} \hookrightarrow W_0^{1,\kappa_1(\cdot)}(\Theta)$ continuously. Thus, assertion (i) is a direct consequence of the classical embedding results for variable Lebesgue and Sobolev spaces due to the boundedness of Q.
So we have continuous embedding $L_{\mathcal{A}}(\Theta) \hookrightarrow L^{r(\cdot)}(\Theta)$, for any measurable $r(x)$ which satisfies (2.10), let $u \in L_{\mathcal{A}}(\Theta)$. We have

$$\int_{\Theta} |u|^{r(x)} \, dx \le \int_{\Theta} \left(|u|^{\kappa_1(x)} + \eta(x) \, |u|^{\kappa_2(x)} \right) dx = \int_{\Theta} \mathcal{A}\,(x,u) \, dx < \infty,$$

and so we have $u \in L^{r(\cdot)}(\Theta)$. This means that $L_{\mathcal{A}}(\Theta) \subset L^{r(\cdot)}(\Theta)$.
Similar also for $\mathbb{H} \hookrightarrow W_0^{1,r(\cdot)}(\Theta)$ is continuous, with $u \in \mathbb{H}$. We obtain

$$\begin{aligned}
&\int_{\Theta} \left| {}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi} u \right|^{r(x)} dx + \int_{\Theta} |u|^{r(x)} \, dx \\
\le &\int_{\Theta} \left(\left| \mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi} u \right|^{\kappa_1(x)} + \eta(x) \left| {}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi} u \right|^{\kappa_2(x)} \right) dx + \int_{\Theta} \left(|u|^{\kappa_1(x)} + \eta(x) \, |u|^{\kappa_2(x)} \right) dx \\
= &\widehat{\rho}_{\mathcal{A}}(u) \\
< &\infty,
\end{aligned}$$

and so we have $u \in W_0^{1,r(\cdot)}(\Theta)$. This means that $W_0^1 L_{\mathcal{A}}(\Theta) \subset W_0^{1,r(\cdot)}(\Theta)$.

2. Since the embedding $\mathbb{H} \hookrightarrow L^{r(\cdot)}(\Theta)$ is continuous, let $\{u_n\}_n \subset \mathbb{H}$ be a bounded sequence. Then $\{u_n\}_n$ is also bounded in $L^{r(\cdot)}(\Theta)$. As $L^{r(\cdot)}(\Theta)$ is reflexive, there exists a subsequence $\{u_{n_k}\}_n$ that converges weakly to some function $u \in L^{r(\cdot)}(\Theta)$. Consequently, the embedding $\mathbb{H} \hookrightarrow L^{r(\cdot)}(\Theta)$ is compact.

3. To this end, let $u \in L_{\mathcal{A}}(\Theta)$, then we have

$$\int_{\Theta} \eta(x) \, |u|^{\kappa_2(x)} \, dx \le \int_{\Theta} \left(|u|^{\kappa_1(x)} + \eta(x) \, |u|^{\kappa_2(x)} \right) dx = \int_{\Theta} \mathcal{A}\,(x, u(x)) \, dx.$$

Since $\int_{\Theta} \mathcal{A}\left(x, \frac{u}{\|u\|_{L_{\mathcal{A}}(\Theta)}}\right) = 1$ whenever $u \neq 0$, we obtain for $u \neq 0$

$$\int_{\Theta} \eta(x) \left(\frac{u}{\|u\|_{L_{\mathcal{A}}(\Theta)}} \right)^{\kappa_2(x)} dx \le 1.$$

Thus

$$\|u\|_{L^{\kappa_2(\cdot)}_{\eta(\cdot)}(\Theta)} \leq \|u\|_{L_{\mathcal{A}}(\Theta)} .$$

5. For this, we take $u \in \mathbb{H}$, then we have

$$\begin{aligned}
&\int_{\Theta} \mathcal{A}\left(x, u(x)\right) dx \\
=&\int_{\Theta} \left(|u|^{\kappa_1(x)} + \eta(x) \, |u|^{\kappa_2(x)}\right) dx \\
\leq&\int_{\Theta} \left(\left|{}^H\mathbb{D}^{\alpha_1,\alpha_2;\varphi}_{+} u\right|^{\kappa_1(x)} + \eta(x) \left|{}^H\mathbb{D}^{\alpha_1,\alpha_2;\varphi}_{+} u\right|^{\kappa_2(x)}\right) dx + \int_{\Theta} \left(|u|^{\kappa_1(x)} + \eta(x) \, |u|^{\kappa_2(x)}\right) dx \\
=&\widehat{\rho}_{\mathcal{A}}(u).
\end{aligned}$$

We have $\widehat{\rho}_{\mathcal{A}}\left(\frac{u}{\|u\|_{L_{\mathcal{A}}(\Theta)}}\right) = 1$ whenever $u \neq 0$, we obtain for $u \neq 0$

$$\int_{\Theta} \mathcal{A}\left(x, \frac{u(x)}{\tau}\right) dx \leq 1 \quad \text{for some} \quad \tau > 0.$$

In addition,

$$\|u\|_{L_{\mathcal{A}}(\Theta)} \leq \|u\|_{{}^H W^1_0 L_{\mathcal{A}}(\Theta)} .$$

□

Proposition 2.5. ([40]). *Consider a real Banach space $\mathbb{Y}$ and its dual $\mathbb{Y}^*$. Assume that $\Lambda \in \mathcal{C}^1(\mathbb{Y}, \mathbb{R})$ satisfies the condition*

$$\max\left(\Lambda(0), \Lambda(e)\right) \leq \nu \leq \inf_{\|u\|_{\mathbb{Y}}=\rho} \Lambda(u),$$

for some $\vartheta < \nu$, $\rho > 0$, and $e \in \mathbb{Y}$ with $\|e\|_{\mathbb{Y}} > \rho$. Let $c_4 \leq \nu$ be such that

$$c_4 = \inf_{\gamma\in\Omega} \max_{\tau\in[0,1]} \Lambda(\gamma(\tau)),$$

where $\Omega = \{\gamma \in \mathcal{C}([0, 1], \mathbb{Y}) : \gamma(0) = \gamma(1) = e\}$ is the set of continuous paths joining 0 and e. Then, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{Y}$ such that

$$\Lambda(u_n) \longrightarrow c_4 \geq \nu \quad \textit{and} \quad (1 + \|u_n\|_{\mathbb{Y}}) \left\|\Lambda'(u_n)\right\|_{\mathbb{Y}^*} \longrightarrow 0.$$

3. Essential hypotheses

This section is devoted to the basic hypothesis on the data to find the required results. Let $g : \Theta \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with the potential $G(x, z) = \int_0^z g(x, t)dt$, that satisfies the hypotheses below.

1. The function g is assumed to satisfy the following two conditions:

(i) There exists a constant $c_5 > 0$ and an exponent q such that $\kappa_2^+ < q < \overline{\kappa_2}(x)$ for every $x \in \Theta$, and for all $(x, s) \in \Theta \times \mathbb{R}$, the following growth condition holds:

$$|g(x, s)| \leq c_5 |s|^q.$$

Additionally, the function g vanishes for non-positive arguments, i.e., $g(x, t) = 0$ for all $x \in \Theta$ and $t \leq 0$, in particular $g(x, 0) = 0$.

(ii) There exist constants $c_6 > 0$ and $\delta > 0$ such that the primitive function $G(x, t) = \int_0^t g(x, \tau)\, d\tau$ satisfies the inequality

$$G(x, t) \geq c_6 |t|^p \quad \text{for all } x \in \Theta \text{ and } |t| < \delta,$$

where the exponent p satisfies $p < \kappa_1^-$.

2. We assume the following assumptions:

$$\lim_{t \to 0} \frac{|g(x, t)|}{|t|^{\kappa_1^- - 1}} = l_1 < \infty, \quad \text{and} \quad \lim_{t \to \infty} \frac{|g(x, t)t|}{|t|^{\kappa_2^+}} = \infty,$$

uniformly for $x \in \Theta$.

3. For almost every where $x \in \Theta$, the function $t \longrightarrow \frac{|g(x,t)|}{|t|^{\kappa_2^+ - 1}}$, is increasing with respect to $t \geq 0$.

4. Assume the following estimation:

$$\lim \sup_{|t| \to +\infty} \left(\frac{G(x, t)}{\frac{1}{\kappa_1(x)} |t|^{\kappa_1(x)} + \frac{\eta(x)}{\kappa_2(x)} |t|^{\kappa_2(x)}} \right) < \zeta_1,$$

uniformly for a.e. $x \in \Theta$, with

$$\zeta_1 = \inf_{u \in \mathbb{H}} \frac{\int_\Theta \frac{1}{\kappa_1(x)} \left(\left| \mathbb{D}_+^{\alpha_1, \alpha_2; \varphi} u \right|^{\kappa_1(x)} + |u|^{\kappa_1(x)} \right) dx + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} \left(\left| \mathbb{D}_+^{\alpha_1, \alpha_2; \varphi} u \right|^{\kappa_2(x)} + |u|^{\kappa_2(x)} \right) dx}{\int_\Theta \frac{1}{\kappa_1(x)} |u|^{\kappa_1(x)}\, dx + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} |u|^{\kappa_2(x)}\, dx} > 0.$$

In what follows we provide the definition of weak solution that will be considered for (1.2).

Definition 3.1. We say that $u \in \mathbb{H}$ is a weak solution of (1.2), if for every $v \in \mathbb{H}$ the following holds

$$\begin{aligned} &\int_\Theta \left| {}^H\mathbb{D}_+^{\alpha_1, \alpha_2; \varphi} u \right|^{\kappa_1(x) - 2} {}^H\mathbb{D}_+^{\alpha_1, \alpha_2; \varphi} u \, {}^H\mathbb{D}_+^{\alpha_1, \alpha_2; \varphi} v \, dx + \int_\Theta |u|^{\kappa_1(x) - 2} u\, v\, dx \\ &+ \int_\Theta \eta(x) \left| {}^H\mathbb{D}_+^{\alpha_1, \alpha_2; \varphi} u \right|^{\kappa_2(x) - 2} {}^H\mathbb{D}_+^{\alpha_1, \alpha_2; \varphi} u \, {}^H\mathbb{D}_{0^+}^{\alpha_1, \alpha_2; \varphi} v \, dx \\ &+ \int_\Theta \eta(x) |u|^{\kappa_2(x) - 2} u\, v\, dx - \int_\Theta g(x, u) v\, dx = 0. \end{aligned}$$

Now, let us introduce the energy functional $\Lambda : \mathbb{H} \longrightarrow \mathbb{R}$ associated to problem (1.2), which is defined as

$$\Lambda(u) = \int_\Theta \frac{1}{\kappa_1(x)} \left| {}^H\mathbb{D}_+^{\alpha_1, \alpha_2; \varphi} u \right|^{\kappa_1(x)} dx + \int_\Theta \frac{1}{\kappa_1(x)} |u|^{\kappa_1(x)}\, dx + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} \left| {}^H\mathbb{D}_+^{\alpha_1, \alpha_2; \varphi} u \right|^{\kappa_2(x)} dx$$

$$+\int_{\Theta}\frac{\eta(x)}{\kappa_2(x)}|u|^{\kappa_2(x)}\,dx-\int_{\Theta}G(x,u)\,dx. \tag{3.1}$$

We have $\Lambda\in C^1(\mathbb{H},\mathbb{R})$ and it is noteworthy that the critical points of Λ correspond to weak solutions of (1.2) and its Gâteaux derivative is

$$\begin{aligned}\langle\Lambda'(u),v\rangle=&\int_{\Theta}\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_1(x)}v\,dx+\int_{\Theta}|u|^{\kappa_1(x)}v\,dx+\int_{\Theta}\eta(x)\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_2(x)}v\,dx\\&+\int_{\Theta}\eta(x)|u|^{\kappa_2(x)}v\,dx-\int_{\Theta}g(x,u)v\,dx.\end{aligned}$$

4. Somme technical lemmas

This section is devoted to establishing several technical lemmas that will be instrumental in the proof of the main results. These intermediate results provide the necessary analytical groundwork for addressing the challenges posed by equation (1.2).

Lemma 4.1. *If conditions 1-3 are satisfied, then the following assertions hold:*
(**i**) *There exists $v\in\mathbb{H}$ with $v>0$ such that $\Lambda(tv)\longrightarrow-\infty$ as $t\longrightarrow\infty$.*
(**ii**) *There exist $\vartheta,\nu>0$ such that $\Lambda(u)\geq\nu$ for all $u\in\mathbb{H}$ with $\|u\|_{\mathbb{H}}=\vartheta$.*

Proof. (**i**) Using these conditions, it can be inferred that for all $c_7>0$, there exist $\delta>0$, such that

$$G(x,t)>c_7|t|^{\kappa_2^+}\quad\text{for all}\quad x\in\Theta,\quad\text{and}\quad|t|>\delta. \tag{4.1}$$

Let $t>1$ large enough and $v\in\mathbb{H}$ with $v>0$. From (4.1), one has

$$\begin{aligned}\Lambda(tv)=&\int_{\Theta}\frac{1}{\kappa_1(x)}\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi}tv\right|^{\kappa_1(x)}dx+\int_{\Theta}\frac{1}{\kappa_1(x)}|tv|^{\kappa_1(x)}dx\\&+\int_{\Theta}\frac{\eta(x)}{\kappa_2(x)}\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi}tv\right|^{\kappa_2(x)}dx+\int_{\Theta}\frac{\eta(x)}{\kappa_2(x)}|tv|^{\kappa_2(x)}dx-\int_{\Theta}G(x,tv)\,dx\\\leq&\frac{t^{\kappa_2^+}}{\kappa_1^-}\left(\int_{\Theta}\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi}v\right|^{\kappa_1(x)}dx+\int_{\Theta}|v|^{\kappa_1(x)}dx+\int_{\Theta}\eta(x)\left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi}v\right|^{\kappa_2(x)}dx\right.\\&\left.+\int_{\Theta}\eta(x)|v|^{\kappa_2(x)}dx\right)-\int_{|tv|\geq\delta}G(x,tv)\,dx-\int_{|tv|<\delta}G(x,tv)\,dx\\\leq&\frac{t^{\kappa_2^+}}{\kappa_1^-}\widehat{\rho}_{\mathcal{A}}(v)-c_7t^{\kappa_2^+}\int_{\Theta}|v|^p\,dx\,dx-\int_{|tv|\leq\delta}G(x,tv)\,dx\\\leq&\frac{t^{\kappa_2^+}}{\kappa_1^-}\widehat{\rho}_{\mathcal{A}}(v)-c_7t^{\kappa_2^+}\int_{\Theta}|v|^p\,dx-c_8\\\leq&c_9t^{\kappa_2^+}-c_{10}t^{\kappa_2^+}-c_8.\end{aligned}$$

Choosing c_{10} to be sufficiently large to guarantee a specific condition

$$c_9t^{\kappa_2^+}-c_{10}t^{\kappa_2^+}-c_8<0,$$

we have that

$$\Lambda(tv)\longrightarrow-\infty\quad\text{as}\quad t\longrightarrow+\infty,$$

which concludes the proof of (**i**).

(**ii**) Thus, for $\|u\|_{\mathbb{H}} < 1$, one has

$$\begin{aligned}\Lambda(u) &= \int_\Theta \frac{1}{\kappa_1(x)} \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_1(x)} dx + \int_\Theta \frac{1}{\kappa_1(x)} |u|^{\kappa_1(x)} dx + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_2(x)} dx \\ &\quad + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} |u|^{\kappa_2(x)} dx - \int_\Theta G(x,u)\,dx \\ &\geq \frac{1}{\kappa_1^+} \int_\Theta \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_1(x)} dx + \frac{1}{\kappa_1^+} \int_\Theta |u|^{\kappa_1(x)} dx + \frac{1}{\kappa_2^+} \int_\Theta \eta(x) \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_2(x)} dx \\ &\quad + \frac{1}{\kappa_2^+} \int_\Theta \eta(x) |u|^{\kappa_2(x)} dx - \int_\Theta G(x,u)\,dx \\ &\geq \frac{1}{\kappa_2^+} \|u\|_{\mathbb{H}}^{\kappa_2^+} - \frac{1}{\kappa_1^-} c_5^{\kappa_1^-} \|u\|_{L^q(\Theta)}^q \\ &\geq c_{11} \|u\|_{\mathbb{H}}^{\kappa_2^+} - c_{12} \|u\|_{\mathbb{H}}^q .\end{aligned}$$

Since $\kappa_2^+ < q$, for sufficiently small values of ϑ, with $c_{12} > 0$, we choose $\nu > 0$ such that

$$\Lambda(u) \geq \nu, \quad \text{for all} \quad u \in \mathbb{H} \quad \text{with} \quad \|u\|_{\mathbb{H}} = \vartheta.$$

Which finished the proof. □

Lemma 4.2. *Given the hypotheses 1 and 3, and a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ such that*

$$\left\langle \Lambda'(u), u \right\rangle \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty, \tag{4.2}$$

then there is a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that for all $t > 0$, it holds

$$\Lambda(tu_n) \leq \frac{t^{\kappa_1^-}}{\kappa_1^-} \left[\frac{1}{n} + \int_\Theta g(x,u_n)u_n\,dx\right] - \int_\Theta G(x,tu_n)\,dx.$$

Proof. Let ψ be a function such that

$$\psi(t) = \frac{t^{\kappa_1^-}}{\kappa_1^-} g(x,u_n)u_n - G(x,tu_n).$$

Thus,

$$\begin{aligned}\psi'(t) &= t^{\kappa_1^- - 1} g(x,u_n)u_n - g(x,tu_n)u_n \\ &= t^{\kappa_1^- - 1} u_n \left(g(x,u_n) - \frac{g(x,tu_n)}{t^{\kappa_1^- - 1}}\right),\end{aligned}$$

which implies that $\psi'(t) \geq 0$ for $t \in]0,1]$, and $\psi'(t) \leq 0$ when $t \geq 1$, which leads to

$$\psi(t) \leq \psi'(1), \quad \text{for all} \quad t > 0. \tag{4.3}$$

According to (4.2), one has

$$\left|\left\langle \Lambda'(u_n), u_n \right\rangle\right| < \frac{1}{n}.$$

Hence,

$$\begin{aligned}
-\frac{1}{n} &< \left\langle \Lambda'(u_n), u_n \right\rangle \\
&= \int_\Theta \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_1(x)} dx + \int_\Theta |u_n|^{\kappa_1(x)} dx + \int_\Theta \eta(x) \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_2(x)} dx \\
&\quad + \int_\Theta \eta(x) |u_n|^{\kappa_2(x)} dx - \int_\Theta g(x,u_n)u_n \, dx \\
&< \frac{1}{n}.
\end{aligned} \tag{4.4}$$

By utilizing (4.3) and (4.4), we derive

$$\begin{aligned}
\Lambda(tu_n) &= \int_\Theta \frac{1}{\kappa_1(x)} \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} tu_n\right|^{\kappa_1(x)} dx + \int_\Theta \frac{1}{\kappa_1(x)} |tu_n|^{\kappa_1(x)} dx \\
&\quad + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} tu_n\right|^{\kappa_2(x)} dx + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} |tu_n|^{\kappa_2(x)} dx - \int_\Theta G(x,tu_n)\, dx \\
&\leq \frac{t^{\kappa_1^-}}{\kappa_1^-} \left[\frac{1}{n} + \int_\Theta g(x,u_n)u_n \, dx\right] - \int_\Theta G(x,tu_n)\, dx,
\end{aligned} \tag{4.5}$$

which completes the proof. □

5. Main result

In what follows, we present the main results of this work, along with their proofs.

Theorem 5.1. **(i)** *Under hypotheses 1-3, problem* (1.2) *admits at least one nontrivial solution in* $\mathbb{H}$ *within the setting of Musielak-Orlicz-Sobolev spaces for double phase problems.*

(ii) *Under hypotheses 1, and 4, problem* (1.2) *also has at least one nontrivial solution in* $\mathbb{H}$ *the same functional space.*

Proof. 3.1 (**i**) If $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ satisfy Proposition 2.5, then

$$\begin{aligned}
\Lambda(u_n) &= \int_\Theta \frac{1}{\kappa_1(x)} \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_1(x)} dx + \int_\Theta \frac{1}{\kappa_1(x)} |u_n|^{\kappa_1(x)} dx \\
&\quad + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_2(x)} dx + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} |u_n|^{\kappa_2(x)} dx - \int_\Theta G(x,u_n)\, dx \\
&= c_4 + o(1),
\end{aligned}$$

and

$$(1 + \|u_n\|_{\mathbb{H}}) \left\|\psi'(u_n)\right\|_{(\mathbb{H})^*} \longrightarrow 0.$$

Hence,

$$\|u_n\|_{\mathbb{H}} - \int_\Theta g(x,u_n)u_n \, dx = o(1).$$

Moreover,

$$\begin{aligned}
&\int_\Theta \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_1(x)-2} {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n \, {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} v \, dx + \int_\Theta |u_n|^{\kappa_1(x)-2} u_n \, v \, dx \\
&+ \int_\Theta \eta(x) \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n\right|^{\kappa_2(x)-2} {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n \, {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} v \, dx \\
&+ \int_\Theta \eta(x) |u_n|^{\kappa_2(x)-2} u_n \, v \, dx - \int_\Theta g(x, u_n) \, v \, dx \\
&= o(1), \quad \text{for all} \quad v \in \mathbb{H}.
\end{aligned}$$

Claim 1. *The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{H}$.*

Let us define

$$t_n = \frac{\left(\kappa_2^+ c_4\right)^{\frac{1}{\kappa_2^+}}}{\|u_n\|_{\mathbb{H}}} > 0 \quad \text{and} \quad \mathbf{v}_n = t_n u_n.$$

Since $\|\mathbf{v}_n\| = \left(\kappa_2^+ c_4\right)^{\frac{1}{\kappa_2^+}}$, then $\mathbf{v}_n$ is bounded in $\mathbb{H}$. Hence, up to a subsequence still denoted by $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$, we have

$$\begin{cases}
\mathbf{v}_n \longrightarrow \mathbf{v} \quad \text{in} \quad \mathbb{H}, \\
\mathbf{v}_n \longrightarrow \mathbf{v} \quad \text{in} \quad L^{r(\cdot)}(\Theta), \quad \text{for} \quad r(x) \in \left(1, \max\left\{\overline{\kappa_2}(x), \kappa_2(x)\right\}\right), \\
\mathbf{v}_n \longrightarrow \mathbf{v} \quad \text{in} \quad \Theta.
\end{cases}$$

If $\|u_n\| \longrightarrow \infty$, we obtain $\mathbf{v} \equiv 0$. In fact, let

$$\Theta_1 = \{x \in \Theta : \mathbf{v}(x) = 0\} \quad \text{and} \quad \Theta_2 = \{x \in \Theta : \mathbf{v}(x) \neq 0\}.$$

Since $|u_n| = |\mathbf{v}_n| \, \|u_n\| \left(\kappa_2^+ c_4\right)^{-\frac{1}{\kappa_2^+}}$, it follows that $|u_n| \longrightarrow \infty$ almost everywhere in Θ_2. Based on hypothesis 2 and for a sufficiently large n, we deduce that

$$\frac{g\left(x, u_n\right) u_n}{|u_n|^{\kappa_2^+}} > c_{13} \quad \text{uniformly} \quad x \in \Theta_2,$$

for a large enough c_{13}. Then,

$$\begin{aligned}
\kappa_2^+ c_4 &= \lim_{n\to\infty} \|\mathbf{v}_n\|^{\kappa_2^+} \\
&= \lim_{n\to\infty} |t_n|^{\kappa_2^+} \|u_n\|^{\kappa_2^+} \\
&= \lim_{n\to\infty} |t_n|^{\kappa_2^+} \int_\Theta \frac{\left|g\left(x, u_n\right) u_n\right|}{|u_n|^{\kappa_2^+}} |u_n|^{\kappa_2^+} \, dx \\
&> c_{13} \lim_{n\to\infty} \int_{Q_2} |\mathbf{v}_n|^{\kappa_2^+} \, dx \\
&= c_{13} \int_{Q_2} |\mathbf{v}_n|^{\kappa_2^+} \, dx.
\end{aligned} \tag{5.1}$$

Given the constant of $\kappa_2^+ c_4$ and the sufficiently large value of c_{13}, we can conclude that $|\Theta_2| = 0$, implying $v \equiv 0$ in Θ. Moreover, with $v = 0$ and considering the continuity of the Nemitskii operator, we obtain

$$G(\cdot, \mathbf{v}_n) \longrightarrow 0 \quad \text{in} \quad L^1(\Theta),$$

which implies that

$$\lim_{n \longrightarrow \infty} G(x, \mathbf{v}_n) = 0.$$

Therefore, since $\{\mathbf{v}_n\}_n$ is a constant sequence, then ${}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} \mathbf{v}_n = 0$ and we deduce that

$$\begin{aligned}
\Lambda(\mathbf{v}_n) &\geq \frac{t_n^{\kappa_1^-}}{\kappa_2^+} \left(\int_\Theta \left| {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n \right|^{\kappa_1(x)} dx + \int_\Theta |u_n|^{\kappa_1(x)} dx \right) \\
&\quad + \frac{t_n^{\kappa_2^-}}{\kappa_2^+} \left(\int_\Theta \eta(x) \left| {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n \right|^{\kappa_2(x)} dx + \int_\Theta \eta(x) |u_n|^{\kappa_2(x)} dx \right) + o(1) \\
&> \frac{t_n^{\kappa_2^-}}{\kappa_2^+} \int_\Theta |u_n|^{\kappa_1(x)} dx + o(1) \\
&> \frac{\kappa_2^+ c_4}{\kappa_2^+} + o(1) \\
&> c_4.
\end{aligned} \tag{5.2}$$

Similarly to (4.4), for some $n > 1$, we find

$$-\frac{1}{n} < \frac{\kappa_1^-}{\kappa_2^+} \left| \langle \Lambda'(u_n), u_n \rangle \right| < \frac{1}{n}.$$

Hence,

$$\begin{aligned}
\Lambda(u_n) &= \int_\Theta \frac{1}{\kappa_1(x)} \left| {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n \right|^{\kappa_1(x)} dx + \int_\Theta \frac{1}{\kappa_1(x)} |u_n|^{\kappa_1(x)} dx \\
&\quad + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} \left| {}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u_n \right|^{\kappa_2(x)} dx + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} |u_n|^{\kappa_2(x)} dx - \int_\Theta G(x, u_n)\, dx \\
&\geq \frac{1}{\kappa_2^+} \frac{\kappa_2^+}{\kappa_1^-} \left(-\frac{1}{n} + \int_\Theta g(x, u_n) u_n\, dx \right) - \int_\Theta G(x, u_n)\, dx,
\end{aligned} \tag{5.3}$$

that is,

$$\Lambda(u_n) + \frac{1}{n\kappa_1^-} \geq \int_\Theta \left(\frac{1}{\kappa_1^-} g(x, u_n) u_n - G(x, u_n) \right) dx. \tag{5.4}$$

Furthermore, according to Lemma 4.2, one has

$$\Lambda(t u_n) \leq \frac{t^{\kappa_1^-}}{n\kappa_1^-} + \int_\Theta \left(\frac{1}{\kappa_1^-} g(x, u_n) u_n - G(x, u_n) \right) dx. \tag{5.5}$$

Due to (5.4) and (5.5), we obtain

$$\Lambda(\mathbf{v}_n) \leq \frac{t^{\kappa_1^-}+1}{n\kappa_1^-} + \psi(t_n) \longrightarrow c_4,$$

that contradicts (5.2). Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{H}$.

Claim 2. *The sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $\mathbb{H}$.*

In fact, given the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{H}$ and the reflexivity of $\mathbb{H}$, there exists $u \in \mathbb{H}$ such that $u_n \longrightarrow u$. Since, the space $\mathbb{H}$ is compactly embedded in $L^{\kappa_1(\cdot)}(\Theta)$, we obtain, for a subsequence still denoted by u_n, that $u_n \longrightarrow u$ in $L^{\kappa_1(\cdot)}(\Theta)$. Then, employing Hölder's inequality, we conclude

$$\lim_{n\to\infty}\int_\Theta |u_n|^{\kappa_1(x)-2}\, u_n(u_n-u)\,dx + \lim_{n\to\infty}\int_\Theta \eta(x)\,|u_n|^{\kappa_2(x)-2}\,u_n(u_n-u)\,dx = 0.$$

On the other hand, utilizing (4.2), yields

$$\lim_{n\to\infty}\left\langle \Lambda'(u_n), u_n-u\right\rangle = 0.$$

Therefore, employing the aforementioned equations, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Theta \left|{}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u_n\right|^{\kappa_1(x)-2}\ {}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u_n\left({}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u_n - {}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u\right)dx\\
&+\lim_{n\to\infty}\int_\Theta \eta(x)\left|{}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u_n\right|^{\kappa_2(x)-2}\ {}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u_n\left({}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u_n - {}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right)dx = 0.
\end{aligned}\tag{5.6}$$

Moreover, (5.6) combined with the weak convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u in $\mathbb{H}$ implies

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Theta \left(\left|{}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u_n\right|^{\kappa_1(x)-2}\ {}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u_n - \left|{}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_1(x)-2}\ {}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u\right)\\
&\times\left({}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u_n - {}^H\mathbb{D}_{0^+}^{\alpha_1,\alpha_2;\varphi}u\right)dx\\
&+\lim_{n\to\infty}\int_\Theta \eta(x)\left(\left|{}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u_n\right|^{\kappa_2(x)-2}\ {}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u_n - \left|{}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_2(x)-2}\ {}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right)\\
&\times\left({}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u_n - {}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right)dx = 0.
\end{aligned}\tag{5.7}$$

Thus, by using Simon inequality [42, 52], we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Theta \left|{}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u_n - {}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_1(x)}dx\\
&+\lim_{n\to\infty}\int_\Theta \eta(x)\left|{}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u_n - {}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_2(x)}dx = 0.
\end{aligned}$$

Hence, $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $\mathbb{H}$. Then, from Lemma 4.1, and 4.2, Claim 1 and Claim 2, it follows that Λ satisfies Mountain-pass geometry lemma.

Subsequently, we obtain

$$\begin{aligned}
&\int_\Theta \frac{1}{\kappa_1(x)}\left|{}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_1(x)}dx + \int_\Theta \frac{1}{\kappa_1(x)}|u|^{\kappa_1(x)}dx + \int_\Theta \frac{\eta(x)}{\kappa_2(x)}\left|{}^H\mathbb{D}_{+}^{\alpha_1,\alpha_2;\varphi}u\right|^{\kappa_2(x)}dx\\
&+\int_\Theta \frac{\eta(x)}{\kappa_2(x)}|u|^{\kappa_2(x)}dx\\
\geq&\frac{1}{\kappa_2^+}\|u\|_{\mathbb{H}}^{\kappa_1^-} - c_{14}.
\end{aligned}$$

The condition holds provided that c_{14} is sufficiently small.

Next, we will prove that Λ is coercive. In fact, for $\|u\|_{\mathbb{H}} > 1$, by considering 1 and 4, in cases (5.8) for $\varepsilon \in \left]\zeta_1 \left(1 - \frac{\kappa_1^-}{\kappa_2^+}\right), \zeta_1\right[$, we have

$$\begin{aligned}\Lambda(u) &= \int_\Theta \frac{1}{\kappa_1(x)} \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_1(x)} dx + \int_\Theta \frac{|u|^{\kappa_1(x)}}{\kappa_1(x)} dx + \int_\Theta \frac{\eta(x)}{\kappa_2(x)} \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_2(x)} dx \\ &\quad + \int_\Theta \eta(x) \frac{|u|^{\kappa_2(x)}}{\kappa_2(x)} dx - \int_\Theta G(x,u)\, dx \\ &\geq \frac{1}{\kappa_1^+} \int_\Theta \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_1(x)} dx + \frac{1}{\kappa_1^+} \int_\Theta |u|^{\kappa_1(x)} dx + \frac{1}{\kappa_2^+} \int_\Theta \eta(x) \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} u\right|^{\kappa_2(x)} dx \\ &\quad + \frac{1}{\kappa_2^+} \int_\Theta \eta(x) |u|^{\kappa_2(x)} dx - (\zeta_1 - \varepsilon) \left(\int_\Theta \frac{|u|^{\kappa_1(x)}}{\kappa_1(x)} dx + \int_\Theta \eta(x) \frac{|u|^{\kappa_2(x)}}{\kappa_2(x)} dx \right) \\ &\geq \left(\frac{\kappa_1^- \zeta_1 - \kappa_2^+ (\zeta_1 - \varepsilon)}{\kappa_2^+ \kappa_1^- \zeta_1} \right) \widehat{\rho}_{\mathcal{A}}(u) \\ &\geq \left(\frac{\kappa_1^- \zeta_1 - \kappa_2^+ (\zeta_1 - \varepsilon)}{\kappa_2^+ \kappa_1^- \zeta_1} \right) \|u\|_{\mathbb{H}}^{\kappa_1^-}.\end{aligned} \tag{5.8}$$

Hence, Λ is coercive and possesses a global minimizer $\overline{u}$, which implies that $\langle \Lambda'(\overline{u}), \overline{u} \rangle = 0$, which is nontrivial. Thus, by considering $v_0 \in {}^H W_0^1 L_{\mathcal{A}}(\Theta)$, $t > 0$ small enough small, and using the inequality $p < \kappa_1^-$, we obtain from 1 that

$$\begin{aligned}\Lambda(tv_0) &\leq \frac{t^{\kappa_1^-}}{\kappa_1^-} \left(\int_\Theta \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} v_0\right|^{\kappa_1(x)} dx + \int_\Theta |v_0|^{\kappa_1(x)} dx \right. \\ &\quad \left. + \int_\Theta \eta(x) \left|{}^H\mathbb{D}_+^{\alpha_1,\alpha_2;\varphi} v_0\right|^{\kappa_2(x)} dx + \int_\Theta \eta(x) |v_0|^{\kappa_2(x)} dx \right) - \int_\Theta G(x, tv_0)\, dx \\ &\leq c_{15} t^{\kappa_1^-} - c_6 t^p \\ &< 0,\end{aligned}$$

which completes the proof. □

To illustrate the theoretical framework developed above, we now present concrete examples that demonstrate the application of φ-Hilfer fractional derivatives with Riemann-Liouville integral boundary conditions in various physical contexts. These examples not only validate our existence and uniqueness theorems but also showcase the flexibility of our approach in modeling complex systems with memory effects.

6. Examples

In this section, we present several illustrative cases of the main problem (1.2). The aim is to highlight how different choices of the variable exponents, the double-phase coefficient, and the nonlinear term $g(x,u)$ influence the structure of the equation and the qualitative behavior of its solutions. These examples demonstrate how the theoretical framework developed previously can be adapted to distinct functional settings and physical interpretations.

Example 6.1 (One-dimensional case with constant exponents)**.** Consider the problem on $\Theta = (0,1)$ with parameters $\kappa_1(x) = p$, $\kappa_2(x) = q$ for $1 < p < q$, a positive parameter $\eta(x) = \lambda$, and a nonlinear source term $g(x,u) = h(x)|u|^{r-2}u$, where h is continuous on $[0,1]$. The fractional double-phase problem with φ-Hilfer derivatives reads:

$$\begin{cases} \quad {}^H\mathbb{D}_{x^-}^{\alpha_1,\alpha_2;\varphi}\Big(|{}^H\mathbb{D}_{x^+}^{\alpha_1,\alpha_2;\varphi}u|^{p-2}\,{}^H\mathbb{D}_{x^+}^{\alpha_1,\alpha_2;\varphi}u + \lambda|{}^H\mathbb{D}_{x^+}^{\alpha_1,\alpha_2;\varphi}u|^{q-2}\,{}^H\mathbb{D}_{x^+}^{\alpha_1,\alpha_2;\varphi}u\Big) \\ \quad +|u|^{p-2}u + \lambda|u|^{q-2}u \\ = h(x)|u|^{r-2}u, \quad x \in (0,1), \\ I_{x^+}^{1-\alpha_2;\varphi}u(0) = I_{x^-}^{1-\alpha_1;\varphi}u(1) = 0. \end{cases}$$

This formulation models a system whose behavior depends on the magnitude of the fractional derivative, switching between p-growth and q-growth regimes. The φ-Hilfer derivatives introduce memory and nonlocal effects, generalizing classical derivatives, while the nonlinear term $h(x)|u|^{r-2}u$ accounts for reactions or external forcing.

This fractional double-phase equation has several physical applications:

Anomalous diffusion: The fractional derivatives describe memory effects in heterogeneous media. The double-phase structure models materials with two different diffusion regimes, such as soft and stiff layers, and the nonlinear term represents sources or sinks.

Nonlinear elasticity: If $u(x)$ represents displacement, the gradient corresponds to strain. The two-phase term captures materials that stiffen for large strains, while the fractional derivative models long-range interactions or viscoelastic memory.

Thermal or electrochemical transport: The gradient-dependent flux law can describe heat or charge flow in fractal or composite media, with the response depending nonlinearly on the gradient magnitude.

Reaction-diffusion systems with memory: The term $h(x)|u|^{r-2}u$ models population growth, chemical reactions, or epidemic spread, while the fractional derivative captures history-dependent feedback effects.

For the particular case when $p = 2$, $q = 4$, $\lambda = 1$, $\alpha_1 = \alpha_2 = 0.8$, $\varphi(x) = x$, $r = 3$, and $h(x) \equiv 1$. The problem reduces to:

$${}^H\mathbb{D}_{x^-}^{0.8,0.8;x}\Big(|{}^H\mathbb{D}_{x^+}^{0.8,0.8;x}u|^0\,{}^H\mathbb{D}_{x^+}^{0.8,0.8;x}u + |{}^H\mathbb{D}_{x^+}^{0.8,0.8;x}u|^2\,{}^H\mathbb{D}_{x^+}^{0.8,0.8;x}u\Big) + u + |u|^2u = |u|u,$$

with corresponding fractional boundary conditions.

In this specific instance, the system exhibits a transition between linear diffusion ($p = 2$) and strongly nonlinear diffusion ($q = 4$) depending on the magnitude of the fractional derivative. The memory effect introduced by the φ-Hilfer derivative ensures that the evolution at a point depends on the entire history of the process. Physically, this can model a viscoelastic material in which small strains propagate diffusively while larger strains experience stronger resistance, or a composite medium where heat or particle fluxes are history-dependent and nonlinearly related to the gradient.

Example 6.2 (Case of variable exponents)**.** Let $\Theta = (0,1)$ and define the variable exponents

$$\kappa_1(x) = 2 + \sin(\pi x), \qquad \kappa_2(x) = 3 + \cos(\pi x),$$

together with $\eta(x) = x(1-x)$. Consider the nonlinear term $g(x,u) = \mu|u|^{s(x)-2}u$, where $s(x) = 2 + \frac{x}{2}$ and $\mu > 0$ is a real parameter. The corresponding fractional double-phase problem driven by φ-Hilfer derivatives is given by

$$\begin{cases} \quad {}^{H}\mathbb{D}_{x^-}^{\alpha_1,\alpha_2;\varphi}\Big(\left| {}^{H}\mathbb{D}_{x^+}^{\alpha_1,\alpha_2;\varphi} u \right|^{\kappa_1(x)-2} {}^{H}\mathbb{D}_{x^+}^{\alpha_1,\alpha_2;\varphi} u + x(1-x) \left| {}^{H}\mathbb{D}_{x^+}^{\alpha_1,\alpha_2;\varphi} u \right|^{\kappa_2(x)-2} {}^{H}\mathbb{D}_{x^+}^{\alpha_1,\alpha_2;\varphi} u \Big) \\ \quad +|u|^{\kappa_1(x)-2}u + x(1-x)|u|^{\kappa_2(x)-2}u \\ = \mu|u|^{s(x)-2}u, \quad x \in (0,1), \\ I_{x^+}^{1-\alpha_2;\varphi}u(0) = I_{x^-}^{1-\alpha_1;\varphi}u(1) = 0. \end{cases}$$

Mathematically, this problem represents a fractional elliptic-type equation with variable exponents and nonlocal operators. The functions $\kappa_1(x)$ and $\kappa_2(x)$ describe spatially dependent growth conditions, producing a heterogeneous double-phase structure where both diffusion and potential terms vary continuously with respect to x. The coefficient $\eta(x) = x(1-x)$ modulates the effect of the higher-phase term, enhancing its influence near the midpoint of the interval and diminishing it near the boundaries. The φ-Hilfer derivative introduces nonlocal and memory effects, bridging Riemann-Liouville and Caputo formulations, and thus allowing the system to capture spatial variability together with historical dependence. The nonlinear source $\mu|u|^{s(x)-2}u$ extends classical reaction terms by incorporating position-dependent nonlinearities.

From a physical standpoint, this equation can model several classes of phenomena characterized by spatial heterogeneity and memory effects. In heterogeneous diffusion, the exponents $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ represent position-dependent diffusion intensities, describing materials whose microstructure or porosity changes along the domain. The double-phase nature reflects the coexistence of two diffusion regimes—dominant near the center and weaker at the boundaries. In nonlinear elasticity, the variable exponents correspond to a medium with spatially varying stiffness, such as a composite bar with alternating soft and rigid regions, while the fractional derivative captures long-range stress interactions typical of viscoelastic or fractal materials. In thermal or electrical transport, the same structure can describe heat or charge conduction in nonhomogeneous or fractal media, where the effective conductivity depends nonlinearly on both position and the magnitude of the flux, and the fractional nature accounts for anomalous diffusion with memory. Finally, in reaction–diffusion systems with spatially variable reaction rates, the term $\mu|u|^{s(x)-2}u$ may describe chemical kinetics, biological growth, or population dynamics influenced by environmental heterogeneity, where the reaction intensity varies continuously with space and the fractional derivative captures the delayed or hereditary response of the system.

This problem describes a nonlocal and spatially heterogeneous process governed by a double-phase fractional operator with variable exponents, providing a realistic mathematical model for heterogeneous diffusion, viscoelasticity, anomalous heat transfer, and reaction–diffusion phenomena with memory effects.

Example 6.3 (Two-dimensional case)**.** Let $\Theta = (0,1)^2$, and consider the parameters $\kappa_1(x) = 2$, $\kappa_2(x) = 4$, and $\eta(x) = x_1^2 + x_2^2$. Let the nonlinear source be given by $g(x,u) = |u|^3u$. The corresponding two-dimensional fractional double-phase problem driven by the φ-Hilfer derivative is formulated as

$$\begin{cases} \quad \displaystyle\sum_{i=1}^{2} {}^{H}\mathbb{D}_{x_i^-}^{\alpha_1,\alpha_2;\varphi}\Big({}^{H}\mathbb{D}_{x_i^+}^{\alpha_1,\alpha_2;\varphi} u + (x_1^2+x_2^2) \left| {}^{H}\mathbb{D}_{x_i^+}^{\alpha_1,\alpha_2;\varphi} u \right|^2 {}^{H}\mathbb{D}_{x_i^+}^{\alpha_1,\alpha_2;\varphi} u \Big) + u + (x_1^2+x_2^2)|u|^2u \\ = |u|^3u, \quad x \in (0,1)^2, \\ I_{x_i^+}^{1-\alpha_2;\varphi}u(x) = I_{x_i^-}^{1-\alpha_1;\varphi}u(x) = 0, \quad x \in \partial\Theta. \end{cases}$$

This problem represents a two-dimensional nonlinear system governed by a fractional double-phase operator. The first term corresponds to a standard fractional diffusion of order (α_1, α_2), while the second one, weighted by $\eta(x) = x_1^2 + x_2^2$, introduces a higher-order nonlinear diffusion regime. The variable weight $\eta(x)$ enhances the influence of the nonlinear phase near the corners far from the origin, where $(x_1^2 + x_2^2)$ is large, and diminishes it near the center of the domain. The fractional derivatives of φ-Hilfer type capture nonlocal interactions and memory effects in both spatial directions x_1 and x_2, making the model suitable for systems in which diffusion or stress propagation depends on past states or long-range correlations. The reaction term $|u|^3 u$ introduces a strongly nonlinear source behavior, describing processes where the response grows rapidly with the amplitude of u.

From a physical perspective, this model can describe heterogeneous diffusion or conduction phenomena in two-dimensional media with memory and spatially varying nonlinear properties. In the context of heat conduction, it may represent a temperature field in a nonhomogeneous plate, where the local conductivity depends quadratically on the spatial position and on the magnitude of the fractional temperature gradient. In nonlinear elasticity, $u(x_1, x_2)$ may denote the displacement field in a thin elastic membrane with position-dependent stiffness; small strains produce linear diffusion, while larger deformations activate the nonlinear phase, amplified by the weight $\eta(x)$. The presence of fractional derivatives accounts for viscoelastic effects, capturing hereditary stress relaxation and long-range mechanical interactions.

In electrochemical transport or charge diffusion in fractal or composite materials, the model characterizes a two-dimensional medium where the charge or energy flux depends on both spatial heterogeneity and memory, with the term $(x_1^2 + x_2^2)$ reflecting the spatial variation of the material's resistivity or conductivity. Finally, in reaction-diffusion systems, the nonlinear source $|u|^3 u$ may describe population or chemical reactions with strong autocatalytic behavior, while the fractional double-phase operator represents a medium with mixed diffusive and reactive dynamics influenced by its geometric configuration.

In summary, this two-dimensional problem illustrates the interplay between nonlocal diffusion, spatial heterogeneity, and nonlinear reaction effects. The combination of φ-Hilfer derivatives, variable weights, and double-phase diffusion provides a versatile framework for modeling memory-dependent processes in heterogeneous plates, membranes, or reactive surfaces with spatially varying material properties.

The one-dimensional example with constant exponents illustrates a fractional double-phase model in which the transition between the two different growth behaviors is regulated by the parameters p and q. The coefficient λ determines the relative influence of the second phase, and therefore plays a central role in the energy distribution of the system.

The second example highlights the effect of spatially varying exponents. Here, the growth properties depend on the point x, which introduces non-uniform diffusion effects. This setting is representative of heterogeneous materials or media where local structural characteristics change gradually across the domain.

The third example incorporates both multi-dimensionality and a spatially dependent double-phase coefficient. The weight $\eta(x) = x_1^2 + x_2^2$ enhances the dominance of the second phase away from the origin. This configuration illustrates how fractional differentiation in multiple directions interacts with the double-phase mechanism, while still remaining within the functional framework previously established.

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