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GLOBAL SOLUTION FOR THE INCOMPRESSIBLE MHD EQUATIONS WITH A CLASS OF LARGE DATA IN $BMO^{-1}(\mathbb{R}^3)$*

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Abstract In [12], Koch & Tataru have proved the global well-posedness of the Navier-Stokes equations with small initial data $u_0 \in BMO^{-1}(\mathbb{R}^n)$, and then the spatial and time analyticity of the Koch & Tataru solution have been presented by Germain-Pavlović-Stffilani [9] and the first author [3] when the initial data $u_0 \in BMO^{-1}(\mathbb{R}^n)$ small enough. Subsequently, the similar results for the incompressible MHD equations have been studied by the first author [4] for $(u_0, b_0) \in BMO^{-1}(\mathbb{R}^n)$ small enough. In this paper, we shall prove the global well-posedness for the incompressible MHD equations with a class of large data $(u_0, b_0) \in BMO^{-1}(\mathbb{R}^n)$. Besides, the space-time regularities also have been proved.

Keywords Global regularity, MHD equations, large data, Koch-Tataru solution.

MSC(2010) 35Q35, 76W05, 35B65, 42B35.

1. Introduction

This paper is devoted to study the periodic solution for the incompressible MHD equations on the domain $(x,t) \in \mathbb{R}^3 \times R^+$.

The incompressible MHD equations have the following form

$$\begin{cases} \dfrac{\partial U}{\partial t} + (U\cdot\nabla)U - \mu\Delta U + \nabla P = (B\cdot\nabla)B, \\ \dfrac{\partial B}{\partial t} + (U\cdot\nabla)B - (B\cdot\nabla)U - \eta\Delta B = 0, \\ \nabla\cdot U = \nabla\cdot B = 0, \\ t = 0, U = u_0, \ B = b_0, \end{cases} \tag{1.1}$$

where $U(x,t)$ and $B(x,t)$ denote the fluid velocity field and the magnetic field respectively, $P = p + \frac{1}{2}|B|^2$ with p is the pressure. $\mu > 0$ is the constant kinematic viscosity, and $\eta > 0$ is the constant magnetic diffusivity. For simplicity, we take $\mu = \eta = 1$ in this paper.

Magnetohydrodynamics (MHD) equation describes the motion of an electrically conducting fluid in the presence of the magnetic field, which essentially needs to consider the interaction between the fluid velocity and the magnetic field. There are a lot of studies on this equation in the literature. Duraut and Lions [6] constructed a class of global weak solution with finite energy. In [25], for the two dimensional case, the smoothness and uniqueness of classical solution have been presented. In [22], the authors have proved the Koch & Tataru [12] type solution

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for for MHD equation (1.1) when initial data $(u_0, b_0) \in BMO^{-1}(\mathbb{R}^n)$. See more efforts in [8, 15, 19, 20, 24, 26, 27] etc.

To make a clearer introduction to the results of this paper, we shall recall some well-posedness results of Navier-Stokes equations firstly.

In [17], Leray proved the local well-posedness for local strong solutions and for any finite square-integrable initial data there exists a (possibly not unique) global in time weak solution in $\mathbb{R}^n$. Moreover, for case of two space dimensions, he proved in [18] the uniqueness of the weak solution. Subsequently, in the work of Fujita-Kato [7], they proved the local well-posedness for strong solutions to the Navier-Stokes equations in a scaling invariant space $\dot{H}^{\frac{n}{2}-1}$. The scaling-invariance in the context of the Navier-Stokes equations is as follows. Define

$$(U_\lambda, P_\lambda)(x,t) = (\lambda U(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t)), \tag{1.2}$$

if the pair $(U(x,t), P(x,t))$ solves the incompressible Navier-Stokes equations, then $(U_\lambda(x,t), P_\lambda(x,t))$ is also a pair of solution to the incompressible Navier-Stokes equations with initial data $(U_\lambda(x,0), P_\lambda(x,0)) = (\lambda u_0(\lambda x), \lambda^2 P_0(\lambda x))$. The spaces which are invariant under such a scaling are also called critical spaces. The study of the Navier-Stokes equations in critical spaces was initiated by Kato [11] and continued by many authors, see [1, 10, 23] etc. In 2001 Koch-Tataru [12] proved the existence of solutions to Navier-Stokes equations in $\mathbb{R}^n$ when the corresponding initial data in BMO^{-1}, see also [16]. The space BMO^{-1} has a special role since it is the largest critical space for Navier-Stokes equations with well-posedness results available. Hereafter, we call the solution presented by Koch & Tataru [12] as Koch-Tataru solution.

The results listed above are most concerned the incompressible N-S equations with the initial data small enough. To prove whether global existence or finite time blow up for a large data is a famous open problem. Recently, in Lei-Lin-Zhou [14] the authors have explored the solution with the initial data close to a Beltrami flow, and proved the global well-posedness with large data in the energy space. Subsequently, in Du-Zhou [5], the global well-posedness for the incompressible N-S equations in $BMO^{-1}(\mathbb{R}^3)$ have been proved. Motivated by the paper of Du-Zhou [5], this paper is to study the global well-posedness for the 3D incompressible MHD equations with general initial data in BMO^{-1}.

The Navier-Stokes equations share some similar structures with MHD equations, therefore, the space analytical to the Koch & Tataru type solution of the MHD equations (1.1) with initial data $(u_0, b_0) \in BMO^{-1}$ have been given by [21]. Here, it is also worth to mention that the MHD equations have their special structures compare to the Navier-Stokes equations. An impression example is that the global regularity to the axially symmetric solutions to Navier-Stokes equations are widely open, however, it had been verified by Lei [15] for the axially symmetric solutions to the ideal MHD equations in three dimensions.

Before stating the main results, we shall give some definitions and notations. Let

$$Q(y_0, r) = B(y_0, r) \times (0, r^2] \tag{1.3}$$

be the space-time ball. For $(x,t) \in Q(y_0, r)$ means $x \in B(y_0, r)$ and $0 < t \leq r^2$, where $B(y_0, r) \subset \mathbb{R}^n$ is a n-dimensional space ball centered at $y_0 \in \mathbb{R}^n$ and radius r.

Definition 1.1. Let f be a tempered distribution, W be the solution of $W_t - \Delta W = 0$ with initial data f. Denote

$$[f]_{BMO(\mathbb{R}^n)} = \sup_{y_0 \in \mathbb{R}^n} \left(r^{-n} \int_{Q(y_0,r)} |\nabla W|^2 dt dy \right)^{\frac{1}{2}},$$

we say the function $f \in L^1_{loc}(\mathbb{R}^n)$ is in BMO if the semi-norm $[f]_{BMO}$ is finite.

If there exist $g_i \in BMO$ and $f = \sum_{i=1}^n \frac{\partial g_i}{\partial x_j}$, denote

$$[f]_{BMO^{-1}(\mathbb{R}^n)} = \inf \sum_{i=1}^n \|g_i\|_{BMO(\mathbb{R}^n)},$$

we say $f \in BMO^{-1}$ if the above norm $[f]_{BMO^{-1}}$ is finite.

Clearly the divergence of a vector field with components in BMO is in BMO^{-1}. See more details in Koch-Tataru [12] and Chap11 in [16].

In Koch& Tataru [12], the following results have been presented:

Theorem 1.1. *If $\|u_0\|_{BMO^{-1}}$ is small enough, then the incompressible Navier Stokes equations admits a unique pair of global solution.*

Our main result is as follows:

Theorem 1.2. *Let λ be a given arbitrary constant and M_0 be a arbitrary positive constant, suppose that the initial data (u_0, b_0) of system (1.1) is a pair of periodic functions with*

$$\int_{\mathbb{T}^3(x)} u_0(y)dy = \int_{\mathbb{T}^3(x)} b_0(y)dy = 0, \tag{1.4}$$

and

$$\|u_0\|_{BMO^{-1}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)} \leq M_0, \tag{1.5}$$

then there exists a constant ϵ_0 which depend on M_0 and λ small enough, the system (1.1) admits a unique global periodic solution if

$$\|\nabla \times u_0 - \lambda u_0\|_{BMO^{-1}(\mathbb{R}^3)} \leq \epsilon_0, \tag{1.6}$$

and

$$\|u_0 - b_0\|_{BMO^{-1}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)} \leq \epsilon_0. \tag{1.7}$$

Remark 1.1. Without loss of generality, in this paper we take $\mathbb{T}^3(x) = [x - \pi, x + \pi]$, x is the center. Actually, the condition (1.6) looks natural. Roughly speaking, the condition (1.6) implies that the initial velocity almost parallel to the initial vorticity, which illumined by the Beltrami flow. The constrain (1.7) means that the initial data b_0 is just similar to u_0.

Furthermore, we also have the following results:

Theorem 1.3. *Under the conditions of Theorem 1.2, then the solution presented by Theorem 1.2 is analytic about the space and time.*

We immediately obtain the following decay estimates.

Corollary 1.1. *Under the condition of Theorem 1.2, then there exists a constant $C(m, k)$, such that the unique solution (U, B) of (1.1) satisfying*

$$\|\partial_t^m \nabla^k U\|_{L^\infty(\mathbb{R}^3)} + \|\partial_t^m \nabla^k B\|_{L^\infty(\mathbb{R}^3)} \leq C t^{-\frac{k+1}{2} - m}, \tag{1.8}$$

for any $t > 0$ and integers $m, k \geq 0$.

Our results do not need $\|u_0\|_{BMO^{-1}}$ and $\|b_0\|_{BMO^{-1}}$small, actually, we can take it as large as we want. Therefore, to prove the large data results for the incompressible MHD equations in BMO^{-1} space, we need to explore the structure of the nonlinear term, where we have used the idea "nonlinear small" from Chemin et al. [2]. It expanded the results of Koch-Tataru [12].

This paper is organized as following: In the next section, we present some preliminaries and some prepare work for the linear homogenous heat equation. The main theorem will be proved in Section 3. Throughout this paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant C.

2. Preliminaries

At the beginning, we recall some properties for the Leary projection operator $\mathbb{P}$ to divergence free vector fields, which is defined by its matrix valued Fourier multiplier $\hat{\mathbb{P}}(\xi) = \delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}$. For any multi-indices α, this symbol satisfies Mihlin-Hormander condition $\sup_{|\xi|\neq 0} |\xi|^{\alpha}|\partial_{\xi}^{\alpha}\hat{\mathbb{P}}(\xi)| \leq C$. Besides, we have the following pointwise bound (see [16]).

Lemma 2.1. *Denote $e^{t\Delta}$ is the heat operator, n is the space dimensions, and $\tilde{\mathbb{P}}(x,t)$ is the kernel of $\nabla^{k+1}\mathbb{P}e^{t\Delta}$, then holds*

$$\tilde{\mathbb{P}}(x,t) \leq C(k)\frac{1}{(\sqrt{t}+|x|)^{n+k+1}}, \tag{2.1}$$

where and $C(k)$ is a constant depending on k.

Lemma 2.2. *Let $\mathbb{K}(x,t) = \frac{1}{\sqrt{t}^n}e^{-\frac{x^2}{4t}}$, then there exists a polynomial $J^{k+2m}(\frac{x}{\sqrt{t}})$ with degree $k+2m$, such that*

$$\partial_t^m \nabla^k \mathbb{K}(x,t) = \frac{1}{t^{m+\frac{k}{2}}}\mathbb{K}(x,t)J^{k+2m}(\frac{x}{\sqrt{t}}). \tag{2.2}$$

Proof. *It can be proved by induction.* □

For completeness, we also give the following well-known equality:

Lemma 2.3. *Let $\nabla \cdot f = 0$ and $\bigtriangledown \cdot g = 0$, then we have*

$$f \cdot \bigtriangledown g + g \cdot \bigtriangledown f = -f \times (\bigtriangledown \times g) - g \times (\bigtriangledown \times f) + \bigtriangledown(f \cdot g), \tag{2.3}$$

and

$$(f \cdot \bigtriangledown)g - (g \cdot \bigtriangledown)f = \nabla \times (g \times f). \tag{2.4}$$

To go ahead, we introduce the following function set:

Definition 2.1. Let f be a function defined on $\mathbb{R}^3 \times [0,T)$ $(0 < T \leq +\infty)$. We say $f \in \mathbb{X}_T$ if

$$\|f\|_{\mathbb{X}_T} \triangleq \sup_{0<t\leq T} t^{\frac{1}{2}}\|f\|_{L^\infty(\mathbb{R}^3)} + \sup_{0<r\leq\sqrt{T}}\left(r^{-3}\int_{Q(y_0,r)}|f|^2 dydt\right)^{\frac{1}{2}} < +\infty. \tag{2.5}$$

We say $f \in \mathbb{Y}_T$ if

$$\|f\|_{\mathbb{Y}_T} \triangleq \sup_{0<t\leq T} t\|f\|_{L^\infty(\mathbb{R}^3)} + \sup_{0<r\leq\sqrt{T}} r^{-3}\int_{Q(y_0,r)}|f|dydt < +\infty. \tag{2.6}$$

At the beginning, we shall give some estimates for the following homogenous heat equation in $\mathbb{R}^n$:

$$\begin{cases} \partial_t u - \Delta u = 0, \\ u|_{t=0} = u_0, \quad \nabla \cdot u_0 = 0. \end{cases} \tag{2.7}$$

We have the following proposition:

Proposition 2.1. *There exists a uniform constant C_0, such that*

$$\|u\|_{X_T} \le C_0 \|u_0\|_{BMO^{-1}}. \tag{2.8}$$

Proof. We write the solution of the linear equation as

$$u = S(t)u_0, \tag{2.9}$$

where $S(t)$ is the heat flow. Then we have:

$$\begin{aligned} t^{\frac{1}{2}} \|u\|_{L^\infty(\mathbb{R}^n)} &= 4t^{-\frac{1}{2}} \left\| \int_{\frac{t}{8}}^{\frac{3t}{8}} S(t-\tau)\big(S(\tau)u_0\big) d\tau \right\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim t^{-\frac{1}{2}} \left\| \int_{\frac{t}{8}}^{\frac{3t}{8}} \left(\int_{\mathbb{R}^3} \frac{1}{\sqrt{t-\tau}^n} e^{-\frac{(y-\tilde{y})^2}{4(t-\tau)}} \big(e^{\tau\Delta} u_0\big)^2 d\tilde{y} \right)^{1/2} d\tau \right\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim t^{-\frac{1}{2}} \left\| \left(\int_{\frac{t}{8}}^{\frac{3t}{8}} \int_{\mathbb{R}^n} \frac{1}{\sqrt{t-\tau}^n} e^{-\frac{(y-\tilde{y})^2}{4(t-\tau)}} \big(e^{\tau\Delta} u_0\big)^2 d\tilde{y} d\tau \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^n)} \cdot \sqrt{t} \\ &\lesssim \left\| \left(\sum_{q=0}^{\infty} e^{-\frac{q^2}{4}} \frac{1}{\sqrt{t}^n} \int_{\frac{t}{8}}^{\frac{3t}{8}} \int_{q \le \frac{|y-\tilde{y}|}{\sqrt{t}} \le q+1} \big(e^{\tau\Delta} u_0\big)^2 d\tilde{y} d\tau \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \sup_{B(y,\sqrt{t}) \subset \mathbb{R}^n} \left(\frac{1}{\sqrt{t}^n} \int_0^t \int_{B(y,\sqrt{t})} \big(e^{\tau\Delta} u_0\big)^2 d\tilde{y} d\tau \right)^{1/2} \\ &\lesssim \|u_0\|_{BMO^{-1}(\mathbb{R}^n)}. \end{aligned} \tag{2.10}$$

We need to estimate the term $\sup\limits_{y_0 \in \mathbb{R}^n} \big(r^{-n} \int_{Q(y_0,r)} u^2 dy d\tau\big)^{1/2}$.

$$\begin{aligned} \sup_{y_0 \in \mathbb{R}^n} \left(r^{-n} \int_{Q(y_0,r)} u^2 dy dt \right)^{1/2} &\lesssim \sup_{y_0 \in \mathbb{R}^n} r \|u\|_{L^\infty(Q(y_0,r))} \\ &\le \sup_{y_0 \in \mathbb{R}^n} \frac{1}{r} \left\| \int_{\frac{r^2}{8}}^{\frac{3r^2}{8}} S(t-\tau)\big(e^{\tau\Delta} u_0\big) d\tau \right\|_{L^\infty(Q(y_0,r))} \\ &\lesssim \sup_{y_0 \in \mathbb{R}^n} \frac{1}{r} \left\| \int_{\frac{r^2}{8}}^{\frac{3r^2}{8}} \left(\int_{\mathbb{R}^n} \frac{e^{-\frac{\tilde{y}^2}{4(t-\tau)}}}{\sqrt{t-\tau}^n} \right. \right. \\ &\qquad \left. \left. \times \big(e^{\tau\Delta} u_0(y-\tilde{y},\tau)\big)^2 d\tilde{y} \right)^{\frac{1}{2}} d\tau \right\|_{L^\infty(Q(y_0,r))} \\ &\lesssim \sup_{y_0 \in \mathbb{R}^n} \left\| \left(\int_{\frac{r^2}{8}}^{\frac{3r^2}{8}} \|e^{\tau\Delta} u_0\|^2_{L^\infty(\mathbb{R}^3)} d\tau \right)^{\frac{1}{2}} \right\|_{L^\infty(Q(y_0,r))}. \end{aligned} \tag{2.11}$$

Recalling (2.10) that $\tau^{\frac{1}{2}}\|e^{\tau\Delta}u_0\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u_0\|_{BMO^{-1}(\mathbb{R}^n)}$, we have

$$\sup_{B(y_0,r)\subset\mathbb{R}^n}\left\|\left(\int_{\frac{r^2}{8}}^{\frac{3r^2}{8}}\|e^{\tau\Delta}u_0\|^2_{L^\infty(\mathbb{T}^3)}d\tau\right)^{\frac{1}{2}}\right\|_{L^\infty(Q(y_0,r))} \lesssim \|u_0\|_{BMO^{-1}(\mathbb{R}^n)}. \tag{2.12}$$

Combining (2.10)-(2.12), we completed this proof. □

Corollary 2.1. *For any integers $m,k\geq 0$ we have*

$$\|t^{\frac{k}{2}+m}\partial_t^m\nabla^k u\|_{X_T}\lesssim \|u_0\|_{BMO^{-1}}. \tag{2.13}$$

Proof. Similar to the proof of Proposition 2.1, by Lemma 2.2, we have:

$$\begin{aligned}
&t^{\frac{k+1}{2}+m}\|\partial_t^m\nabla^k u\|_{L^\infty(\mathbb{R}^n)}\\
&=4t^{\frac{k-1}{2}+m}\left\|\int_{\frac{t}{8}}^{\frac{3t}{8}}\partial_t^m\nabla^k S(t-\tau)\big(e^{\tau\Delta}u_0\big)d\tau\right\|_{L^\infty(\mathbb{R}^n)}\\
&\lesssim t^{\frac{k-1}{2}+m}\left\|\int_{\frac{t}{8}}^{\frac{3t}{8}}\left(\int_{\mathbb{R}^n}\frac{J^{k+2m}(\frac{y-\tilde{y}}{\sqrt{t-\tau}})}{\sqrt{t-\tau}^n}e^{-\frac{(y-\tilde{y})^2}{4(t-\tau)}}\big(e^{\tau\Delta}u_0\big)^2d\tilde{y}\right)^{1/2}\frac{1}{\sqrt{t-\tau}^{k+2m}}d\tau\right\|_{L^\infty(\mathbb{R}^n)}\\
&\lesssim \left\|\left(\sum_{q=0}^{\infty}J^{k+2m}(q)e^{-\frac{q^2}{4}}\frac{1}{\sqrt{t}^n}\int_{\frac{t}{8}}^{\frac{3t}{8}}\int_{q\leq\frac{|y-\tilde{y}|}{\sqrt{t}}\leq q+1}\big(e^{\tau\Delta}u_0\big)^2d\tilde{y}d\tau\right)^{1/2}\right\|_{L^\infty(\mathbb{R}^n)}\\
&\leq C_0\|u_0\|_{BMO^{-1}}.
\end{aligned}\tag{2.14}$$

We still need to estimate the term $\sup\limits_{y_0\in\mathbb{R}^n,r>0}\big(r^{-n}\int_{Q(y_0,r)}(t^{\frac{k}{2}+m}\partial_t^m\nabla^k u)^2dydt\big)^{1/2}$. Here, if we estimate it directly as in (2.11), then we will have a non-integrable factor. By using $\partial_t^m u=\partial_t^{m-1}\partial_t u=\Delta^m u$, we get

$$\begin{aligned}
&\sup_{y_0\in\mathbb{R}^n,r>0}\left(r^{-n}\int_{Q(y_0,r)}\big(t^{\frac{k}{2}+m}\partial_t^m\nabla^k u\big)^2dydt\right)^{1/2}\\
&\lesssim\sup_{y_0\in\mathbb{R}^n,r>0}\left(r^{-n}\int_{Q(y_0,r)}\big(t^{\frac{k}{2}+m}\nabla^{k+2m}u\big)^2dydt\right)^{1/2}.
\end{aligned}\tag{2.15}$$

Therefore, it is sufficient to estimate the term $\sup\limits_{y_0\in\mathbb{R}^n,r>0}\big(r^{-n}\int_{Q(y_0,r)}(t^{\frac{k}{2}}\nabla^k u)^2dydt\big)^{1/2}$ for any integer $k>0$.

$$\begin{aligned}
&\sup_{y_0\in\mathbb{R}^n,r>0}\left(r^{-n}\int_{Q(y_0,r)}\big(t^{\frac{k}{2}}\nabla^k u\big)^2dydt\right)^{1/2}\\
&\lesssim\sup_{y_0\in\mathbb{R}^n,r>0}r\|t^{\frac{k}{2}}\nabla^k u\|_{L^\infty(Q(y_0,r))}\\
&\leq\sup_{y_0\in\mathbb{R}^n,r>0}\frac{1}{r}\left\|t^{\frac{k}{2}}\int_{\frac{r^2}{8}}^{\frac{3r^2}{8}}S(t-\tau)\nabla^k\big(e^{\tau\Delta}u_0\big)d\tau\right\|_{L^\infty(Q(y_0,r))}\\
&\lesssim\sup_{y_0\in\mathbb{R}^n,0<t\leq r^2}\left\|t^{\frac{k}{2}}\left(\int_{\frac{r^2}{8}}^{\frac{3r^2}{8}}\|\nabla^k e^{\tau\Delta}u_0\|^2_{L^\infty(\mathbb{R}^3)}d\tau\right)^{\frac{1}{2}}\right\|_{L^\infty(Q(y_0,r))}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sup_{y_0\in\mathbb{R}^n, 0<t\le r^2} t^{\frac{k}{2}}\left(\int_{\frac{r^2}{8}}^{\frac{3r^2}{8}} \frac{\left(\tau^{k/2}\|\nabla^k e^{\tau\Delta}u_0\|_{L^\infty(\mathbb{R}^3)}\right)^2}{\tau^{k/2}} d\tau\right)^{\frac{1}{2}} \\
&\lesssim \|u_0\|_{BMO^{-1}}.
\end{aligned} \tag{2.16}
$$

In the last inequality above, we have used (2.14). □

Proposition 2.2. *Let G_1 be a tensor, and V_1 be the solution of the following system,*

$$
\begin{cases}
V_{1t} - \triangle V_1 + \nabla P_1 = \nabla \cdot G_1, \\
\nabla \cdot V_1 = 0, \\
t = 0: \;\; V_1 = 0,
\end{cases} \tag{2.17}
$$

then we have

$$
\|V_1\|_{X_T} \lesssim \|G_1\|_{Y_T}. \tag{2.18}
$$

Proof. First, we rewrite the system (2.17) as an integral equation

$$
V_1(t) = \int_0^t S(t-\tau)\mathbb{P}\nabla\cdot G_1 d\tau. \tag{2.19}
$$

When $0 \le \tau \le t/2$, by using Lemma 2.1, we get:

$$
\begin{aligned}
&t^{\frac{1}{2}}\left\|\int_0^{t/2} S(t-\tau)\mathbb{P}\nabla\cdot G_1\, d\tau\right\|_{L^\infty(\mathbb{R}^n)} \\
&\lesssim t^{\frac{1}{2}}\left\|\int_0^{t/2}\int_{\mathbb{R}^n}\frac{1}{(\sqrt{t-\tau}+|y-\tilde{y}|)^{n+1}} G_1\, d\tilde{y}\, d\tau\right\|_{L^\infty(\mathbb{R}^n)} \\
&\lesssim \left\|\frac{1}{\sqrt{t}^n}\int_0^{t/2}\int_{\mathbb{R}^n}\frac{1}{(1+\frac{|y-\tilde{y}|}{\sqrt{t-\tau}})^{n+1}}|G_1|\, d\tilde{y}\, d\tau\right\|_{L^\infty(\mathbb{R}^n)} \\
&\lesssim \left\|\frac{1}{\sqrt{t}^n}\sum_{q=0}^{\infty}\int_0^{t/2}\int_{q\le\frac{|y-\tilde{y}|}{\sqrt{t-\tau}}\le q+1}\frac{|G_1|}{(1+q)^{n+1}}\, d\tilde{y}\, d\tau\right\|_{L^\infty(\mathbb{R}^n)} \\
&\lesssim \left\|\sum_{q=0}^{\infty}\frac{q^{n-1}}{(1+q)^{n+1}}\frac{1}{\sqrt{t}^n}\int_0^{t/2}\int_{B(y,\sqrt{t})}|G_1|\, d\tilde{y}\, d\tau\right\|_{L^\infty(\mathbb{R}^n)} \\
&\lesssim \|G_1\|_{Y_T}.
\end{aligned} \tag{2.20}
$$

If $\frac{t}{2} \le \tau \le t$, we get

$$
|S(t-\tau)\mathbb{P}\nabla\cdot G_1| \lesssim \left|\int_{\mathbb{R}^n}\frac{1}{(\sqrt{t-\tau}+|y-\tilde{y}|)^{n+1}} d\tilde{y}\right| \|G_1\|_{L^\infty(\mathbb{R}^n)} \lesssim \frac{\|G_1\|_{L^\infty(\mathbb{R}^n)}}{\sqrt{t-\tau}}. \tag{2.21}
$$

Therefore we have

$$
t^{\frac{1}{2}}\left\|\int_{t/2}^t S(t-\tau)\mathbb{P}\nabla\cdot G_1\, d\tau\right\|_{L^\infty(\mathbb{R}^n)} \lesssim \int_{t/2}^t \frac{1}{\sqrt{\tau}\sqrt{t-\tau}}\tau\|G_1\|_{L^\infty(\mathbb{R}^n)}\, d\tau \lesssim \|G_1\|_{Y_T}. \tag{2.22}
$$

We still need to estimate the term $\sup\limits_{y_0\in\mathbb{R}^n, r>0} \frac{1}{r^n}\int_0^{r^2}\int_{B(y_0,r)} |V_1(s,y)|^2 dyds$. For any given $y_0 \in \mathbb{R}^n$ and $r > 0$, take a smooth cut-off function

$$\chi\left(\frac{y}{r}\right) = \begin{cases} 1, & |y-y_0| \le 3r, \\ 0, & |y-y_0| \ge 5r. \end{cases} \tag{2.23}$$

Then, it is sufficient to estimate the following term

$$\begin{aligned} I = & \sup_{y_0\in\mathbb{R}^n, r>0} \left(r^{-n} \int_{Q(y_0,r)} \left(\nabla \int_0^t S(t-\tau)\mathbb{P}\chi G_1 \, d\tau \right)^2 dydt \right)^{\frac{1}{2}} \\ & + \sup_{y_0\in\mathbb{R}^n, r>0} \left(r^{-n} \int_{Q(y_0,r)} \left(\nabla \int_0^t S(t-\tau)\mathbb{P}(1-\chi) G_1 \, d\tau \right)^2 dydt \right)^{\frac{1}{2}} \\ & \triangleq I_1 + I_2. \end{aligned} \tag{2.24}$$

To deal with I_1, we will drop the projector $\mathbb{P}$, which is a bounded operator in L^2 and which commutes with $S(t-\tau)$, ∇ and the integral about t. We set up a heat function

$$W_t - \Delta W = \chi G_1, \qquad W\big|_{t=0} = 0. \tag{2.25}$$

Then $I_1 = \sup\limits_{y_0\in\mathbb{R}^n, r>0} \left(r^{-n} \|\nabla W\|^2_{L^2(Q(y_0,r))} \right)^{\frac{1}{2}}$. By Lemma 2.3, we get

$$\|\nabla W\|^2_{L^2(Q(y_0,r))} \le \int_0^{r^2} \int_{\mathbb{R}^n} (\nabla W)^2 dyd\tau \lesssim \int_0^{r^2} \int_{\mathbb{R}^n} |\chi G_1 W| dyd\tau. \tag{2.26}$$

For the term $\int_0^{r^2}\int_{\mathbb{R}^n} |\chi G_1 W| dydt$, recalling that $\chi = \chi(y/r)$ and $t \le r^2$, we have

$$\int_0^{r^2} \int_{\mathbb{R}^n} |\chi G_1 W| dyd\tau \lesssim \|W\|_{L^\infty(Q(y_0,5r))} \int_0^{r^2} \int_{\mathbb{R}^n} |\chi G_1| dyd\tau, \tag{2.27}$$

where

$$\|W\|_{L^\infty(Q(y_0,5r))} = \left\| \int_0^{\tau/2} S(\tau-s)\chi G_1 ds + \int_{\tau/2}^{\tau} S(\tau-s)\chi G_1 ds \right\|_{L^\infty(Q(y_0,5r))}. \tag{2.28}$$

We shall study the above inequality separately.

By Lemma 2.2, we have

$$\begin{aligned} & \left\| \int_0^{\tau/2} S(\tau-s) G_1 \, ds \right\|_{L^\infty(Q(y_0,5r))} \\ \lesssim & \left\| \frac{1}{\sqrt{\tau}^n} \int_0^{\tau/2} \sum_{q=0}^{\infty} \int_{q \le \frac{|y-\tilde{y}|}{\sqrt{\tau}} \le q+1} e^{-q^2} G_1 \, d\tilde{y} ds \right\|_{L^\infty(Q(y_0,5r))} \\ \lesssim & \sup_{y\in\mathbb{R}^n, 0<\tau\le T^*} \frac{1}{\sqrt{\tau}^n} \int_0^{\tau} \int_{B(y,\sqrt{\tau})} |G_1| \, d\tilde{y} ds \\ \lesssim & \|G_1\|_{Y_{T^*}}. \end{aligned} \tag{2.29}$$

Recalling the definition of the cut-off function χ, we have

$$\begin{aligned}
&\Big\| \int_{\tau/2}^{\tau}\int_{\mathbb{R}^n} \frac{1}{\sqrt{\tau-s}^n} e^{-\frac{(y-\tilde{y})^2}{4(\tau-s)}} \chi G_1\, d\tilde{y}ds \Big\|_{L^\infty(Q(y_0,5r))} \\
\lesssim &\int_{\tau/2}^{\tau}\int_{\mathbb{R}^n} \frac{1}{\sqrt{\tau-s}^n} e^{-\frac{(y-\tilde{y})^2}{4(\tau-s)}} s^{-1}\, d\tilde{y}ds \cdot \tau \|G_1\|_{L^\infty(Q(y_0,5r))} \\
\lesssim &\|G_1\|_{Y_{T^*}} \int_{\tau/2}^{\tau} \frac{1}{s} ds \\
\lesssim &\|G_1\|_{Y_{T^*}}. \qquad (2.30)
\end{aligned}$$

To finish the estimate of I_1, from (2.27), we still need to estimate

$$r^{-n}\int_0^{r^2}\int_{\mathbb{R}^n} |\chi G_1| dy d\tau \lesssim r^{-n}\int_0^{r^2}\int_{B(y_0,5r)} |G_1| dy d\tau \lesssim \|G_1\|_{Y_T}. \qquad (2.31)$$

Combining (2.26)-(2.31), we get

$$I_1 \lesssim \|G_1\|_{Y_T}. \qquad (2.32)$$

As to the term I_2, we have

$$I_2^2 = \sup_{0<r\le\sqrt{T}} r^2 \Big\| \nabla \int_0^t S(t-\tau)\mathbb{P}(1-\chi)G_1\, d\tau \Big\|^2_{L^\infty(Q(y_0,r))}. \qquad (2.33)$$

When $0\le\tau\le t\le (r/2)^2$, noting the cut-off function $1-\chi$ and Lemma 2.1, we have

$$\begin{aligned}
&\sup_{y_0\in\mathbb{R}^n, 0<r\le\sqrt{T}} r^2 \Big\| \nabla \int_0^t S(t-\tau)\mathbb{P}(1-\chi)G_1\, d\tau \Big\|^2_{L^\infty(Q(y_0,r))} \\
\lesssim &\sup_{y_0\in\mathbb{R}^n, 0<r\le\sqrt{T}} r^2 \Big\| \int_0^t \int_{\mathbb{R}^n} \frac{1-\chi(\frac{\tilde{y}}{r})}{(\sqrt{t-\tau}+|y-\tilde{y}|)^{n+1}} |G_1|\, d\tilde{y}d\tau \Big\|^2_{L^\infty(Q(y_0,r))} \\
\lesssim &\sup_{y_0\in\mathbb{R}^n, 0<r\le\sqrt{T}} r^2 \Big\| \int_0^t \sum_{q=1}^{\infty} \int_{qr\le|y-\tilde{y}|\le(q+1)r} \frac{|G_1|}{(qr)^{n+1}}\, d\tilde{y}d\tau \Big\|^2_{L^\infty(Q(y_0,r))} \\
\lesssim &\sup_{y_0\in\mathbb{R}^n, 0<r\le\sqrt{T}} \left(\frac{1}{r^n} \int_0^{r^2} \int_{B(y,r)} |G_1|\, d\tilde{y}d\tau \right)^2 \\
\lesssim &\|G_1\|^2_{Y_T}. \qquad (2.34)
\end{aligned}$$

For the remaining part $(r/2)^2 < t < r^2$, we shall divide into two parts to estimate.

(i): When $(r/2)^2 < t < r^2$ and $0\le\tau\le t/2$, by Lemma 2.1, we have

$$\begin{aligned}
&\sup_{y_0\in\mathbb{R}^n, 0<r\le\sqrt{T}} r^2 \Big\| \nabla \int_0^{t/2} S(t-\tau)\mathbb{P}(1-\chi)G_1\, d\tau \Big\|^2_{L^\infty(Q(y_0,r))} \\
\lesssim &\sup_{y_0\in\mathbb{R}^n, 0<r\le\sqrt{T}} r^2 \Big\| \int_0^{t/2} \int_{\mathbb{R}^n} \frac{(1-\chi)}{(\sqrt{t-\tau}+|y-\tilde{y}|)^{n+1}} |G_1(\tau,\tilde{y})|\, d\tilde{y}d\tau \Big\|^2_{L^\infty(Q(y_0,r))} \\
\lesssim &\sup_{y_0\in\mathbb{R}^n, 0<r\le\sqrt{T}} \Big\| \frac{1}{\sqrt{t}^n} \int_0^{t/2} \sum_{q=0}^{\infty} \int_{q\le\frac{|y-\tilde{y}|}{\sqrt{t}}\le q+1} \frac{|G_1(\tau,\tilde{y})|}{(1+q)^{n+1}}\, d\tilde{y}d\tau \Big\|^2_{L^\infty(Q(y_0,r))}
\end{aligned}$$

$$\lesssim \sup_{y_0\in\mathbb{R}^n, 0<r\le\sqrt{T}} \left\| \frac{1}{\sqrt{t}^n} \int_0^{t/2} \sum_{q=0}^{\infty} \frac{q^{n-1}}{(1+q)^{n+1}} \int_{B(y,\sqrt{t})} |G_1(\tau,\tilde{y})|\, d\tilde{y} d\tau \right\|^2_{L^\infty(Q(y_0,r))}$$
$$\lesssim \|G_1\|^2_{Y_T}. \tag{2.35}$$

(ii): When $(r/2)^2 < t < r^2$ and $t/2 \le \tau \le t$, we have

$$|r(1-\chi)G_1| \lesssim \sqrt{\tau}|G_1| \le \frac{1}{\sqrt{\tau}}\|G_1\|_{Y_T}. \tag{2.36}$$

Then by (2.36) and Lemma 2.1, we get

$$\sup_{y_0,r>0} r^2 \left\|\nabla\cdot \int_{t/2}^t S(t-\tau)\mathbb{P}(1-\chi)G_1\, d\tau\right\|^2_{L^\infty(Q(y_0,r))}$$
$$\lesssim \|G_1\|^2_{Y_T}\left(\int_{t/2}^t \frac{1}{\sqrt{\tau}}\int_{\mathbb{R}^n} \frac{1}{(\sqrt{t-\tau}+|\tilde{y}|)^{n+1}}\, d\tilde{y} d\tau\right)^2$$
$$\lesssim \|G_1\|^2_{Y_T}\left(\int_{t/2}^t \frac{1}{\sqrt{\tau}}\int_{\mathbb{R}^n} \frac{1}{\sqrt{t-\tau}^{\,n+1}(1+\frac{|\tilde{y}|}{\sqrt{t-\tau}})^{n+1}}\, d\tilde{y} d\tau\right)^2$$
$$\lesssim \|G_1\|^2_{Y_T}\left(\int_{t/2}^t \frac{1}{\sqrt{\tau}}\frac{1}{\sqrt{t-\tau}}\, d\tau \int_{\mathbb{R}^n} \frac{1}{(1+|\bar{y}|)^{n+1}}\, d\bar{y}\right)^2$$
$$\lesssim \|G_1\|^2_{Y_T}. \tag{2.37}$$

Then from (2.33)-(2.37), we can get

$$I_2 \lesssim \|G_1\|^2_{Y_T}. \tag{2.38}$$

□

Corollary 2.2. *Under the same conditions of Proposition 2.2, for any integers $m,k\ge 0$ we have*

$$\|t^{\frac{k}{2}+m}\partial_t^m\nabla^k V_1\|_{X_{T^*}} \lesssim \|G_1\|_{Y_{T^*}}. \tag{2.39}$$

Proof. This result can be verified similarly to Corollary 2.1. See also [9] and [3]. □

Proposition 2.3. *Let $\int_{\mathbb{T}^3(x)} G_2(y,\cdot)\,dy = 0$, and V_2 be the solution of the following system:*

$$\begin{cases} V_{2t} - \Delta V_2 + \nabla P_2 = G_2, \\ \nabla\cdot V_2 = 0, \\ t=0:\ \ V_2 = 0, \end{cases} \tag{2.40}$$

then for any positive integers m and k, we have

$$\|t^{\frac{m}{2}+k}\partial_t^k\nabla^m V_2\|_{X_T} \lesssim \|t^{\frac{m}{2}+k}\partial_t^k\nabla^m G_2\|_{Y_T}. \tag{2.41}$$

Proof. It is sufficient to prove the case when $m=k=0$. For any given $r^*>0$, we have

$$G_2(x,t) = \fint_{\mathbb{T}^n(x)} \big(G_2(t,x)-G_2(t,y)\big)dy$$
$$= -\fint_{\mathbb{T}^n(0)}\int_0^1 \frac{d}{ds}G(t,x+sy)\, dsdy$$

$$= -\nabla \cdot \left(\int_0^1 \fint_{\mathbb{T}^n(0)} G(t, x+sz) z \, dz ds \right)$$
$$\triangleq \nabla \cdot \tilde{G}_2(x,t), \tag{2.42}$$

where

$$\tilde{G}_2 = -\int_0^1 \fint_{\mathbb{T}^n(0)} G_2(x+sz) z \, dz ds. \tag{2.43}$$

Now by using Proposition 2.2, we get

$$\|V_2\|_{X_T} \lesssim \|\tilde{G}_2\|_{Y_T}. \tag{2.44}$$

For the term $\|\tilde{G}_2\|_{Y_T}$, we have

$$\begin{aligned} \sup_{0<t\leq T} t\|\tilde{G}_2\|_{L^\infty(\mathbb{R}^n)} &= \sup_{0<t\leq T} t \left\| \int_0^1 \fint_{\mathbb{T}^n(0)} G_2(t, x+sz) z \, dz ds \right\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \sup_{0<t\leq T} t\|G_2\|_{L^\infty(\mathbb{R}^n)}, \end{aligned} \tag{2.45}$$

and

$$\begin{aligned} &\sup_{0<r\leq\sqrt{T}} \frac{1}{r^n} \int_0^{r^2} \int_{B(x,r)} |\tilde{G}_2| \, dy d\tau \\ &= \sup_{0<r\leq\sqrt{T}} \frac{1}{r^n} \int_0^{r^2} \int_{B(y_0,r)} \left(\int_0^1 \fint_{\mathbb{T}^n(0)} |G_2(\tau, x+sz) z| \, dz ds \right) dx d\tau \\ &= \sup_{0<r\leq\sqrt{T}} \int_0^1 \frac{1}{r^n} \int_0^{r^2} \fint_{\mathbb{T}^n(0)} |z| \int_{B(x,r)} |G_2(\tau, x+sz)| \, dx dz d\tau ds \\ &\lesssim \sup_{0<r\leq\sqrt{T}} \frac{1}{r^n} \int_0^{r^2} \int_{B(x,r)} |G_2(\tau, y)| \, dy d\tau. \end{aligned} \tag{2.46}$$

Combining (2.44)–(2.46), this proposition is proved. □

3. Proof of the Theorem 1.2

Write the solution U of (1.1) as

$$U = u + v, \quad B = b + r, \tag{3.1}$$

where $u = e^{t\Delta} u_0$ and $b = e^{t\Delta} b_0$. Then from the system (1.1) we get

$$\begin{aligned} v_t - \Delta v + \nabla p = &-(v \cdot \nabla) v - u \cdot \nabla v - v \cdot \nabla u - u \cdot \nabla u \\ &+ b \cdot \nabla b + b \cdot \nabla r + r \cdot \nabla b + r \cdot \nabla r, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} r_t - \Delta r = &-u \cdot \nabla b - u \cdot \nabla r - v \cdot \nabla b - v \cdot \nabla r \\ &+ b \cdot \nabla u + b \cdot \nabla v + r \cdot \nabla u + r \cdot \nabla v. \end{aligned} \tag{3.3}$$

Now, we are ready to prove the main results of our paper. We shall study our problem by the classical fixed point argument. First, we introduce the following space:

Definition 3.1. Let $0 < \epsilon_0 \ll \epsilon \ll 1$ and $T > 0$ be given constants. We say $(f, g) \in \mathbb{E}_{T,\epsilon}$ if the following hold:

(i) $f(x)$ and $g(x)$ are periodic functions on $\mathbb{T}^3$;

(ii) $\|(f,g)\|_{\mathbb{E}_{T,\epsilon}} \triangleq \|f\|_{\mathbb{X}_T} + \|g\|_{\mathbb{X}_T} < \epsilon$;

(iii) $\nabla \cdot f = \nabla \cdot g = 0 \quad \text{and} \quad \|\nabla \times f - \lambda f\|_{\mathbb{X}_T}, \|\nabla \times g - \lambda g\|_{\mathbb{X}_T}, \|f - g\|_{\mathbb{X}_T} < \epsilon_0.$

We define $v = \mathfrak{F}\tilde{v}$ and $r = \mathfrak{F}\tilde{r}$ by solving the following linear equations with $\tilde{v}, \tilde{r} \in \mathbb{E}_{T,\epsilon}$:

$$\begin{aligned}
&v_t - \Delta v + \nabla p \\
&= -u \cdot \nabla u - \tilde{v} \cdot \nabla \tilde{v} - u \cdot \nabla \tilde{v} - \tilde{v} \cdot \nabla u + b \cdot \nabla b + b \cdot \nabla \tilde{r} + \tilde{r} \cdot \nabla b + \tilde{r} \cdot \nabla \tilde{r} \\
&= u \times (\nabla \times u - \lambda_1 u) - \nabla \cdot (\tilde{v} \otimes \tilde{v} + u \otimes \tilde{v} + \tilde{v} \otimes u) - b \times (\nabla \times b - \lambda_2 b) \\
&\quad + \nabla \cdot (\tilde{r} \otimes \tilde{r} + b \otimes \tilde{r} + \tilde{r} \otimes b) - \nabla \left(\frac{|u|^2 - |b|^2}{2} \right) \\
&\triangleq F_1 - \nabla \left(\frac{|u|^2 - |b|^2}{2} \right),
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
r_t - \Delta r &= -u \cdot \nabla b - u \cdot \nabla \tilde{r} - \tilde{v} \cdot \nabla b - \tilde{v} \cdot \nabla \tilde{r} + b \cdot \nabla u + b \cdot \nabla \tilde{v} + \tilde{r} \cdot \nabla u + \tilde{r} \cdot \nabla \tilde{v} \\
&= \nabla \cdot \big(b \otimes u - u \otimes b + b \otimes \tilde{v} - \tilde{v} \otimes b + \tilde{r} \otimes u - u \otimes \tilde{r} + \tilde{r} \otimes \tilde{v} - \tilde{v} \otimes \tilde{r} \big) \\
&= -\nabla \times [(u + \tilde{v}) \times (b + \tilde{r})] \\
&\triangleq F_2.
\end{aligned} \tag{3.5}$$

The proof of our main theorem.

Proof. We shall prove the main theorem by using the fixed point argument. For simplicity, we write the solution of (3.4)-(3.5) as $(v, r) = \mathfrak{F}(\tilde{v}, \tilde{r})$. Before that, we give the following proposition.

Proposition 3.1. *When $(\tilde{v}, \tilde{r}) \in \mathbb{E}_{T,\epsilon}$, then there exists a uniform constant C_0, for $T_0 = \frac{1}{1+4M_0^2C_0^2}$, we have*

$$\|v - r\|_{X_{T_0}} < \epsilon_0. \tag{3.6}$$

Proof. By using (3.4) and (3.5), we have

$$\begin{aligned}
&\partial_t (v - r) - \Delta (v - r) + \nabla P \\
&= -[(u + \tilde{v}) \cdot \nabla][(u + \tilde{v}) - (b + \tilde{r})] - [(b + \tilde{r}) \cdot \nabla][(u + \tilde{v}) - (b + \tilde{r})] \\
&= -(u \cdot \nabla)(u - b) - (u \cdot \nabla)(\tilde{v} - \tilde{r}) - (\tilde{v} \cdot \nabla)(u - b) - (\tilde{v} \cdot \nabla)(\tilde{v} - \tilde{r}) \\
&\quad - (b \cdot \nabla)(u - b) - (b \cdot \nabla)(\tilde{v} - \tilde{r}) - (\tilde{r} \cdot \nabla)(u - b) - (\tilde{r} \cdot \nabla)(\tilde{v} - \tilde{r}).
\end{aligned} \tag{3.7}$$

Then by Proposition 2.2, we get

$$\begin{aligned}
\|\tilde{v} - r\|_{X_T} &\lesssim \|(|u| + |b|)(|u - b| + |\tilde{v} - \tilde{r}|)\|_{Y_T} + \|(|\tilde{v}| + |\tilde{r}|)(|u - b| + |\tilde{v} - \tilde{r}|)\|_{Y_T} \\
&\lesssim T^{\frac{1}{2}} (\|u\|_{L^\infty} + \|b\|_{L^\infty})(\|u - b\|_{X_T} + \|\tilde{v} - \tilde{r}\|_{X_T}) \\
&\quad + (\|\tilde{v}\|_{X_T} + \|\tilde{r}\|_{X_T})(\|u - b\|_{X_T} + \|\tilde{v} - \tilde{r}\|_{X_T}) \\
&\lesssim \epsilon\epsilon_0 + T^{\frac{1}{2}} M_0 \epsilon_0 \\
&\le C_0 \epsilon\epsilon_0 + C_0 T^{\frac{1}{2}} M_0 \epsilon_0,
\end{aligned} \tag{3.8}$$

where C_0 is a uniform constant. By taking ϵ small enough and $T_0 = \frac{1}{1+4M_0^2C_0^2}$, we proved this proposition. □

3.1. Estimating the bound for $(v, r) = \mathfrak{F}(\tilde{v}, \tilde{r})$

At the beginning, we give the following prepare work.

Proposition 3.2. *When $(\tilde{v}, \tilde{r}) \in \mathbb{E}_{T,\epsilon}$, then we have*

$$\int_{\mathbb{T}^3} F_1(x)dx = \int_{\mathbb{T}^3} F_2(x)dx = 0. \tag{3.9}$$

Proof. By noting the periodic boundary condition, it is sufficient to estimate $\int_{\mathbb{T}^3} u \times (\nabla \times u - \lambda u)dx$.

$$\int_{\mathbb{T}^3} u \times (\nabla \times u - \lambda u)dx = \int_{\mathbb{T}^3} \nabla \frac{|u|^2}{2} - \nabla \cdot (u \otimes u)dx = 0. \tag{3.10}$$

□

Recall that the initial data (u_0, b_0) satisfying (1.6) and (1.7), for $\forall T > 0$, $(\tilde{v}, \tilde{r}) \in \mathbb{E}_{T,\epsilon}$, by using Proposition 2.3 and Proposition 3.2, we have

$$\begin{aligned}
\|v\|_{X_T} &\lesssim \|u \times (\nabla \times u - \lambda u)\|_{Y_T} + \|b \times (\nabla \times \tilde{v} - \lambda \tilde{v})\|_{Y_T} \\
&\quad + \|\tilde{v} \times (\nabla \times \tilde{v} - \lambda \tilde{v})\|_{Y_T} + \|u \times (\nabla \times \tilde{v} - \lambda \tilde{v})\|_{Y_T} \\
&\quad + \|\tilde{r} \times (\nabla \times \tilde{r} - \lambda \tilde{r})\|_{Y_T} + \|\tilde{v} \times (\nabla \times u - \lambda u)\|_{Y_T} \\
&\quad + \|b \times (\nabla \times \tilde{r} - \lambda \tilde{r})\|_{Y_T} + \|\tilde{r} \times (\nabla \times b - \lambda b)\|_{Y_T} \\
&\lesssim (\|u\|_{X_T} + \|\tilde{v}\|_{X_T})(\|\nabla \times u - \lambda u\|_{X_T}) \\
&\quad + (\|u\|_{X_T} + \|\tilde{v}\|_{X_T} + \|b\|_{X_T})(\|\nabla \times \tilde{v} - \lambda \tilde{v}\|_{X_T}) \\
&\quad + (\|\tilde{r}\|_{X_T} + \|b\|_{X_T})(\|\nabla \times \tilde{r} - \lambda \tilde{r}\|_{X_T}) \\
&\quad + \|\tilde{r}\|_{X_T} \|\nabla \times b - \lambda b\|_{X_T}.
\end{aligned} \tag{3.11}$$

To estimate the term $\|\nabla \times u - \lambda_1 u\|_{X_T}$, noting that

$$\nabla \times u - \lambda_1 u = e^{t\Delta}(\nabla \times u_0 - \lambda u_0), \tag{3.12}$$

we set up the system as

$$\begin{cases} g_t - \Delta g = 0, \\ t = 0: \ \ g = \nabla \times u_0 - \lambda u_0, \end{cases} \tag{3.13}$$

then following Proposition 2.1, we have

$$\|\nabla \times u - \lambda u\|_{X_T} \lesssim \|\nabla \times u_0 - \lambda u_0\|_{\mathrm{BMO}^{-1}(\mathbb{R}^3)} \lesssim \epsilon_0. \tag{3.14}$$

Similarly, we have

$$\|\nabla \times b - \lambda b\|_{X_T} \lesssim \|\nabla \times b_0 - \lambda b_0\|_{\mathrm{BMO}^{-1}(\mathbb{R}^3)} \lesssim \epsilon_0. \tag{3.15}$$

Combining (3.11)–(3.15), we get the bound as

$$\|v\|_{X_T} \lesssim \epsilon_0 M_0 + \epsilon\epsilon_0 < \epsilon, \tag{3.16}$$

where we take ϵ_0 small enough such that $M_0\epsilon_0 \leq \epsilon/2$.

Similarly, we can get

$$\|r\|_{X_T} \lesssim \|u \times b\|_{Y_T} + \|u \times \tilde{r} + \tilde{v} \times b\|_{Y_T} + \|\tilde{v} \times \tilde{r}\|_{Y_T}$$

$$\lesssim \|(u-b)\times b\|_{Y_T} + \|(u-b)\times \tilde{r}\|_{Y_T} + \|(\tilde{v}-\tilde{r})\times b\|_{Y_T} + \|(\tilde{v}-\tilde{r})\times \tilde{r}\|_{Y_T}$$
$$\lesssim (\|\tilde{v}-\tilde{r}\|_{X_T} + \|u-b\|_{X_T})(\|b\|_{X_T} + \|\tilde{r}\|_{X_T}). \tag{3.17}$$

Similar to (3.13)–(3.14), we get

$$\|u-b\|_{X_T} = \|e^{t\Delta}(u_0-b_0)\|_{X_T} \lesssim \|u_0-b_0\|_{\mathrm{BMO}^{-1}(\mathbb{R}^3)} \lesssim \epsilon_0. \tag{3.18}$$

Combining (3.17)–(3.18), by taking ϵ_0 small enough, we get

$$\|r\|_{X_T} < \epsilon. \tag{3.19}$$

3.2. Proving that $(v,r)=\mathfrak{F}(\tilde{v},\tilde{r})$ is a contracting map

By a similar process, let (v_1,r_1) be the corresponding solution to (3.4)–(3.5) when $(\tilde{v}_1,\tilde{r}_1)\in E_{T,\epsilon}$ and (v_2,r_2) be the solution with $(\tilde{v}_2,\tilde{r}_2)\in E_{T,\epsilon}$.

Denote $\bar{\tilde{v}}=\tilde{v}_1-\tilde{v}_2$, $\bar{\tilde{r}}=\tilde{r}_1-\tilde{r}_2$, $\bar{v}=v_1-v_2$, and $\bar{r}=r_1-r_2$; then there holds:

$$\bar{v}_t - \Delta\bar{v} = -(\tilde{v}_2\cdot\nabla)\bar{\tilde{v}} - (\bar{\tilde{v}}\cdot\nabla)\tilde{v}_1 - (u\cdot\nabla)\bar{\tilde{v}} - (\bar{\tilde{v}}\cdot\nabla)u$$
$$+ (b\cdot\nabla)\bar{\tilde{r}} + (\bar{\tilde{r}}\cdot\nabla)b + (\bar{\tilde{r}}\cdot\nabla)\tilde{r}_1 + (\tilde{r}_2\cdot\nabla)\bar{\tilde{r}}. \tag{3.20}$$

By using Proposition 2.2, we have

$$\|\bar{v}\|_{X_T} \lesssim (\|\tilde{v}_2\|_{X_T} + \|\tilde{v}_1\|_{X_T} + \|\tilde{r}_2\|_{X_T} + \|\tilde{r}_1\|_{X_T}$$
$$+ T^{1/2}(\|u\|_{L^\infty} + \|b\|_{L^\infty}))(\|\bar{\tilde{r}}\|_{X_T} + \|\bar{\tilde{r}}\|_{X_T})$$
$$\le C_1(\epsilon + T^{1/2}M_0)(\|\bar{\tilde{r}}\|_{X_T} + \|\bar{\tilde{r}}\|_{X_T}), \tag{3.21}$$

where C_1 is a uniform constant.

Similarly, we also have

$$\|\bar{r}\|_{X_T} \lesssim (\|\tilde{v}_2\|_{X_T} + \|\tilde{v}_1\|_{X_T} + \|\tilde{r}_2\|_{X_T} + \|\tilde{r}_1\|_{X_T}$$
$$+ T^{1/2}(\|u\|_{L^\infty} + \|b\|_{L^\infty}))(\|\bar{\tilde{r}}\|_{X_T} + \|\bar{\tilde{r}}\|_{X_T})$$
$$\le C_1(\epsilon + T^{1/2}M_0)(\|\bar{\tilde{r}}\|_{X_T} + \|\bar{\tilde{r}}\|_{X_T}). \tag{3.22}$$

For ϵ small enough, we take

$$T_1 = \frac{1}{1+4M_0^2C_1^2}, \tag{3.23}$$

then for $t\in[0,T_1]$, from (3.21) and (3.22), we have

$$\|\bar{r}\|_{X_t} + \|\bar{v}\|_{X_t} \le \|\bar{\tilde{v}}\|_{X_t} + \|\bar{\tilde{r}}\|_{X_t}. \tag{3.24}$$

3.3. Estimate of $\|\nabla\times v-\lambda v\|_{X_T}$ and $\|\nabla\times r-\lambda r\|_{X_T}$

By using (3.4), we get

$$(\nabla\times v-\lambda v)_t - \Delta(\nabla\times v-\lambda v) - \lambda\nabla\left(p + \frac{|u|^2}{2} + \frac{|\tilde{v}|^2}{2} + u\cdot\tilde{v} - \frac{|b|^2}{2} - \frac{|\tilde{r}|^2}{2} - b\cdot\tilde{r}\right)$$
$$= \nabla\times\Big(u\times(\nabla\times u-\lambda u) + \tilde{v}\times(\nabla\times\tilde{v}-\lambda\tilde{v}) + u\times(\nabla\times\tilde{v}-\lambda\tilde{v})$$

$$
\begin{aligned}
&+ \tilde{v} \times (\nabla \times u - \lambda u) - b \times (\nabla \times b - \lambda b) - b \times (\nabla \times \tilde{r} - \lambda \tilde{r}) \\
&- \tilde{r} \times (\nabla \times b - \lambda b) + \tilde{r} \times (\nabla \times \tilde{r} - \lambda \tilde{r})\Big) \\
&- \lambda\Big(u \times (\nabla \times u - \lambda u) + \tilde{v} \times (\nabla \times \tilde{v} - \lambda \tilde{v}) + u \times (\nabla \times \tilde{v} - \lambda \tilde{v}) \\
&+ \tilde{v} \times (\nabla \times u - \lambda u) - b \times (\nabla \times b - \lambda b) - b \times (\nabla \times \tilde{r} - \lambda \tilde{r}) \\
&- \tilde{r} \times (\nabla \times b - \lambda b) + \tilde{r} \times (\nabla \times \tilde{r} - \lambda \tilde{r})\Big),
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
&(\nabla \times r - \lambda r)_t - \Delta(\nabla \times r - \lambda r) \\
&= \nabla \times \Big(u \times (b-u) + u \times \tilde{r} - (b-u) \times \tilde{v} - u \times \tilde{r} + (\tilde{v} - \tilde{r}) \times \tilde{r}\Big).
\end{aligned}
\tag{3.26}
$$

Following Proposition 2.2 and Proposition 2.3, we get

$$
\begin{aligned}
&\|\nabla \times v - \lambda v\|_{X_T} \\
&\lesssim (1+|\lambda|)\|u \times (\nabla \times u - \lambda u) + \tilde{v} \times (\nabla \times \tilde{v} - \lambda \tilde{v}) + u \times (\nabla \times \tilde{v} - \lambda \tilde{v})\|_{Y_T} \\
&\leq C_2(1+|\lambda|)(M_0 T^{\frac{1}{2}} + \epsilon)\epsilon_0,
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
&\|\nabla \times r - \lambda r\|_{X_T} \\
&\lesssim (1+|\lambda|)\|u \times (b-u) + u \times \tilde{r} - (b-u) \times \tilde{v} - u \times \tilde{r} + (\tilde{v} - \tilde{r}) \times \tilde{r}\|_{Y_T} \\
&\leq C_2(1+|\lambda|)(M_0 T^{\frac{1}{2}} + \epsilon)\epsilon_0,
\end{aligned}
\tag{3.28}
$$

where C_2 is a uniform constant.

For ϵ small enough, we take

$$
T_2 = \frac{1}{1 + 4\lambda^2 M_0^2 C_2^2},
\tag{3.29}
$$

then when $T < T_3$, we have

$$
\|\nabla \times v - \lambda v\|_{X_T}, \|\nabla \times r - \lambda r\|_{X_T} < \epsilon_0.
\tag{3.30}
$$

Taking

$$
T = \min\{T_0, T_1, T_2\},
\tag{3.31}
$$

then from (3.16), (3.19), (3.24), and (3.30), the system (3.2)–(3.3) admits a unique pair of solutions on $[0, T]$.

3.4. Extension of the local solution

Based on the computations above, at time $t = T$, we have

(i) $(v(x,T), r(x,T))$ is a pair of periodic functions on $\mathbb{T}^3$;

(ii) $\|(v(x,T), r(x,T))\|_{\mathrm{BMO}^{-1}} < \epsilon$;

(iii) $\nabla \cdot v = \nabla \cdot r = 0$ and $\|\nabla \times v - \lambda v\|_{\mathrm{BMO}^{-1}}$, $\|\nabla \times r - \lambda r\|_{\mathrm{BMO}^{-1}}$, $\|v - r\|_{\mathrm{BMO}^{-1}} < \epsilon_0$.

Then we take $(v(x,T), r(x,T))$ as the initial data and repeat the process above; we can thus obtain the global well-posedness. □

4. Time and spatial analyticity

In this section, we prove Theorem 1.3 and Corollary 1.1. The argument follows a very similar procedure to the proof of Theorem 1.2. Before proceeding, we first define the following:

Definition 4.1. Let g be a function defined on $\mathbb{R}^n \times [0, T^*)$ $(0 < T^* \leq +\infty)$. For any integers $M, K \geq 0$, we say $g \in \mathbb{X}_{T^*}^{M,K}$ if

$$\|g\|_{\mathbb{X}_{T^*}^{M,K}} = \sum_{m=0}^{M}\sum_{k=0}^{K}\Bigg(\sup_t t^{\frac{k+1}{2}+m}\|\partial_t^m\nabla^k g\|_{L^\infty(\mathbb{R}^3)} + \sup_{y_0\in\mathbb{R}^3, r>0}\left(r^{-3}\int_{Q(y_0,r)}\left|t^{\frac{k}{2}+m}\partial_t^m\nabla^k g\right|^2 dydt\right)^{\frac{1}{2}}\Bigg) < +\infty. \tag{4.1}$$

For $M, K = 0$, $\mathbb{X}^{0,0}$ (simply denoted as $\mathbb{X}$) is the Koch-Tataru space; for $M = 0, K \geq 0$, a similar definition can be found in [9]. It is obvious that $\mathbb{X}_{T^*}^{M,K}$ for $M, K \geq 0$ is a Banach space. Moreover, we have that $g \in \mathbb{X}_{T^*}^{M,K}$ if and only if $t^{M+\frac{K}{2}}\partial_t^M\nabla^K g \in \mathbb{X}_{T^*}$ for any g.

Subsequently, we introduce the following function spaces:

Definition 4.2. Let $0 < \epsilon \ll 1$ and $T > 0$ be given constants. For any integers $M, K > 0$, we say $f \in \mathbb{E}_{T,\epsilon}^{M,K}$ if the following conditions hold:

(i) $f(x)$ is a periodic function on $\mathbb{T}^3$;

(ii) $\|f\|_{\mathbb{E}_{T,\epsilon}^{M,K}} \triangleq \|f\|_{\mathbb{X}_T^{M,K}} < \epsilon$.

Repeating the process in Section 3, we can prove Theorem 1.3 and Corollary 1.1. For simplicity, we omit the details here.

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