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HYPERSTABILITY ANALYSIS FOR APOLLONIUS FUNCTIONAL EQUATIONS WITH FIXED POINT THEORY BASED ON A QUASI SPACE

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Abstract This paper investigates a fixed-point theorem within the framework of quasi-(m, β)-normed spaces, extending the results of Brzdek [5] and El-Fassi [6] to a more generalized setting. The extension is utilized to study the hyperstability of Apollonius-type functional equations, emphasizing the role of inequalities in stability analysis. By employing advanced fixed-point methods, we provide a comprehensive framework that highlights the interplay between inequalities and stability phenomena in functional equations. These findings contribute to the growing body of research on inequalities in functional analysis and their applications in diverse mathematical contexts.

Keywords Apollonius type functional equation, hyperstability, (m, β)-normed space, fixed point theorem.

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1. Introduction

A review of the evolution of stability theory for functional equations is essential as a starting point. The foundation of this theory dates back to 1925, when Pólya and Szegő's [27] pioneering work presented a significant stability result. Since then, numerous researchers have contributed to the development and deepening of this theory, exploring various aspects and generalizations.

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Notably, many authors have investigated the stability of functional equations under different conditions and in diverse mathematical structures [1, 11, 13, 22, 29, 30]. The question "if a solution to a slightly perturbed version of a functional equation is found, under what circumstances can we guarantee that this solution remains close to the exact solution of the original equation?" about stability of functional equation was raised by the Ulam [31] the case of group homomorphisms in 1940. In 1941, Hyers [12] provided an initial answer to Ulam's question regardg the stability of functional equations in Banach spaces. Rassias [28] extended this foundational work in 1978 and introduced a generalized version of Hyers' theorem by considering additive mappings with unbounded control functions. Rassias' extension significantly contributed to developing what is now called Hyers-Ulam-Rassias stability for functional equations. Over time, several mathematicians have explored related problems various spaces, we refer [14–19, 27] for more details. However, a number of researchers [2,5,6,24] have already used fixed point theorems to the theory of Hyers-Ulam Rassis stability, and it appears that Baker [3] is the first to employ this approach in this area of study. In the context of fixed point methods applied to stability theory, Brzdęk et al. [6] and Dung et al. [7] obtained notable results in metric and quasi-Banach spaces, while El-Fassi [8] found hyperstability results for radical-type functional equations in quasi-$(2, \beta)$-normed spaces. These contributions, while important, are often restricted to particular functional equations or certain normed spaces. In contrast, our work generalizes the fixed point framework to quasi-(m, β)-normed spaces, which include and extend many previously studied cases. Moreover, unlike earlier works that dealt with radical-type or classical additive equations, we investigate the Apollonius-type additive functional equation, for which no hyperstability results currently exist. This establishes our research as a methodological and structural extension of the existing literature.

To the best of our knowledge, the initial hyperstability finding appeared in [4] and related to ring homomorphisms. However, it appears that the word hyperstability was first used in [23] (commonly mistaken with superstability, which accepts bounded functions).

In 2008, Park and T. M. Rassias [26] introduced the following Apollonius type additive functional equation

$$f(u_1 + u_2) + 2f(u_3 - u_1) + 2f(u_3 + u_2) = 4f\left(u_3 - \frac{u_1 + u_2}{4}\right).$$

Kim and J. M. Rassias [21] investigated the functional equation in modular space and fuzzy Banach spaces. El-Fassi [8] found hyperstability result for the Apollonius equation on a restricted domain in normed space. Although, there is no result on the hyperstability of Apollonius type additive functional equation. Thus, we investigate the hyperstability of Apollonius type additive functional equation in quasi (m, β)-normed space.

In 2018, the fixed point theorem of Brzdek et al [6] was extended by Dung et al. [7] in metric space to quasi-Banach spaces and also found the hyperstability for the general linear equation. In 2020, El Fassi extended the fixed point theorem in quasi-$(2, \beta)$-normed space and found the hyperstability of radical-type functional equation. As far as we know, there is no result on fixed point theorem in quasi (m, β)-normed space, thus we extend the Brzdek et al and El Fassi results on fixed point theorem.

A. Misiak [25] defined m-normed spaces in 1989 as a generalisation of the concept of 2-normed spaces, which was first proposed by S. Gähler [9, 10] 25 years prior. In 2015, Yang et al. [32] provide the notion of (m, β)-normed space. Here, we generalize the concept of (m, β)-normed space and provide the notion of quasi-(m, β)-normed space.

The aim of this research is to find the notion of quasi-(m,β)-normed space. Later, we investigates the validity of a fixed point theorem with the framework of quasi-(m,β)-normed spaces, extending the foundational results established by Brzdek and El-Fassi. At last, using the results of fixed point theory, we study the hyperstability for Apollonius type additive functional equation in quasi (m,β)-normed space.

Throughout this paper, $\mathbb{N}_0$ represents the set of whole numbers, $\mathbb{N}$ represents the set of natural numbers, $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_0$ denotes the set of real numbers excludg zero, $\mathbb{R}_+$ denotes the set of positive real numbers and the parameters $m, \beta, \mathfrak{p}$ and $\mathfrak{k}$ are used with the following meanings: $m \in \mathbb{N}$ denotes the number of vectors in the m-tuple norm structure, $\beta > 0$ represents the weight or scaling parameter in the (m,β) norm and $\mathfrak{k}$ is the modulus of convexity parameter. $\mathfrak{p} > 0$ is the order parameter that determines the degree of the quasi-norm, controlling how sizes and distances are aggregated in our stability and fixed-point framework.

Now, we generalize the basic definitions, terminology, and typical characteristics of quasi-(m,β)-NS.

Definition 1.1. [20] Let $\mathfrak{U}(\mathbb{R})(dim\mathfrak{U} \geq n)$ be a vector space and $||.,...,.||_\beta : \mathfrak{U}^n \to \mathbb{R}$ be a mapping holds below conditions

1. $||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\beta = 0 \leftrightarrow u_1, ..., u_n$ are linearly dependent,
2. $||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\beta$ is invariant under permutations of $\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}$,
3. $||c\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\beta = |c|^\beta ||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\beta$,
4. $||\mathfrak{u}+\mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\beta \leq ||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\beta + ||\mathfrak{u}+\mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\beta$

for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1} \in \mathfrak{U}$, $c \in \mathbb{R}$ and $0 < \beta \leq 1$.

The mapping $||.,...,.||_\beta$ is known as (m,β)-norm and the pair $(\mathfrak{U}, ||.,...,.||_\beta)$ is known as (m,β)-NS.

Definition 1.2. Let $\mathfrak{U}(\mathbb{R})(dim\mathfrak{U} \geq n)$ be a vector space and $||.,...,.|| : \mathfrak{U}^n \to \mathbb{R}$ be a mapping holds below conditions

1. $||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\mathfrak{q} = 0 \leftrightarrow u_1, ..., u_n$ are linearly dependent,
2. $||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\mathfrak{q}$ is invariant under permutations of $\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}$,
3. $||c\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\mathfrak{q} = |c|||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\mathfrak{q}$,
4. $||\mathfrak{u}+\mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\mathfrak{q} \leq \mathfrak{k}(||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\mathfrak{q} + ||\mathfrak{u}+\mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_\mathfrak{q})$

for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1} \in \mathfrak{U}$, $c \in \mathbb{R}$ and $0 < \beta \leq 1$, and $\mathfrak{k}$ is the modulus of concavity.

The mapping $||.,...,.||_\beta$ is known as quasi-m-norm and the pair $(\mathfrak{U}, ||.,...,.||_\beta)$ is known as quasi-m-NS.

The main difference between quasi-m-norm and m-norm is modulus of concavity. In quasi-m-norm the value of the modulus of concavity is greater than or equal to 1 and in m-norm the value of the modulus of concavity is equal to 1.

Definition 1.3. Let $\mathfrak{U}(\mathbb{R})(dim\mathfrak{U} \geq n)$ be a vector space and $||.,...,.|| : \mathfrak{U}^n \to \mathbb{R}$ be a mapping holds below conditions

1. $||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta} = 0 \leftrightarrow u_1, ..., u_n$ are linearly dependent,
2. $||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta}$ is invariant under permutations of $\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}$,

3. $||c\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta} = |c|^{\beta}||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta}$,
4. $||\mathfrak{u} + \mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta} \leq \mathfrak{k}(||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta} + ||\mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta})$

for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1} \in \mathfrak{U}$, $c \in \mathbb{R}$ and $0 < \beta \leq 1$, and $\mathfrak{k}$ is the modulus of concavity.

The mapping $||., ..., .||_{\mathfrak{q},\beta}$ is known as quasi-(m, β)-norm and the pair $(\mathfrak{U}, ||., ..., .||_{\mathfrak{q},\beta})$ is known as quasi-(m, β)-NS.

Example 1.1. Let $\mathfrak{U} = \mathbb{R}^n$ with $m = n$ and $0 < \beta \leq 1$. For any vectors $\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1} \in \mathbb{R}^n$, define the mapping $||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta} = |\det\{\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\}|^{\beta}$, where $\det\{\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\}$ denotes the determinant of the $n \times n$ matrix whose columns are the given vectors. Then the pair $(\mathbb{R}^n, ||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta})$ is a quasi-(m, β)-NS.

Lemma 1.1. *Suppose* $(\mathfrak{U}, ||., ..., .||_{\mathfrak{q},\beta})$ *is a quasi-*(m, β)*-NS,* $m \geq 2, 0 < \beta \leq 1$. *If* $\mathfrak{v} \in \mathfrak{U}$ *and* $||\mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta} = 0$ *for all linearly independent vectors* $\mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1} \in \mathfrak{U}$*, then* $\mathfrak{v} = 0$.

Proof. Let us suppose that $\mathfrak{v} \neq 0$. Also, it is given that $\dim(\mathfrak{U}) = m$, so choose a set $\{\mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\}$ of linearly independent vectors from $\mathfrak{U}$. Now using definition(1.3), we find that $||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}||_{\mathfrak{q},\beta} \neq 0$ which is contradiction, hence $\mathfrak{v} = 0$. □

The advantage of this work lies in that it generalizes the fixed-point approaches to quasi-(m, β)-normed spaces, yielding more flexiblility than conventional normed structures and enabling a more efficient framework of hyperstability analysis. Innovation lies in the construction of a new family of fixed-point theorems for contractive-type mappings in the generalized setup and applying them to establish the hyperstability of Apollonius-type functional equations, which has not been carried out previously in literature. The motivation is the rising appeal to generalized spaces as universal tools for the description of non-standard convergence and stability behaviors and for expanding the usage of the fixed-point methodologies in functional equations. One of the weaknesses of the present work, however, is that the results are restricted to quasi-(m, β)-normed spaces; generalizing these conceptions to other spaces such as modular spaces or the variable exponent Lebesgue spaces is an open pathway for future study.

2. Main results

In this part, we discuss our results, starting with the formulation of an equivalent quasi-m norm in a quasi-m normed space. In Theorem 2.1, we prove that every quasi-m normed space admits an equivalent quasi-m norm having the same topology. Theorem 2.2 shows that this equivalent norm preserves the main inequalities and structure of the space, sometimes with better bounds. Theorem 2.3 applies these results to stability theory, proving that stability and continuity properties with respect to the original norm also hold for the equivalent norm.

Theorem 2.1. *Let* $(\mathfrak{U}, \|., ..., .\|_{\mathfrak{q}}, \mathfrak{k})$ *be a quasi-m-NS. There is an equivalent quasi-m-norm* $|||., ..., .|||_{\mathfrak{q}}$ *on* $\mathfrak{U}$ *holds*

$$|||\mathfrak{u} + \mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}} \leq |||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}} + |||\mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}} \tag{2.1}$$

for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1} \in \mathfrak{U}$ *and* $\mathfrak{p}$ $(0, 1]$*, with*

$$|||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}} = f\left\{\left(\sum_{i}^{n} \|\mathfrak{u}_{\mathfrak{i}}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}}\right\}, \tag{2.2}$$

where, $\mathfrak{u} = \sum_{i=1}^{n} \mathfrak{u}_{\mathfrak{i}}$*,* $\mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}$ $\mathfrak{U}$ *and* $\mathfrak{p} = log_{2\mathfrak{k}}m$.

Proof. Let $\mathfrak{k}$ be the modulus of concavity of $\|.,...,.\|_{\mathfrak{q}}$ and $m^{\frac{1}{\mathfrak{p}}} = 2\mathfrak{k}$. Let $\mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1} \in \mathfrak{U}$ be fixed and define a new quasi-m-norm by

$$|||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}} = f\left\{\left(\sum_{i}^{n} \|\mathfrak{u}_{\mathfrak{i}}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}}\right\},\ \mathfrak{u} = \sum_{i=1}^{n} \mathfrak{u}_{\mathfrak{i}}. \tag{2.3}$$

It is clear that $|||\alpha\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}} = |\alpha||||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}$, $\|\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}} \leq |||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}$ and fulfils the inequality requirement. Now, we prove that

$$\|\sum_{i=1}^{n} \mathfrak{u}_{\mathfrak{i}}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} \leq m^{\frac{2}{\mathfrak{p}}}\left(\|\sum_{i=1}^{n} \mathfrak{u}_{\mathfrak{i}}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}} \tag{2.4}$$

for all $\{\mathfrak{u}_{\mathfrak{i}}\}_{i=1}^{n}$. This means that $\|\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} \leq m^{\frac{2}{\mathfrak{p}}}|||\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}$ and concludes the result. Now, for every $\mathfrak{u} \in \mathfrak{U}$ take $\mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u}) = m^{\frac{\mathfrak{t}}{\mathfrak{p}}}$, where the integer $\mathfrak{t}$ is taken such that

$$m^{\frac{\mathfrak{t}-1}{\mathfrak{p}}} < \|\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} \leq m^{\frac{\mathfrak{t}}{\mathfrak{p}}}.$$

The inequality (2.1) is proved by showing that

$$\|\sum_{i=1}^{n} \mathfrak{u}_{\mathfrak{i}}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} \leq m^{\frac{1}{\mathfrak{p}}}\left(\sum_{i=1}^{n} \mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u}_{\mathfrak{i}})^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}}. \tag{2.5}$$

Suppose there are two $\mathfrak{u}_{\mathfrak{i}}$'s say $\mathfrak{u}$ and $\mathfrak{v}$ such that $\mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u}) = \mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{v}) = m^{\frac{\mathfrak{t}}{\mathfrak{p}}}$ then, we can replace the pair $\mathfrak{u}$, $\mathfrak{v}$ in the proof of (2.5) by their sum. Certainly,

$$\|\mathfrak{u} + \mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} \leq \mathfrak{k}\|\mathfrak{u}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} + \mathfrak{k}\|\mathfrak{v}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} \leq 2\mathfrak{k}m^{\frac{\mathfrak{t}}{\mathfrak{p}}}$$

by the choice of $\mathfrak{p}$. Thus

$$\mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u} + \mathfrak{v})^{\mathfrak{p}} = m^{\mathfrak{t}+1} = \mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u})^{\mathfrak{p}} + \mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{v})^{\mathfrak{p}}. \tag{2.6}$$

Using (2.6) and reorganising the $\mathfrak{u}_{\mathfrak{i}}$'s, let the sequence $\{\mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u}_{\mathfrak{i}})^{\mathfrak{p}}\}_{i=1}^{n}$ be a strictly decreasing sequence. So,

$$\|\mathfrak{u}_{\mathfrak{i}}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} \leq \mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u}_{\mathfrak{i}}) \leq m^{\frac{-(\mathfrak{i}-1)}{\mathfrak{p}}}\mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u}_1)$$

for all $\mathfrak{i}(1 \leq \mathfrak{i} \leq n)$. The iterative use of the concavity modulus definition and the definition of $\mathfrak{p}$ provides

$$\begin{aligned}
\|\sum_{i=1}^{n} \mathfrak{u}_{\mathfrak{i}}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} &\leq \mathfrak{k}\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} + \mathfrak{k}^2\|\mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} + ... + \mathfrak{k}^{\mathfrak{n}}\|\mathfrak{u}_{\mathfrak{n}}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} \\
&\leq \sum_{i=1}^{n} \mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u}_1)\mathfrak{k}^{i}m^{\frac{-(\mathfrak{i}-1)}{\mathfrak{p}}} \\
&= m^{\frac{1}{\mathfrak{p}}}\mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}\sum_{i=1}^{n}\frac{1}{2^{\mathfrak{i}}} \leq m^{\frac{1}{\mathfrak{p}}}\mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}} \\
&\leq m^{\frac{1}{\mathfrak{p}}}\left(\sum_{i=1}^{n} \mathcal{N}_{\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}}(\mathfrak{u}_{\mathfrak{i}})^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}}
\end{aligned} \tag{2.7}$$

as required. □

Theorem 2.2. *If* $(\mathfrak{U}, \|.,...,.\|_{\mathfrak{q}})$ *is a quasi-m-norm space then* $(\mathfrak{U}, \|.,...,.\|_{\mathfrak{q},\beta})$ *is a quasi-*(m,β)*-norm space, i.e.*

$$\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} = \|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}^{\beta} \ \forall\ \mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\ \mathfrak{U},\ 0 < \beta \leq 1.$$

Proof. Since $\mathfrak{U}$ is a quasi-m-norm space, then

(1) $\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} = 0 \iff \|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} = 0 \iff \{\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\}$ is linearly dependent.

(2) $\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}$ is invariant under permutations of $\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}$.

(3) $\|\lambda\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} = \|\lambda\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}^{\beta} = |\lambda|^{\beta}\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}$.

(4)

$$\begin{aligned}\|\mathfrak{u}_1 + \mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} &= \|\mathfrak{u}_1 + \mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}^{\beta}\\ &\leq \mathfrak{k}^{\beta}(\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} + \|\mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}})^{\beta}\\ &\leq \mathfrak{k}^{\beta}(\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} + \|\mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}})^{\beta}\\ &\leq \mathfrak{k}^{\beta}(\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}^{\beta} + \|\mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}^{\beta})\\ &= \mathfrak{k}^{\beta}(\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} + \|\mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}).\end{aligned}$$

Sce $\mathfrak{k}^{\beta} \geq 1$, then $(\mathfrak{U}, \|.,...,.\|_{\mathfrak{q},\beta}, \mathfrak{k}^{\beta})$ is a quasi-(m,β)-normed space. □

Theorem 2.3. *Consider the quasi-*(m,β)*-normed space* $(\mathfrak{U}, \|.,...,.\|_{\mathfrak{q},\beta}, \mathfrak{k})$*, where* $0 < \beta \leq 1$*. There is an equivalent quasi-m-norm* $|||.,...,.|||_{\mathfrak{q},\beta}$ *on* $\mathfrak{U}$ *holds*

$$|||\mathfrak{u}_1 + \mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}^{\mathfrak{p}} \leq |||\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}^{\mathfrak{p}} + |||\mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}^{\mathfrak{p}}$$

for all $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1} \in \mathfrak{U}$, $\mathfrak{p} \in (0,1]$ *with*

$$|||\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta} = \inf\left\{\left(\sum_{i=1}^{n}\|\mathfrak{u}_{1\mathfrak{i}}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\frac{\mathfrak{p}}{\beta}}\right)^{\frac{1}{\mathfrak{p}}} : \ \mathfrak{u}_1 = \sum_{i=1}^{n}\mathfrak{u}_{1\mathfrak{i}},\ \mathfrak{u}_{1\mathfrak{i}} \in \mathfrak{U}, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1} \in \mathfrak{U}\right\}$$

and $\mathfrak{p} = \beta log_{2\mathfrak{k}} m$.

Proof. We will show that $\|.,...,.\|_{\mathfrak{q}} := \|.,...,.\|_{\mathfrak{q},\beta}^{\frac{1}{\beta}}$ is a quasi-m-norm on $\mathfrak{U}$ with the modulus of concavity $\frac{(2\mathfrak{k})^{\frac{1}{\beta}}}{m}$.To do this, we assume that $\|.,...,.\|_{\mathfrak{q},\beta}$ is a quasi-(m,β)-norm on $\mathfrak{U}$. Then for each $\mathfrak{u}_1, \mathfrak{u}_2\mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}$, we have

$$\begin{aligned}\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} = 0 &\iff \|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\frac{1}{\beta}} = 0\\ &\iff \|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} = 0\\ &\iff \{\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\} \text{ is linearly dependents.}\end{aligned}$$

Also, $\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}$ is invariant under permutations of $\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}$, and

$$\|\lambda\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} = \|\lambda\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\frac{1}{\beta}} = |\lambda|\|\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}},$$

and

$$\|\mathfrak{u}_1 + \mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}} = \|\mathfrak{u}_1 + \mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\frac{1}{\beta}}$$

$$\leq \mathfrak{k}^{\frac{1}{\beta}}(\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}+\|\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta})^{\frac{1}{\beta}}. \tag{2.8}$$

Suppose $\mathfrak{g}(\mathfrak{t})=\mathfrak{t}^{\mathfrak{r}}\ \forall\ \mathfrak{t}>0$ with $\mathfrak{r}\geq 1$ is a convex function. we have

$$\mathfrak{g}\left(\frac{\mathfrak{t}_1+\mathfrak{t}_2+\ldots+\mathfrak{t}_{\mathfrak{m}}}{m}\right)\leq \frac{1}{m}(\mathfrak{g}(\mathfrak{t}_1)+\mathfrak{g}(\mathfrak{t}_2)+\ldots+\mathfrak{g}(\mathfrak{t}_{\mathfrak{m}}))\ \forall\ \mathfrak{t}_1,\mathfrak{t}_2,\ldots,\mathfrak{t}_{\mathfrak{m}}>0.$$

This implies

$$(\mathfrak{t}_1+\mathfrak{t}_2+\ldots+\mathfrak{t}_{\mathfrak{m}})^{\mathfrak{r}}\leq m^{\mathfrak{r}-1}(\mathfrak{t}_1{}^{\mathfrak{r}}+\mathfrak{t}_2{}^{\mathfrak{r}}+\ldots+\mathfrak{t}_{\mathfrak{m}}{}^{\mathfrak{r}}).$$

Using above inequality, (2.8) becomes

$$\begin{aligned}\|\mathfrak{u}_1+\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}&\leq \frac{(m\mathfrak{k})^{\frac{1}{\beta}}}{m}\left(\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\frac{1}{\beta}}+\|\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\frac{1}{\beta}}\right)\\&=\frac{(m\mathfrak{k})^{\frac{1}{\beta}}}{m}\left(\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}+\|\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}\right).\end{aligned}$$

This implies that $(\mathfrak{U},\|.,\ldots,.\|_{\mathfrak{q}},\frac{(m\mathfrak{k})^{\frac{1}{\beta}}}{m}$ is a quasi-space with a m-norm. We obtain a quasi-m-norm $|||.,\ldots,.|||_{\mathfrak{q}}$ from Theorem 2.1, on $\mathfrak{U}$ satisfies

$$|||\mathfrak{u}_1+\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}}\leq|||\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}}+|||\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}},$$

and there is $\nu_1,\nu_2>0$ such that

$$\begin{aligned}\mu_1\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\frac{1}{\beta}}&=\mu_1\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}\\&\leq|||\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}\\&\leq\mu_2\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q}}\\&=\mu_2\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\frac{1}{\beta}}.\end{aligned}$$

Now, from Theorem 2.2, we get $|||.,\ldots,.|||_{\mathfrak{q},\beta}=|||.,\ldots,.|||_{\mathfrak{q}}^{\beta}$ is also a quasi-(m,β)-norm on $\mathfrak{U}$ and

$$\begin{aligned}|||\mathfrak{u}_1+\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}^{\mathfrak{p}}&=|||\mathfrak{u}_1+\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}\beta}\\&\leq(|||\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}}+|||\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}})^{\beta}\\&\leq|||\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}\beta}+|||\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\mathfrak{p}\beta}\\&=|||\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}^{\mathfrak{p}}+|||\mathfrak{u}_2,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}^{\mathfrak{p}}.\end{aligned}$$

Also, we have

$$\nu_1^{\beta}\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}\leq|||\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\beta}\leq\nu_2^{\beta}\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}.$$

Hence,

$$\mathfrak{M}_1\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}\leq|||\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q}}^{\beta}\leq\mathfrak{M}_2\|\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \tag{2.9}$$

where $\mathfrak{M}_1=\nu_1^{\beta},\mathfrak{M}_2=\nu_2^{\beta}$. □

Next, we show that the fixed point Theorem is still applicable the quasi-(m,β)-Banach space. We shall provide the subsequent four assumptions:

1. (H1): $\mathfrak{W}$ is a non-empty set. $(\mathfrak{U}, \|., ..., .\|_{\mathfrak{q},\beta}, \mathfrak{k})$ is a quasi-(m, β)-Banach space.
2. (H2): $\mathfrak{g}_\mathfrak{i} : \mathfrak{W} \to \mathfrak{W}$ and $\mathfrak{L}_\mathfrak{i} : \mathfrak{W} \times \mathfrak{U} \times \mathfrak{U} \times ... \times \mathfrak{U} \to \mathbb{R}_+$ are given maps for $i = 1, ..., r$.
3. (H3): $\mathfrak{T} : \mathfrak{U}^{\mathfrak{W}} \to \mathfrak{U}^{\mathfrak{W}}$ is an operator follows the inequality

$$\begin{aligned}&\|(\mathfrak{T}\psi_1)(\mathfrak{u}_1) - (\mathfrak{T}\psi_2)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}\\ &\le \sum_{i=1}^{r} \mathfrak{L}_\mathfrak{i}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\|\psi_1(\mathfrak{g}_\mathfrak{i}(\mathfrak{u}_1)) - \psi_2(\mathfrak{g}_\mathfrak{i}(\mathfrak{u}_1)), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}\end{aligned}$$

for all $\psi_1, \psi_2 \in \mathfrak{U}^{\mathfrak{W}}$ and $(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \in \mathfrak{W} \times \mathfrak{U} \times \mathfrak{U} \times ... \times \mathfrak{U}$.
4. (H4): $\Omega : {\mathbb{R}_+}^{\mathfrak{W}\times\mathfrak{U}\times\mathfrak{U}\times...\times\mathfrak{U}} \to {\mathbb{R}_+}^{\mathfrak{W}\times\mathfrak{U}\times\mathfrak{U}\times...\times\mathfrak{U}}$ is a linear operator defined as

$$\begin{aligned}&(\Omega\delta)(\mathfrak{W} \times \mathfrak{U} \times \mathfrak{U} \times ... \times \mathfrak{U})\\ &:= \sum_{i=1}^{r} \mathfrak{L}_\mathfrak{i}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\delta(\mathfrak{g}_\mathfrak{i}(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}),\ \ \delta \in {\mathbb{R}_+}^{\mathfrak{W}\times\mathfrak{U}\times\mathfrak{U}\times...\times\mathfrak{U}}.\end{aligned}$$

Theorem 2.1. *Let assumptions (H1)-(H4) be true and let $\phi : \mathfrak{W}\times\mathfrak{U}\times\mathfrak{U}\times...\times\mathfrak{U} \to \mathbb{R}_+, \xi : \mathfrak{W} \to \mathfrak{U}$ fulfil the conditions*

$$\|(\mathfrak{T}\xi)(\mathfrak{u}_1) - \xi(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_\beta \le \phi(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}), \tag{2.10}$$

$$\phi^*(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) = \sum_{n=0}^{\infty}(\Omega^n\phi)^\theta(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) < \infty \tag{2.11}$$

where $\theta = \beta \log_{2\mathfrak{k}} m$, then limit

$$\varphi(\mathfrak{u}_1) = \lim_{n\to\infty} \mathfrak{T}^n\xi(\mathfrak{u}_1), \tag{2.12}$$

exists and the fixed point of $\mathfrak{T}$ is a function $\zeta : \mathfrak{W} \to \mathfrak{U}$ defined by (2.12) fulfills

$$\|\xi(\mathfrak{u}_1) - \zeta(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|^\theta_{\mathfrak{q},\beta} \le \mathfrak{C}\phi^*(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \tag{2.13}$$

for some constant $\mathfrak{C} > 0$. Moreover, if

$$\phi^*(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \le \left(\mathfrak{M}\sum_{n=0}^{\infty}(\Omega^n\phi)(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\right)^\theta < \infty \tag{2.14}$$

for $\mathfrak{M} > 0$, then $\mathfrak{T}$ has unique fixed point ζ satisfies (2.13).

Proof. By induction, it is simple to show that, for any $n \in \mathbb{N}_0$

$$\|(\mathfrak{T}^{\mathfrak{n}+1}\xi)(\mathfrak{u}_1) - (\mathfrak{T}^\mathfrak{n}\xi)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_\beta \le (\Omega^n\phi)(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}). \tag{2.15}$$

Using (2.10), we get

$$\|(\mathfrak{T}\xi)(\mathfrak{u}_1) - \xi(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_\beta \le (\Omega^0\phi)(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}). \tag{2.16}$$

Hence, the inequality is true for $n = 1$. Now, suppose that equality (2.15) is true for $n \in \mathbb{N}_0$. Now, by using (H3) and (H4), we have

$$\|(\mathfrak{T}^{\mathfrak{n}+2}\xi)(\mathfrak{u}_1) - (\mathfrak{T}^{\mathfrak{n}+1}\xi)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}$$

$$
\begin{aligned}
&= \|\mathfrak{T}(\mathfrak{T}^{\mathfrak{n}+1}\xi)(\mathfrak{u}_1) - \mathfrak{T}(\mathfrak{T}^{\mathfrak{n}}\xi)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \\
&\leq \sum_{i=1}^{r} \mathfrak{L}_{\mathfrak{i}}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\|\mathfrak{T}^{\mathfrak{n}}\xi(\mathfrak{g}_{\mathfrak{i}}(\mathfrak{u}_1)) - \mathfrak{T}^{\mathfrak{n}+1}\xi(\mathfrak{g}_{\mathfrak{i}}(\mathfrak{u}_1)), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \\
&\leq \sum_{i=1}^{r} \mathfrak{L}_{\mathfrak{i}}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})(\Omega^n\phi)(\mathfrak{g}_{\mathfrak{i}}(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&= (\Omega^{n+1}\phi)(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}).
\end{aligned}
$$

Hence, the inequality (2.15) is true for all $n \in \mathbb{N}_0$. Now, by using (2.15), (2.9) and Theorem 2.1, we have

$$
\begin{aligned}
&|||(\mathfrak{T}^{\mathfrak{n}}\xi)(\mathfrak{u}_1) - (\mathfrak{T}^{\mathfrak{n}+\mathfrak{t}}\xi)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||^{\theta}_{\mathfrak{q},\beta} \\
&= |||\sum_{i=0}^{\mathfrak{t}-1}(\mathfrak{T}^{\mathfrak{n}+\mathfrak{i}}\xi)(\mathfrak{u}_1) - (\mathfrak{T}^{\mathfrak{n}+\mathfrak{i}+1}\xi)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||^{\theta}_{\mathfrak{q},\beta} \\
&\leq \sum_{i=0}^{\mathfrak{t}-1} |||(\mathfrak{T}^{\mathfrak{n}+\mathfrak{i}}\xi)(\mathfrak{u}_1) - (\mathfrak{T}^{\mathfrak{n}+\mathfrak{i}+1}\xi)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||^{\theta}_{\mathfrak{q},\beta} \\
&\leq \mathfrak{M}_2{}^{\theta} \sum_{i=0}^{\mathfrak{t}-1} |||(\mathfrak{T}^{\mathfrak{n}+\mathfrak{i}}\xi)(\mathfrak{u}_1) - (\mathfrak{T}^{\mathfrak{n}+\mathfrak{i}+1}\xi)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||^{\theta}_{\mathfrak{q},\beta} \\
&\leq \mathfrak{M}_2{}^{\theta} \sum_{i=0}^{\mathfrak{t}-1}(\Omega^{n+i}\phi)^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&= \mathfrak{M}_2{}^{\theta} \sum_{i=n}^{\mathfrak{n}+\mathfrak{t}-1}(\Omega^{n+i}\phi)^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&\leq \mathfrak{M}_2{}^{\theta}\phi^*(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}).
\end{aligned} \tag{2.17}
$$

It implies from (3.17) and the convergence of the series $\sum(\Omega^n\phi)^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})$ that the sequence $\{\mathfrak{T}^{\mathfrak{n}}\xi(\mathfrak{u}_1)\}_{n\in\mathbb{N}}$ is Cauchy in $(\mathfrak{U}, |||., ..., .|||_{\mathfrak{q},\beta})$. Also, by using Theorem 2.3, we get that the sequence $\{\mathfrak{T}^{\mathfrak{n}}\xi(\mathfrak{u}_1)\}_{n\in\mathbb{N}}$ is also a Cauchy in the quasi-(m, β)-Banach space $(\mathfrak{U}, \|., ..., .\|_{\mathfrak{q},\beta,\mathfrak{k}})$, hence convergent, then the limit $\varphi(\mathfrak{u}_1) = \lim_{k\to\infty} \mathfrak{T}^k\xi(\mathfrak{u}_1)$ exists for all $\mathfrak{u}_1 \in \mathfrak{W}$, therefore (2.12) holds. Taking $n = 0$ and $\mathfrak{t} \to \infty$ (2.17), we have

$$
|||\xi(\mathfrak{u}_1) - \varphi(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||^{\theta}_{\mathfrak{q},\beta} \leq \mathfrak{M}_2{}^{\theta}\phi^*(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}). \tag{2.18}
$$

Using the inequality (2.9),

$$
\begin{aligned}
\|\xi(\mathfrak{u}_1) - \varphi(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|^{\theta}_{\mathfrak{q},\beta} &\leq \mathfrak{M}_1{}^{-\theta}|||\xi(\mathfrak{u}_1) - \varphi(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|||^{\theta}_{\mathfrak{q},\beta} \\
&\leq \left(\frac{\mathfrak{M}_2}{\mathfrak{M}_1}\right)^{\theta}\phi^*(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}),
\end{aligned} \tag{2.19}
$$

therefore (2.13) holds with $\mathcal{C} = \left(\frac{\mathfrak{M}_2}{\mathfrak{M}_1}\right)^{\theta}$. Now, applying (H3), (2.12) and (2.9), we have

$$
\begin{aligned}
&\|(\mathfrak{T}^{\mathfrak{n}+1}\xi)(\mathfrak{u}_1) - (\mathfrak{T}\varphi)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \\
&\leq \sum_{i=1}^{r} \mathfrak{L}_{\mathfrak{i}}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\|(\mathfrak{T}^{\mathfrak{n}}\xi)(\mathfrak{g}_{\mathfrak{i}}(\mathfrak{u}_1)) - \varphi(\mathfrak{g}_{\mathfrak{i}}(\mathfrak{u}_1)), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}
\end{aligned}
$$

$$\leq \mathfrak{M_1}^{-1}\sum_{i=1}^{r}\mathfrak{L}_\mathfrak{i}(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}})\||(\mathfrak{T}^\mathfrak{n}\xi)(\mathfrak{g_i}(\mathfrak{u_1}))-\varphi(\mathfrak{g_i}(\mathfrak{u_1})),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\||_{\mathfrak{q},\beta}. \tag{2.20}$$

Taking $n\to\infty$ (2.20), we get

$$\lim_{n\to\infty}\|(\mathfrak{T}^{\mathfrak{n+1}}\xi)(\mathfrak{u_1})-(\mathfrak{T}\varphi)(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\|_{\mathfrak{q},\beta}=0,$$

this proves $\lim_{n\to\infty}(\mathfrak{T}^{\mathfrak{n+1}}\xi)(\mathfrak{u_1})=(\mathfrak{T}\varphi)(\mathfrak{u_1})$ for all $\mathfrak{u_1}\in\mathcal{W}$, this implies $\mathfrak{T}\varphi=\varphi$. Therefore, φ is a fixed point of $\mathfrak{T}$ that satisfies (2.13). Next, we prove the φ is unique. For this, we assume that $\mathfrak{T}$ has another fixed point φ' satisfies (2.13). First we prove that

$$\begin{aligned}\|\varphi(\mathfrak{u_1})-\varphi'(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\|_{\mathfrak{q},\beta}&=\|(\mathfrak{T}^k\varphi)(\mathfrak{u_1})-(\mathfrak{T}^k\varphi')(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\|_{\mathfrak{q},\beta}\\&\leq(m\mathfrak{M})^{\frac{1}{\theta}}\mathcal{C}\sum_{i=k}^{\infty}(\Omega^i\phi)(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}}).\end{aligned} \tag{2.21}$$

Infact, for $k=0$ and using (2.18), we get

$$\begin{aligned}&\||\varphi(\mathfrak{u_1})-\varphi'(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\||^\theta_{\mathfrak{q},\beta}\\&\leq\|\varphi(\mathfrak{u_1})-\xi(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\|^\theta_{\mathfrak{q},\beta}+\|\xi(\mathfrak{u_1})-\varphi'(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\|^\theta_{\mathfrak{q},\beta}\\&\leq m\mathfrak{M_2}^\theta\phi^*(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}}).\end{aligned}$$

Using inequality (2.9) and (2.14), we get

$$\begin{aligned}\||\varphi(\mathfrak{u_1})-\varphi'(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\||^\theta_{\mathfrak{q},\beta}&\leq\mathfrak{M_1}^{-\theta}\||\varphi(\mathfrak{u_1})-\varphi'(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\||^\theta_{\mathfrak{q},\beta}\\&\leq\left(\frac{\mathfrak{M_2}}{\mathfrak{M_1}}\right)^\theta\phi^*(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}})\\&\leq m\mathcal{C}\left(\mathfrak{M}\sum_{n=0}^{\infty}(\Omega^n\phi)(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}})\right)^\theta.\end{aligned}$$

Hence, (2.21) is true for $k=0$. Now, let us suppose that (2.21) is true for some $k\in\mathbb{N}$, by using (H3), we have

$$\begin{aligned}&\|(\mathfrak{T}^{k+1}\varphi)(\mathfrak{u_1})-(\mathfrak{T}^{k+1}\varphi')(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\|_{\mathfrak{q},\beta}\\&\leq\sum_{i=1}^{r}\mathfrak{L}_\mathfrak{i}(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}})\|(\mathfrak{T}^k\varphi)(\mathfrak{g_i}(\mathfrak{u_1}))-(\mathfrak{T}^k\varphi')(\mathfrak{g_i}(\mathfrak{u_1})),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\|_{\mathfrak{q},\beta}\\&\leq\sum_{i=1}^{r}\mathfrak{L}_\mathfrak{i}(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}})(m\mathcal{C})^{\frac{1}{\theta}}\mathfrak{M}\sum_{i=k}^{\infty}(\Omega^n\phi)(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}})\\&=(m\mathcal{C})^{\frac{1}{\theta}}\mathfrak{M}\sum_{i=k+1}^{\infty}(\Omega^n\phi)(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}}).\end{aligned}$$

Hence, (2.21) is true for all $k\in\mathbb{N}_0$. From (2.9) and (2.21), we get

$$\begin{aligned}\||(\mathfrak{T}^k\varphi)(\mathfrak{u_1})-(\mathfrak{T}^k\varphi')(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\||^\theta_{\mathfrak{q},\beta}&\leq\mathfrak{M_2}^\theta\|(\mathfrak{T}^k\varphi)(\mathfrak{u_1})-(\mathfrak{T}^k\varphi')(\mathfrak{u_1}),\mathfrak{z_1},...,\mathfrak{z_{m-1}}\|^\theta_{\mathfrak{q},\beta}\\&\leq m\mathcal{C}\mathfrak{M_2}^\theta\left(\mathfrak{M}\sum_{n=0}^{\infty}(\Omega^n\phi)(\mathfrak{u_1},\mathfrak{z_1},...,\mathfrak{z_{m-1}})\right)^\theta.\end{aligned} \tag{2.22}$$

Let $k \to \infty$ (2.22) and using (2.14), we have

$$\||(\mathfrak{T}^k \varphi)(\mathfrak{u}_1) - (\mathfrak{T}^k \varphi')(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}|\|_{\mathfrak{q},\beta} = 0 \implies \varphi \equiv \varphi'.$$

Hence, $\mathfrak{T}$ has unique fixed point φ. □

Theorem 2.2. *Suppose that* $(\mathfrak{U}, \|., ..., .\|_{\mathfrak{q},\beta}, \mathfrak{k})$ *is a quasi-*(m, β)*-normed space where* $0 < \beta \le 1$ *and* $\mathfrak{h}_\mathfrak{i} : \mathbb{R} \times \mathfrak{U} \times ... \times \mathfrak{U} \to \mathbb{R}_+$ *are given functions for* $i \in \{1, 2, 3\}$. *Assume*

$$\mathcal{M}_0 = \{n \in \mathbb{N} : \mathfrak{k}(2\mathfrak{s}_1(1 + a)\mathfrak{s}_2(1 + a)\mathfrak{s}_3(1 + a) + \mathfrak{s}_1(1 + 2a)\mathfrak{s}_2(1 + 2a)\mathfrak{s}_3(1 + 2a)) < 1\}$$

is an infite set, where

$$\begin{aligned}
\mathfrak{s}_1(\rho) &= \inf\{\mathfrak{t} \in \mathbb{R}_+ : \mathfrak{h}_1(\rho\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&\quad \le \mathfrak{t}\mathfrak{h}_1(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \text{ for all } (\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \in \mathbb{R} \times \mathfrak{U} \times ... \times \mathfrak{U}\}, \\
\mathfrak{s}_2(\rho) &= \inf\{\mathfrak{t} \in \mathbb{R}_+ : \mathfrak{h}_2(\rho\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&\quad \le \mathfrak{t}\mathfrak{h}_2(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \text{ for all } (\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \in \mathbb{R} \times \mathfrak{U} \times ... \times \mathfrak{U}\}, \\
\mathfrak{s}_3(\rho) &= \inf\{\mathfrak{t} \in \mathbb{R}_+ : \mathfrak{h}_3(\rho\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&\quad \le \mathfrak{t}\mathfrak{h}_3(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \text{ for all } (\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \in \mathbb{R} \times \mathfrak{U} \times ... \times \mathfrak{U}\},
\end{aligned}$$

$\rho \in \mathbb{R}_0$, *and* $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ *satisfy the followg two conditions:*

$$\begin{aligned}
&(1)\ (\lim_{\rho\to\infty} \mathfrak{s}_1(\pm\rho)\mathfrak{s}_2(\pm\rho)\mathfrak{s}_3(\pm\rho) = 0. \\
&(2)\ \lim_{\rho\to\infty} \mathfrak{s}_1(\rho) = 0 \text{ or } \lim_{\rho\to\infty} \mathfrak{s}_2(\rho) = 0 \text{ or } \lim_{\rho\to\infty} \mathfrak{s}_3(\rho) = 0.
\end{aligned}$$

If the function $f : \mathfrak{U} \to \mathcal{V}$ *follows the following inequality*

$$\begin{aligned}
&\|f(\mathfrak{u}_1 + \mathfrak{u}_2) + 2f(\mathfrak{u}_3 - \mathfrak{u}_1) + 2f(\mathfrak{u}_3 + \mathfrak{u}_2) - 4f\left(\mathfrak{u}_3 - \frac{\mathfrak{u}_1 + \mathfrak{u}_2}{4}\right), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \\
&\le \mathfrak{h}_1(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_2, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_3, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}),
\end{aligned} \tag{2.23}$$

then f *satisfies the equation*

$$f(\mathfrak{u}_1 + \mathfrak{u}_2) + 2f(\mathfrak{u}_3 - \mathfrak{u}_1) + 2f(\mathfrak{u}_3 + \mathfrak{u}_2) = 4f\left(\mathfrak{u}_3 - \frac{\mathfrak{u}_1 + \mathfrak{u}_2}{4}\right). \tag{2.24}$$

Proof. Obviously, for $i \in \{1, 2, 3\}$, we obtain

$$\mathfrak{h}_\mathfrak{i}(\rho\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \le \mathfrak{s}_\mathfrak{i}(\rho)\mathfrak{h}_\mathfrak{i}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}). \tag{2.25}$$

Changing $(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)$ by $(a\mathfrak{u}_1, -a\mathfrak{u}_1, a\mathfrak{u}_1 + \mathfrak{u}_1)$ in (2.23), we have

$$\begin{aligned}
&\|2f(\mathfrak{u}_1) + 2f((1 + 2a)\mathfrak{u}_1) - 4f((1 + a)\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \\
&\le \mathfrak{h}_1(a\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(-a\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(a\mathfrak{u}_1 + \mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}).
\end{aligned}$$

Then

$$\| - f(\mathfrak{u}_1) - f((1 + 2a)\mathfrak{u}_1) + 2f((1 + a)\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}$$

$$\leq \left(\frac{1}{2}\right)^{\beta} \mathfrak{h}_1(a\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(-a\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(a\mathfrak{u}_1 + \mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}). \tag{2.26}$$

Now, we will define an operator $\mathcal{T}_a : \mathcal{V}^{\mathfrak{U}^\star} \to \mathcal{V}^{\mathfrak{U}^\star}$ by

$$\mathcal{T}_a\xi(\mathfrak{u}_1) = 2\xi((1+a)\mathfrak{u}_1) - \xi((1+2a)\mathfrak{u}_1),\ \xi \in \mathfrak{U}^\star.$$

Now, put

$$\epsilon_a(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) = \left(\frac{1}{2}\right)^{\beta} \mathfrak{h}_1(a\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(-a\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(a\mathfrak{u}_1 + \mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}).$$

Then by (2.25), we have

$$\begin{aligned} &\epsilon_a(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\ &\leq \mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)\left(\frac{1}{2}\right)^{\beta} \mathfrak{h}_1(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}). \end{aligned} \tag{2.27}$$

Then the equality (2.26) takes the form

$$\|(\mathcal{T}_a f)(\mathfrak{u}_1) - f(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \leq \epsilon_a(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}).$$

Now, we observe that the operator $\Omega_a : \mathbb{R}_+{}^{\mathbb{R}_0\times\mathfrak{U}\times...\times\mathfrak{U}} \to \mathbb{R}_+{}^{\mathbb{R}_0\times\mathfrak{U}\times...\times\mathfrak{U}}$ defined by

$$(\Omega_a\delta)(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) = 2\mathfrak{k}\delta((1+a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) + \mathfrak{k}\delta((1+2a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})$$

has the shape given in (H4) with $i = 2$,

$$\mathfrak{g}_1(\mathfrak{u}_1) \equiv (1+a)\mathfrak{u}_1, \quad \mathfrak{g}_1(\mathfrak{u}_1) \equiv (1+2a)\mathfrak{u}_1, \quad \mathfrak{L}_1(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) = 2^{\beta}\mathfrak{k}, \quad \mathfrak{L}_1(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) = \mathfrak{k}.$$

Furthermore, for every $\psi_1, \psi_2 \in \mathbb{R}_+{}^{\mathbb{R}_0\times\mathfrak{U}\times...\times\mathfrak{U}}$ and $(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \in \mathbb{R}_0 \times \mathfrak{U} \times ... \times \mathfrak{U}$, we get

$$\begin{aligned} &\|(\mathfrak{T}_a\psi_1)(\mathfrak{u}_1) - (\mathfrak{T}_a\psi_2)(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \\ &= \|2\psi_1((1+a)\mathfrak{u}_1) - \psi_1((1+2a)\mathfrak{u}_1) - 2\psi_2((1+a)\mathfrak{u}_1) + \psi_2((1+2a)\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \\ &\times 2^{\beta}\mathfrak{k}\|\psi_1((1+a)\mathfrak{u}_1) - \psi_2((1+a)\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} + \mathfrak{k}\|\psi_1((1+2a)\mathfrak{u}_1) \\ &-\psi_2((1+2a)\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta} \\ &\times \sum_{i=1}^{2} \mathfrak{L}_{\mathfrak{i}}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\|\psi_1(\mathfrak{g}_{\mathfrak{i}}(\mathfrak{u}_1)) - \psi_2(\mathfrak{g}_{\mathfrak{i}}(\mathfrak{u}_1)), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}. \end{aligned}$$

Therefore, (H3) is hold for $\mathfrak{T}_a$ with $a \in \mathbb{N}_2$. Next, we have to prove that

$$\begin{aligned} &\Omega_a^n\epsilon_a(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\ &\leq \left(\frac{1}{2}\right)^{\beta} \mathfrak{k}^n\mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a) \\ &+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^n \\ &\times\mathfrak{h}_1(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}). \end{aligned} \tag{2.28}$$

The equality (2.28) is true for $n = 0$. Now, assume that inequality (2.28) holds for $n = k, n \in \mathbb{N}_0$. Then, we prove that result is true for $n = k + 1$.

$$\Omega_a^{k+1}\epsilon_a(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})$$

$$= \Omega_a(\Omega_a^k \epsilon_a(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}))$$
$$= 2\mathfrak{k}\Omega_a^k \epsilon_a((1+a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) + \mathfrak{k}\Omega_a^k \epsilon_a((1+2a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})$$
$$= 2\mathfrak{k}\left(\frac{1}{2}\right)^{\beta} \mathfrak{k}^k \mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a) + \mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^k$$
$$\times \mathfrak{h}_1((1+a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2((1+a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3((1+a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})$$
$$+ \mathfrak{k}\left(\frac{1}{2}\right)^{\beta} \mathfrak{k}^k \mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a) + \mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^k$$
$$\times \mathfrak{h}_1((1+2a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2((1+2a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3((1+2a)\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})$$
$$\leq 2\mathfrak{k}\left(\frac{1}{2}\right)^{\beta} \mathfrak{k}^k \mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a) + \mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^k$$
$$\times \mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)\mathfrak{h}_1(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})$$
$$+ \mathfrak{k}\left(\frac{1}{2}\right)^{\beta} \mathfrak{k}^k \mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a) + \mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^k$$
$$\times \mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a)\mathfrak{h}_1(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})$$
$$\leq \left(\frac{1}{2}\right)^{\beta} \mathfrak{k}^{k+1} \mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a) + \mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^{k+1}$$
$$\times \mathfrak{h}_1(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}).$$

This demonstrates that (2.28) holds for every $n \in \mathbb{N}$. Now, by using the definition of M_0, we find that

$$\begin{aligned}
&\epsilon_a^{\star}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&= \sum_{n=0}^{\infty} (\Omega_a^n \epsilon_a)^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&\leq \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{\theta\beta} \mathfrak{k}^{\theta n} {\mathfrak{s}_1}^{\theta}(a){\mathfrak{s}_2}^{\theta}(-a){\mathfrak{s}_3}^{\theta}(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a) \\
&\quad + \mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^{\theta n} \\
&\quad \times {\mathfrak{h}_1}^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}){\mathfrak{h}_2}^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}){\mathfrak{h}_3}^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&= \left(\frac{1}{2}\right)^{\theta\beta} \frac{{\mathfrak{s}_1}^{\theta}(a){\mathfrak{s}_2}^{\theta}(-a){\mathfrak{s}_3}^{\theta}(1+a){\mathfrak{h}_1}^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}){\mathfrak{h}_2}^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}){\mathfrak{h}_3}^{\theta}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})}{1 - (2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a) + \mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^{\theta}}.
\end{aligned}$$

As a result of Theorem 2.4, the operator $\mathfrak{T}_a$ has a fixed point $O_a : \mathbb{R}_0 \to \mathfrak{U}$ that satisfies

$$\begin{aligned}
&\|f(\mathfrak{u}_1) - O_a(\mathfrak{u}_1), \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\theta} \\
&\leq \mathfrak{C}\epsilon_a^{\star}(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1}) \\
&\leq \mathfrak{C}\left(\frac{1}{2}\right)^{\theta\beta} \frac{{\mathfrak{s}_1}^{\theta}(a){\mathfrak{s}_2}^{\theta}(-a){\mathfrak{s}_3}^{\theta}(1+a){\mathfrak{h}_1}^{\theta}(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}){\mathfrak{h}_2}^{\theta}(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}){\mathfrak{h}_3}^{\theta}(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1})}{1-(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^{\theta}}. \quad (2.29)
\end{aligned}$$

That is,

$$\mathcal{O}_a(\mathfrak{u}_1) = 2O_a((1+a)\mathfrak{u}_1) - O_a((1+2a)\mathfrak{u}_1)$$

and (2.29) holds for all $(\mathfrak{u}_1, \mathfrak{z}_1, ..., \mathfrak{z}_{\mathfrak{m}-1})$ $\mathbb{R} \times \mathfrak{U} \times ... \times \mathfrak{U}$. Moreover,

$$O_a(\mathfrak{u}_1) = \lim_{n\to\infty} \mathfrak{T}_a^n f(\mathfrak{u}_1). \quad (2.30)$$

Now, we prove that

$$\begin{aligned}
&\|\mathfrak{T}_a^n f(\mathfrak{u}_1+\mathfrak{u}_2)+2\mathfrak{T}_a^n f(\mathfrak{u}_3-\mathfrak{u}_1)+2\mathfrak{T}_a^n f(\mathfrak{u}_3+\mathfrak{u}_2)-4\mathfrak{T}_a^n f\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right),\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}\\
&\le \mathfrak{k}^n\mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^n\\
&\quad\times\mathfrak{h}_1(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1}). \qquad (2.31)
\end{aligned}$$

Indeed, if $n=0$, then (2.31) follows by (2.23). Assume that (2.31) is true for some n. By the help of induction, we prove that (2.31) is true for all $n\in\mathbb{N}$. Using definition of $\mathfrak{T}_a$, we get

$$\begin{aligned}
&\|\mathfrak{T}_a^{n+1} f(\mathfrak{u}_1+\mathfrak{u}_2)+2\mathfrak{T}_a^{n+1} f(\mathfrak{u}_3-\mathfrak{u}_1)+2\mathfrak{T}_a^{n+1} f(\mathfrak{u}_3+\mathfrak{u}_2)\\
&\quad-4\mathfrak{T}_a^{n+1} f\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right),\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}\\
&=\|2\mathfrak{T}_a^n f((1+a)(\mathfrak{u}_1+\mathfrak{u}_2))-\mathfrak{T}_a^n f((1+2a)(\mathfrak{u}_1+\mathfrak{u}_2))\\
&\quad+4\mathfrak{T}_a^n f((1+a)(\mathfrak{u}_3-\mathfrak{u}_1))-2\mathfrak{T}_a^n f((1+2a)(\mathfrak{u}_3-\mathfrak{u}_1))\\
&\quad+4\mathfrak{T}_a^n f((1+a)(\mathfrak{u}_3+\mathfrak{u}_2))-2\mathfrak{T}_a^n f((1+2a)(\mathfrak{u}_3+\mathfrak{u}_2))-8\mathfrak{T}_a^n f\left((1+a)\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right)\right)\\
&\quad+4\mathfrak{T}_a^n f\left((1+2a)\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right)\right),\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}\\
&\le \mathfrak{k}2^\beta\|\mathfrak{T}_a^n f((1+a)(\mathfrak{u}_1+\mathfrak{u}_2))+2\mathfrak{T}_a^n f((1+a)(\mathfrak{u}_3-\mathfrak{u}_1))+2\mathfrak{T}_a^n f((1+a)(\mathfrak{u}_3+\mathfrak{u}_2))\\
&\quad-4\mathfrak{T}_a^n f\left((1+a)\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right)\right),\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}+\mathfrak{k}\|\mathfrak{T}_a^n f((1+2a)(\mathfrak{u}_1+\mathfrak{u}_2))\\
&\quad+2\mathfrak{T}_a^n f((1+2a)(\mathfrak{u}_3-\mathfrak{u}_1))+2\mathfrak{T}_a^n f((1+2a)(\mathfrak{u}_3+\mathfrak{u}_2))\\
&\quad-4\mathfrak{T}_a^n f\left((1+2a)\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right)\right),\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}\\
&\le 2\mathfrak{k}\left(\frac{1}{2}\right)^\beta\mathfrak{k}^n\mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^n\\
&\quad\times\mathfrak{h}_1((1+a)\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2((1+a)\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3((1+a)\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\\
&\quad+\mathfrak{k}\left(\frac{1}{2}\right)^\beta\mathfrak{k}^n\mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^n\\
&\quad\times\mathfrak{h}_1((1+2a)\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2((1+2a)\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3((1+2a)\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\\
&\le 2\mathfrak{k}\left(\frac{1}{2}\right)^\beta\mathfrak{k}^n\mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^n\\
&\quad\times\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)\mathfrak{h}_1(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\\
&\quad+\mathfrak{k}\left(\frac{1}{2}\right)^\beta\mathfrak{k}^n\mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^n\\
&\quad\times\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a)\mathfrak{h}_1(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\\
&\le\left(\frac{1}{2}\right)^\beta\mathfrak{k}^{n+1}\mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^{n+1}\\
&\quad\times\mathfrak{h}_1(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1,\mathfrak{z}_1,\dots,\mathfrak{z}_{\mathfrak{m}-1}).
\end{aligned}$$

Hence, with the help of induction, we have proved that (2.31) is true for all $n\in\mathbb{N}_0$. Now, from

(2.31) and (2.22), we get

$$\begin{aligned}
&|||\mathfrak{T}_a^n f(\mathfrak{u}_1+\mathfrak{u}_2)+2\mathfrak{T}_a^n f(\mathfrak{u}_3-\mathfrak{u}_1)+2\mathfrak{T}_a^n f(\mathfrak{u}_3+\mathfrak{u}_2)-4\mathfrak{T}_a^n f\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}^{\theta}\\
&\le \mathfrak{M}_2{}^{\theta}\|\mathfrak{T}_a^n f(\mathfrak{u}_1+\mathfrak{u}_2)+2\mathfrak{T}_a^n f(\mathfrak{u}_3-\mathfrak{u}_1)+2\mathfrak{T}_a^n f(\mathfrak{u}_3+\mathfrak{u}_2)-4\mathfrak{T}_a^n f\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\theta}\\
&\le \mathfrak{M}_2{}^{\theta}\left(\frac{1}{2}\right)^{\theta\beta}\mathfrak{k}^{\theta n}\mathfrak{s}_1{}^{\theta}(a)\mathfrak{s}_2{}^{\theta}(-a)\mathfrak{s}_3{}^{\theta}(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)\\
&\quad+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^{\theta n}\\
&\quad\times\mathfrak{h}_1{}^{\theta}(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2{}^{\theta}(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3{}^{\theta}(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}). \qquad (2.32)
\end{aligned}$$

Since $|||.,...,.|||^{\theta}$ is continuous, taking limit $n\to\infty$ (2.32) and using (2.30), we get

$$\begin{aligned}
&|||O_a(\mathfrak{u}_1+\mathfrak{u}_2)+2O_a(\mathfrak{u}_3-\mathfrak{u}_1)+2O_a(\mathfrak{u}_3+\mathfrak{u}_2)-4O_a\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}^{\theta}\\
&=\lim_{n\to\infty}|||\mathfrak{T}_a^n f(\mathfrak{u}_1+\mathfrak{u}_2)+2\mathfrak{T}_a^n f(\mathfrak{u}_3-\mathfrak{u}_1)+2\mathfrak{T}_a^n f(\mathfrak{u}_3+\mathfrak{u}_2)-4\mathfrak{T}_a^n f\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}^{\theta}\\
&=0. \qquad (2.33)
\end{aligned}$$

This implies that

$$O_a(\mathfrak{u}_1+\mathfrak{u}_2)+2O_a(\mathfrak{u}_3-\mathfrak{u}_1)+2O_a(\mathfrak{u}_3+\mathfrak{u}_2)-4O_a\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right)=0. \qquad (2.34)$$

Now, we prove that $O_a:\mathbb{R}_0\to\mathfrak{U}$ is a unique mapping. Let $O_a':\mathbb{R}_0\to\mathfrak{U}$ be a the solution of (2.34) and

$$\begin{aligned}
&\|f(\mathfrak{u}_1)-O_a'(\mathfrak{u}_1),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}^{\theta}\\
&\le\mathfrak{C}\left(\frac{1}{2}\right)^{\theta\beta}\frac{\mathfrak{s}_1{}^{\theta}(a)\mathfrak{s}_2{}^{\theta}(-a)\mathfrak{s}_3{}^{\theta}(1+a)\mathfrak{h}_1{}^{\theta}(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2{}^{\theta}(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3{}^{\theta}(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1})}{1-(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^{\theta}}. \qquad (2.35)
\end{aligned}$$

Now, replacing $(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)$ by $(a\mathfrak{u}_1,-a\mathfrak{u}_1,a\mathfrak{u}_1+\mathfrak{u}_1)$ in (2.34), we get $\mathfrak{T}_aO_a'(\mathfrak{u}_1)=O_a'(\mathfrak{u}_1)\;\forall\,\mathfrak{u}_1\in\mathbb{R}_0$. Moreover, we have

$$\begin{aligned}
&|||O_a(\mathfrak{u}_1)-O_a'(\mathfrak{u}_1),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}\\
&\le\mathfrak{M}_2\|O_a(\mathfrak{u}_1)-O_a'(\mathfrak{u}_1),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}\\
&\le\mathfrak{k}\mathfrak{M}_2(\|O_a(\mathfrak{u}_1)-f(\mathfrak{u}_1),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}+\|f(\mathfrak{u}_1)-O_a'(\mathfrak{u}_1),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta})\\
&\le\mathfrak{C}^{\frac{1}{\theta}}\left(\frac{1}{2}\right)^{\beta}2\mathfrak{k}\mathfrak{M}_2\frac{\mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)\mathfrak{h}_1(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_2(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1,\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1})}{(1-(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^{\theta})^{\frac{1}{\theta}}}.
\end{aligned}$$

Now, we can simply show by induction on n that

$$\begin{aligned}
&|||O_a(\mathfrak{u}_1)-O_a'(\mathfrak{u}_1),\mathfrak{z}_1,...,\mathfrak{z}_{\mathfrak{m}-1}|||_{\mathfrak{q},\beta}\\
&\le\mathfrak{C}^{\frac{1}{\theta}}\left(\frac{1}{2}\right)^{\beta}2\mathfrak{k}\mathfrak{M}_2\mathfrak{k}^n\mathfrak{s}_1(a)\mathfrak{s}_2(-a)\mathfrak{s}_3(1+a)(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)\\
&\quad+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^n
\end{aligned}$$

$$\times \frac{1}{(1-(2\mathfrak{s}_1(1+a)\mathfrak{s}_2(1+a)\mathfrak{s}_3(1+a)+\mathfrak{s}_1(1+2a)\mathfrak{s}_2(1+2a)\mathfrak{s}_3(1+2a))^{\theta})^{\frac{1}{\theta}}}\mathfrak{h}_1(\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1})$$
$$\times\mathfrak{h}_2(\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1})\mathfrak{h}_3(\mathfrak{u}_1,\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}). \tag{2.36}$$

Taking $n\to\infty$ in (2.36), we get $O_a \equiv O'_a$, hence O_a is unique fixed of $\mathfrak{T}_a$. Now, taking $a\to\infty$ in (2.29), we get

$$\lim_{a\to\infty}\|f(\mathfrak{u}_1)-O_a(\mathfrak{u}_1),\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}\|_{\mathfrak{q},\beta}=0 \implies \lim_{a\to\infty}O_a(\mathfrak{u}_1)=f(\mathfrak{u}_1). \tag{2.37}$$

Also, taking $a\to\infty$ in (2.33) and using (2.37), we get

$$\begin{aligned}
&\||f(\mathfrak{u}_1+\mathfrak{u}_2)+2f(\mathfrak{u}_3-\mathfrak{u}_1)+2f(\mathfrak{u}_3+\mathfrak{u}_2)-4f\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right),\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|\|_{\mathfrak{q},\beta}\\
&=\lim_{a\to\infty}\||O_a(\mathfrak{u}_1+\mathfrak{u}_2)+2O_a(\mathfrak{u}_3-\mathfrak{u}_1)+2O_a(\mathfrak{u}_3+\mathfrak{u}_2)-4O_a\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right),\mathfrak{z}_1,\ldots,\mathfrak{z}_{\mathfrak{m}-1}|\|_{\mathfrak{q},\beta}\\
&=0,
\end{aligned}$$

this implies that

$$f(\mathfrak{u}_1+\mathfrak{u}_2)+2f(\mathfrak{u}_3-\mathfrak{u}_1)+2f(\mathfrak{u}_3+\mathfrak{u}_2)-4f\left(\mathfrak{u}_3-\frac{\mathfrak{u}_1+\mathfrak{u}_2}{4}\right)=0$$

for all $\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3\in\mathbb{R}_0$. This demonstrates that f is a solution to equation (2.24). □

Conclusion

In this work, we validated a fixed-point theorem within the broader framework of quasi-(m,β)-normed spaces, extending the foundational contributions of Brzdek and El-Fassi. This extension facilitated the analysis of the hyperstability of Apollonius-type functional equations, demonstrating the effectiveness of fixed-point methods in addressing stability phenomena under generalized normed structures.

The results presented underscore the essential role of inequalities in understanding stability in functional equations and highlight the applicability of fixed-point techniques in addressing advanced mathematical problems. Future research could explore the hyperstability of other classes of functional equations, such as Jensen-type, quadratic, or mixed-type equations, further advancing the integration of inequalities and fixed-point methodologies in stability theory.

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References

[1] M. R. Abdollahpour, A. Najati and Y. Khedmati Yengejeh, *Stability results for some functional equations on 2-Banach spaces with restricted domains*, J. Math., 2025, 2025, 8405645, 9.

[2] H. Azadi Kenary, *Hyers–Ulam–Rassias stability of set-valued functional equations: A fixed point approach*, Math. Anal. Differ. Equ. Appl., 2024, 3, 97–109.

[3] J. A. Baker, *The stability of certain functional equations*, Proc. Am. Math. Soc., 1991, 112, 729–732.

[4] D. G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J., 1949, 16, 385–397.

[5] J. Brzdęk, J. Chudziak and Z. Páles, *A fixed point approach to the stability of functional equations*, Nonlinear Anal., 2011, 17, 6728–6732.

[6] J. Brzdęk and K. Ciepliński, *On a fixed point theorem in 2-Banach spaces and some of its applications*, Acta Math. Sci. Ser. B Engl. Ed., 2018, 38, 377–390.

[7] N. V. Dung and V. T. L. Hang, *The generalized hyperstability of general linear equations in quasi-Banach spaces*, J. Math. Anal. Appl., 2018, 562, 131–147.

[8] E. Fassi, *A new hyperstability result for the Apollonius equation on a restricted domain and some applications*, J. Fixed Point Theory Appl., 2018, 20, 1–16.

[9] S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr., 1964, 28, 1–43.

[10] S. Gähler, *Untersuchungen über verallgemeerte m-metrische Räume. I.*, Math. Nachr., 1969, 40, 165–189.

[11] K. Hattaf, *On the stability and numerical scheme of fractional differential equations with application to biology*, Computation, 2022, 10, 97.

[12] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, 1941, 27, 222–224.

[13] K. Indira, P. Agilan, M. Suresh, M. Balamurugan, S. Karthikeyan and R. Sakthi, *Solution and Fuzzy stability of n-dimensional quadratic functional equation*, Int. J. Anal. Appl., 2025, 23, 83.

[14] J. Jakhar, D. Baleanu, S. Sharma, M. A. Yousif, J. Jakhar, N. Chorfi and P. O. Mohammed, *Comparative results on stability analysis for three dimensional functional equations*, Eur. J. Pure Appl. Math., 2025, 18, 6098.

[15] J. Jakhar, R. Chugh and J. Jakhar, *Solution and intuitionistic Fuzzy stability of 3-dimensional cubic functional equation: Using two different methods*, J. Math. Comput. Sci., 2022, 25, 103–114.

[16] J. Jakhar, R. Chugh and J. Jakhar, *Stability of various iterative type functional equations in Menger-ϕ normed space*, Bull. Math. Anal. Appl., 2021, 13, 106–120.

[17] J. Jakhar, J. Jakhar and R. Chugh, *Fuzzy stability of mixed type functional equations in Modular spaces*, Math. Found. Comput., 2024, 7, 419–428.

[18] J. Jakhar, S. Sharma, J. Jakhar, M. A. Yousif, P. O. Mohammed, N. Chorfi and M. Vivas-Cortez, *Hyers–Ulam–Rassias stability of functional equations with integrals in B-metric frameworks*, Symmetry, 2025, 17, 168.

[19] J. Jakhar, S. Sharma, J. Jakhar, M. A. Yousif, P. O. Mohammed, A. A. Lupas and N. Chorfi, *Orthogonal stability and solution of a three-variable functional equation in extended Banach spaces*, Mathematics, 2024, 12, 2868.

[20] Jyotsana, R. Chugh, S. Jaiswal and R. Dubey, *Stability of functional equations in non-Archimedean (n, β)-normed spaces*, J. Anal., 2022, 30, 1653–1669.

[21] S. Kim and J. M. Rassias, *Stability of the Apollonius-type additive functional equation in modular spaces and fuzzy Banach spaces*, Mathematics, 2019, 7, 1125.

[22] Z. Li, B. Yang and L. Xu, *η-stability of uncertain functional differential equations*, Int. J. Control, 2025, 1, 1–8.

[23] G. Maksa and Z. Páles, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Paedagog. Nyházi., 2001, 17, 107–112.

[24] M. Mirzavaziri and M. S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc., 2006, 37, 361–376.

[25] A. Misiak, *n-inner product spaces*, Math. Nachr., 1989, 140, 299–319.

[26] C. Park and T. M. Rassias, *Homomorphism in $C^{\star}$-ternary algebras and $JB^{\star}$-triples*, J. Math. Anal. Appl., 2008, 337, 13–20.

[27] G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Springer: Berlin, 1925.

[28] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc., 1978, 72, 297–300.

[29] M. Sarfraz, J. Zhou, M. Islam and Y. Li, *Isometries to analyze the stability of norm-based functional equations in p-uniformly convex spaces*, Symmetry, 2024, 16, 825.

[30] Y. Sayyari, M. Dehghanian, A. Mohammadhasani and M. J. Nassiri, *Hyers–Ulam stability of the drygas type functional equation*, J. Anal., 2025, 33, 225–233.

[31] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience: New York, 1960.

[32] X. Yang, L. Chang, G. Liu and G. Shen, *Stability of functional equations in (n, β)-normed spaces*, J. Inequal. Appl., 2015, 1, 112–130.

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