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L^p-NULL CONTROLLABILITY OF A ψ-CAPUTO FRACTIONAL EVOLUTION SYSTEM*

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Abstract This article intends to investigate the L^p-null controllability of an abstract system involving the ψ-Caputo fractional derivative in a Banach space. By leveraging the operator L_0^{-1} and integrating the improved approximation solvability technique with the resolvent approach, we first explore this problem without relying on the compactness of the semigroup, Lipschitz continuity of the perturbation item or noncompactness measure conditions. Subsequently, by introducing suitable conditions and combining Banach's fixed point theorem with the integral contractor method, we tackle this problem independent of L_0^{-1}. Finally, we offer two concrete examples to confirm the practical applicability of our mentioned findings.

Keywords ψ-Caputo fractional derivative, L^p-null controllability, resolvent, approximation solvability method, integral contractor.

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1. Introduction

Fractional evolution equations have held a prominent position based on their compelling mathematical framework for modeling memory-dependent physical systems as evidenced by studies [4, 6, 28, 29, 37, 38]. Among these equations, Riemann-Liouville, Caputo and Hilfer fractional systems have received the most extensive investigation. In [1], Almeida initially introduced and investigated a ψ-Caputo type fractional derivative, which generalizes the Caputo fractional derivative, allows superior flexibility in the selection of the function ψ and enhances the accuracy of objective modeling. Subsequently, many researchers have broadened the utilization of this derivative in evolution models, see [7, 12, 19, 20, 30–32]. In [19, 20], Liang et al. investigated ψ-Caputo fractional Sobolev-type integro-differential nonlocal equations. Sharma et al. analyzed a neutral-type ψ-Caputo fractional delayed stochastic differential inclusion [30] and further extended their work to neutral-type fractional Sobolev-type stochastic systems [31] and neutral-type fractional stochastic systems with the instantaneous impulses [32].

The exploration of controllability plays a pivotal role in advancing the development of evolution equations and control theory. Several distinct notions of controllability have been extensively studied, including exact controllability, null controllability and approximate controllability. In real-world applications, exact controllability proves valuable in rocket launches and null controllability holds particular significance in population dynamics (see [18]). Consequently, many scholars have shown keen interest in these two concepts, see [8, 14, 17, 18, 23, 31]. For

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instance, Sharma et al. [31] established the controllability of a ψ-Caputo fractional Sobolev type stochastic delay system by utilizing the inverse of the controllability operator. Under the condition $R(f) \subseteq R(B)$, the integral contractor method was employed in [8] to investigate the controllability of Hilfer fractional evolution equations. Recently, with the aid of the inverse of the controllability operator, Malaguti et al. [23] analyzed the L^p-controllability of a nonlocal semilinear differential equation involving controls in the Banach space $L^p(J, U)$ with a broader range of p-values. Subsequently, Kumbhakar and Pandey [18] investigated the L^p-null controllability of a nonlocal semilinear differential inclusion. Furthermore, in certain infinite dimensional systems, null controllability is attainable while exact controllability is unachievable (see [11, 18]). We thereby prefer L^p-null controllability over exact controllability and approximate controllability.

The compactness of the semigroup or measure of noncompactness condition was traditionally indispensable in many studies of evolution equations. Recently, [5] introduced a novel approximation solvability approach to bypass these constraints. We outline this approach: First, by employing the projection operator $\mathbb{P}_m$ and Yosida's approximation operator A_n, the authors constructed the approximation system in H_m (a finite dimensional subspace of a Hilbert space H), thereby eliminating the need for the compactness of the semigroup. Then, by leveraging the approximation properties of A_n and the weak sequential compactness of $L^1(J, H)$, they obtained the solution to the system under study. Subsequently, many authors developed and improved this idea to explore the solvability and controllability of broader classes of abstract systems. Xu et al. [35] developed the approach by replacing the operator A_n with $A_m = \mathbb{P}_m A$ and exploiting the weak sequential compactness of $C(J, H)$. In [39, 40, 42, 43], we extended this method to fractional evolution systems by utilizing resolvent theory. The authors in [18, 23] further generalized the technique to Banach spaces by employing the natural projection operator $\mathbb{P}_m$ and the weak sequential compactness of $L^p(J, X)$.

However, until now, the L^p-null controllability of ψ-Caputo fractional control systems in Banach spaces without invoking compactness of the semigroup or measure of noncompactness condition remains unexplored. Let $J' = (0, b]$, $J = [0, b]$, X be a reflexive separable Banach space and U be a uniformly convex Banach space. We thereby center on the L^p-null controllability of the following ψ-Caputo fractional control system:

$$\begin{cases} D^{\alpha,\psi}x(t) = Ax(t) + f(t, x(t)) + Bu(t), t \in J', \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Here $\alpha \in (0, 1)$, A generates a semigroup $\{T(t)\}_{t\geq 0}$, $f : J \times X \to X$, $B \in \mathscr{L}(U, X)$.

When $\alpha = 1,\ \psi(t) = t,\ A = \Delta$, system (1.1) reduces to the classical parabolic models. Such models can describe a wide range of physical phenomena, see [41].

This work exhibits the following key features:

(i) We develop the approximation solvability approach in Theorem 3.1 to address the L^p-null controllability of system (1.1) without relying on the compactness of $\{T(t)\}_{t\geq 0}$ or measure of noncompactness conditions. We outline our idea: By employing L_0^{-1} (the inverse of control operator L_0) and the natural projection operator $\mathbb{P}_m$, we first construct the auxiliary operator in the subspace X_m to yield the state-control approximation sequence pair (x_m, u_m) satisfying $x_m(b) = 0$. Subsequently, by leveraging the properties of $\mathbb{P}_m$ and the weak sequential compactness of $C(J, X)$, we set up a control function $u \in L^p(J, U)$, a state function x and a subsequence (still written $\{x_m\}_{m>0}$) extracted from $\{x_m\}_{m>0}$ to ensure that $x_m \rightharpoonup x,\ x \in S(u)$ and $x(b) = 0$. We emphasize that we eliminate the use of A_n in [5] and drop the operator

$A_m = \mathbb{P}_m A$ in [35, 39, 40, 42, 43]. In addition, we replace the weak sequential compactness of $L^p(J, X)$ in [18, 23] with the weak sequential compactness of $C(J, X)$ to construct the weakly convergent subsequence.

(ii) By utilizing a weaker condition (Hc) (instead of $R(f) \subseteq R(B)$ as in [8]), we integrate Banach's fixed point method with the integral contractor approach in Theorem 4.2 to investigate the L^p-null controllability of system (1.1) without relying on the operator L_0^{-1} or the compactness of $\{T(t)\}_{t\geq 0}$. We outline our technique: With the help of the regular integral contractor condition, we first construct the iteration sequences to establish the existence result of system (1.1). Then, we derive the required control function by using Banach's fixed point theorem. Emphasis here is that our approach remains valid under the condition $R(f) \subseteq R(B)$.

(iii) We present a meticulous analysis of two concrete examples to demonstrate the feasibility of our assumptions in our mentioned abstract findings.

We provide a breakdown of the residual parts. The next Section incorporates some helpful symbols and important known facts. Section 3 discusses the case involving the operator L_0^{-1}. Section 4 examines the case independent of L_0^{-1}. Section 5 contains two examples to validate our outcomes. Finally, in Section 6, we offer our conclusion and point out our future work.

2. Preliminaries

We begin by gathering some symbols. Let X^* and $\{e_m\}_{m\in\mathbb{N}}$ denote the dual space and a monotone Schauder basis of X, respectively. We signify by $\mathbb{P}_m : X \to X_m$ the natural projection operator from X onto the subspace $X_m = \text{span}\{e_1, e_2, \cdots, e_m\}$. Let $C(J, X)$ be the space of all continuous functions from J to X with the sup-norm $\|x\| = \sup\limits_{t\in J} \|x(t)\|$ and $L^p([0, b], X)$ the space of all X-valued Bochner integrable functions on $[0, b]$ with the norm $\|x\|_{L^p} = \big(\int_0^b \|x(t)\|^p \mathrm{d}t\big)^{\frac{1}{p}}$. Moreover, we define

$$C^1(J, \mathbb{R}) = \{\psi \in C(J, \mathbb{R}) | \psi' \in C(J, \mathbb{R})\}$$

with the norm $\|\psi\|_{C^1} = \sup\limits_{t\in J}(|\psi(t)| + |\psi'(t)|)$. We set

$$\mathscr{L}(X, U) = \{f : X \to U |\ f \text{ is linear and bounded}\}$$

and abbreviate $\mathscr{L}(X, X)$ to $\mathscr{L}(X)$. Additionally, throughout this paper, we assume that $\frac{1}{\alpha} < p < +\infty$.

Lemma 2.1. [18, 23] *The operator $\mathbb{P}_m$ holds the following outstanding features:*

(i) $\mathbb{P}_m$ is linear and bounded;

(ii) $\|\mathbb{P}_m\| = 1$;

(iii) as $m \to \infty$, $\mathbb{P}_m y \to y$ for any $y \in X$;

(iv) as $m \to \infty$, if $y_m \rightharpoonup y$ in X, then $\mathbb{P}_m y_m \rightharpoonup y$ in X;

(v) as $k \to \infty$, if $y_k \rightharpoonup y$ in X, then $\mathbb{P}_m y_k \to \mathbb{P}_m y$ and $\mathbb{P}_m y_k \rightharpoonup y$ in X.

Let $\psi \in C^1(J, \mathbb{R})$ satisfy $\psi'(t) > 0$ and $\psi(0) = 0$. We then revisit some notions associated with this function ψ.

Definition 2.1. [15] The α-order ψ-Riemann-Liouville fractional integral operator $I^{\alpha,\psi}$ is defined by

$$I^{\alpha,\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(\tau))^{\alpha-1} \psi'(\tau) g(\tau) \mathrm{d}\tau,\ t > 0,\ \alpha > 0.$$

Definition 2.2. [1] The α-order ψ-Caputo type fractional derivative operator $D^{\alpha,\psi}$ is defined by

$$D^{\alpha,\psi}g(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (\psi(t)-\psi(\tau))^{-\alpha}g'(\tau)\mathrm{d}\tau,\ t>0,\ 0<\alpha<1.$$

Remark 2.1. When $\psi(t)=t$, $D^{\alpha,\psi}$ turns into the well-known Caputo type fractional derivative operator, see [25].

Definition 2.3. [19] The ψ-Laplace transform and the Laplace transform of a function g are defined by

$$\mathcal{L}_\psi[g](\lambda) = \int_0^\infty e^{-\lambda\psi(t)}\psi'(t)g(t)\mathrm{d}t$$

and

$$\mathcal{L}[g](\lambda) = \int_0^\infty e^{-\lambda t}g(t)\mathrm{d}t,$$

respectively.

Next, we summarize the concepts of fractional resolvent and solution operator, respectively.

Definition 2.4. [3] By a fractional resolvent generated by A, we mean the family $\{S_\alpha(t)\}_{t>0} \subseteq \mathscr{L}(X)$ satisfying that for $x\in X$, $S_\alpha(\cdot)x \in C(\mathbb{R}^+, X)$ and there exists $\omega>0$ such that

$$(\lambda^\alpha \mathrm{I}-A)^{-1}x = \mathcal{L}[S_\alpha(t)x](\lambda),\ Re\lambda>\omega.$$

Definition 2.5. [4] By a solution operator generated by A, we mean the family $\{T_\alpha(t)\}_{t>0} \subseteq \mathscr{L}(X)$ satisfying that for $x\in X$, $T_\alpha(\cdot)x\in C(\mathbb{R}^+,X)$ and there exist two constants $M>0$ and $\omega>0$ such that $\|T_\alpha(t)\| \le Me^{\omega t}$ and

$$\lambda^{\alpha-1}(\lambda^\alpha \mathrm{I}-A)^{-1}x = \mathcal{L}[T_\alpha(t)x](\lambda),\ Re\lambda>\omega.$$

Let

$$S_\alpha(t) = t^{\alpha-1}\int_0^\infty \alpha\tau\Psi_\alpha(\tau)T(t^\alpha\tau)\mathrm{d}\tau,\ T_\alpha(t) = \int_0^\infty \Psi_\alpha(\tau)T(t^\alpha\tau)\mathrm{d}\tau,$$

where $\Psi_\alpha(t)$ is defined by (see [22, 44])

$$\Psi_\alpha(t) = \frac{1}{\alpha}t^{-1-1/\alpha}\omega_\alpha(t^{-1/\alpha}),$$
$$\omega_\alpha(t) = \frac{1}{\pi}\sum_{m=0}^\infty (-1)^m t^{-(m+1)\alpha-1}\frac{\Gamma((m+1)\alpha+1)}{(m+1)!}\sin\left((m+1)\pi\alpha\right),\ t\in\mathbb{R}^+.$$

We can obtain the following subordination principles:

Lemma 2.2. [4, 39, 40] *Let A generate an analytic semigroup $\{T(t)\}_{t\ge0}$ with $\|T(t)\|\le M$. Then*

(a) $\{S_\alpha(t)\}_{t>0}$ is a fractional resolvent satisfying that $\{t^{1-\alpha}S_\alpha(t)\}_{t\ge0}$ is equicontinuous and $\|t^{1-\alpha}S_\alpha(t)\| \le \frac{M}{\Gamma(\alpha)}$;

(b) $\{T_\alpha(t)\}_{t\ge0}$ is a solution operator satisfying that $\{T_\alpha(t)\}_{t\ge0}$ is equicontinuous and $\|T_\alpha(t)\| \le M$.

Define an operator $\widetilde{B} : L^p(J,U) \to L^p(J,X)$ by $(\widetilde{B}u)(t) = Bu(t)$. By applying $B \in \mathscr{L}(U,X)$, we obtain $\widetilde{B} \in \mathscr{L}(L^p(J,U), L^p(J,X))$. To promote establishment the concepts of mild solution to (1.1) and L^p-null controllability, respectively, we replace B by $\widetilde{B}$ and treat the ensuing system:

$$\begin{cases} D^{\alpha,\psi}x(t) = Ax(t) + g(t) + (\widetilde{B}u)(t), \ t \in (0,b], \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $g \in L^p(J,X)$.

Based on (2.1) and $I^{\alpha,\psi}D^{\alpha,\psi}x(t) = x(t) - x_0$, we get

$$x(t) = x_0 + I^{\alpha,\psi}Ax(t) + I^{\alpha,\psi}g(t) + I^{\alpha,\psi}(\widetilde{B}u)(t).$$

By employing ψ-Laplace transform and

$$\mathcal{L}_\psi[I^{\alpha,\psi}g](\lambda) = \frac{1}{\lambda^\alpha}\mathcal{L}_\psi[g](\lambda),$$

we obtain

$$\mathcal{L}_\psi[x](\lambda) = \lambda^{\alpha-1}(\lambda^\alpha \mathrm{I} - A)^{-1}x_0 + (\lambda^\alpha \mathrm{I} - A)^{-1}(\mathcal{L}_\psi[g](\lambda) + \mathcal{L}_\psi[\widetilde{B}u](\lambda)).$$

Direct calculation gives

$$\mathcal{L}_\psi[T_\alpha(\psi(t))x_0](\lambda) = \mathcal{L}[T_\alpha(t)x_0](\lambda)$$

and

$$\mathcal{L}_\psi\left[\int_0^t S_\alpha(\psi(t)-\psi(s))g(s)\psi'(s)\mathrm{d}s\right](\lambda) = (\lambda^\alpha \mathrm{I} - A)^{-1}\mathcal{L}_\psi[g](\lambda).$$

Hence, we get

$$x(t) = T_\alpha(\psi(t))x_0 + \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)(g(\tau) + (\widetilde{B}u)(\tau))\mathrm{d}\tau, \ t \in J'.$$

Definition 2.6. By the mild solution to system (1.1), we understand the function $x \in C(J,X)$ satisfying that for $t \in J'$,

$$x(t) = T_\alpha(\psi(t))x_0 + \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)(f(\tau,x(\tau)) + Bu(\tau))\mathrm{d}\tau. \tag{2.2}$$

To facilitate subsequent investigation, in what follows, I always substitute $\widetilde{B}$ for B and set

$$S(u) = \big\{x(\cdot,u,f) \in C(J,X) : x(\cdot,u,f) = x(\cdot) \text{ satisfies (2.2)}\big\}.$$

Define two operators $L_0^b : L^p(J,U) \to X$ and $N_0^b : X \times L^p(J,X) \to X$ by

$$L_0^b(u) = \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)(\widetilde{B}u)(\tau)\mathrm{d}\tau,$$

and

$$N_0^b(x_0,f) = T_\alpha(\psi(b))x_0 + \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)f(\tau)\mathrm{d}\tau,$$

respectively.

Definition 2.7. [10, 13] System (2.1) is null controllable on J, if $\mathrm{Im}L_0^b \supset \mathrm{Im}N_0^b$ or there is a state-control pair (x, u) such that $u \in L^p(J, U)$ and

$$x(b) = L_0^b(u) + N_0^b(x_0, f) = 0.$$

Let $Ker(L_0^b) = \{u \in L^p(J, U) : L_0^b(u) = 0\}$. Relying on the linearity and boundedness of L_0^b, we can introduce a Banach space $L^p(J, U)/Ker(L_0^b)$ with the norm $\|[u]\| = \inf_{v \in [u]} \|v\|_{L^p}$, where

$$[u] = \{v \in L^p(J, U) : L_0^b(v) = L_0^b(u)\} = u + Ker(L_0^b).$$

Define an operator $L_0 : L^p(J, U)/KerL_0^b \to \mathrm{Im}L_0^b$ by

$$L_0([u]) = L_0^b(u).$$

Then Banach's inverse theorem yields a linear bounded operator (the inverse of L_0) L_0^{-1} : $\mathrm{Im}L_0^b \to L^p(J, U)/Ker(L_0^b)$.

Thanks to [23], we can consider a continuous operator

$$\Pi : L^p(J, U)/Ker(L_0^b) \to L^p(J, U)$$

satisfying $L_0^b(\Pi([u])) = L_0^b(u)$ and

$$\|\Pi([u])\|_{L^p} = \min\{\|v\|_{L^p} : L_0^b(u) = L_0^b(v),\ v \in L^p(J, U)\}.$$

In addition, the definition of the norm $\|[u]\|$ enables us to assume that

$$\|\Pi([u])\|_{L^p} \leq 2\|[u]\|.$$

Definition 2.8. If there is a state-control pair (x, u) such that $u \in L^p(J, U)$, $x \in S(u)$ and $x(b) = 0$, system (1.1) is L^p-null controllable on J.

Subsequently, we propose a notion of the regular integral contractor.

Definition 2.9. By the bounded regular integral contractor of the function f, we understand the operator $\gamma : J \times X \to \mathscr{L}(C(J, X))$ satisfying that for all $t \in J$ and $x,\ y \in C(J, X)$,

$$\begin{aligned}\Bigg\| f\bigg(t, x(t) + y(t) + \int_0^t S_\alpha(\psi(t) - \psi(\tau))\psi'(\tau)(\gamma(\tau, x(\tau))y)(\tau)\mathrm{d}\tau\bigg) \\ - f(t, x(t)) - (\gamma(t, x(t))y)(t)\Bigg\| \leq \lambda\|y(t)\|\end{aligned} \tag{2.3}$$

with $\lambda > 0$ and for any $x,\ z \in C(J, X)$, the equation

$$y(t) + \int_0^t S_\alpha(\psi(t) - \psi(\tau))\psi'(\tau)(\gamma(\tau, x(\tau)y)(\tau)\mathrm{d}\tau = z(t)$$

has a solution $y \in C(J, X)$.

Remark 2.2. For the integral contractor γ, we note:

(i) due to the boundedness of γ, we can suppose that

$$\|(\gamma(t, x(t))y)(t)\| \leq \rho\|y(t)\|,$$

(ii) if $f : J \times X \to X$ satisfies $\|f(t, x(t)) - f(t, y(t))\| \leq \lambda\|x(t) - y(t)\|$, then f holds a regular integral contractor $\gamma = 0$.

Afterward, we propose the Gronwall type inequality in the form:

Lemma 2.3. *Let $\rho > 0$ and $\lambda > 0$. If for $t \in [0, b]$,*

$$z(t) \leq \lambda + \rho \int_0^t (\psi(t) - \psi(\tau))^{\alpha-1} \psi'(\tau) z(\tau) \mathrm{d}\tau,$$

with a locally integrable nonnegative function $z(t)$ on $[0, b]$, then

$$z(t) \leq \lambda E_\alpha(\rho \Gamma(\alpha) \psi^\alpha(b)).$$

Proof. Let $Hz(t) = \rho \int_0^t (\psi(t) - \psi(\tau))^{\alpha-1} \psi'(\tau) z(\tau) \mathrm{d}\tau$. We estimate that

$$z(t) \leq \lambda \sum_{k=0}^{n-1} \frac{(\rho \Gamma(\alpha) \psi^\alpha(b))^k}{\Gamma(k\alpha + 1)} + H^n z(t).$$

Employing induction gives

$$H^n z(t) \leq \int_0^t \frac{(\rho \Gamma(\alpha))^n}{\Gamma(n\alpha)} (\psi(t) - \psi(\tau))^{n\alpha-1} \psi'(\tau) z(\tau) \mathrm{d}\tau.$$

Since for any $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\alpha(\rho \Gamma(\alpha) t^\alpha) = \sum_{n=1}^{\infty} \frac{(\rho \Gamma(\alpha))^n t^{n\alpha-1}}{\Gamma(n\alpha)},$$

we can conclude that $H^n z(t) \to 0$, as $n \to \infty$, hence that

$$z(t) \leq \lambda E_\alpha(\rho \Gamma(\alpha) \psi^\alpha(b)).$$

□

We end up with two important theoretical tools:

Theorem 2.1. [35] *Let D be a bounded equicontinuous subset of $C(J, X)$. Then D is relatively weakly sequentially compact in $C(J, X)$.*

Theorem 2.2. [2] *Let E be a convex closed subset of X and $T : E \times [0, 1] \to X$ be a compact map with a closed graph. If $T(E, 0) \subset \mathring{E}$ and for all $\lambda \in [0, 1)$, $T(\cdot, \lambda)$ does not hold fixed points on ∂E, then $T(y, 1) = y$ has fixed points.*

3. Controllability result invoking the inverse operator

By relying on the operator L_0^{-1} and integrating the resolvent method with the approximation solvability trick, we here deal with the L^p-null controllability of system (1.1). We list our prerequisites:

(HA) $\{T(t)\}_{t \geq 0}$ is analytic with $\sup\limits_{t \in J} \|T(t)\| \leq M$.

(Hf) $f : J \times X \to X$ satisfies

- (i) for every $x \in X$, $f(\cdot, x) : J \to X$ is measurable;
- (ii) for a.e. $\tau \in J$, $f(\tau, \cdot) : X \to X$ is weak-to-weak continuous;
- (iii) for every $x \in X$ and $\tau \in J$, $\|f(\tau, x)\| \leq \rho(\tau)\|x\|$ with $\rho \in L^p(J, X)$.

(Hl) System (2.1) is null controllable.

Remark 3.1. Under condition (HA), according to Lemma 2.2, A can generate a fractional resolvent $\{S_\alpha(t)\}_{t>0}$ and a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ satisfying that $\{t^{1-\alpha}S_\alpha(t)\}_{t\geq 0}$ and $\{T_\alpha(t)\}_{t\geq 0}$ are equicontinuous, $\{S_\alpha(t)\}_{t>0} \subseteq \mathscr{L}(X)$, $\{T_\alpha(t)\}_{t\geq 0} \subseteq \mathscr{L}(X)$, $\|t^{1-\alpha}S_\alpha(t)\| \leq \frac{M}{\Gamma(\alpha)}$ and $\|T_\alpha(t)\| \leq M$. In what follows, we analyze the L^p-null controllability of system (1.1) by applying the fractional resolvent $\{S_\alpha(t)\}_{t>0}$ and the solution operator $\{T_\alpha(t)\}_{t\geq 0}$.

We start with the following two weak convergence results:

Lemma 3.1. *Let (HA) hold, $h \in L^p(J,X)$ and $h_m \rightharpoonup h$ in $L^p(J,X)$. Then*

$$\int_0^t \mathbb{P}_m S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)h_m(\tau)\mathrm{d}\tau \rightharpoonup \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)h(\tau)\mathrm{d}\tau$$

in X.

Proof. Based on Remark 3.1, Hölder's inequality and the conditions imposed on the function ψ, we have

$$\begin{aligned}
&\left\|\int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)h(\tau)\mathrm{d}\tau\right\| \\
&\leq \frac{M}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(\tau))^{\alpha-1}\psi'(\tau)\|h(\tau)\|\mathrm{d}\tau \\
&\leq \frac{M}{\Gamma(\alpha)}\|\psi\|_{C^1}^{\frac{1}{p}}\int_0^t (\psi(t)-\psi(\tau))^{\alpha-1}(\psi'(\tau))^{\frac{p-1}{p}}\|h(\tau)\|\mathrm{d}\tau \\
&\leq \frac{M}{\Gamma(\alpha)}\|\psi\|_{C^1}^{\alpha}\|h\|_{L^p}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}.
\end{aligned}$$

For $x^* \in X^*$, we thus can easily obtain a bounded linear operator $\Phi : L^p(J,X) \to \mathbb{R}$ defined by

$$\Phi(h) = x^*\left(\int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)h(\tau)\mathrm{d}\tau\right),$$

which indicates that $\Phi \in (L^p(J,X))^*$. By means of $h_m \rightharpoonup h$ in $L^p(J,X)$, we have $\Phi(h_m) \to \Phi(h)$. Thus,

$$\int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)h_m(\tau)\mathrm{d}\tau \rightharpoonup \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)h(\tau)\mathrm{d}\tau.$$

Thanks to Lemma 2.1, we arrive at our needed result. □

Lemma 3.2. *Let (HA) hold and $x_m \rightharpoonup x$ in X. Then*

$$\int_0^t \mathbb{P}_m S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)\mathbb{P}_m x_m\mathrm{d}\tau \rightharpoonup \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)x\mathrm{d}\tau.$$

Proof. Due to Lemma 2.1 and $x_m \rightharpoonup x$, we obtain $\mathbb{P}_m x_m \rightharpoonup x$. By Remark 3.1 and the conditions imposed on the function ψ, we thereby have

$$S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)\mathbb{P}_m x_m \rightharpoonup S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)x.$$

Let $x^* \in X^*$. We get

$$x^*\left(\int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)\mathbb{P}_m x_m\mathrm{d}\tau\right)$$

$$
\begin{aligned}
&= \int_0^t x^* \left(S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)\mathbb{P}_m x_m \right) \mathrm{d}\tau \\
&\to \int_0^t x^* \left(S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)x \right) \mathrm{d}\tau \\
&= x^* \left(\int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)x\mathrm{d}\tau \right).
\end{aligned}
$$

Hence, we have

$$
\int_0^t \mathbb{P}_m S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)\mathbb{P}_m x_m \mathrm{d}\tau \rightharpoonup \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)x\mathrm{d}\tau.
$$

Thanks to Lemma 2.1, our desired assertion can be obtained. □

Let $h \in L^p(J,X)$. Define a map $\Lambda : L^p(J,X) \to C(J,X_m)$ by

$$
\Lambda(h)(\cdot) = \int_0^\cdot \mathbb{P}_m S_\alpha(\psi(\cdot)-\psi(\tau))\psi'(\tau)h(\tau)\mathrm{d}\tau.
$$

We then check the equicontinuity of the map Λ.

Lemma 3.3. *Let condition* (HA) *hold. The map* Λ *is equicontinuous.*

Proof. Let $h \in L^p(J,X)$ with $\|h\|_{L^p} \le r$. By Remark 3.1, Hölder's inequality and the conditions imposed on the function ψ, an analysis analogous to Lemma 3.1 gives

$$
\|\Lambda(h)(t)\| \le \frac{Mr}{\Gamma(\alpha)}\psi^{\alpha-\frac{1}{p}}(t)\|\psi\|_{C^1}^{\frac{1}{p}} \left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}.
$$

Thus, we get $\|\Lambda(h)(0)\| = \lim_{t\to 0^+} \|\Lambda(h)(t)\| = 0$ and $\|\Lambda(h)(t) - \Lambda(h)(0)\| \to 0,\ t \to 0^+$.

Let $t_1, t_2 \in J$ with $0 < t_1 < t_2 \le b$. For abbreviation, we write

$$
\begin{aligned}
&\widetilde{S_\alpha}(\psi(t)-\psi(\tau)) = (\psi(t)-\psi(\tau))^{1-\alpha} S_\alpha(\psi(t)-\psi(\tau)), \\
&\Phi(t_2,t_1) = \widetilde{S}_\alpha(\psi(t_2)-\psi(\tau)) - \widetilde{S}_\alpha(\psi(t_1)-\psi(\tau)), \\
&g(t_2,t_1) = (\psi(t_1)-\psi(\tau))^{\alpha-1} - (\psi(t_2)-\psi(\tau))^{\alpha-1}.
\end{aligned}
$$

Let $0 < \varepsilon < t_1$. Based on Remark 3.1, Hölder's inequality and the conditions imposed on the function ψ, we have

$$
\begin{aligned}
&\|\Lambda(h)(t_2) - \Lambda(h)(t_1)\| \\
\le &\frac{M}{\Gamma(\alpha)} \int_{t_1}^{t_2} (\psi(t_2)-\psi(\tau))^{\alpha-1}\psi'(\tau)\|h(\tau)\|\mathrm{d}\tau \\
&+ \int_0^{t_1-\varepsilon} (\psi(t_2)-\psi(\tau))^{\alpha-1}\psi'(\tau)\|\Phi(t_2,t_1)\|\|h(\tau)\|\mathrm{d}\tau \\
&+ \int_{t_1-\varepsilon}^{t_1} (\psi(t_2)-\psi(\tau))^{\alpha-1}\psi'(\tau)\|\Phi(t_2,t_1)\|\|h(\tau)\|\mathrm{d}\tau \\
&+ \int_0^{t_1} g(t_2,t_1)\psi'(\tau)\|\widetilde{S_\alpha}(\psi(t_1)-\psi(\tau))\|\|h(\tau)\|\mathrm{d}\tau
\end{aligned}
$$

$$\le \frac{Mr}{\Gamma(\alpha)}(\psi(t_2)-\psi(t_1))^{\alpha-\frac{1}{p}}\|\psi\|_{C^1}^{\frac{1}{p}}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}$$
$$+\sup_{\tau\in[0,t_1-\varepsilon]}\left\|\Phi(t_2,t_1)\right\|\|\psi\|_{C^1}^{\alpha} r\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}$$
$$+\frac{2Mr}{\Gamma(\alpha)}\|\psi\|_{C^1}^{\frac{1}{p}}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\left[(\psi(t_2)-\psi(t_1-\varepsilon))^{\alpha-\frac{1}{p}}-(\psi(t_2)-\psi(t_1))^{\alpha-\frac{1}{p}}\right]$$
$$+\frac{Mr}{\Gamma(\alpha)}\left(\int_0^{t_1}\left(g(t_2,t_1)\psi'(\tau)\right)^q \mathrm{d}\tau\right)^{1/q}.$$

Thereby, by leveraging p-mean continuity (refer to Problem 23.9 in [36]), Remark 3.1 and the arbitrariness of ε, Λ is equicontinuous. □

We now are able to establish the L^p-null controllability result for system (1.1).

Theorem 3.1. *Under conditions (HA), (Hf) and (Hl), system (1.1) is L^p-null controllable if*

$$\frac{M\|\psi\|_{C^1}^{\alpha}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\|\rho\|_{L^p}\Big(\frac{2M\|L_0^{-1}\|\|\widetilde{B}\|\|\psi\|_{C^1}^{\alpha}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}+1\Big)<1. \tag{3.1}$$

Proof. The main idea of this proof is to construct a trajectory-control pair (x,u) satisfying that $u\in L^p(J,U)$, $x\in S(u)$ and $x(b)=0$. We proceed in two procedures.

Step 1. By employing the projection operator $\mathbb{P}_m$, we construct the auxiliary operator in the subspace X_m to yield a state-control approximation sequence pair (x_m,u_m) satisfying $x_m(b)=0$.

Let $R>0$, $Q_R=\{x\in C(J,X)|\ \|x\|<R\}$, $Q^{(m)}=Q_R\bigcap C(J,X_m)$, $q\in Q^{(m)}$ and $\lambda\in[0,1]$. We set up the following operator $\Sigma: Q^{(m)}\times[0,1]\to C(J,X_m)$:

$$\begin{aligned}\Sigma(q,\lambda)(t)=&\lambda\mathbb{P}_mT_\alpha(\psi(t))\mathbb{P}_mx_0\\&+\lambda\int_0^t\mathbb{P}_mS_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)((\widetilde{B}u_q)(\tau)+\mathbb{P}_mf(\tau,q(\tau)))\mathrm{d}\tau,\end{aligned}$$

where

$$\begin{aligned}u_q&=\Pi L_0^{-1}N_0^b(\mathbb{P}_mx_0,\mathbb{P}_mf(\tau,q(\tau)))\\&=\Pi L_0^{-1}\Big(-T_\alpha(\psi(b))\mathbb{P}_mx_0-\int_0^bS_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)\mathbb{P}_mf(\tau,q(\tau))\mathrm{d}\tau\Big).\end{aligned}$$

Below, we employ Theorem 2.2 to analyze the operator Σ.

We first examine the compactness of Σ. Put

$$\begin{aligned}\Sigma(Q^{(m)}\times[0,1])=&\bigcup_{q\in Q^{(m)},\lambda\in[0,1]}\lambda\Big(\mathbb{P}_mT_\alpha(\psi(\cdot))\mathbb{P}_mx_0\\&+\int_0^{\cdot}\mathbb{P}_mS_\alpha(\psi(\cdot)-\psi(\tau))\psi'(\tau)((\widetilde{B}u_q)(\tau)+\mathbb{P}_mf(\tau,q(\tau)))\mathrm{d}\tau\Big).\end{aligned}$$

Based on (Hf), Remark 3.1, the conditions imposed on the function ψ, the boundedness of operator L_0^{-1} and Hölder's inequality, we get

$$\|u_q\|_{L^p}\le 2\|L_0^{-1}\|\|N_0^b(\mathbb{P}_mx_0,\mathbb{P}_mf(\tau,q(\tau)))\|$$

$$
\begin{aligned}
&\leq 2\|L_0^{-1}\| \left(M\|x_0\| + \frac{MR}{\Gamma(\alpha)} \|\psi\|_{C^1}^{\alpha} \left(\frac{p-1}{\alpha p-1} \right)^{1-\frac{1}{p}} \|\rho\|_{L^p} \right) \\
&:= d. \qquad (3.2)
\end{aligned}
$$

We thereby have

$$
\|\Sigma(q,\lambda)(t)\| \leq M \left(\|x_0\| + \frac{\|\psi\|_{C^1}^{\alpha}}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1} \right)^{1-\frac{1}{p}} (\|\widetilde{B}\|d + \|\rho\|_{L^p} R) \right),
$$

which gives the boundedness of $\Sigma(Q^{(m)} \times [0,1])$. In addition, thanks to Lemma 3.3, we can infer that $\Sigma(Q^{(m)} \times [0,1])$ is equicontinuous. Thus, Σ is compact.

We then prove that Σ meets a closed graph. Let $\{q_k\} \subseteq Q^{(m)}$ with $q_k \to q$ and $\lambda_k \subseteq [0,1]$ with $\lambda_k \to \lambda$. Relying on $q_k \to q$ and (Hf), we get

$$
f(\tau, q_k(\tau)) \rightharpoonup f(\tau, q(\tau)) \text{ and } \mathbb{P}_m f(\tau, q_k(\tau)) \to \mathbb{P}_m f(\tau, q(\tau)).
$$

Due to the continuity of operators Π and L_0^{-1}, and

$$
\begin{aligned}
&\|N_0^b(\mathbb{P}_m x_0, \mathbb{P}_m f(\tau, q_k(\tau))) - N_0^b(\mathbb{P}_m x_0, \mathbb{P}_m f(\tau, q(\tau)))\| \\
\leq &\frac{M}{\Gamma(\alpha)} \int_0^b (\psi(t) - \psi(\tau))^{\alpha-1} \psi'(\tau) \|\mathbb{P}_m f(\tau, q_k(\tau)) - \mathbb{P}_m f(\tau, q(\tau))\| \mathrm{d}\tau,
\end{aligned}
$$

we can assert that $u_{q_k} \to u_q$ in $L^p(J,U)$. In view of Remark 3.1 and the conditions imposed on the function ψ, we get

$$
\begin{aligned}
&\left\|\Sigma(q_k, \lambda_k)(t) - \Sigma(q,\lambda)(t)\right\| \\
\leq &|\lambda_k - \lambda| \|\Sigma(q,1)(t)\| + \frac{M\|\psi\|_{C^1}^{\alpha}}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1} \right)^{1-\frac{1}{p}} \|\widetilde{B}\| \|u_{q_k} - u_q\|_{L^p} \\
&+ \frac{M}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(\tau))^{\alpha-1} \psi'(\tau) \|\mathbb{P}_m (f(\tau, q_k(\tau)) - f(\tau, q(\tau)))\| \mathrm{d}\tau.
\end{aligned}
$$

Thus, Σ possesses a closed graph.

Next, it is straightforward to observe that $\Sigma(q,0) = 0 \in \mathring{Q}^{(m)}$.

We end by testifying that $\Sigma(\cdot,\lambda)$ does not have fixed points on $\partial Q^{(m)}$. Let $q = \Sigma(q,\lambda)$ and $\lambda \in (0,1)$. Based on (3.1) and (3.2), we can select an appropriate R such that

$$
M \left(\|x_0\| + \frac{\|\psi\|_{C^1}^{\alpha}}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1} \right)^{1-\frac{1}{p}} (\|\widetilde{B}\|d + \|\rho\|_{L^p} R) \right) < R,
$$

which means that $q = \Sigma(q,\lambda)$ does not have fixed points on $\partial Q^{(m)}$.

Consequently, by Theorem 2.2, $q = \Sigma(q,1)$ admits a fixed point x_m, i.e.,

$$
\begin{aligned}
x_m(t) = &\mathbb{P}_m T_\alpha(\psi(t)) \mathbb{P}_m x_0 + \int_0^t \mathbb{P}_m S_\alpha(\psi(t) - \psi(\tau)) \psi'(\tau) (\widetilde{B} u_{x_m})(\tau) \mathrm{d}\tau \\
&+ \int_0^t \mathbb{P}_m S_\alpha(\psi(t) - \psi(\tau)) \psi'(\tau) \mathbb{P}_m f(\tau, x_m(\tau)) \mathrm{d}\tau,
\end{aligned}
$$

where

$$\begin{aligned} u_{x_m} &= \Pi L_0^{-1} N_0^b(\mathbb{P}_m x_0, \mathbb{P}_m f(\tau, x_m(\tau))) \\ &= \Pi L_0^{-1}\Big(-T_\alpha(\psi(b))\mathbb{P}_m x_0 - \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)\mathbb{P}_m f(\tau, x_m(\tau))\mathrm{d}\tau\Big). \end{aligned}$$

Let

$$z_{x_m} = -T_\alpha(\psi(b))\mathbb{P}_m x_0 - \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)\mathbb{P}_m f(\tau, x_m(\tau))\mathrm{d}\tau$$

and $[u_{x_m}] = L_0^{-1} z_{x_m}$. We have

$$L_0^b(\Pi([u_{x_m}])) = L_0^b(u_{x_m}) = L_0([u_{x_m}]) = z_{x_m}.$$

Thus,

$$\mathbb{P}_m\Big\{T_\alpha(\psi(b))\mathbb{P}_m x_0 + \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)\mathbb{P}_m f(\tau, x_m(\tau))\mathrm{d}\tau + L_0^b(u_{x_m})\Big\} = 0,$$

i.e., $x_m(b) = 0$.

Step 2. We seek a control function $u \in L^p(J, U)$, a state function x and a subsequence (still written $\{x_m\}_{m>0}$) extracted from $\{x_m\}_{m>0}$ to ensure that

$$x_m \rightharpoonup x,\ x \in S(u),\ x(b) = 0.$$

Since Step 1 offers the boundedness and equicontinuity of $\{x_m\}_{m>0}$, according to Theorem 2.1, we can assume that $x_m \rightharpoonup x$, $m \to \infty$, in $C(J, X)$. We thus get $x_m(t) \rightharpoonup x(t)$ for any $t \in J$. Relying on (Hf) and Lemma 3.2, we have

$$\begin{aligned} &\int_0^t \mathbb{P}_m S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)\mathbb{P}_m f(\tau, x_m(\tau))\mathrm{d}\tau \\ &\rightharpoonup \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau) f(\tau, x(\tau))\mathrm{d}\tau. \end{aligned}$$

Based on (3.2) and the reflexivity of $L^p(J, U)$, we can extract a subsequence from $\{u_{x_m}\}_{m>0}$ (still written $\{u_{x_m}\}_{m>0}$) such that $u_{x_m} \rightharpoonup u$ for some $u \in L^p(J, U)$. Relying on $\widetilde{B} \in \mathscr{L}(L^p(J, U), L^p(J, X))$, we obtain $\widetilde{B}u_{x_m} \rightharpoonup \widetilde{B}u$. Thereby, according to Lemma 3.1, we can assert that

$$\int_0^t \mathbb{P}_m S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)(\widetilde{B}u_{x_m})(\tau)\mathrm{d}\tau \rightharpoonup \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)(\widetilde{B}u)(\tau)\mathrm{d}\tau.$$

In view of Lemma 2.1 and the strong continuity of $\{T_\alpha(t)\}_{t\geq 0}$, we get

$$T_\alpha(\psi(t))\mathbb{P}_m x_0 \to T_\alpha(\psi(t))x_0,$$

and

$$\mathbb{P}_m T_\alpha(\psi(t))\mathbb{P}_m x_0 \rightharpoonup T_\alpha(\psi(t))x_0.$$

Therefore, we arrive at

$$x(t)=T_\alpha(\psi(t))x_0+\int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)((\widetilde{B}u)(\tau)+f(\tau,x(\tau)))\mathrm{d}\tau,$$

i.e., $x\in S(u)$.

Since Step 1 provides $x_m(b)=0$, we have $x_m(b)\rightharpoonup 0$. Thus, $x(b)=0$, i.e., system (1.1) is L^p-null controllable. □

Remark 3.2. In this work, by leveraging the operator L_0^{-1}, we have developed an improved approximation solvability technique to seek the desired state-control pair (x,u).

Our method avoids reliance on the Yosida approximation operator A_n in [5] or the operator $A_m=\mathbb{P}_mA$ in [35, 39, 40, 42]. Additionally, we have relaxed the compactness requirement of $\{T(t)\}_{t\geq 0}$ and the Lipschitz assumption on f. We emphasize that the key distinction between our approach and that in [18] lies in the construction of a weakly convergent subsequence of $\{x_m\}_{m>0}$. Specifically, we replace the weak sequential compactness of $L^p(J,X)$ in [18, 23] with the weak sequential compactness of $C(J,X)$ to achieve the construction. Furthermore, by appropriately modifying the expression of the operator Σ in Theorem 3.1, the same approach can be extended to investigate the L^p-null controllability of ψ-Hilfer fractional evolution systems.

4. Controllability result without invoking the inverse operator

Without relying on the operator L_0^{-1}, we, in this section, treat the L^p-null controllability of system (1.1) by integrating Banach's fixed point theorem approach with the integral contractor method. We list our prerequisites:

(Hf') $f:J\times X\to X$ satisfies

(i) $x\to f(t,x)$ is continuous for a.e. $t\in J$;

(ii) $t\to f(t,x)$ is measurable for all $x\in X$;

(iii) for a.e. $t\in J$ and all $x\in X$, $\|f(t,x)\|\leq k+\varrho\|x\|$ with $\varrho,\ k>0$;

(iv) $f:J\times X\to X$ holds a bounded regular integral contractor γ.

(Hl') System

$$\begin{cases} D^{\alpha,\psi}x(t)=Ax(t)+(\widetilde{B}u)(t),\ t\in J',\\ x(0)=x_0\end{cases}$$

is L^p-null controllable.

(Hc) For any $h\in L^p(J,X)$, there is a $u\in L^p(J,U)$ such that $G(\widetilde{B}u)=Gh$, where $G:L^p(J,X)\to X$ is a linear map defined by

$$Gh=\int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)h(\tau)\mathrm{d}\tau,\ h\in L^p(J,X).$$

Due to condition (Hf'), we define a Nemytskil operator $\mathscr{F}:C(J,X)\to L^p(J,X)$ by

$$(\mathscr{F}x)(t)=f(t,x(t)).$$

Thanks to (Hc) and Lemma 2 in [27], we can choose a continuous operator $\varphi:L^p(J,X)\to L^p(J,U)$ such that for any $h\in L^p(J,X)$,

$$G(\widetilde{B}(\varphi h))+Gh=0 \tag{4.1}$$

and

$$\|\varphi h\|_{L^p(J,U)} \le d\|h\|_{L^p(J,X)}, \tag{4.2}$$

where d is a positive constant satisfying $d > 0$ and

$$L := (\lambda+\rho)db^{\frac{1}{p}}\frac{M\|\widetilde{B}\|\|\psi\|_{C^1}^{\alpha}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}} E_\alpha(M\lambda\psi^\alpha(b)) < 1. \tag{4.3}$$

Theorem 4.1. *Let $u \in L^p(J,U)$. If conditions (HA) and (Hf') hold, then system (1.1) admits a unique solution.*

Proof. By applying the integral contractor method, we first verify the existence of solutions to system (1.1). We set up two sequences $\{y_n\}$ and $\{z_n\}$ in $C(J,X)$ in the form:

$$y_0(t) = T_\alpha(\psi(t))x_0 + \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)(\widetilde{B}u)(\tau)\mathrm{d}\tau,$$

$$z_n(t) = y_n(t) - y_0(t) - \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)f(\tau, y_n(\tau))\mathrm{d}\tau, \tag{4.4}$$

$$y_{n+1}(t) = y_n(t) - z_n(t) - \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)(\gamma(\tau, y_n(\tau))z_n)(\tau)\mathrm{d}\tau. \tag{4.5}$$

Based on (4.4) and (4.5), we obtain

$$\begin{aligned} y_{n+1}(t) = y_0(t) &+ \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)f(\tau, y_n(\tau))\mathrm{d}\tau \\ &- \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)(\gamma(\tau, y_n(\tau))z_n)(\tau)\mathrm{d}\tau. \end{aligned} \tag{4.6}$$

On account of (4.4)-(4.6), we get

$$\begin{aligned} z_{n+1}(t) = &\int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)f(\tau, y_n(\tau))\mathrm{d}\tau \\ &- \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)(\gamma(\tau, y_n(\tau))z_n)(\tau)\mathrm{d}\tau \\ &- \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)f\Bigg(\tau, y_n(\tau) - z_n(\tau) \\ &- \int_0^\tau S_\alpha(\psi(\tau)-\psi(\theta))\psi'(\theta)(\gamma(\theta, y_n(\theta))z_n)(\theta)\mathrm{d}\theta\Bigg)\mathrm{d}\tau. \end{aligned}$$

Leveraging Remarks 2.2 and 3.1, and (2.3) with $x = y_n$ and $y = -z_n$ gives

$$\|z_{n+1}(t)\| \le \frac{M\lambda}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(\tau))^{\alpha-1}\psi'(\tau)\|z_n(\tau)\|\mathrm{d}\tau.$$

By induction, we get

$$\|z_n\| \le \frac{(M\lambda\psi^\alpha(b))^n}{\Gamma(n\alpha+1)}\|z_0\|. \tag{4.7}$$

Due to Hölder's inequality, an analysis analogous to Lemma 3.1 gives

$$\|y_0\| \leq M\|x_0\| + \frac{M}{\Gamma(\alpha)}\|\psi\|_{C^1}^{\alpha}\|\widetilde{B}u\|_{L^p}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}.$$

Thus, by condition $(Hf')(iii)$, it is straightforward to observe that

$$\|z_0\| \leq \frac{M\psi^{\alpha}(b)}{\Gamma(\alpha+1)}\left(k + M\varrho\|x_0\| + \frac{M\varrho}{\Gamma(\alpha)}\|\psi\|_{C^1}^{\alpha}\|\widetilde{B}u\|_{L^p}\left(\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}}\right).$$

Hence the convergence of

$$E_{\alpha}(\psi^{\alpha}(b)M\lambda) = \sum_{n=0}^{\infty}\frac{(\psi^{\alpha}(b)M\lambda)^n}{\Gamma(n\alpha+1)}$$

yields $\lim\limits_{n\to\infty} z_n = 0$ in $C(J,X)$.

By employing Remarks 2.2 and 3.1, (4.5) and (4.7), we obtain

$$\begin{aligned}
&\|y_{n+1}(t) - y_n(t)\| \\
&\leq \|z_n(t)\| + \frac{M\rho}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(\tau))^{\alpha-1}\psi'(\tau)\|z_n(\tau)\|\mathrm{d}\tau \\
&\leq \left(1 + \frac{M\rho\psi^{\alpha}(b)}{\Gamma(\alpha+1)}\right)\|z_n\| \\
&\leq \left(1 + \frac{M\rho\psi^{\alpha}(b)}{\Gamma(\alpha+1)}\right)\frac{(M\lambda\psi^{\alpha}(b))^n}{\Gamma(n\alpha+1)}\|z_0\|.
\end{aligned}$$

Thereby

$$\|y_n - y_m\| \leq \sum_{k=m}^{n-1}\|y_{k+1} - y_k\| \leq \left(1 + \frac{M\rho\psi^{\alpha}(b)}{\Gamma(\alpha+1)}\right)\sum_{k=m}^{n-1}\frac{(M\lambda\psi^{\alpha}(b))^k}{\Gamma(k\alpha+1)}\|z_0\|$$

for $n > m \geq 0$, which means that $\{y_n\}$ is a Cauchy sequence in $C(J,X)$. We thus can assume that $y_n \to y^*$ in $C(J,X)$ for some y^* in $C(J,X)$, as $n \to \infty$. Therefore, by utilizing condition (Hf') and (4.4), we get that for $t \in J'$,

$$y^*(t) = T_{\alpha}(\psi(t))x_0 + \int_0^t S_{\alpha}(\psi(t)-\psi(\tau))\psi'(\tau)(f(\tau, y^*(\tau)) + (\widetilde{B}u)(\tau))\mathrm{d}\tau,$$

i.e., $y^* \in S(u)$.

We then demonstrate the uniqueness by employing the Gronwall type inequality in Lemma 2.3. Let $y_1 \in S(u)$ and $y_2 \in S(u)$. Then

$$y_2(t) - y_1(t) = \int_0^t S_{\alpha}(\psi(t)-\psi(\tau))\psi'(\tau)[f(\tau, y_2(\tau)) - f(\tau, y_1(\tau))]\mathrm{d}\tau.$$

Due to Definition 2.9, the equation

$$z(t) + \int_0^t S_{\alpha}(\psi(t)-\psi(\tau))\psi'(\tau)(\gamma(\tau, y_1(\tau))z(\tau)\mathrm{d}\tau = y_2(t) - y_1(t)$$

possesses a solution $z \in C(J, X)$. Thus, we obtain

$$z(t) = \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)\bigg[-f(\tau, y_1(\tau)) - (\gamma(\tau, y_1(\tau))z)(\tau) \\ +f\bigg(\tau, y_1(\tau)+z(\tau)+\int_0^\tau S_\alpha(\psi(\tau)-\psi(\theta))\psi'(\phi)(\gamma(\theta, y_1(\theta))z)(\theta)\mathrm{d}\theta\bigg)\bigg]\mathrm{d}\tau.$$

Applying (2.3) and Remark 3.1 gives

$$\|z(t)\| \le \frac{M\lambda}{\Gamma(\alpha)}\int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)\|z(\tau)\|\mathrm{d}\tau.$$

By employing Lemma 2.3, one can infer that $\|z\| = 0$. Since for any $t \in J$,

$$\begin{aligned}&\|y_2(t)-y_1(t)\| \\ \le\ & \|z\| + \frac{M}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(\tau))^{\alpha-1}\psi'(\tau)\|(\gamma(\tau, y_1(\tau))z)(\tau)\|\mathrm{d}\tau \\ \le\ & \left(1+\frac{M\rho\psi^\alpha(b)}{\Gamma(\alpha+1)}\right)\|z\|,\end{aligned}$$

we have $y_1 = y_2$. Thus, the uniqueness is garnered. □

Theorem 4.2. *If conditions (HA), (Hf'), (Hl') and (Hc) hold, then system (1.1) is L^p-null controllable.*

Proof. For any $u_0 \in L^p(J, U)$, we define a map $Q : L^p(J, U) \to L^p(J, U)$ by

$$Q(u) = \varphi \circ \mathscr{F} \circ S(u_0 + u),\ u \in L^p(J, U).$$

We proceed in two steps.

Step 1. We first verify that the map Q admits a fixed point by applying Banach's fixed point theorem. Let $y_1 \in S(u_0 + u_1)$ and $y_2 \in S(u_0 + u_2)$. We have

$$y_2(t) - y_1(t) = \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)[f(\tau, y_2(\tau)) - f(\tau, y_1(\tau))]\mathrm{d}\tau \\ + \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)[(\widetilde{B}u_2)(\tau) - (\widetilde{B}u_1)(\tau)]\mathrm{d}\tau.$$

Definition 2.9 implies that the equation

$$z(t) + \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)(\gamma(\tau, y_1(\tau))z)(\tau)\mathrm{d}\tau = y_2(t) - y_1(t)$$

admits a solution $z \in C(J, X)$, hence that

$$z(t) = \int_0^t S_\alpha(\psi(t)-\psi(\tau))\psi'(\tau)\bigg[(\widetilde{B}u_2)(\tau) - (\widetilde{B}u_1)(\tau) \\ -f(\tau, y_1(\tau)) - (\gamma(\tau, y_1(\tau))z)(\tau) \\ +f\bigg(\tau, y_1(\tau)+z(\tau)+\int_0^\tau S_\alpha(\psi(\tau)-\psi(\theta))\psi'(\phi)(\gamma(\theta, y_1(\theta))z)(\theta)\mathrm{d}\theta\bigg)\bigg]\mathrm{d}\tau.$$

By utilizing (2.3) and Remark 3.1, we thus get

$$\|z(t)\| \leq \frac{M}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(\tau))^{\alpha-1} \psi'(\tau) \Big[\lambda \|z(\tau)\| + \|(\widetilde{B}u_2)(\tau) - (\widetilde{B}u_1)(\tau)\|\Big] \mathrm{d}\tau.$$

Thanks to Lemma 2.3 and the conditions imposed on the function ψ, one can conclude that

$$\|z\| \leq \frac{M\|\widetilde{B}\|\|\psi\|_{C^1}^{\alpha}}{\Gamma(\alpha)} \|u_2 - u_1\|_{L^p} \left(\frac{p-1}{\alpha p - 1}\right)^{1-\frac{1}{p}} E_\alpha(M\lambda\psi^\alpha(b)),$$

hence that

$$\begin{aligned}
&\|f(t, y_2(t)) - f(t, y_1(t))\| \\
&\leq \|f(t, y_2(t)) - f(t, y_1(t)) - (\gamma(t, y_1(t))z)(t)\| + \|(\gamma(t, y_1(t))z)(t)\| \\
&\leq (\lambda + \rho)\|z(t)\| \\
&\leq (\lambda + \rho) \frac{M\|\widetilde{B}\|\|\psi\|_{C^1}^{\alpha}}{\Gamma(\alpha)} \|u_2 - u_1\|_{L^p} \left(\frac{p-1}{\alpha p - 1}\right)^{1-\frac{1}{p}} E_\alpha(M\lambda\psi^\alpha(b)).
\end{aligned}$$

By (4.2), we get

$$\begin{aligned}
&\|Q(u_2) - Q(u_1)\|_{L^p} \\
&= \|\varphi \circ \mathscr{F} \circ S(u_0 + u_2) - \varphi \circ \mathscr{F} \circ S(u_0 + u_1)\|_{L^p} \\
&\leq d\|\mathscr{F}(y_2) - \mathscr{F}(y_1)\|_{L^p} \\
&\leq L\|u_2 - u_1\|_{L^p}.
\end{aligned}$$

Thus, we can assert from (4.3) that Q holds a fixed point $\overline{u}$.

Step 2. Subsequently, we demonstrate the L^p-null controllability of system (1.1) by applying (Hl') and utilizing the property (4.1) of the operator φ. Due to (Hl'), we can choose a $u_0 \in L^p(J, U)$ such that

$$T_\alpha(\psi(b))x_0 + G(\widetilde{B}u_0) = 0.$$

For this u_0 and $y \in S(u_0 + \overline{u})$, due to (4.1), we derive

$$\begin{aligned}
y(b, u_0 + \overline{u}, f) &= T_\alpha(\psi(b))x_0 + G\left(\widetilde{B}(u_0 + \overline{u})\right) + G \circ \mathscr{F}(y) \\
&= T_\alpha(\psi(b))x_0 + G(\widetilde{B}u_0) + G\left(\widetilde{B}\overline{u} + \mathscr{F}(y)\right) \\
&= 0 + G\left(\widetilde{B} \circ \varphi \circ \mathscr{F}(y) + \mathscr{F}(y)\right) \\
&= 0.
\end{aligned}$$

Hence, system (1.1) is L^p-null controllable. □

Remark 4.1. In Theorem 3.1, we employ the operator L_0^{-1} and the developed approximation solvability technique to construct the required state-control pair for achieving the L^p-null controllability. Meanwhile, in Theorem 4.2, we utilize the condition (Hc) and Banach's fixed point theorem to derive the required control function that ensures the L^p-null controllability.

Remark 4.2. Under the condition $R(f) \subset R(B)$, [8] investigated the exact controllability of a Hilfer fractional evolution system. Meanwhile, under the condition $R(f) \subset \overline{R(B)}$, [34] examined the approximate controllability of a Hilfer fractional semilinear control system, [9] analyzed the approximate controllability of a ψ-Hilfer fractional control semilinear system and [33] investigated the interior approximate controllability of a second-order system semilinear control system.

In Theorem 4.2, without invoking L_0^{-1}, we establish the L^p-null controllability result by leveraging (Hc) and Banach's fixed point approach. We emphasize that under the condition $R(f) \subset R(B)$, by setting $h = \mathscr{F}(x)$, it is straightforward to select a $u \in L^p(J, X)$ such that $G(\widetilde{B}u) = Gh$. Consequently, our method remains valid for this condition. Furthermore, by appropriately modifying the expression of the operator G in (Hc), the same approach can be extended to investigate the L^p-null controllability of ψ-Hilfer fractional evolution systems. In future, under condition $R(f) \subset \overline{R(B)}$, we will seek appropriate methods to investigate the L^p-null controllability problem.

5. Examples

We here opt for two examples to validate the applicability of above-obtained abstract findings.

Example 5.1. Let Ω be a bounded region in $\mathbb{R}^n$ $(n \geq 2)$ with a Lipschitz boundary $\partial\Omega$. We pay attention to the following model:

$$\begin{cases} D^{\frac{3}{4},\ln(1+t)}x(t,z) = \Delta x(t,z) + u(t,z) + f(t, x(t,z))), \text{ on } (0,b] \times \Omega, \\ x(t,z) = 0, \text{ on } (0,b) \times \partial\Omega, \\ x(0,z) = x_0(z), \quad z \in \Omega. \end{cases} \tag{5.1}$$

Here $D^{\frac{3}{4},\ln(1+t)}$ is a $\frac{3}{4}$-order $\ln(1+t)$-Caputo type fractional derivative operator.

Let $H = U = L^2(\Omega)$. Determine an operator A as

$$Ax = \Delta x, \ x \in D(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

With the help of $\{-\lambda_n, \xi_n\}_{n=1}^{\infty}$ (the eigensystem of A), it is well-known that A can generate a contractive self-adjoint analytic semigroup $\{T(t)\}_{t\geq 0}$ (refer to [24]):

$$T(t)x = \sum_{n=1}^{\infty} e^{-\lambda_n t}\langle x, \xi_n\rangle \xi_n, \ \lambda_n > 0.$$

Then (HA) holds with $M = 1$. Due to Lemma 2.2, A can also generate a fractional resolvent $\{S_\alpha(t)\}_{t>0}$:

$$S_\alpha(t)x = \sum_{n=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\langle x, \xi_n\rangle \xi_n$$

and a solution operator $\{T_\alpha(t)\}_{t\geq 0}$:

$$T_\alpha(t)x = \sum_{n=1}^{\infty} E_\alpha(-\lambda_n t^\alpha)\langle x, \xi_n\rangle \xi_n,$$

respectively.

Let $\alpha=\frac{3}{4}$, $\psi(t)=\ln(1+t)$, $x(t)(z)=x(t,z)$, $B=I$ and $u(t)(z)=u(t,z)$. Define a function $f: J\times X\to X$ by $f(t,x(t))(z)=f(t,x(t,z))$. Then system (5.1) can be converted into system (1.1).

Based on

$$\begin{aligned}
&\int_0^b \|S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)x\|^2\mathrm{d}\tau\\
&=\sum_{k=1}^{\infty}\int_0^b(\psi(b)-\psi(\tau))^{2\alpha-2}E_{\alpha,\alpha}^2(-\lambda_k(\psi(b)-\psi(\tau))^\alpha)\psi'^2(\tau)\mathrm{d}\tau\langle x,\xi_k\rangle^2\\
&\ge\frac{1}{b}\sum_{k=1}^{\infty}\Big(\int_0^b(\psi(b)-\psi(\tau))^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(\psi(b)-\psi(\tau))^\alpha)\psi'(\tau)\mathrm{d}\tau\Big)^2\langle x,\xi_k\rangle^2\\
&=\sum_{k=1}^{\infty}\frac{1}{b\Gamma^2(\alpha)}\Big(\int_0^b(\psi(b)-\psi(\tau))^{\alpha-1}E_{\alpha}(-\lambda_k(\psi(\tau))^\alpha)\psi'(\tau)\mathrm{d}\tau\Big)^2\langle x,\xi_k\rangle^2\\
&\ge\frac{\psi^{2\alpha}(b)}{b\Gamma^2(\alpha+1)}\sum_{k=1}^{\infty}E_\alpha^2(-\lambda_k\psi^\alpha(b))\langle x,\xi_k\rangle^2\\
&=\frac{\psi^{2\alpha}(b)}{b\Gamma^2(\alpha+1)}\|T_\alpha(\psi(b))x\|^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^b \|S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)x\|^2\mathrm{d}\tau\\
&\ge\frac{\psi^{2\alpha}(b)}{b\Gamma^2(\alpha+1)+\psi^{2\alpha}(b)}\bigg(\|T_\alpha(\psi(b))x\|^2+\int_0^b\|S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)x\|^2\mathrm{d}\tau\bigg).
\end{aligned}$$

Thereby, the linear system of (5.1) is null controllable, that is, (Hl) holds.

Thus, Theorem 3.1 yields the following result:

Theorem 5.1. *Let (Hf) hold. Then system (5.1) is L^2-null controllable if*

$$\frac{\|\psi\|_{C^1}^\alpha}{\Gamma(\alpha)}\sqrt{\frac{1}{2\alpha-1}}\|\rho\|_{L^2}\Big(\frac{2\|L_0^{-1}\|\|\psi\|_{C^1}^\alpha}{\Gamma(\alpha)}\sqrt{\frac{1}{2\alpha-1}}+1\Big)<1.$$

Example 5.2. Consider the control model in the form:

$$\begin{cases}D^{\alpha,\psi}y(t,z)=\dfrac{\partial^2}{\partial z^2}y(t,z)+Bu(t,z)+f(t,y(t,z)),\ \text{on } (0,1]\times(0,1),\\ y(t,0)=y(t,1)=0,\\ y(0,z)=y_0(z),\quad z\in(0,1).\end{cases}\tag{5.2}$$

Let $J=[0,1]$, $X=L^2(0,1)$, $e_m(z)=\sqrt{2}\sin(m\pi z),\ m=1,2,\cdots$. Determine an operator A as

$$A\xi=\xi'',\ \xi\in D(A)=\{\xi\in X:\xi',\ \xi''\in X \text{ and } \xi(0)=\xi(1)=0\}.$$

A can generate a self-adjoint analytic semigroup $\{T(t)\}_{t\geq 0}$ (see [24]):

$$T(t)y_0 = \sum_{m=1}^{\infty} e^{-m^2\pi^2 t}\langle y_0, e_m\rangle e_m.$$

Obviously, $\|T(t)\| \leq 1$, that is, (HA) holds.

Let

$$U = \left\{ v : v = \sum_{m=2}^{\infty} v_m e_m,\ \sum_{m=2}^{\infty} |v_m|^2 < \infty \right\}$$

with

$$\|v\| = \left(\sum_{m=2}^{\infty} |v_m|^2 \right)^{1/2}.$$

Consider two operators $B : U \to X$ and $\widetilde{B} : L^2(J,U) \to L^2(J,X)$ formulated by

$$Bv = 2v_2e_1 + \sum_{m=2}^{\infty} v_m e_m,\ v \in U$$

and

$$(\widetilde{B}u)(t) = Bu(t),\ u \in L^2(J,U),$$

respectively.

For any $u \in L^2([0,1],U)$, we have

$$u(t) = \sum_{m=2}^{\infty} u_m(t)e_m.$$

Thereby,

$$\begin{aligned}
\|\widetilde{B}u\|_{L^2}^2 &= \int_0^1 \|(\widetilde{B}u)(\tau)\|^2 \mathrm{d}\tau \\
&= \int_0^1 \|Bu(\tau)\|^2 \mathrm{d}\tau \\
&= \int_0^1 \left\| 2u_2(\tau)e_1 + \sum_{m=2}^{\infty} u_m(\tau)e_m \right\|^2 \mathrm{d}\tau \\
&= \int_0^1 \left(4\|u_2(\tau)\|^2 + \sum_{m=2}^{\infty} \|u_m(\tau)\|^2 \right) \mathrm{d}\tau \\
&\leq 5 \int_0^1 \sum_{m=2}^{\infty} \|u_m(\tau)\|^2 \mathrm{d}\tau \\
&\leq 5\|u\|_{L^2}^2.
\end{aligned}$$

Hence, $\widetilde{B} \in \mathscr{L}(L^2(J,U), L^2(J,X))$ and $B \in \mathscr{L}(U,X)$.

Let

$$y(t)(z) = y(t,z) = \sum_{m=1}^{\infty} y_m(t)e_m(z),$$
$$u(t)(z) = u(t,z) = \sum_{m=1}^{\infty} u_m(t)e_m(z)$$

and

$$y_0(z) = \sum_{m=1}^{\infty} c_m e_m(z).$$

We can convert the linear system of (5.2) into

$$\begin{cases} D^{\alpha,\psi} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} -\pi^2 & 0 \\ 0 & -4\pi^2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u_2(t),\ t \in (0,1], \\ \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{cases} \tag{5.3}$$

and for $n = 3, 4, \cdots$,

$$\begin{cases} D^{\alpha,\psi} y_n(t) = -n^2\pi^2 y_n(t) + u_n(t),\ t \in (0,1], \\ y_n(0) = c_n. \end{cases}$$

For convenience, put

$$A_1 = \begin{pmatrix} -\pi^2 & 0 \\ 0 & -4\pi^2 \end{pmatrix},\ B_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

A_1 can generate a fractional resolvent

$$\overline{S_\alpha}(t) = \begin{pmatrix} t^{\alpha-1}E_{\alpha,\alpha}(-\pi^2 t^\alpha) & 0 \\ 0 & t^{\alpha-1}E_{\alpha,\alpha}(-4\pi^2 t^\alpha) \end{pmatrix}.$$

For any $q \in L^2(J,X)$, set

$$q(\tau) = \sum_{m=1}^{\infty} q_m(\tau)e_m,\ q_m(\tau) = \langle q(\tau), e_m \rangle$$

and

$$h = \int_0^1 S_\alpha(\psi(1)-\psi(\tau))\psi'(\tau)q(\tau)\mathrm{d}\tau = \sum_{m=1}^{\infty} h_m e_m,\ h_m = \langle h, e_m \rangle.$$

We have

$$\int_0^1 (\psi(1)-\psi(\tau))^{\alpha-1}E_{\alpha,\alpha}(-n^2\pi^2(\psi(1)-\psi(\tau))^{\alpha})\psi'(\tau)q_m(\tau)\mathrm{d}\tau = h_m.$$

Let

$$\widehat{\mathcal{W}} = \int_0^1 \overline{S_\alpha}(\psi(1)-\psi(\tau))B_1B_1^{\mathrm{T}}\overline{S_\alpha}^{\mathrm{T}}(\psi(1)-\psi(\tau))\psi'^2(\tau)\mathrm{d}\tau.$$

Arguments similar to those in [16, Theorem 2.1], [21, Theorem 1] and [26] give:

Lemma 5.1. *The following statements are equivalent:*

(i) system (5.3) is controllable.

(ii) $R[B_1, A_1B_1] = 2$, where $R[B_1, A_1B_1]$ stands for the rank of $[B_1, A_1B_1]$.

(iii) $\widehat{\mathcal{W}}^{-1}$ exists.

Moreover, if $R[B_1, A_1B_1] = 2$, then for any $h \in \mathbb{R}^2$, the control function

$$u(\tau) = B_1^{\mathrm{T}}\overline{S_\alpha}^{\mathrm{T}}(\psi(1)-\psi(\tau))\psi'(\tau)\widehat{\mathcal{W}}^{-1}h$$

possesses the minimum norm property:

$$\|u\|_{L^2} = \inf\left\{\|v\|_{L^2} : \int_0^1 \overline{S_\alpha}(\psi(1)-\psi(\tau))\psi'(\tau)B_1v(\tau)\mathrm{d}\tau = h\right\}.$$

Lemma 5.2. *For any $q \in L^2(J, X)$, there is a control function $u \in L^2(J, U)$ such that $Gq = G(\widetilde{B}u)$, where $G : L^2(J, X) \to X$ is a linear map given by*

$$Gq = \int_0^1 S_\alpha(\psi(1)-\psi(\tau))\psi'(\tau)q(\tau)\mathrm{d}\tau.$$

Proof. Based on $R[B_1, A_1B_1] = 2$ and Lemma 5.1, we can define a control function $u_2 \in L^2(J, \mathbb{R})$ by

$$u_2(\tau) = B_1^{\mathrm{T}}\overline{S_\alpha}^{\mathrm{T}}(\psi(1)-\psi(\tau))\psi'(\tau)\widehat{\mathcal{W}}^{-1}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Let

$$\mathcal{W} = \int_0^1 (\psi(1)-\psi(\tau))^{2\alpha-2}E^2_{\alpha,\alpha}(-n^2\pi^2(\psi(1)-\psi(\tau))^{\alpha})\psi'^2(\tau)\mathrm{d}\tau.$$

Due to

$$\begin{aligned}\mathcal{W} &\geq \left(\int_0^1 (\psi(1)-\psi(\tau))^{\alpha-1}E_{\alpha,\alpha}(-n^2\pi^2(\psi(1)-\psi(\tau))^{\alpha})\psi'(\tau)\mathrm{d}\tau\right)^2 \\ &= \psi^{\alpha}(1)E^2_{\alpha,\alpha+1}(-n^2\pi^2\psi^{\alpha}(1)) \\ &> 0,\end{aligned}$$

we can define a control sequence $\{u_m\}_{m\geq 3}$ by

$$u_m(\tau) = \begin{cases} (\psi(1)-\psi(\tau))^{\alpha-1}E_{\alpha,\alpha}(-n^2\pi^2(\psi(1)-\psi(\tau))^{\alpha})\psi'(\tau)\mathcal{W}^{-1}h_m, \ \tau\in[0,1), \\ 0, \ \tau = 1. \end{cases}$$

Thanks to the minimum norm property (see Lemma 5.2), we obtain

$$\int_0^1 |u_m(\tau)|^2 \mathrm{d}\tau \leq \int_0^1 |q_m(\tau)|^2 \mathrm{d}\tau.$$

Set $u(t) = \sum\limits_{m=2}^{\infty} u_m(t)e_m$. We can easily see that $u(t)\in U$,

$$\begin{aligned}\|u\|_{L^2}^2 &= \sum_{m=2}^{\infty}\int_0^1 |u_m(\tau)|^2\mathrm{d}\tau \\ &= \int_0^1 |u_2(\tau)|^2\mathrm{d}\tau + \sum_{m=3}^{\infty}\int_0^1 |u_m(\tau)|^2\mathrm{d}\tau \\ &\leq \|u_2\|_{L^2}^2 + \sum_{m=1}^{\infty}\int_0^1 |q_m(\tau)|^2\mathrm{d}\tau \\ &= \|u_2\|_{L^2}^2 + \|q\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 S_\alpha(\psi(1)-\psi(\tau))\psi'(\tau)(\widetilde{B}u)(\tau)\mathrm{d}\tau \\ =&\int_0^1 S_\alpha(\psi(1)-\psi(\tau))\psi'(\tau)Bu(\tau)\mathrm{d}\tau \\ =&\int_0^1\sum_{m=2}^{\infty}(\psi(1)-\psi(\tau))^{\alpha-1}E_{\alpha,\alpha}(-n^2\pi^2(\psi(1)-\psi(\tau))^{\alpha})\psi'(\tau)\langle Bu(\tau),e_m\rangle e_m\mathrm{d}\tau \\ =&(e_1,e_2)\int_0^1 \overline{S_\alpha}(\psi(1)-\psi(\tau))\psi'(\tau)B_1u_2(\tau)\mathrm{d}\tau \\ &+\sum_{m=3}^{\infty}\int_0^1(\psi(1)-\psi(\tau))^{\alpha-1}E_{\alpha,\alpha}(-n^2\pi^2(\psi(1)-\psi(\tau))^{\alpha})\psi'(\tau)u_m(\tau)e_m\mathrm{d}\tau \\ =&\sum_{m=1}^{\infty}h_me_m \\ =&\int_0^1 S_\alpha(\psi(1)-\psi(\tau))\psi'(\tau)q(\tau)\mathrm{d}\tau.\end{aligned}$$

Thus, (Hc) holds. □

Following the approach in Example 5.1, it is straightforward to verify that the linear system of (5.2) is L^2-null controllable, i.e., (Hl') is satisfied. Thus, Theorem 4.2 forces the following result:

Theorem 5.2. *If (Hf') holds and $\frac{1}{2}<\alpha<1$, then system (5.2) is L^2-null controllable.*

6. Conclusion

In this paper, we have established two novel L^p-null controllability results for ψ-Caputo fractional evolution systems. By leveraging the operator L_0^{-1} and integrating the approximation solvability approach with the resolvent method, we first have achieved the L^p-null controllability without relying on the compactness of the semigroup $\{T(t)\}_{t\geq 0}$, Lipschitz continuity of the function f or noncompactness measure conditions. Subsequently, by introducing a weaker condition (specifically (Hc) instead of $R(f) \subseteq R(B)$ as in [8]) and the regular contractor condition, we have demonstrated the L^p-null controllability independent of L_0^{-1}.

The main contribution of this work lies in eliminating the operator A_n introduced in [5] and removing the operator $A_m = \mathbb{P}_m A$ employed in [35, 39, 40, 42, 43] within Theorem 3.1. Furthermore, we replace the weak sequential compactness of $L^p(J, X)$ in [18, 23] with that of $C(J, X)$ to construct the weakly convergent subsequence. Additionally, our method in Theorem 4.2 remains applicable under the condition $R(f) \subseteq R(B)$ used in [8].

In future work, under the condition $R(f) \subset \overline{R(B)}$, we attempt to develop appropriate methods to investigate the L^p-null controllability problem. Additionally, we wish to generalize the above-obtained results to broader classes of abstract systems and seek more natural conditions for exploring controllability problems.

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