

OPIAL-TYPE INEQUALITIES IN THE FRAMEWORK OF CONFORMABLE FRACTIONAL CALCULUS*

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Abstract This article investigates fractional Opial-type inequalities within the framework of conformable calculus. The results are established using conformable fractional analogues of Hölder's inequality, the integration by parts formula, and the chain rule. Several classical integral Opial-type inequalities are derived as special cases of the main results, demonstrating the generality and applicability of the proposed approach. Notably, the paper highlights a structural symmetry between the new inequalities and those previously established in the literature.

Keywords Generalized inequalities, integral inequalities, Opial-type inequalities, conformable fractional calculus, chain rule.

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1. Introduction

Opial-type inequalities and their extensions and generalizations, present an elementary role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations in addition to difference equations. There are a large number of papers that have been obtained with the simple proofs during the last twenty years, for more information, see [2, 7, 8]. The Opial inequality has great importance in differential and difference equations, and other mathematical courses.

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Opial in [15], showed that if $\psi(q) > 0$ and $\psi \in C^1[\varsigma_1, \varsigma_2]$ s.t. $\psi(\varsigma_1) = \psi(\varsigma_2) = 0$, then

$$\int_{\varsigma_1}^{\varsigma_2} |\psi(q) \psi'(q)| dq \leq \frac{\varsigma_2 - \varsigma_1}{4} \int_{\varsigma_1}^{\varsigma_2} (\psi'(q))^2 dq, \tag{1.1}$$

where $(\varsigma_2 - \varsigma_1)/4$ is sharp and if $\varsigma_1 = 0$, then (1.1) becomes

$$\int_0^{\varsigma_2} |\psi(q) \psi'(q)| dq \leq \frac{\varsigma_2}{2} \int_0^{\varsigma_2} (\psi'(q))^2 dq. \tag{1.2}$$

Also, Opial [15] established that if ψ is absolutely continuous on $[0, \varsigma_1]$ s.t. $\psi(0) = 0$, then

$$\int_0^{\varsigma_1} (|\psi(q)|)^\beta (|\psi'(q)|)^\gamma dq \leq \frac{\gamma}{\beta + \gamma} \varsigma_1^\beta \int_0^{\varsigma_1} (|\psi'(q)|)^{\beta + \gamma} dq, \text{ for } \beta \geq 0 \text{ and } \gamma \geq 1. \tag{1.3}$$

In the same work [15], he demonstrated that if $\psi(0) = \psi(\varsigma_1) = 0$, then

$$\int_0^{\varsigma_1} (|\psi(q)|)^\beta (|\psi'(q)|)^\gamma dq \leq \frac{\gamma}{\beta + \gamma} \left(\frac{\varsigma_1}{2}\right)^\beta \int_0^{\varsigma_1} (|\psi'(q)|)^{\beta + \gamma} dq, \text{ for } \beta \geq 0 \text{ and } \gamma \geq 1. \tag{1.4}$$

In [9], Hua proved an extension of (1.2) as follows:

$$\int_{\varsigma_1}^{\varsigma_2} (|\psi(q)|)^\beta |\psi'(q)| dq \leq \frac{(\varsigma_2 - \varsigma_1)^\beta}{\beta + 1} \int_{\varsigma_1}^{\varsigma_2} (|\psi'(q)|)^{\beta + 1} dq, \tag{1.5}$$

where $\beta > 0$ is integer and ψ is an absolutely continuous such that $\psi(\varsigma_1) = 0$ and if $\beta = 1$, then (1.5) becomes

$$\int_{\varsigma_1}^{\varsigma_2} |\psi(q)| |\psi'(q)| dq \leq \frac{\varsigma_2 - \varsigma_1}{2} \int_{\varsigma_1}^{\varsigma_2} (|\psi'(q)|)^2 dq. \tag{1.6}$$

In [24], Yong demonstrated the following inequality

$$\int_{\varsigma_1}^{\varsigma_2} (|\psi(q)|)^\beta (|\psi'(q)|)^\gamma dq \leq \frac{\gamma}{\beta + \gamma} (\varsigma_2 - \varsigma_1)^\beta \int_{\varsigma_1}^{\varsigma_2} (|\psi'(q)|)^{\beta + \gamma} dq, \text{ for } \beta, \gamma \geq 1, \tag{1.7}$$

where ψ is absolutely continuous on $[\varsigma_1, \varsigma_2]$ with $\psi(\varsigma_1) = 0$.

Note that (1.7) is proved also for $\beta \geq 0, \gamma \geq 1$, and if $\beta = \gamma = 1, \varsigma_1 = 0$, we have the Opial inequality (1.2).

In [6], Boyd and Wong proved an extension of (1.5) as follows:

$$\int_0^{\varsigma_1} \theta(q) (|\psi(q)|)^\beta |\psi'(q)| dq \leq \frac{1}{\gamma(\beta + 1)} \int_0^{\varsigma_1} \zeta(q) (|\psi'(q)|)^{\beta + 1} dq \text{ for } \beta > 0, \tag{1.8}$$

where $\theta \geq 0, \zeta \in C^1[0, \varsigma_1]$ and γ is the smallest eigenvalue of

$$\left(\zeta(q) (g'(q))^\beta\right)' = \epsilon \theta'(q) g^\beta(q),$$

with $\zeta(\varsigma_1) (g'(\varsigma_1))^\beta = \epsilon \theta(\varsigma_1) g^\beta(\varsigma_1)$ s.t. $g' > 0$ in $[0, \varsigma_1]$ and $g(0) = 0$.

In [5], Beesack and Das obtained that

$$\int_{\varsigma_1}^{\varsigma_2} \theta(q) (|\psi(q)|)^\beta (|\psi'(q)|)^\gamma dq \leq k(\beta, \gamma) \int_{\varsigma_1}^{\varsigma_2} \zeta(q) (|\psi'(q)|)^{\beta + \gamma} dq, \tag{1.9}$$

where $\theta, \zeta \geq 0$ are measurable functions on $[\varsigma_1, \varsigma_2]$, ψ is absolutely continuous on $[\varsigma_1, \varsigma_2]$, $\beta\gamma > 0$ and $\beta + \gamma < 0$ or $\beta + \gamma \geq 1$, $k(\beta, \gamma)$ is a sharp constant which depends on ζ, θ, β and γ .

In [25], Yong proved that if ψ is absolutely continuous on $[\varsigma_1, \varsigma_2]$ s.t. $\psi(\varsigma_1) = 0$ and $\zeta(\eta) > 0$ is bounded, then for $\beta \geq 0$ and $\gamma \geq 1$,

$$\int_{\varsigma_1}^{\varsigma_2} \zeta(q) (|\psi(q)|)^\beta (|\psi'(q)|)^\gamma dq \leq \frac{\gamma}{\beta + \gamma} (\varsigma_2 - \varsigma_1)^\beta \int_{\varsigma_1}^{\varsigma_2} \zeta(q) (|\psi'(q)|)^{\beta + \gamma} dq. \quad (1.10)$$

We refer the reader to [4, 16, 17, 22, 26], for additional information on Opial-type inequalities and other types.

In recent years, various authors have demonstrated a number of integral inequalities using conformable fractional calculus, such as Hermite-Hadamard-type inequalities [13, 14], Chebyshev-type inequalities [3, 20] and Steffensen-type inequalities [19]. Also, in a recent study, Kashuri et al. [11] introduced and examined a novel class of functions termed n -polynomial harmonically t gs-convex functions. The work establishes several integral inequalities of both integer and fractional orders for this class, employing classical integral formulations alongside Riemann-Liouville fractional integrals. It's important to highlight that in this paper, we will uncover the symmetry between the obtained results and those proven in previous studies. This symmetry can be obtained by setting $\alpha = 1$ in the basic results.

The objectives of the study are to present some Opial-type inequalities by using conformable fractional calculus. We recover a few classical integral Opial-type inequalities that have been documented in the literature as particular cases of the main results. In particular, this paper aims to generalize Opial-type inequalities proven by Opial [15], Hua [9], and Yong [25] to more comprehensive ones via conformable fractional calculus. The paper contains two parts: The first part consists of an introduction about fundamentals of conformable fractional calculus, and the second part contains our main results.

2. Preliminaries

Conformable fractional calculus is a significant and accessible tool in scientific research that underpins innovative work across multiple scientific and engineering fields. Its importance is evident in the development of novel solution techniques for a wide range of problems. For instance, Jassim et al. [10] successfully applied a hybrid method to solve fractional partial differential equations, while Wang et al. [23] developed a meshless approach for three-dimensional fractional models. Further demonstrating the versatility of these concepts, Shah et al. [21] conducted a comparative analysis on the fractional gas dynamics equation. Now we will discuss briefly some of the essential definitions of the conformable fractional calculus, which is introduced by Khalil et al. [12], and some of the fundamental concepts and lemmas, such as chain rule and integration by parts on conformable fractional calculus which can help us to obtain our results, see [1, 18].

Definition 2.1. [12] The conformable fractional derivative of order α for ψ as:

$$T_\alpha \psi(e) = \lim_{\epsilon \rightarrow 0} \frac{\psi(e + \epsilon e^{1-\alpha}) - \psi(e)}{\epsilon}, \quad \forall e > 0, \alpha \in (0, 1]. \quad (2.1)$$

If ψ is α -differentiable on $(0, \varsigma_1)$, $\varsigma_1 > 0$ and $\lim_{e \rightarrow 0^+} \psi(e)$ exists, then we can define $T_\alpha \psi(0) = \lim_{e \rightarrow 0^+} T_\alpha \psi(e)$.

Definition 2.2. [12] The conformable integral of order α for ψ as:

$$(I_{\alpha}^{\varsigma_1} \psi) (\eta) = \int_{\varsigma_1}^{\eta} \psi(z) d_{\alpha} z = \int_{\varsigma_1}^{\eta} z^{\alpha-1} \psi(z) dz, \alpha \in (0, 1]. \tag{2.2}$$

2.1. Properties of conformable derivative

Theorem 2.1 (Properties of conformable derivatives). [12] Let ψ and \mathfrak{R} be α^{th} -differentiable with respect to $e > 0$. Then for $\alpha \in (0, 1]$, we have

1. $T_{\alpha} (\varsigma_1 \psi + \varsigma_2 \mathfrak{R}) (e) = \varsigma_1 T_{\alpha} \psi (e) + \varsigma_2 T_{\alpha} \mathfrak{R} (e)$.
2. $T_{\alpha} (\psi^q) (e) = q \psi^{q-\alpha} (e), \forall q \in \mathbb{R}$.
3. $T_{\alpha} (\eta) = 0$, for all constant functions $\psi (e) = \eta$.
4. $T_{\alpha} (\psi \mathfrak{R}) (e) = \psi T_{\alpha} \mathfrak{R} (e) + \mathfrak{R} T_{\alpha} \psi (e)$.
5. $T_{\alpha} \left(\frac{\psi}{\mathfrak{R}} \right) (e) = \frac{\mathfrak{R} T_{\alpha} \psi (e) - \psi T_{\alpha} \mathfrak{R} (e)}{\mathfrak{R}^2}$.
6. $T_{\alpha} \psi (e) = e^{1-\alpha} \frac{d\psi}{de}$ if ψ is differentiable.

Lemma 2.1 (Chain rule). [1] Let \mathfrak{R} be α^{th} -differentiable w.r.t. e and ψ be differentiable w.r.t. \mathfrak{R} . Then

$$T_{\alpha} \psi (\mathfrak{R} (e)) = \mathfrak{R}^{\alpha-1} (e) (T_{\alpha} \psi) (\mathfrak{R} (e)) T_{\alpha} \mathfrak{R} (e). \tag{2.3}$$

Lemma 2.2 (Fractional integration by parts). [1] Let ψ and \mathfrak{R} be α^{th} -differentiable w.r.t. e on $[\varsigma_1, \varsigma_2]$. Then

$$\int_{\varsigma_1}^{\varsigma_2} (T_{\alpha} \psi (e)) \mathfrak{R} (e) d_{\alpha} e = [\psi (e) \mathfrak{R} (e)]_{\varsigma_1}^{\varsigma_2} - \int_{\varsigma_1}^{\varsigma_2} \psi (e) (T_{\alpha} \mathfrak{R} (e)) d_{\alpha} e. \tag{2.4}$$

Lemma 2.3 (Hölder inequality). [18] Let $0 < \alpha \leq 1$ and $\psi, \mathfrak{R} : [\varsigma_1, \varsigma_2] \rightarrow \mathbb{R}$. Then

$$\int_{\varsigma_1}^{\varsigma_2} |\psi (e) \mathfrak{R} (e)| d_{\alpha} e \leq \left(\int_{\varsigma_1}^{\varsigma_2} |\psi (e)|^{\beta} d_{\alpha} e \right)^{\frac{1}{\beta}} \left(\int_{\varsigma_1}^{\varsigma_2} |\mathfrak{R} (e)|^{\gamma} d_{\alpha} e \right)^{\frac{1}{\gamma}}, \tag{2.5}$$

where $1/\beta + 1/\gamma = 1, \beta > 1$.

3. Main results

In this section, we will present new generalizations of fractional Opial-type inequalities utilizing the fundamental concepts of the conformable fractional calculus that were presented in the previous section, therefore we will obtain classical formulae of Opial when $\alpha = 1$.

Theorem 3.1. Let $\zeta \geq 0$ be non-increasing on $[\varsigma_1, w]$ with $\varsigma_1, w \in \mathbb{R}$ and $\delta > 1$. If $\mathfrak{R} : [\varsigma_1, w] \rightarrow \mathbb{R}$ s.t. $\mathfrak{R} (\varsigma_1) = 0$ and $\alpha \in (0, 1]$, then for $\varpi \geq 0$

$$\int_{\varsigma_1}^w \zeta (e) (|\mathfrak{R} (e)|)^{\varpi} (|T_{\alpha} \mathfrak{R} (e)|)^{\delta} d_{\alpha} e \leq \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{w^{\alpha} - \varsigma_1^{\alpha}}{\alpha} \right)^{\varpi} \int_{\varsigma_1}^w \zeta (e) (|T_{\alpha} \mathfrak{R} (e)|)^{\varpi + \delta} d_{\alpha} e. \tag{3.1}$$

Proof. Suppose that $z(e) = \int_{\varsigma_1}^e (\zeta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta$ for $e \in [\varsigma_1, w]$, then

$$T_\alpha z(e) = (\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(e)|)^\delta > 0 \text{ and } z(\varsigma_1) = 0. \tag{3.2}$$

Therefore,

$$|\mathfrak{R}(e)| = \left| \int_{\varsigma_1}^e T_\alpha \mathfrak{R}(\eta) d_\alpha \eta \right| \leq \int_{\varsigma_1}^e |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta = \int_{\varsigma_1}^e (\zeta(\eta))^{\frac{-1}{\varpi+\delta}} (\zeta(\eta))^{\frac{1}{\varpi+\delta}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta.$$

Applying (2.5) on $|\mathfrak{R}(e)|$ with $\beta = \delta/(\delta - 1)$ and $\gamma = \delta$, we find that

$$\begin{aligned} |\mathfrak{R}(e)| &\leq \left(\int_{\varsigma_1}^e \left((\zeta(\eta))^{\frac{-1}{\varpi+\delta}} \right)^{\frac{\delta}{\delta-1}} d_\alpha \eta \right)^{\frac{\delta-1}{\delta}} \left(\int_{\varsigma_1}^e (\zeta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta \right)^{\frac{1}{\delta}} \\ &\leq (\zeta(e))^{\frac{-1}{\varpi+\delta}} \left(\int_{\varsigma_1}^e d_\alpha \eta \right)^{\frac{\delta-1}{\delta}} \left(\int_{\varsigma_1}^e (\zeta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta \right)^{\frac{1}{\delta}} \\ &= (\zeta(e))^{\frac{-1}{\varpi+\delta}} \left(\int_{\varsigma_1}^e d_\alpha \eta \right)^{\frac{\delta-1}{\delta}} (z(e))^{\frac{1}{\delta}}. \end{aligned} \tag{3.3}$$

Since $T_\alpha \left(\frac{\eta^\alpha}{\alpha} \right) = 1$, then

$$\int_{\varsigma_1}^e d_\alpha \eta = \int_{\varsigma_1}^e T_\alpha \left(\frac{\eta^\alpha}{\alpha} \right) d_\alpha \eta = \frac{e^\alpha - \varsigma_1^\alpha}{\alpha}. \tag{3.4}$$

From (3.3) and (3.4), we get that

$$|\mathfrak{R}(e)| \leq (\zeta(e))^{\frac{-1}{\varpi+\delta}} \left(\frac{e^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\delta-1}{\delta}} (z(e))^{\frac{1}{\delta}},$$

then

$$(\zeta(e))^{\frac{\varpi}{\varpi+\delta}} (|\mathfrak{R}(e)|)^\varpi \leq \left(\frac{e^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(e))^{\frac{\varpi}{\delta}}. \tag{3.5}$$

Using (2.3), we obtain

$$\begin{aligned} T_\alpha \left((z(e))^{\frac{\varpi+\delta}{\delta}} \right) &= \left(T_\alpha z^{\frac{\varpi+\delta}{\delta}} \right) (z(e)) (T_\alpha z(e)) (z(e))^{\alpha-1} \\ &= \left(\frac{\varpi + \delta}{\delta} \right) (z(e))^{\frac{\varpi+\delta}{\delta}-\alpha} (z(e)) (T_\alpha z(e)) (z(e))^{\alpha-1} \\ &= \left(\frac{\varpi + \delta}{\delta} \right) (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e). \end{aligned} \tag{3.6}$$

Therefore, from (3.2), (3.5) and (3.6), we have

$$\begin{aligned} &\int_{\varsigma_1}^w \zeta(e) (|\mathfrak{R}(e)|)^\varpi (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \\ &= \int_{\varsigma_1}^w (\zeta(e))^{\frac{\varpi}{\varpi+\delta}} (|\mathfrak{R}(e)|)^\varpi (\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{\varsigma_1}^w \left(\frac{e^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e) d_\alpha e \\
 &\leq \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \int_{\varsigma_1}^w (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e) d_\alpha e \\
 &\leq \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \int_{\varsigma_1}^w T_\alpha \left((z(e))^{\frac{\varpi+\delta}{\delta}} \right) d_\alpha e \\
 &= \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(w))^{\frac{\varpi+\delta}{\delta}}. \tag{3.7}
 \end{aligned}$$

Now, using (3.4) and applying (2.5) on $z(w)$ with $\beta = (\varpi + \delta) / \varpi$ and $\gamma = (\varpi + \delta) / \delta$, we find that

$$\begin{aligned}
 z(w) &= \int_{\varsigma_1}^w (\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \Re(e)|)^\delta d_\alpha e \\
 &\leq \left(\int_{\varsigma_1}^w d_\alpha e \right)^{\frac{\varpi}{\varpi+\delta}} \left(\int_{\varsigma_1}^w \left((\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \Re(e)|)^\delta \right)^{\frac{\varpi+\delta}{\delta}} d_\alpha e \right)^{\frac{\delta}{\varpi+\delta}} \\
 &= \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi}{\varpi+\delta}} \left(\int_{\varsigma_1}^w \zeta(e) (|T_\alpha \Re(e)|)^{\varpi+\delta} d_\alpha e \right)^{\frac{\delta}{\varpi+\delta}}. \tag{3.8}
 \end{aligned}$$

Then, from (3.7) and (3.8), we get that

$$\int_{\varsigma_1}^w \zeta(e) (|\Re(e)|)^\varpi (|T_\alpha \Re(e)|)^\delta d_\alpha e \leq \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^\varpi \int_{\varsigma_1}^w \zeta(e) (|T_\alpha \Re(e)|)^{\varpi+\delta} d_\alpha e,$$

which is (3.1). The proof is now complete. The obtained result present a novel conformable fractional analogue of Opial’s inequality. \square

Remark 3.1. In Theorem 3.1, if $\alpha = 1$, then

$$\int_{\varsigma_1}^w \zeta(e) (|\Re(e)|)^\varpi (|\Re'(e)|)^\delta de \leq \frac{\delta(w - \varsigma_1)^\varpi}{\varpi + \delta} \int_{\varsigma_1}^w \zeta(e) (|\Re'(e)|)^{\varpi+\delta} de,$$

which is (1.10). If $\delta = \varpi = 1$ and $\zeta(e) = 1$ then

$$\int_{\varsigma_1}^w |\Re(e)| |\Re'(e)| de \leq \frac{w - \varsigma_1}{2} \int_{\varsigma_1}^w (|\Re'(e)|)^2 de,$$

which is (1.6).

Example 3.1. In Theorem 3.1, if $\zeta(e) = 1$ and $\varsigma_1 = 0$, then

$$\int_0^w (|\Re(e)|)^\varpi (|T_\alpha \Re(e)|)^\delta d_\alpha e \leq \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{w^\alpha}{\alpha} \right)^\varpi \int_0^w (|T_\alpha \Re(e)|)^{\varpi+\delta} d_\alpha e,$$

which is the fractional version of Opial-type inequality (1.3).

Theorem 3.2. Let $0 \leq \zeta \in C([\varsigma_1, w], \mathbb{R}^+)$ be non-increasing on $[\varsigma_1, w]$, $\varsigma_1, w \in \mathbb{R}$. If $\Re : [\varsigma_1, w] \rightarrow \mathbb{R}$ s.t. $\Re(\varsigma_1) = 0$ and $\alpha \in (0, 1]$, then for $\varpi \geq 0$

$$\int_{\varsigma_1}^w \zeta(e) (|\Re(e)|)^\varpi |T_\alpha \Re(e)| d_\alpha e \leq \left(\frac{1}{\varpi + 1} \right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^\varpi \int_{\varsigma_1}^w \zeta(e) (|T_\alpha \Re(e)|)^{\varpi+1} d_\alpha e. \tag{3.9}$$

Proof. Suppose that $z(e) = \int_{s_1}^e (\zeta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta$ for $e \in [s_1, w]$, then

$$T_\alpha z(e) = (\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| > 0 \text{ and } z(s_1) = 0.$$

Therefore,

$$\begin{aligned} |\mathfrak{R}(e)| &= \left| \int_{s_1}^e T_\alpha \mathfrak{R}(\eta) d_\alpha \eta \right| \\ &\leq \int_{s_1}^e |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\ &= \int_{s_1}^e (\zeta(\eta))^{\frac{-1}{\varpi+1}} (\zeta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\ &\leq (\zeta(e))^{\frac{-1}{\varpi+1}} \int_{s_1}^e (\zeta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\ &= (\zeta(e))^{\frac{-1}{\varpi+1}} z(e). \end{aligned} \tag{3.10}$$

By utilizing (2.3), we find that

$$T_\alpha \left((z(e))^{\varpi+1} \right) = (T_\alpha z^{\varpi+1})(z(e)) (T_\alpha z(e)) (z(e))^{\alpha-1} = (\varpi + 1) (z(e))^\varpi T_\alpha z(e). \tag{3.11}$$

Then, from (3.10) and (3.11), we have that

$$\begin{aligned} \int_{s_1}^w \zeta(e) (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e &= \int_{s_1}^w (\zeta(e))^{\frac{\varpi}{\varpi+1}} (|\mathfrak{R}(e)|)^\varpi (\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| d_\alpha e \\ &\leq \int_{s_1}^w (z(e))^\varpi T_\alpha z(e) d_\alpha e \\ &= \frac{1}{\varpi + 1} \int_{s_1}^w T_\alpha \left((z(e))^{\varpi+1} \right) d_\alpha e \\ &= \frac{1}{\varpi + 1} (z(w))^{\varpi+1}. \end{aligned} \tag{3.12}$$

Using (3.4) and applying (2.5) on $z(w)$ with $\beta = (\varpi + 1) / \varpi$ and $\gamma = \varpi + 1$, we get that

$$\begin{aligned} z(w) &= \int_{s_1}^w (\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| d_\alpha e \\ &\leq \left(\int_{s_1}^w d_\alpha e \right)^{\frac{\varpi}{\varpi+1}} \left(\int_{s_1}^w \left((\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| \right)^{\varpi+1} d_\alpha e \right)^{\frac{1}{\varpi+1}} \\ &\leq \left(\frac{w^\alpha - s_1^\alpha}{\alpha} \right)^{\frac{\varpi}{\varpi+1}} \left(\int_{s_1}^w \left((\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| \right)^{\varpi+1} d_\alpha e \right)^{\frac{1}{\varpi+1}} \\ &= \left(\frac{w^\alpha - s_1^\alpha}{\alpha} \right)^{\frac{\varpi}{\varpi+1}} \left(\int_{s_1}^w \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e \right)^{\frac{1}{\varpi+1}}. \end{aligned} \tag{3.13}$$

Then, from (3.12) and (3.13), we get that

$$\int_{s_1}^w \zeta(e) (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e \leq \left(\frac{1}{\varpi + 1} \right) \left(\frac{w^\alpha - s_1^\alpha}{\alpha} \right)^\varpi \int_{s_1}^w \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e,$$

which is (3.9). Thus, the proof is finished, yielding a new conformable fractional versions of Opial-type inequality that extend and generalize classical results in the literature. \square

Remark 3.2. In Theorem 3.2, if $\alpha = 1$ and $\varpi = 1$, then

$$\int_{\varsigma_1}^w \zeta(e) |\mathfrak{R}(e)| |\mathfrak{R}'(e)| de \leq \frac{w - \varsigma_1}{2} \int_{\varsigma_1}^w \zeta(e) (|\mathfrak{R}'(e)|)^2 de. \tag{3.14}$$

If $\alpha = 1$ and $\zeta(e) = 1$, then (3.9) becomes

$$\int_{\varsigma_1}^w (|\mathfrak{R}(e)|)^\varpi |\mathfrak{R}'(e)| de \leq \frac{(w - \varsigma_1)^\varpi}{\varpi + 1} \int_{\varsigma_1}^w (|\mathfrak{R}'(e)|)^{\varpi+1} de,$$

which is (1.5).

Example 3.2. In Theorem 3.2, if $\zeta(e) = 1$, then

$$\int_{\varsigma_1}^w (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e \leq \left(\frac{1}{\varpi + 1}\right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha}\right)^\varpi \int_{\varsigma_1}^w (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e,$$

which is the fractional version of inequality (1.5).

Theorem 3.3. Let $\zeta \geq 0$ be non-decreasing on $[w, \varsigma_2]$ with $w, \varsigma_2 \in \mathbb{R}$ and $\delta > 1$. If $\mathfrak{R} : [w, \varsigma_2] \rightarrow \mathbb{R}$ s.t. $\mathfrak{R}(\varsigma_2) = 0$ and $\alpha \in (0, 1]$, then for $\varpi \geq 0$

$$\int_w^{\varsigma_2} \zeta(e) (|\mathfrak{R}(e)|)^\varpi (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \leq \left(\frac{\delta}{\varpi + \delta}\right) \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha}\right)^\varpi \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+\delta} d_\alpha e. \tag{3.15}$$

Proof. Suppose that $z(e) = \int_e^{\varsigma_2} (\zeta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta$ for $e \in [w, \varsigma_2]$, then

$$T_\alpha z(e) = -(\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(e)|)^\delta < 0 \text{ and } z(\varsigma_2) = 0. \tag{3.16}$$

Using (3.4) and applying (2.5) on $|\mathfrak{R}(e)|$ with $\beta = \delta/(\delta - 1)$ and $\gamma = \delta$, we obtain

$$\begin{aligned} |\mathfrak{R}(e)| &= \left| \int_e^{\varsigma_2} T_\alpha \mathfrak{R}(\eta) d_\alpha \eta \right| \\ &\leq \int_e^{\varsigma_2} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\ &= \int_e^{\varsigma_2} (\zeta(\eta))^{\frac{-1}{\varpi+\delta}} (\zeta(\eta))^{\frac{1}{\varpi+\delta}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\ &\leq \left(\int_e^{\varsigma_2} \left((\zeta(\eta))^{\frac{-1}{\varpi+\delta}} \right)^{\frac{\delta}{\delta-1}} d_\alpha \eta \right)^{\frac{\delta-1}{\delta}} \left(\int_e^{\varsigma_2} (\zeta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta \right)^{\frac{1}{\delta}} \\ &\leq (\zeta(e))^{\frac{-1}{\varpi+\delta}} \left(\int_e^{\varsigma_2} d_\alpha \eta \right)^{\frac{\delta-1}{\delta}} \left(\int_e^{\varsigma_2} (\zeta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta \right)^{\frac{1}{\delta}} \\ &\leq (\zeta(e))^{\frac{-1}{\varpi+\delta}} \left(\frac{\varsigma_2^\alpha - e^\alpha}{\alpha} \right)^{\frac{\delta-1}{\delta}} (z(e))^{\frac{1}{\delta}}. \end{aligned}$$

Therefore,

$$(\zeta(e))^{\frac{\varpi}{\varpi+\delta}} (|\mathfrak{R}(e)|)^\varpi \leq \left(\frac{\varsigma_2^\alpha - e^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(e))^{\frac{\varpi}{\delta}}. \tag{3.17}$$

Using (3.16) and (2.3), we obtain

$$\begin{aligned}
 -T_\alpha \left((z(e))^{\frac{\varpi+\delta}{\delta}} \right) &= - \left(T_\alpha z^{\frac{\varpi+\delta}{\delta}} \right) (z(e)) (T_\alpha z(e)) (z(e))^{\alpha-1} \\
 &= - \left(\frac{\varpi + \delta}{\delta} \right) (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e).
 \end{aligned}
 \tag{3.18}$$

Then, from (3.16), (3.17) and (3.18), we get that

$$\begin{aligned}
 &\int_w^{\varsigma_2} \zeta(e) (|\Re(e)|)^\varpi (|T_\alpha \Re(e)|)^\delta d_\alpha e \\
 &= \int_w^{\varsigma_2} (\zeta(e))^{\frac{\varpi}{\varpi+\delta}} (|\Re(e)|)^\varpi (\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \Re(e)|)^\delta d_\alpha e \\
 &\leq \int_w^{\varsigma_2} \left(\frac{\varsigma_2^\alpha - e^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(e))^{\frac{\varpi}{\delta}} (-T_\alpha z(e)) d_\alpha e \\
 &\leq \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \int_w^{\varsigma_2} \left(- (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e) \right) d_\alpha e \\
 &\leq \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \int_w^{\varsigma_2} -T_\alpha \left((z(e))^{\frac{\varpi+\delta}{\delta}} \right) d_\alpha e \\
 &= \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(w))^{\frac{\varpi+\delta}{\delta}}.
 \end{aligned}
 \tag{3.19}$$

Now, using (3.4) and applying (2.5) on $z(w)$ with $\beta = (\varpi + \delta) / \varpi$ and $\gamma = (\varpi + \delta) / \delta$, we have that

$$\begin{aligned}
 z(w) &= \int_w^{\varsigma_2} (\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \Re(e)|)^\delta d_\alpha e \\
 &\leq \left(\int_w^{\varsigma_2} d_\alpha e \right)^{\frac{\varpi}{\varpi+\delta}} \left(\int_w^{\varsigma_2} \left((\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \Re(e)|)^\delta \right)^{\frac{\varpi+\delta}{\delta}} d_\alpha e \right)^{\frac{\delta}{\varpi+\delta}} \\
 &\leq \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi}{\varpi+\delta}} \left(\int_w^{\varsigma_2} \zeta(e) (|T_\alpha \Re(e)|)^{\varpi+\delta} d_\alpha e \right)^{\frac{\delta}{\varpi+\delta}}.
 \end{aligned}
 \tag{3.20}$$

Then, from (3.19) and (3.20), we get

$$\int_w^{\varsigma_2} \zeta(e) (|\Re(e)|)^\varpi (|T_\alpha \Re(e)|)^\delta d_\alpha e \leq \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^\varpi \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \Re(e)|)^{\varpi+\delta} d_\alpha e,$$

which is (3.15). Thus, the proof is finished, yielding a new conformable fractional version of Opial-type inequality that extend and generalize classical results in the literature. \square

Remark 3.3. In Theorem 3.3, if $\alpha = 1$, then

$$\int_w^{\varsigma_2} \zeta(e) (|\Re(e)|)^\varpi (|\Re'(e)|)^\delta de \leq \frac{\delta(\varsigma_2 - w)^\varpi}{\varpi + \delta} \int_w^{\varsigma_2} \zeta(e) (|\Re'(e)|)^{\varpi+\delta} de,$$

which is (1.10). If $\varpi = \delta = 1$ and $\zeta(e) = 1$, then

$$\int_w^{\varsigma_2} |\Re(e)| |\Re'(e)| de \leq \frac{\varsigma_2 - w}{2} \int_w^{\varsigma_2} (|\Re'(e)|)^2 de,$$

which is (1.6).

Example 3.3. In Theorem 3.3, if $\zeta(e) = 1$ and $\varpi = \delta = 1$, then

$$\int_w^{\varsigma_2} (|\mathfrak{R}(e)|)(|T_\alpha \mathfrak{R}(e)|)d_\alpha e \leq \left(\frac{\varsigma_2^\alpha - w^\alpha}{2\alpha}\right)^\varpi \int_w^{\varsigma_2} (|T_\alpha \mathfrak{R}(e)|)^2 d_\alpha e,$$

which is the fractional version of inequality (1.6).

Theorem 3.4. Let $0 \leq \zeta \in C([w, \varsigma_2], \mathbb{R}^+)$ be non-decreasing on $[w, \varsigma_2]$, $w, \varsigma_2 \in \mathbb{R}$. If $\mathfrak{R} : [w, \varsigma_2] \rightarrow \mathbb{R}$ s.t. $\mathfrak{R}(\varsigma_2) = 0$ and $\alpha \in (0, 1]$, then for $\varpi \geq 0$

$$\int_w^{\varsigma_2} \zeta(e) (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e \leq \left(\frac{1}{\varpi + 1}\right) \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha}\right)^\varpi \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e. \tag{3.21}$$

Proof. Suppose that $z(e) = \int_e^{\varsigma_2} (\zeta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta$ for $e \in [w, \varsigma_2]$, then

$$T_\alpha z(e) = -(\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| < 0 \text{ and } z(\varsigma_2) = 0.$$

Therefore,

$$\begin{aligned} |\mathfrak{R}(e)| &= \left| \int_e^{\varsigma_2} T_\alpha \mathfrak{R}(\eta) d_\alpha \eta \right| \\ &\leq \int_e^{\varsigma_2} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\ &= \int_e^{\varsigma_2} (\zeta(\eta))^{\frac{-1}{\varpi+1}} (\zeta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\ &\leq (\zeta(e))^{\frac{-1}{\varpi+1}} \int_e^{\varsigma_2} (\zeta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\ &\leq (\zeta(e))^{\frac{-1}{\varpi+1}} z(e). \end{aligned} \tag{3.22}$$

Using (2.3), we get that

$$\begin{aligned} -T_\alpha \left((z(e))^{\varpi+1} \right) &= - (T_\alpha z^{\varpi+1}) (z(e)) (T_\alpha z(e)) (z(e))^{\alpha-1} \\ &= -(\varpi + 1) (z(e))^\varpi T_\alpha z(e). \end{aligned} \tag{3.23}$$

Then, from (3.22) and (3.23), we have that

$$\begin{aligned} \int_w^{\varsigma_2} \zeta(e) (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e &= \int_w^{\varsigma_2} (\zeta(e))^{\frac{\varpi}{\varpi+1}} (|\mathfrak{R}(e)|)^\varpi (\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| d_\alpha e \\ &\leq - \int_w^{\varsigma_2} (z(e))^\varpi T_\alpha z(e) d_\alpha e \\ &\leq \frac{-1}{\varpi + 1} \int_w^{\varsigma_2} T_\alpha \left((z(e))^{\varpi+1} \right) d_\alpha e \\ &= \frac{1}{\varpi + 1} (z(w))^{\varpi+1}. \end{aligned} \tag{3.24}$$

Using (3.4) and applying (2.5) on $z(w)$ with $\beta = (\varpi + 1) / \varpi$ and $\gamma = \varpi + 1$, we find that

$$z(w) = \int_w^{\varsigma_2} (\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| d_\alpha e$$

$$\begin{aligned}
 &\leq \left(\int_w^{\varsigma_2} d_\alpha e \right)^{\frac{\varpi}{\varpi+1}} \left(\int_w^{\varsigma_2} \left((\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| \right)^{\varpi+1} d_\alpha e \right)^{\frac{1}{\varpi+1}} \\
 &\leq \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi}{\varpi+1}} \left(\int_w^{\varsigma_2} \left((\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| \right)^{\varpi+1} d_\alpha e \right)^{\frac{1}{\varpi+1}} \\
 &= \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi}{\varpi+1}} \left(\int_w^{\varsigma_2} \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e \right)^{\frac{1}{\varpi+1}}. \tag{3.25}
 \end{aligned}$$

Then, from (3.24) and (3.25), we have that

$$\int_w^{\varsigma_2} \zeta(e) (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e \leq \frac{1}{\varpi + 1} \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^\varpi \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e,$$

which is (3.21). Hence, the theorem is proved, presenting a new Opial-type inequality via conformable fractional calculus. □

Remark 3.4. In Theorem 3.4, if $\alpha = 1$ and $\varpi = 1$, then

$$\int_w^{\varsigma_2} \zeta(e) |\mathfrak{R}(e)| |\mathfrak{R}'(e)| de \leq \frac{\varsigma_2 - w}{2} \int_w^{\varsigma_2} \zeta(e) (|\mathfrak{R}'(e)|)^2 de. \tag{3.26}$$

If $\alpha = 1$ and $\zeta(e) = 1$, then (3.21) becomes

$$\int_w^{\varsigma_2} (|\mathfrak{R}(e)|)^\varpi |\mathfrak{R}'(e)| de \leq \frac{(\varsigma_2 - w)^\varpi}{\varpi + 1} \int_w^{\varsigma_2} (|\mathfrak{R}'(e)|)^{\varpi+1} de,$$

which is (1.5).

Theorem 3.5. Let $\theta \in C([s_1, w], \mathbb{R}^+)$ be non-increasing on $[s_1, w]$ and $\zeta \geq 0$ on (s_1, w) s.t. $\int_{s_1}^w (\zeta(e))^{\frac{-1}{\varpi+\delta}} d_\alpha e < \infty$, $s_1, w \in \mathbb{R}$ and $\delta > 1$. If $\mathfrak{R} : [s_1, w] \rightarrow \mathbb{R}$ with $\mathfrak{R}(s_1) = 0$ and $\alpha \in (0, 1]$, then for $\varpi \geq 0$

$$\int_{s_1}^w \theta(e) (|\mathfrak{R}(e)|)^\varpi (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \leq k_1(s_1, w, \varpi, \delta) \int_{s_1}^w \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+\delta} d_\alpha e, \tag{3.27}$$

where

$$k_1(s_1, w, \varpi, \delta) = \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{w^\alpha - s_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \left(\int_{s_1}^w \left(\frac{\theta(e)}{\zeta(e)} \right)^{\frac{\delta}{\varpi}} d_\alpha e \right)^{\frac{\varpi}{\delta}}.$$

Proof. Suppose that $z(e) = \int_{s_1}^e (\theta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta$ for $e \in [s_1, w]$, then

$$T_\alpha z(e) = (\theta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(e)|)^\delta > 0 \text{ and } z(s_1) = 0. \tag{3.28}$$

Applying (2.5) on $|\mathfrak{R}(e)|$ with $\beta = \delta/(\delta - 1)$ and $\gamma = \delta$, then

$$\begin{aligned}
 |\mathfrak{R}(e)| &= \left| \int_{s_1}^e T_\alpha \mathfrak{R}(\eta) d_\alpha \eta \right| \\
 &\leq \int_{s_1}^e |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\
 &= \int_{s_1}^e (\theta(\eta))^{\frac{-1}{\varpi+\delta}} (\theta(\eta))^{\frac{1}{\varpi+\delta}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta
 \end{aligned}$$

$$\begin{aligned} &\leq \left(\int_{\varsigma_1}^e \left((\theta(\eta))^{\frac{-1}{\varpi+\delta}} \right)^{\frac{\delta}{\delta-1}} d_\alpha \eta \right)^{\frac{\delta-1}{\delta}} \left(\int_{\varsigma_1}^e (\theta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta \right)^{\frac{1}{\delta}} \\ &\leq (\theta(e))^{\frac{-1}{\varpi+\delta}} \left(\int_{\varsigma_1}^e d_\alpha \eta \right)^{\frac{\delta-1}{\delta}} \left(\int_{\varsigma_1}^e (\theta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta \right)^{\frac{1}{\delta}}. \end{aligned} \tag{3.29}$$

Since $T_\alpha \left(\frac{\eta^\alpha}{\alpha} \right) = 1$, then

$$\int_{\varsigma_1}^e d_\alpha \eta = \int_{\varsigma_1}^e T_\alpha \left(\frac{\eta^\alpha}{\alpha} \right) d_\alpha \eta = \frac{e^\alpha - \varsigma_1^\alpha}{\alpha}. \tag{3.30}$$

From (3.29), we get that

$$|\mathfrak{R}(e)| \leq (\theta(e))^{\frac{-1}{\varpi+\delta}} \left(\frac{e^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\delta-1}{\delta}} (z(e))^{\frac{1}{\delta}}.$$

Therefore,

$$(\theta(e))^{\frac{\varpi}{\varpi+\delta}} (|\mathfrak{R}(e)|)^\varpi \leq \left(\frac{e^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(e))^{\frac{\varpi}{\delta}}. \tag{3.31}$$

Using (2.3), we get

$$T_\alpha \left((z(e))^{\frac{\varpi+\delta}{\delta}} \right) = \left(T_\alpha z^{\frac{\varpi+\delta}{\delta}} \right) (z(e)) (T_\alpha z(e)) (z(e))^{\alpha-1} = \left(\frac{\varpi + \delta}{\delta} \right) (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e). \tag{3.32}$$

Then, from (3.28), (3.31) and (3.32), we obtain that

$$\begin{aligned} &\int_{\varsigma_1}^w \theta(e) (|\mathfrak{R}(e)|)^\varpi (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \\ &= \int_{\varsigma_1}^w (\theta(e))^{\frac{\varpi}{\varpi+\delta}} (|\mathfrak{R}(e)|)^\varpi (\theta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \\ &\leq \int_{\varsigma_1}^w \left(\frac{e^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e) d_\alpha e \\ &\leq \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \int_{\varsigma_1}^w (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e) d_\alpha e \\ &\leq \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \int_{\varsigma_1}^w T_\alpha \left((z(e))^{\frac{\varpi+\delta}{\delta}} \right) d_\alpha e \\ &= \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(w))^{\frac{\varpi+\delta}{\delta}}. \end{aligned}$$

Using (3.30) and applying (2.5) on $\int_{\varsigma_1}^w \theta(e) (|\mathfrak{R}(e)|)^\varpi (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e$, with $\beta = (\varpi + \delta) / \varpi$ and $\gamma = (\varpi + \delta) / \delta$, we find that

$$\begin{aligned} &\int_{\varsigma_1}^w \theta(e) (|\mathfrak{R}(e)|)^\varpi (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \\ &\leq \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(w))^{\frac{\varpi+\delta}{\delta}} \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{\delta}{\varpi + \delta}\right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha}\right)^{\frac{\varpi(\delta-1)}{\delta}} \\
 &\quad \times \left(\int_{\varsigma_1}^w (\theta(e))^{\frac{\delta}{\varpi+\delta}} (\zeta(e))^{-\frac{\delta}{\varpi+\delta}} (\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e\right)^{\frac{\varpi+\delta}{\delta}} \\
 &\leq \left(\frac{\delta}{\varpi + \delta}\right) \left(\frac{w^\alpha - \varsigma_1^\alpha}{\alpha}\right)^{\frac{\varpi(\delta-1)}{\delta}} \left(\int_{\varsigma_1}^w \left(\frac{\theta(e)}{\zeta(e)}\right)^{\frac{\delta}{\varpi}} d_\alpha e\right)^{\frac{\varpi}{\delta}} \\
 &\quad \times \int_{\varsigma_1}^w \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\delta+\varpi} d_\alpha e \\
 &= k_1(\varsigma_1, w, \varpi, \delta) \int_{\varsigma_1}^w \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\delta+\varpi} d_\alpha e,
 \end{aligned}$$

which is (3.27). Hence, the theorem is proved, presenting a new Opial-type inequality via conformable fractional calculus. □

Remark 3.5. In Theorem 3.5, if $\theta(e) = \zeta(e)$, then the inequality (3.27) reduce to (3.1).

Theorem 3.6. Let $\theta \in C([\varsigma_1, w], \mathbb{R}^+)$ be non-increasing on $[\varsigma_1, w]$ and $\zeta \geq 0$ on (ς_1, w) s.t. $\int_{\varsigma_1}^w (\zeta(e))^{-\frac{1}{\varpi+1}} d_\alpha e < \infty$, $\varsigma_1, w \in \mathbb{R}$. If $\mathfrak{R} : [\varsigma_1, w] \rightarrow \mathbb{R}$ with $\mathfrak{R}(\varsigma_1) = 0$ and $\alpha \in (0, 1]$, then for $\varpi \geq 0$

$$\int_{\varsigma_1}^w \theta(e) (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e \leq k_2(\varsigma_1, w, \varpi) \int_{\varsigma_1}^w \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e, \tag{3.33}$$

where

$$k_2(\varsigma_1, w, \varpi) = \frac{1}{\varpi + 1} \left(\int_{\varsigma_1}^w \left(\frac{\theta(e)}{\zeta(e)}\right)^{\frac{1}{\varpi}} d_\alpha e\right)^\varpi.$$

Proof. Suppose that $z(e) = \int_{\varsigma_1}^e (\theta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta$ for $e \in [\varsigma_1, w]$, then

$$T_\alpha z(e) = (\theta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| > 0 \text{ and } z(\varsigma_1) = 0.$$

Therefore,

$$\begin{aligned}
 |\mathfrak{R}(e)| &= \left| \int_{\varsigma_1}^e T_\alpha \mathfrak{R}(\eta) d_\alpha \eta \right| \\
 &\leq \int_{\varsigma_1}^e |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\
 &= \int_{\varsigma_1}^e (\theta(\eta))^{\frac{-1}{\varpi+1}} (\theta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\
 &\leq (\theta(e))^{\frac{-1}{\varpi+1}} \int_{\varsigma_1}^e (\theta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\
 &\leq (\theta(e))^{\frac{-1}{\varpi+1}} z(e).
 \end{aligned} \tag{3.34}$$

Using (2.3), we get that

$$T_\alpha \left((z(e))^{\varpi+1}\right) = (T_\alpha z^{\varpi+1})(z(e)) (T_\alpha z(e)) (z(e))^{\alpha-1} = (\varpi + 1) (z(e))^\varpi T_\alpha z(e). \tag{3.35}$$

Then, from (3.34) and (3.35), we obtain that

$$\begin{aligned}
 \int_{\varsigma_1}^w \theta(e) (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e &= \int_{\varsigma_1}^w (\theta(e))^{\frac{1}{\varpi+1}} (|\mathfrak{R}(e)|)^\varpi (\theta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| d_\alpha e \\
 &\leq \int_{\varsigma_1}^w (z(e))^\varpi T_\alpha z(e) d_\alpha e \\
 &\leq \frac{1}{\varpi+1} \int_{\varsigma_1}^w T_\alpha \left((z(e))^{\varpi+1} \right) d_\alpha e \\
 &= \frac{1}{\varpi+1} (z(w))^{\varpi+1}.
 \end{aligned} \tag{3.36}$$

Applying (2.5) on $z(w)$ with $\beta = (\varpi + 1) / \varpi$ and $\gamma = \varpi + 1$, we have

$$\begin{aligned}
 z(w) &= \int_{\varsigma_1}^w (\theta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| d_\alpha e \\
 &= \int_{\varsigma_1}^w (\theta(e))^{\frac{1}{\varpi+1}} (\zeta(e))^{\frac{-1}{\varpi+1}} (\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| d_\alpha e \\
 &\leq \left(\int_{\varsigma_1}^w \left((\theta(e))^{\frac{1}{\varpi+1}} (\zeta(e))^{\frac{-1}{\varpi+1}} \right)^{\frac{\varpi+1}{\varpi}} d_\alpha e \right)^{\frac{\varpi}{\varpi+1}} \left(\int_{\varsigma_1}^w \left((\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| \right)^{\varpi+1} d_\alpha e \right)^{\frac{1}{\varpi+1}} \\
 &\leq \left(\int_{\varsigma_1}^w \left(\frac{\theta(e)}{\zeta(e)} \right)^{\frac{1}{\varpi}} d_\alpha e \right)^{\frac{\varpi}{\varpi+1}} \left(\int_{\varsigma_1}^w \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e \right)^{\frac{1}{\varpi+1}}.
 \end{aligned} \tag{3.37}$$

Then, from (3.36) and (3.37), we have

$$\begin{aligned}
 \int_{\varsigma_1}^w \theta(e) (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e &\leq \frac{1}{\varpi+1} \left(\int_{\varsigma_1}^w \left(\frac{\theta(e)}{\zeta(e)} \right)^{\frac{1}{\varpi}} d_\alpha e \right)^\varpi \int_{\varsigma_1}^w \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e \\
 &= k_2(\varsigma_1, w, \varpi) \int_{\varsigma_1}^w \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e,
 \end{aligned}$$

which is (3.33). This completes the proof of the theorem, establishing a new generalized form of Opial-type inequality via conformable fractional calculus. □

Remark 3.6. In Theorem 3.6, if $\theta(e) = \zeta(e)$, then the inequality (3.33) reduce to (3.9).

Theorem 3.7. Let $\theta \in C([w, \varsigma_2], \mathbb{R}^+)$ be non-decreasing on $[w, \varsigma_2]$ and $\zeta \geq 0$ on (w, ς_2) s.t. $\int_w^{\varsigma_2} (\zeta(e))^{\frac{-1}{\varpi+\delta}} d_\alpha e < \infty$, $w, \varsigma_2 \in \mathbb{R}$ and $\delta > 1$. If $\mathfrak{R} : [w, \varsigma_2] \rightarrow \mathbb{R}$ with $\mathfrak{R}(\varsigma_2) = 0$ and $\alpha \in (0, 1]$, then for $\varpi \geq 0$

$$\int_w^{\varsigma_2} \theta(e) (|\mathfrak{R}(e)|)^\varpi (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \leq k_3(\varsigma_2, w, \varpi, \delta) \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+\delta} d_\alpha e, \tag{3.38}$$

where

$$k_3(\varsigma_2, w, \varpi, \delta) = \left(\frac{\delta}{\varpi + \delta} \right) \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \left(\int_w^{\varsigma_2} \left(\frac{\theta(e)}{\zeta(e)} \right)^{\frac{\delta}{\varpi}} d_\alpha e \right)^{\frac{\varpi}{\delta}}.$$

Proof. Suppose that $z(e) = \int_e^{\varsigma_2} (\theta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(\eta)|)^\delta d_\alpha \eta$ for $e \in [w, \varsigma_2]$, then

$$T_\alpha z(e) = -(\theta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(e)|)^\delta < 0 \text{ and } z(\varsigma_2) = 0.$$

Applying (2.5) on $|\Re(e)|$ with $\beta = \delta$ and $\gamma = \delta / (\delta - 1)$, then we have

$$\begin{aligned}
 |\Re(e)| &= \left| \int_e^{\varsigma_2} T_\alpha \Re(\eta) d_\alpha \eta \right| \\
 &\leq \int_e^{\varsigma_2} |T_\alpha \Re(\eta)| d_\alpha \eta \\
 &= \int_e^{\varsigma_2} (\theta(\eta))^{\frac{-1}{\varpi+\delta}} (\theta(\eta))^{\frac{1}{\varpi+\delta}} |T_\alpha \Re(\eta)| d_\alpha \eta \\
 &\leq \left(\int_e^{\varsigma_2} \left((\theta(\eta))^{\frac{-1}{\varpi+\delta}} \right)^{\frac{\delta}{\delta-1}} d_\alpha \eta \right)^{\frac{\delta-1}{\delta}} \left(\int_e^{\varsigma_2} (\theta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \Re(\eta)|)^\delta d_\alpha \eta \right)^{\frac{1}{\delta}} \\
 &\leq (\theta(e))^{\frac{-1}{\varpi+\delta}} \left(\int_e^{\varsigma_2} d_\alpha \eta \right)^{\frac{\delta-1}{\delta}} \left(\int_e^{\varsigma_2} (\theta(\eta))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \Re(\eta)|)^\delta d_\alpha \eta \right)^{\frac{1}{\delta}}. \tag{3.39}
 \end{aligned}$$

Since $T_\alpha \left(\frac{\eta^\alpha}{\alpha} \right) = 1$, then

$$\int_e^{\varsigma_2} d_\alpha \eta = \int_e^{\varsigma_2} T_\alpha \left(\frac{\eta^\alpha}{\alpha} \right) d_\alpha \eta = \frac{\varsigma_2^\alpha - e^\alpha}{\alpha}. \tag{3.40}$$

From (3.39) and (3.40), we get that

$$(\theta(e))^{\frac{\varpi}{\varpi+\delta}} (|\Re(e)|)^\varpi \leq \left(\frac{\varsigma_2^\alpha - e^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(e))^{\frac{\varpi}{\delta}}. \tag{3.41}$$

Using (2.3), we have

$$T_\alpha \left((z(e))^{\frac{\varpi+\delta}{\delta}} \right) = \left(T_\alpha z^{\frac{\varpi+\delta}{\delta}} \right) (z(e)) (T_\alpha z(e)) (z(e))^{\alpha-1} = \left(\frac{\varpi + \delta}{\delta} \right) (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e). \tag{3.42}$$

Then, from (3.41) and (3.42), we have that

$$\begin{aligned}
 &\int_w^{\varsigma_2} \theta(e) (|\Re(e)|)^\varpi (|T_\alpha \Re(e)|)^\delta d_\alpha e \\
 &= \int_w^{\varsigma_2} (\theta(e))^{\frac{\varpi}{\varpi+\delta}} (|\Re(e)|)^\varpi (\theta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \Re(e)|)^\delta d_\alpha e \\
 &\leq \int_w^{\varsigma_2} \left(\frac{\varsigma_2^\alpha - e^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \left(- (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e) \right) d_\alpha e \\
 &\leq \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \int_w^{\varsigma_2} \left(- (z(e))^{\frac{\varpi}{\delta}} T_\alpha z(e) \right) d_\alpha e \\
 &\leq \frac{\delta}{\varpi + \delta} \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \int_w^{\varsigma_2} T_\alpha \left(- (z(e))^{\frac{\varpi+\delta}{\delta}} \right) d_\alpha e \\
 &= \frac{\delta}{\varpi + \delta} \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} (z(w))^{\frac{\varpi+\delta}{\delta}}. \tag{3.43}
 \end{aligned}$$

Using (3.41) and applying (2.5) on $\int_w^{\varsigma_2} \theta(e) (|\Re(e)|)^\varpi (|T_\alpha \Re(e)|)^\delta d_\alpha e$, with $\beta = (\varpi + \delta) / \varpi$ and $\gamma = (\varpi + \delta) / \delta$, we find that

$$\int_w^{\varsigma_2} \theta(e) (|\Re(e)|)^\varpi (|T_\alpha \Re(e)|)^\delta d_\alpha e$$

$$\begin{aligned}
 &\leq \frac{\delta}{\varpi + \delta} \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \left(\int_w^{\varsigma_2} (\theta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \right)^{\frac{\varpi+\delta}{\delta}} \\
 &= \frac{\delta}{\varpi + \delta} \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \\
 &\quad \times \left(\int_w^{\varsigma_2} (\theta(e))^{\frac{\delta}{\varpi+\delta}} (\zeta(e))^{\frac{-\delta}{\varpi+\delta}} (\zeta(e))^{\frac{\delta}{\varpi+\delta}} (|T_\alpha \mathfrak{R}(e)|)^\delta d_\alpha e \right)^{\frac{\varpi+\delta}{\delta}} \\
 &\leq \frac{\delta}{\varpi + \delta} \left(\frac{\varsigma_2^\alpha - w^\alpha}{\alpha} \right)^{\frac{\varpi(\delta-1)}{\delta}} \left(\int_w^{\varsigma_2} \left(\frac{\theta(e)}{\zeta(e)} \right)^{\frac{\delta}{\varpi}} d_\alpha e \right)^{\frac{\varpi}{\delta}} \\
 &\quad \times \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\delta+\varpi} d_\alpha e \\
 &= k_3(\varsigma_2, w, \varpi, \delta) \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\delta+\varpi} d_\alpha e,
 \end{aligned}$$

which is (3.38). This completes the proof of the theorem, establishing a new generalized form of Opial-type inequality via conformable fractional calculus. \square

Remark 3.7. In Theorem 3.7, if $\theta(e) = \zeta(e)$, then the inequality (3.38) reduce to (3.15).

Theorem 3.8. Let $\theta \in C([w, \varsigma_2], \mathbb{R}^+)$ be non-decreasing on $[w, \varsigma_2]$ and $\zeta \geq 0$ on (w, ς_2) s.t. $\int_w^{\varsigma_2} (\zeta(e))^{\frac{-1}{\varpi+\delta}} d_\alpha e < \infty$, $w, \varsigma_2 \in \mathbb{R}$. If $\mathfrak{R} : [w, \varsigma_2] \rightarrow \mathbb{R}$ with $\mathfrak{R}(\varsigma_2) = 0$ and $\alpha \in (0, 1]$, then for $\varpi \geq 0$

$$\int_w^{\varsigma_2} \theta(e) (|\mathfrak{R}(e)|)^\varpi |T_\alpha \mathfrak{R}(e)| d_\alpha e \leq k_4(\varsigma_2, w, \varpi) \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \mathfrak{R}(e)|)^{\varpi+1} d_\alpha e, \tag{3.44}$$

where

$$k_4(\varsigma_2, w, \varpi) = \left(\frac{1}{\varpi + 1} \right) \left(\int_w^{\varsigma_2} \left(\frac{\theta(e)}{\zeta(e)} \right)^{\frac{1}{\varpi}} d_\alpha e \right)^\varpi.$$

Proof. Suppose that $z(e) = \int_e^{\varsigma_2} (\theta(\eta))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta$ for $e \in [w, \varsigma_2]$, then

$$T_\alpha z(e) = -(\theta(e))^{\frac{1}{\varpi+1}} |T_\alpha \mathfrak{R}(e)| < 0 \text{ and } z(\varsigma_2) = 0.$$

Therefore,

$$\begin{aligned}
 |\mathfrak{R}(e)| &= \left| \int_e^{\varsigma_2} T_\alpha \mathfrak{R}(\eta) d_\alpha \eta \right| \\
 &\leq \int_e^{\varsigma_2} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\
 &= \int_e^{\varsigma_2} (\theta(\eta))^{\frac{-1}{\varpi+\delta}} (\theta(\eta))^{\frac{1}{\varpi+\delta}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\
 &\leq (\theta(e))^{\frac{-1}{\varpi+\delta}} \int_e^{\varsigma_2} (\theta(\eta))^{\frac{1}{\varpi+\delta}} |T_\alpha \mathfrak{R}(\eta)| d_\alpha \eta \\
 &\leq (\theta(e))^{\frac{-1}{\varpi+\delta}} z(e).
 \end{aligned} \tag{3.45}$$

Using (2.3), we get

$$T_\alpha \left((z(e))^{\varpi+1} \right) = (T_\alpha z)^{\varpi+1} (z(e)) (T_\alpha z(e)) (z(e))^{\alpha-1} = (\varpi + 1) (z(e))^\varpi T_\alpha z(e). \tag{3.46}$$

Then, from (3.45) and (3.46), we obtain that

$$\begin{aligned} & \int_w^{\varsigma_2} \theta(e) (|\Re(e)|)^\varpi |T_\alpha \Re(e)| d_\alpha e \\ &= \int_w^{\varsigma_2} (\theta(e))^{\frac{\varpi}{\varpi+1}} (|\Re(e)|)^\varpi (\theta(e))^{\frac{1}{\varpi+1}} |T_\alpha \Re(e)| d_\alpha e \\ &\leq \int_w^{\varsigma_2} (-z(e))^\varpi T_\alpha z(e) d_\alpha e \\ &\leq \frac{1}{\varpi+1} \int_w^{\varsigma_2} T_\alpha \left(-z(e)^{\varpi+1}\right) d_\alpha e \\ &= \frac{1}{\varpi+1} (z(w))^{\varpi+1}. \end{aligned}$$

Applying (2.5) on $\int_w^{\varsigma_2} \theta(e) (|\Re(e)|)^\varpi |T_\alpha \Re(e)| d_\alpha e$, with $\beta = (\varpi + 1) / \varpi$ and $\gamma = \varpi + 1$, we get that

$$\begin{aligned} & \int_w^{\varsigma_2} \theta(e) (|\Re(e)|)^\varpi |T_\alpha \Re(e)| d_\alpha e \\ &\leq \frac{1}{\varpi+1} \left(\int_w^{\varsigma_2} (\theta(e))^{\frac{1}{\varpi+1}} |T_\alpha \Re(e)| d_\alpha e \right)^{\varpi+1} \\ &= \frac{1}{\varpi+1} \left(\int_w^{\varsigma_2} (\theta(e))^{\frac{1}{\varpi+1}} (\zeta(e))^{\frac{-1}{\varpi+1}} (\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \Re(e)| d_\alpha e \right)^{\varpi+1} \\ &\leq \frac{1}{\varpi+1} \left(\int_w^{\varsigma_2} \left(\frac{\theta(e)}{\zeta(e)}\right)^{\frac{1}{\varpi+1}} (\zeta(e))^{\frac{1}{\varpi+1}} |T_\alpha \Re(e)| d_\alpha e \right)^{\varpi+1} \\ &\leq \frac{1}{\varpi+1} \left(\int_w^{\varsigma_2} \left(\frac{\theta(e)}{\zeta(e)}\right)^{\frac{1}{\varpi}} d_\alpha e \right)^\varpi \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \Re(e)|)^{\varpi+1} d_\alpha e \\ &= k_4(\varsigma_2, w, \varpi) \int_w^{\varsigma_2} \zeta(e) (|T_\alpha \Re(e)|)^{\varpi+1} d_\alpha e, \end{aligned}$$

which is (3.44). The proof is now complete. The obtained result present a novel conformable fractional analogue of Opial’s inequality. □

Remark 3.8. In Theorem 3.8, if $\theta(e) = \zeta(e)$, then the inequality (3.44) reduce to (3.21).

4. Conclusions and future work

In this paper, we established certain fractional inequalities of Opial-type by employing the conformable calculus. The technique is based on the applications of well-known inequalities and new tools from conformable calculus. The inequalities obtained in this paper will lead to the classical inequalities which are established earlier by Hua in [9] by putting $\alpha = 1$. The results established in this paper give some contribution in the field of fractional calculus and Opial fractional integral inequalities. From these results, some work directions remain open, for example: Extending these results to other types of integral fractional operators, which contains as particular cases many of those reported in the literature. Obtain new results for other well-known inequalities such as Hardy, Copson, Hilbert, among others and make connections between them in fractional calculus or classical mathematics. It will be interesting to find further Opial fractional integral inequalities and their reverse versions in extended and generalized forms.

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Conflicts of interest

The authors declare that there are no conflicts of interest with respect to the authorship and/or publication of this article.

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