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EXISTENCE OF TIME PERIODIC SOLUTIONS TO PARABOLIC DELAYED EVOLUTION EQUATION IN ORDERED BANACH SPACE*

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Abstract The purpose of the article is to systematically research time ω-periodic problem to the general parabolic delayed evolution equation. First of all, by applying theory of operator semigroup, the Schauder fixed point theorem, and monotone iterative method, the existence and uniqueness results of ω-periodic mild solutions and positive mild solutions to the abstract evolution equation with delay are gained when nonlinear term satisfies some suitable ordered conditions. Next, we use the abstract results to get similar results of periodic problem to the parabolic evolution equation with finite delay, which improves and generalizes the related work of this fields. Ultimately, two examples to parabolic delayed evolution equation are presented.

Keywords Parabolic delayed evolution equation, periodic solution, existence and uniqueness, Schauder fixed point, ordered Banach space.

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1. Introduction

Parabolic partial differential equations have been rapidly developed due to their applications in physics, population system, chemistry, medicine and other fields, see [1, 9, 11, 26] and relevant references. For a number of actual mathematical models, the existence problems of periodic solutions to such equations are of great significance. Thus, most of the work in the past has focused on the time periodic problem of parabolic partial differential equations, see [2, 4, 16–18, 22–24, 27] and so on. By using monotone iterative method, Pao [24] investigated the periodic problem to a set of nonlinear parabolic equations without time lag and a sufficient condition of the stability to periodic solution was also presented. The work by [17] was concerned with the abstract equation

$$x'(t) + Ax(t) = f(t,\ x(t)), \quad t \in \mathbb{R} \tag{1.1}$$

in X. By applying semigroup theory, estimation methods for noncompact measures, and fixed point technique, the uniqueness result to periodic mild solution to (1.1) was obtained.

At present, partial differential equation with delay can describe the natural phenomena more truly. Therefore, the periodic problems to parabolic delayed partial equations are a crucial field. Particularly, the periodic questions to parabolic delayed evolution equations has catched the researchers' attention, see [3, 7, 10, 12, 14, 15, 19, 20, 25, 28, 29, 31–33]. Burton and Zhang in [3] discussed an abstract evolution equation with infinite delay. By using Granas's fixed theorem,

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they got the existence conclusion of periodic problem. In [10,15,25,29], the periodic problems to more general equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + A(x,D)y = g(x,t,y(x,t),y(x,t-\tau)), \ x \in \Omega, \ t \in \mathbb{R}, \\ By = 0, \ x \in \partial\Omega \end{cases}$$

were investigated, where $\Omega \subset R^N$ is a bounded region with a sufficiently smooth boundary $\partial\Omega$ and

$$A(x,D)y = -\sum_{i,j=1}^{N} a_{ij}(x)D_iD_jy + \sum_{i=1}^{N} b_i(x)D_iy + c(x)y$$

is uniformly elliptic differential operator of non-divergence form in $\overline{\Omega}$, whose coefficients $a_{ij}(x)$, $b_i(x)$ $(i,j = 1,2,\ldots,N)$ and $c(x) \geqslant 0$ are Hölder-continuous on $\overline{\Omega}$. Suppose that $B = B(x,D)$ is a bounded operator on $\partial\Omega$ given by

$$Bu := d(x)y + \mu\frac{\partial y}{\partial \beta},$$

where B is of either Dirichlet type $(\mu = 0, d(x) \equiv 1)$ or of Neumann-Robin type $(\mu = 1, d(x) \geqslant 0)$ with $\frac{\partial}{\partial\beta}$ denoting the outward normal (or conormal) derivative on $\partial\Omega$. Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function whose periodic is ω in t. It is noteworthy that almost no one has systematically considered time periodic problem and positive periodic problem to delayed parabolic evolution equation

For the past few years, monotone iterative method of upper and lower solutions has been a rather practical way that was used to study the periodic problems to some parabolic equations or systems, see [24,25,29,30] and the references therein. For the above work, a necessary condition is that the problem studied has a pair of ordered lower and upper solutions. Therefore, by theory of operator semigroup, the Schauder fixed point method and monotone iterative technique of upper and lower solution, we systematically research time ω-periodic and positive periodic problems to the parabolic evolution equation with finite delay

$$\begin{cases} \dfrac{\partial y}{\partial t} + A(x,D)y = g(x,t,y(x,t),y(x,t-\tau)), \ x \in \Omega, \ t \in \mathbb{R}, \\ y|_{\partial\Omega} = 0 \end{cases} \tag{1.2}$$

in the absence of existing a pair of ordered upper and lower solutions when the nonlinear term g satisfies suitable ordered conditions, where $g : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function whose periodic is ω in t and constant $\tau > 0$ is defined as time lag. Eq. (1.2) is widely used to explain and predict periodic and repetitive dynamic phenomena in the fields of nature and engineering, Thus, it is valuable.

2. Preliminaries

In this section, some different notations, definitions and lemmas are presented.

Let $X = L^p(\Omega)(n < p < \infty)$ denote a Banach space of the L^p-norm $\|\cdot\|$. Assume that $K = \{y \in X | y(x) \geqslant 0, x \in \Omega\}$. Obviously, K is a normal cone that introduces a partial "$\geqslant$" denoted by

$$y \geqslant v \Leftrightarrow y - v \in K$$

and it's normal constant is N_0. So X is an ordered Banach space with the partial order "$\geqslant$".

Let $C_\omega(\mathbb{R}, X)$ define the Banach space $\{y \in C(\mathbb{R}, X) | y(t+\omega) = y(t),\ t \in \mathbb{R}\}$ with maximum norm $\| y \|_C = \max\limits_{t\in I} \| y(t) \|$, where $I = [0, \omega]$. It is clear that $C_\omega(\mathbb{R}, X)$ is also an ordered Banach space with the partial order "$\leqslant$" defined by the positive cone $K_C = \{y \in C_\omega(\mathbb{R}, X) | y(t) \in K,\ t \in \mathbb{R}\}$. K_C is also normal and it's normal constant is also N_0. For each $\eta, \xi \in C_\omega(\mathbb{R}, X)$ with $\eta \leqslant \xi$, we define $[\eta, \xi]$ as the ordered interval $\{y | \eta \leqslant y \leqslant \xi\}$ in $C_\omega(\mathbb{R}, X)$ and $[\eta(t), \xi(t)]$ as the ordered interval $\{y(t) | \eta(t) \leqslant y(t) \leqslant \xi(t), t \in \mathbb{R}\}$ in X.

A linear operator $A : \mathcal{D}(A) \subset X \to X$ is defined by

$$\mathcal{D}(A) = \{y \in W^{2,p}(\Omega) | y(x) = 0, x \in \partial\Omega\},\ Ay = A(x, D)y. \tag{2.1}$$

Since $c(x) \geqslant 0$, the semigroup $T(t)(t \geqslant 0)$ generated by $-A$ is exponentially stable and analytic, see [1]. Let λ_1 be the first eigenvalue of the elliptic operator $A(x, D)$ under the Dirichlet boundary condition $y|_{\partial\Omega} = 0$, As we all know, $\lambda_1 > 0$. Thus, for $\lambda \in (0, \lambda_1)$, there exists a constant $M_0 \geqslant 1$ such that

$$\| T(t) \| \leqslant M_0 e^{-\lambda t},\ \lambda \in (0, \lambda_1),\ t \geqslant 0. \tag{2.2}$$

From the maximum principle of $A(x, D)$, $\lambda I + A$ has positive bounded inverse operator $(\lambda I + A)^{-1}$ for $\lambda > 0$. Therefore, $T(t)(t \geqslant 0)$ is positive. By means of A has compact resolvent in X and the analyticity of $T(t)(t \geqslant 0)$, we get that $T(t)(t \geqslant 0)$ is compact, see [11]. From the above discussion, we recall some definitions and properties of C_0-semigroup $T(t)(t \geqslant 0)$.

Definition 2.1. Let $T(t)(t \geqslant 0)$ is C_0-semigroup in Banach space X. Then, there are $M_0 \geqslant 1$ and $\sigma \in \mathbb{R}$ such that $\| T(t) \| \leqslant M_0 e^{\sigma t},\ t \geqslant 0$, we say that $T(t)(t \geqslant 0)$ is exponentially bounded.

Definition 2.2. Let $T(t)(t \geqslant 0)$ is C_0-semigroup in Banach space X. Then, there are $M_0 \geqslant 1$ and $\nu > 0$ such that $\| T(t) \| \leqslant M_0 e^{-\nu t},\ t \geqslant 0$, we say that $T(t)(t \geqslant 0)$ is exponentially stable.

Definition 2.3. Let $T(t)(t \geqslant 0)$ is C_0-semigroup in Banach space X. Then, we say that

$$\nu_0 = \inf\{\sigma \in \mathbb{R} | \ \| T(t) \| \leqslant M_0 e^{\sigma t},\ t \geqslant 0\}$$

is the growth index of $T(t)(t \geqslant 0)$. Moreover, ν_0 is also expressed by

$$\nu_0 = \limsup_{t\to+\infty} \frac{\ln \| T(t) \|}{t}.$$

By [11,26], we further get that the growth index of $T(t)(t \geqslant 0)$ is $\nu_0 = -\lambda_1$ and $T(t)(t \geqslant 0)$ satisfies

$$\| T(t) \| \leqslant M_0 e^{-\lambda_1 t},\ t \geqslant 0. \tag{2.3}$$

For more detailed properties of operator semigroups, we refer to [11,26].

Now, we need to define $f : \mathbb{R} \times X \times X \to X$ as

$$f(t,\ v,\ w) = g(\cdot,\ t,\ v(\cdot),\ w(\cdot)).$$

Evidently, we get that

(H1) Mapping $f : \mathbb{R} \times X \times X \to X$ is continuous function which is ω-periodic in t.

Hence, the time ω-periodic problem for delayed parabolic equation (1.2) is rewritten as the ω-periodic problem to abstract evolution equation

$$y'(t) + Ay(t) = f(t,\ y(t),\ y(t-\tau)),\ t \in \mathbb{R}. \tag{2.4}$$

Let $J=[0,+\infty)$ and $h:J\to X$. Considering the initial value problem to the linear equation

$$\begin{cases} y'(t)+Ay(t)=h(t),\ t\geqslant 0, \\ y(0)=x_0, \end{cases} \tag{2.5}$$

by [26], if $x_0\in X_1$ and $h\in C^1(J,X)$, then the (2.5) has only one classical solution $y\in C^1(J,X)\cap C(J,X_1)$ given by

$$y(t)=T(t)x_0+\int_0^t T(t-s)h(s)ds, \tag{2.6}$$

where $X_1=\mathcal{D}(A)$ is Banach space with the graph norm $\|\cdot\|_1=\|\cdot\|+\|A\cdot\|$. Universally, if $x_0,\ h\in C(J,X)$, $y\in C(J,X)$ is mild solution of the (2.5).

From the literature [16], for every $h\in C_\omega(\mathbb{R},X)$, some lemmas related the linear evolution equation

$$y'(t)+Ay(t)=h(t),\ t\in\mathbb{R} \tag{2.7}$$

are presented below.

Lemma 2.1 ([16]). *Suppose that A is denoted by (2.1), thus the semigroup $T(t)(t\geqslant 0)$ generated by $-A$ is positive, exponentially stable and analytic in X. Then for each $h\in C_\omega(\mathbb{R},X)$, (2.7) has only one ω-periodic mild solution $y\in C_\omega(\mathbb{R},X)$ expressed by*

$$y(t)=(I-T(\omega))^{-1}\int_{t-\omega}^t T(t-s)h(s)ds:=Ph(t),\ t\in\mathbb{R}, \tag{2.8}$$

where $P:C_\omega(\mathbb{R},X)\to C_\omega(\mathbb{R},X)$ is positive bounded linear operator and $r(P)\leqslant\frac{1}{\lambda_1}$.

Lemma 2.2. [26] *Suppose that the semigroup $T(t)(t\geqslant 0)$ generated by $-A$ is positive, exponentially stable and analytic in X. Then for each $h\in C_\omega^1(\mathbb{R},K)$, ω-periodic mild solution $y=Ph$ to (2.7) is positive classical solution $y\in C_\omega^1(\mathbb{R},X)\cap C_\omega(\mathbb{R},X_1)$.*

Now, study delayed linear evolution equation

$$y'(t)+Ay(t)+by(t-\tau)=h(t),\ t\in\mathbb{R}. \tag{2.9}$$

By Lemma 2.1, we obtain following result.

Lemma 2.3. *Suppose that the semigroup $T(t)(t\geqslant 0)$ generated by $-A$ is positive, exponentially stable and analytic in X. If $0<b<\lambda_1$, then for each $h\in C_\omega(\mathbb{R},K)$, the Eq. (2.9) has just one positive ω-periodic mild solution $y=Sh$, moreover, $S:C_\omega(\mathbb{R},K)\to C_\omega(\mathbb{R},X)$ is a positive bounded linear operator.*

Proof. Denote operator $\mathfrak{B}:C_\omega(\mathbb{R},K)\to C_\omega(\mathbb{R},K)$ by given

$$\mathfrak{B}y(t)=by(t-\tau),\ y\in C_\omega(\mathbb{R},K),\ t\in\mathbb{R}.$$

Obviously, $\mathfrak{B}$ is bounded linear operator and it's norm $\|\mathfrak{B}\|=b$. Thus, it follows from Lemma 2.1 that

$$(I-P\mathfrak{B})y(t)=Ph(t),\ t\in\mathbb{R}.$$

Since $r(P\mathfrak{B}) \leqslant r(P)\cdot \parallel \mathfrak{B} \parallel \leqslant \frac{b}{\lambda_1} < 1$, it follows from the perturbation theorem of unit operator that $I - P\mathfrak{B}$ has bounded inverse operator $(I - P\mathfrak{B})^{-1}$. Thus Eq. (2.9) has unique ω-periodic mild solution $y(t)$ by given

$$y(t) = (I - P\mathfrak{B})^{-1}Ph(t) := Sh(t). \tag{2.10}$$

Because $(I - P\mathfrak{B})^{-1}Ph = (\Sigma_{n=0}^{\infty}(P\mathfrak{B})^n)Ph$, obviously, $S : C_\omega(\mathbb{R}, K) \to C_\omega(\mathbb{R}, K)$ and $\parallel S \parallel \leqslant \frac{1}{\lambda_1 - b}$. □

Theorem 2.1. *Please state your theorem here.*

3. Abstract results

Now, we study the existence of ω-periodic mild solution to Eq. (2.4). The first thing is that we need to define the lower and upper ω-periodic solutions of Eq. (2.4).

Definition 3.1. If a function $\eta_0 \in C^1_\omega(\mathbb{R}, X) \cap C_\omega(\mathbb{R}, X_1)$ satisfies

$$\eta_0'(t) + A\eta_0(t) \leqslant f(t,\ \eta_0(t),\ \eta_0(t-\tau)),\ t \in \mathbb{R}, \tag{3.1}$$

then $\eta_0(t)$ is said to be an lower ω-periodic solution of Eq. (2.4). If the inequality of Eq. (2.4) is inverted, then it is said to be an upper ω-periodic solution of Eq. (2.4).

Similarly to definition of the (2.7), we will give definition of ω-periodic mild solution to Eq. (2.4) (or Eq. (1.2)).

Definition 3.2. If $y \in C_\omega(\mathbb{R}, X)$ satisfies

$$y(t) = T(t)B(y) + \int_0^t T(t-s)f(s,\ y(s),\ y(s-\tau))ds,\ t \geqslant 0, \tag{3.2}$$

where $B(y) = (I - T(\omega))^{-1}\int_0^\omega T(\omega - s)f(s,\ y(s),\ y(s-\tau))ds$. Then y is called a ω-periodic mild solution of (2.4).

Firstly, we will apply the Schauder fixed point method and monotone iterative technique of upper and lower solutions to investigate the existence of ω-periodic mild solution to Eq. (2.4).

Theorem 3.1. *Assume that X is an ordered Banach space, the positive cone $K \subset X$ is a normal regeneration cone and $-A$ generates a positive and compact analytic semigroup $T(t)(t \geqslant 0)$. If f satisfies (H1) and following hypotheses:*

(H2) there are a constant $0 < a < \lambda_1$ and function $h \in C^1_\omega(\mathbb{R}, K)$ such that

$$f(t,\ v,\ w) \leqslant av + h(t),\ f(t,\ -v,\ -w) \geqslant -av - h(t),\ t \in \mathbb{R},$$

for almost all $v,\ w \geqslant \theta$ with $v,\ w \in X$;

(H3) there is a constant $N > 0$ such that

$$f(t,\ v_2,\ w_2) - f(t,\ v_1,\ w_1) \geqslant -N(v_2 - v_1),\ t \in \mathbb{R},$$

for all $v_1 \leqslant v_2,\ w_1 \leqslant w_2$ with $v_i, w_i \in X (i = 1, 2)$;

then there is ω-periodic mild solution $y \in C_\omega(\mathbb{R}, X)$ to Eq.(2.4).

Proof. In (H2), if $h \in C^1_\omega(\mathbb{R}, K)$, consider Eq.

$$y'(t) + (A - aI)y(t) = h(t),\ t \in \mathbb{R}. \tag{3.3}$$

Since $a < \lambda_1$, $-(A - aI)$ generates a positive and exponentially stable strongly continuous semigroup $T_1(t) = T(t)e^{at}(t \geqslant 0)$ and $\| T_1(t) \| \leqslant M_0 e^{-(\lambda_1 - a)t}$. Thus, by lemma 2.2), there is only one positive ω-periodic solution $\xi_0 \in C^1_\omega(\mathbb{R}, X) \cap C_\omega(\mathbb{R}, X_1)$ to the Eq.(3.3). Because $\xi_0(t)$ satisfies

$$\xi_0'(t) + A\xi_0(t) = h(t) + a\xi_0(t) \geqslant f(t,\ \xi_0(t),\ \xi_0(t-\tau)),$$

so, ξ_0 is a upper solution of (2.4). Set $\eta_0 = -\xi_0$, we get that

$$\eta_0'(t) + A\eta_0(t) = -h(t) + a\eta_0(t) \leqslant f(t,\ \eta_0(t),\ \eta_0(t-\tau)),$$

so, η_0 is a lower solution of (2.4). It is easily seen that the periodic problem of Eq.(2.4) is replaced by the periodic problem

$$y'(t) + Ay(t) + Ny(t) = f(t,\ y(t),\ y(t-\tau)) + Ny(t),\ t \in \mathbb{R}. \tag{3.4}$$

Since $N > 0$, then $-N - \lambda_1 < 0$, the semigroup $T_N(t) = e^{-Nt}T(t)(t \geqslant 0)$ generated by $-(A + NI)$ is positive, exponentially stable and compact in X. Set $D = [\eta_0, \xi_0] \subset C_\omega(\mathbb{R}, X)$. By lemma 2.1, for each $h \in D$, the Eq.

$$y'(t) + Ay(t) + Ny(t) = f(t,\ h(t),\ h(t-\tau)) + Nh(t),\ t \in \mathbb{R} \tag{3.5}$$

has just one ω-periodic mild solution

$$\begin{aligned} y(t) &= T_N(t)B_1(h) + \int_0^t T_N(t-s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s))ds \\ &= P_N(f(s,\ h(s),\ h(s-\tau)) + Nh(s)),\ t \in \mathbb{R}, \end{aligned}$$

where

$$B_1(h) = (I - T_N(\omega))^{-1} \int_0^\omega T_N(\omega - s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s))ds.$$

Hence, define mapping $Q : D \to C_\omega(\mathbb{R}, X)$ expressed by

$$Qh(t) = T_N(t)B_1(h) + \int_0^t T_N(t-s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s))ds,\ t \in \mathbb{R}. \tag{3.6}$$

Obviously, the mild solution to Eq. (2.4) is equivalent to the fixed point to Q.

Step 1. We show that Q is an increasing operator.

As a matter of fact, if $h_1, h_2 \in D$ with $h_1 \leqslant h_2$, then $\eta_0(t) \leqslant h_1(t) \leqslant h_2(t) \leqslant \xi_0(t)$ and $\eta_0(t-\tau) \leqslant h_1(t-\tau) \leqslant h_2(t-\tau) \leqslant \xi_0(t-\tau)$ for every $t \in \mathbb{R}$. From (H3), we know that

$$f(t,\ h_1(t),\ h_1(t-\tau)) + Nh_1(t) \leqslant f(t,\ h_2(t),\ h_2(t-\tau)) + Nh_2(t),\ t \in \mathbb{R}.$$

Since $T_N(t)(t \geqslant 0)$ is positive strongly continuous semigroup, we get

$$\int_0^\omega T_N(\omega - s)(f(s,\ h_1(s),\ h_1(s-\tau)) + Nh_1(s))ds$$

$$\leqslant \int_0^{\omega} T_N(\omega - s)(f(s,\ h_2(s),\ h_2(s-\tau)) + Nh_2(s))ds.$$

Owing to $(I - T_N(\omega))^{-1} = \sum_0^{\infty} T_N(n\omega)$, we deduce that $B_1(h_1) \leqslant B_1(h_2)$. Thus, By (3.6), we have

$$Qh_1 \leqslant Qh_2.$$

Namely, operator $Q : D \to C_\omega(\mathbb{R}, X)$ is growing.

Step 2. We verify that $\eta_0 \leqslant Q\eta_0$ and $Q\xi_0 \leqslant \xi_0$.

Since $\eta_0 \in C^1_\omega(\mathbb{R}, X) \bigcap C_\omega(\mathbb{R}, X_1)$ is lower ω-periodic solution to (2.4), we get that

$$\eta_0'(t) + A\eta_0(t) \leqslant f(t,\ \eta_0(t),\ \eta_0(t-\tau)),\ t \in \mathbb{R}.$$

So, we further get that

$$\eta_0'(t) + A\eta_0(t) + N\eta_0(t) \leqslant f(t,\ \eta_0(t),\ \eta_0(t-\tau)) + N\eta_0(t),\ t \in \mathbb{R}.$$

Set $m(t) = \eta_0'(t) + A\eta_0(t) + N\eta_0(t)$, from lemma 2.1 and definition 3.1, one has

$$\begin{aligned}\eta_0(t) &= (I - T_N(\omega))^{-1} \int_{t-\omega}^{t} T_N(t-\omega) m(s)ds \\ &\leqslant (I - T_N(\omega))^{-1} \int_{t-\omega}^{t} T_N(t-\omega)(f(s,\ \eta_0(s),\ \eta_0(s-\tau)) + N\eta_0(s))ds \\ &= (Q\eta_0)(t),\ t \in \mathbb{R},\end{aligned}$$

namely, $\eta_0 \leqslant Q\eta_0$. Similarly, it can be proven that $Q\xi_0 \leqslant \xi_0$. Hence, $Q : D \to D$ is continuous.

Step 3. It remains to prove that $Q(D)$ is relatively compact in $C_\omega(\mathbb{R}, X)$.

Set

$$(Wh)(t) = \int_0^t T_N(t-s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s))ds,\ h \in D, t \in \mathbb{R}, \tag{3.7}$$

then

$$(Qh)(t) = T_N(t)B_1(h) + (Wh)(t),\ t \in \mathbb{R}.$$

Firstly, we prove that the $\{(Qh)(t) : h \in D\}$ is relatively compact in X for all $t \in \mathbb{R}$. Set $z(t) = \{(Wh)(t) : h \in D\}$ for all $t \in \mathbb{R}$. For any $\varepsilon \in (0, t)$, set

$$\begin{aligned}&(W_\varepsilon h)(t) \\ =&\int_0^{t-\varepsilon} T_N(t-s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s))ds \\ =&T_N(\varepsilon) \int_0^{t-\varepsilon} T_N(t-\varepsilon-s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s))ds,\ h \in D,\ t \in \mathbb{R}.\end{aligned} \tag{3.8}$$

For every $h \in D$, by (H3), we know that

$$\begin{aligned}&f(t, \eta_0(t), \eta_0(t-\tau)) + N\eta_0(t) \\ \leqslant&f(t,\ h(t),\ h(t-\tau)) + Nh(t) \\ \leqslant&f(t,\ \xi_0(t),\ \xi_0(t-\tau)) + N\xi_0(t),\ t \in \mathbb{R}.\end{aligned}$$

Thus

$$\begin{aligned}0 \leqslant &\| f(t,\ h(t),\ h(t-\tau)) + Nh(t) - (f(t,\ \eta_0(t),\ \eta_0(t-\tau)) + N\eta_0(t)) \| \\ \leqslant &\| f(t,\ \xi_0(t),\ \xi_0(t-\tau)) + N\xi_0(t) - (f(t,\ \eta_0(t),\ \eta_0(t-\tau)) + N\eta_0(t)) \|,\ t \in \mathbb{R}.\end{aligned}$$

By normality of K, we ulteriorly deduce that

$$\begin{aligned}&\| f(t,\ h(t),\ h(t-\tau)) + Nh(t) \| - \| f(t,\ \eta_0(t),\ \eta_0(t-\tau)) + N\eta_0(t) \| \\ \leqslant &\| f(t,\ h(t),\ h(t-\tau)) + Nh(t) - (f(t,\ \eta_0(t),\ \eta_0(t-\tau)) + N\eta_0(t)) \| \\ \leqslant &N_0 \| f(t,\ \xi_0(t),\ \xi_0(t-\tau)) + N\xi_0(t) - (f(t,\ \eta_0(t),\ \eta_0(t-\tau)) + N\eta_0(t)) \| \\ := &M_1,\end{aligned}$$

where N_0 is normal constant. So, there is a constant $M_2 > 0$ such that

$$\| f(t,\ h(t),\ h(t-\tau)) + Nh(t) \| \leqslant M_2,\ h \in D,\ t \in \mathbb{R}. \tag{3.9}$$

By compactness of $T_N(\varepsilon)$, we know that $z_\varepsilon(t) = \{(W_\varepsilon h)(t) : h \in D\}$ is relatively compact in X. It follows from (3.7), (3.8) and (3.9) that

$$\begin{aligned}\| (Wh)(t) - (W_\varepsilon h)(t) \| = &\| \int_0^t T_N(t-s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s))ds \\ &- \int_0^{t-\varepsilon} T_N(t-s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s))ds \| \\ = &\| \int_{t-\varepsilon}^t T_N(t-s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s))ds \| \\ \leqslant &\int_{t-\varepsilon}^t \| T_N(t-s)(f(s,\ h(s),\ h(s-\tau)) + Nh(s)) \| ds \\ = &\int_0^\varepsilon \| T_N(s)(f(t-s,\ h(t-s),\ h(t-s-\tau)) \\ &+ Nh(t-s)) \| ds \\ \leqslant &M_0 M_2 \varepsilon.\end{aligned}$$

Therefore, $z(t)$ is a completely bounded set, and we further get that $z(t)$ is relatively compact in X. Obviously, $z(\omega)$ is relatively compact in X. Since $B_1(D) = \{(I - T_N(\omega))^{-1}(Wh)(\omega) : h \in D\} = (I - S(\omega))^{-1} z(\omega)$, we easily know that $B_1(D)$ is relatively compact in X. Hence, $T_N(t)B_1(D)$ is relatively compact in X. Consequently, $\{(Qh)(t) : h \in D\}$ is relatively compact in X for all $t \in \mathbb{R}$.

Next, we demonstrate that $Q(D)$ is equicontinuous in $C_\omega(\mathbb{R}, X)$. For each $h \in D$ and $0 \leqslant t_1 \leqslant t_2 \leqslant \omega$, from (3.7), (3.9), the continuity of operator norm for semigroup $T_N(t)$ in $(0, \infty)$ and Lebesgue's bounded convergence theorem, we get that

$$\begin{aligned}&\| (Wh)(t_2) - (Wh)(t_1) \| \\ = &\| \int_0^{t_2} T_N(t_2 - s)(f(s, h(s), h(s-\tau)) + Nh(s))ds \\ &- \int_0^{t_1} T_N(t_1 - s)(f(s, h(s), h(s-\tau)) + Nh(s))ds \|\end{aligned}$$

$$\begin{aligned}
&= \| \int_{t_1}^{t_2} T_N(t_2-s)(f(s,h(s),h(s-\tau))+Nh(s))ds \\
&\quad + \int_0^{t_1} (T_N(t_2-s)-T_N(t_1-s))(f(s,h(s),h(s-\tau))+Nh(s))ds \| \\
&\leqslant M_0M_2(t_2-t_1)+M_2\int_0^{t_1} \| T_N(t_2-s)-T_N(t_1-s) \| ds \\
&= M_0M_2(t_2-t_1)+M_2\int_0^{\omega} \| T_N(t_2-t_1+s)-T_N(s) \| ds \\
&\to 0 \ (t_2-t_1\to 0).
\end{aligned}$$

Thus, the set $W(D)$ is equicontinuous in $C_\omega(\mathbb{R},X)$. Further, we deduce that $Q(D)$ is equicontinuous in $C_\omega(\mathbb{R},X)$. By Ascoli-Arzela theorem, $Q(D)$ is relatively compact in $C_\omega(\mathbb{R},X)$. So $Q:D\to D$ is completely continuous. Therefore, by applying the Schauder fixed point theorem, it is Q that has fixed point $y\in C_\omega(\mathbb{R},X)$ which is ω-periodic mild solution to Eq. (2.4). □

Our second result establishes the uniqueness of ω-periodic solution to Eq. (2.4).

Theorem 3.2. *Assume that all assumptions of the theorem 3.1 hold, and f satisfies hypotheses:*

(H4) there is a sufficiently small constant $L_1>0$ such that

$$v_2-v_1\geqslant L_1(w_2-w_1),\ t\in\mathbb{R},$$

for all $v_1\leqslant v_2,\ w_1\leqslant w_2$ with $v_i\ w_i\in X(i=1,2)$;

(H5) there are constants $L_2,\ L_3>0$ satisfying $L_2+\frac{L_3}{L_1}<\lambda_1$ such that

$$f(t,\ v_2,\ w_2)-f(t,\ v_1,\ w_1)\leqslant L_2(v_2-v_1)+L_3(w_2-w_1),\ t\in\mathbb{R},$$

for all $v_1\leqslant v_2,\ w_1\leqslant w_2$ with $v_i\ w_i\in X(i=1,2)$;

then there is just one ω-periodic mild solution $y^*\in C_\omega(\mathbb{R},X)$ to Eq. (2.4).

Proof. From the proof procedure of theorem 3.1, it is easy to deduce that $Q:D\to D$ is a continuous and increasing. Now, define two sequences $\{\eta_n\}$ and $\{\xi_n\}$ and are given the form

$$\eta_n=Q\eta_{n-1},\ \xi_n=Q\xi_{n-1},\ n=1,2,\ldots. \tag{3.10}$$

By the monotonicity of the Q, we get that

$$\eta_0\leqslant\eta_1\leqslant\eta_2\leqslant\cdots\leqslant\eta_n\leqslant\cdots\leqslant\xi_n\leqslant\cdots\leqslant\xi_2\leqslant\xi_1\leqslant\xi_0. \tag{3.11}$$

Next, we verify that $\{\eta_n\}$ and $\{\xi_n\}$ are convergent in $C_\omega(\mathbb{R},X)$. We only prove that $\{\eta_n\}$ is convergent in $C_\omega(\mathbb{R},X)$. The convergence of the sequence $\{\xi_n\}$ can be similarly proven. By virtue of operator Q defined, it follows from (3.6), (3.7) and (3.10) that

$$\begin{aligned}
\eta_n(t)&=(Q\eta_{n-1})(t)\\
&=T_N(t)B_1(\eta_{n-1})+\int_0^t T_N(t-s)(f(s,\ \eta_{n-1}(s),\ \eta_{n-1}(s-\tau))+N\eta_{n-1}(s))ds\\
&=T_N(t)B_1(\eta_{n-1})+(W\eta_{n-1})(t),\ t\in\mathbb{R}.
\end{aligned}$$

By the proof process of theorem 3.1, we clear that $\{T_N(t)B_1(\eta_{n-1}):\eta_{n-1}\in D\}$ and $\{(W\eta_{n-1})(t):\eta_{n-1}\in D\}$ are relatively compact in X. Thus, $\{(Q\eta_{n-1})(t):\eta_{n-1}\in D\}$ is relatively compact in X. Furthermore, $\{\eta_n(t)\}$ is relatively compact in X, so there is convergent subsequence in $\{\eta_n(t)\}$. Combining the monotonicity (3.11) and the normality of the K, it is easy to show that $\{\eta_n(t)\}$ themselves is convergent in X. By assumptions (H4-H5), (3.10) and (3.11), we have that

$$\begin{aligned} 0 &\leqslant \xi_n(t) - \eta_n(t) \\ &= Q\xi_{n-1}(t) - Q\eta_{n-1}(t) \\ &= P_N(f(t, \xi_{n-1}(t), \xi_{n-1}(t-\tau)) + N\xi_{n-1}(t) \\ &\quad - (f(t, \eta_{n-1}(t), \eta_{n-1}(t-\tau)) + N\eta_{n-1}(t))) \\ &\leqslant (L_2 + \frac{L_3}{L_1} + N)P_N(\xi_{n-1}(t) - \eta_{n-1}(t)) \\ &\leqslant \cdots \\ &\leqslant (L_2 + \frac{L_3}{L_1} + N)^n P_N^n(\xi_0(t) - \eta_0(t)). \end{aligned}$$

By the normality of the K_C, we easily obtain that

$$\| \xi_n - \eta_n \|_C \leqslant N_0(L_2 + \frac{L_3}{L_1} + N)^n \| P_N^n \| \cdot \| \xi_0 - \eta_0 \|_C . \tag{3.12}$$

Since $0 < L_2 + \frac{L_3}{L_1} + N < \lambda_1 + N$, there exists a constant $\varepsilon > 0$ such that $L_2 + \frac{L_3}{L_1} + N + \varepsilon < \lambda_1 + N$. According to the spectral radius of Gelfand formula $\lim_{n\to\infty} \sqrt[n]{\| P_N^n \|} = r(P_N) = \frac{1}{\lambda_1+N} < \frac{1}{L_2+\frac{L_3}{L_1}+N+\varepsilon}$, there exists a positive integer N_1 such that $\| P_N^n \| < \frac{1}{(L_2+\frac{L_3}{L_1}+N+\varepsilon)^n}$ when $n \geqslant N_1$. Therefore, by (3.12), when $n \geqslant N_1$

$$\| \xi_n - \eta_n \|_C \leqslant N_0\Big(\frac{L_2 + \frac{L_3}{L_1} + N}{L_2 + \frac{L_3}{L_1} + N + \varepsilon}\Big)^n \| \xi_0 - \eta_0 \|_C \to 0 \ (n \to \infty). \tag{3.13}$$

Therefore,

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \eta_n = y^*.$$

Namely, Eq. (2.4) has only one ω-periodic mild solution $y^* \in C_\omega(\mathbb{R}, X)$. □

Next, it is the existence and uniqueness conclusions of positive ω-periodic mild solution to Eq. (2.4) that will be given.

Theorem 3.3. *Assume that X is an ordered Banach space, the positive cone $K \subset X$ is a normal regeneration cone and $-A$ generates a positive analytic semigroup $T(t)(t \geqslant 0)$. Supposed that f satisfies (H1) with $f(t,\ \theta,\ \theta) \geqslant \theta,\ t \in \mathbb{R}$ and hypotheses*

(H3)′ for each $R > 0$, there is a constant $N = N(R) > 0$ such that

$$f(t,\ v_2,\ w_2) - f(t,\ v_1,\ w_1) \geqslant -N(v_2 - v_1),\ t \in \mathbb{R},$$

for all $\theta \leqslant v_1 \leqslant v_2,\ \theta \leqslant w_1 \leqslant w_2$ with $\| v_i \|,\ \| w_i \| \leqslant R(i = 1, 2)$;

(H4)′ there are constant $L < \lambda_1$ and $0 < b < \lambda_1 - L$ such that

$$f(t,\ v_2,\ w_2) - f(t,\ v_1,\ w_1) \leqslant L(v_2 - v_1) - b(w_2 - w_1),\ t \in \mathbb{R},$$

for all for $\theta \leqslant v_1 \leqslant v_2,\ \theta \leqslant w_1 \leqslant w_2$;

then there exists only one positive ω-periodic mild solution $y^* \in C_\omega(\mathbb{R}, X)$ to Eq. (2.4).

Proof. Set $h_0(t) = f(t,\ \theta,\ \theta)$, we get that $h_0 \in C_\omega(\mathbb{R}, K)$. Consider the linear delayed evolution equation

$$y'(t) + (A - LI)y(t) + by(t-\tau) = h_0(t),\ t \in \mathbb{R}. \tag{3.14}$$

Operator $-(A - LI)$ generates positive C_0-semigroup $T(t)e^{Lt}$ whose growth index $L - \lambda_1 < 0$. So, by Lemma 2.3, (3.14) has just one positive ω-periodic mild solution $\xi_0 \in C_\omega(\mathbb{R}, X)$.

Set $R_0 = N_0 \parallel \xi_0 \parallel_C +1$, $N = N(R_0) > 0$ is the corresponding constant in hypothesis (H3)$'$. We can assume $N > -L$, otherwise, we will replace N with $N+ \mid L \mid$. Obviously, $-(A + NI)$ generates positive C_0-semigroup $T_N(t) = T(t)e^{-Nt}(t \geqslant 0)$ whose growth index $-\lambda_1 - N < 0$ and $b < \lambda_1 + N$. Thus, by Lemma 2.3, the linear evolution equation with delay

$$y'(t) + (A + NI)y(t) + by(t-\tau) = h(t),\ t \in \mathbb{R}, \tag{3.15}$$

has only one ω-periodic mild solution $y = (I - P_N\mathfrak{B})^{-1}P_N h$ and $P_N : C_\omega(\mathbb{R}, X) \to C_\omega(\mathbb{R}, X)$ is positive bounded linear operator satisfying $r(P_N) < \frac{1}{\lambda_1+N}$.

Define operator $F : C_\omega(\mathbb{R}, X) \to C_\omega(\mathbb{R}, X)$ given by $F(y)(t) = f(t,\ y(t),\ y(t-\tau)) + Ny(t) + by(t-\tau)$, it is easy to see that F is continuous and $F(\theta) = f(t,\ \theta,\ \theta) \geqslant \theta$. Denote $D = [\theta, \xi_0]$ is ordered interval in $C_\omega(\mathbb{R}, X)$, by (H3)$'$, F is an increasing operator restricted to the interval D. Let $\eta_0 \equiv \theta$, we define two iterative sequences

$$\eta_n = S_N F(\eta_{n-1}),\ \xi_n = S_N F(\xi_{n-1}),\ n = 1, 2, \cdots, \tag{3.16}$$

where $S_N = (I - P_N\mathfrak{B})^{-1}P_N$. The $\xi_0(t)$ is ω-periodic mild solution of (3.14), namely,

$$\xi_0'(t) + (A - LI)\xi_0(t) + b\xi_0(t-\tau) = h_0(t),\ t \in \mathbb{R}.$$

After adding $L\xi_0(t) + N\xi_0(t)$ to both sides of the above equation, we can easily know that $\xi_0(t)$ is also periodic mild solution of Eq. (3.15) for $h(t) = h_0(t) + L\xi_0(t) + N\xi_0(t)$. It follows from the definition of P_N that

$$\xi_0 = (I - P_N\mathfrak{B})^{-1}P_N(h_0 + L\xi_0 + N\xi_0). \tag{3.17}$$

In (H4)$'$, let $v_1,\ w_1 = \theta,\ v_2 = \xi_0(t),\ w_2 = \xi_0(t-\tau)$, we obtain that

$$f(t,\ \xi_0(t),\ \xi_0(t-\tau)) \leqslant h_0(t) + L\xi_0(t) - b\xi_0(t-\tau),\ t \in \mathbb{R}.$$

Add $N\xi_0(t) + b\xi_0(t-\tau)$ to both sides of the above inequality, we have

$$f(t,\ \xi_0(t),\ \xi_0(t-\tau)) + N\xi_0(t) + b\xi_0(t-\tau) \leqslant h_0(t) + L\xi_0(t) + N\xi_0(t),\ t \in \mathbb{R}.$$

By the increment of F, it easily is got that

$$\theta \leqslant F(\theta)(t) \leqslant F(\xi_0)(t) \leqslant h_0(t) + L\xi_0(t) + N\xi_0(t).$$

Apply S_N to the above inequality, by the positivity of S_N and (3.17), we deduce that

$$\theta \leqslant S_N \circ F(\theta) \leqslant S_N \circ F(\xi_0) \leqslant S_N(h_0 + L\xi_0 + N\xi_0).$$

Namely,

$$\theta = \eta_0 \leqslant \eta_1 \leqslant \xi_1 \leqslant \xi_0.$$

Since $S_N \circ F$ is increasing operator in D, by applying repeatedly $S_N \circ F$ to the above formula, we obtain that

$$\theta \leqslant \eta_1 \leqslant \eta_2 \leqslant \cdots \leqslant \eta_n \leqslant \cdots \leqslant \xi_n \leqslant \cdots \leqslant \xi_2 \leqslant \xi_1 \leqslant \xi_0. \tag{3.18}$$

Thus,

$$\begin{aligned}\theta \leqslant & \xi_n - \eta_n \\ = & S_N(F(\xi_{n-1}) - F(\eta_{n-1})) \\ \leqslant & S_N((L+N)(\xi_{n-1} - \eta_{n-1})) \\ = & (L+N)S_N(\xi_{n-1} - \eta_{n-1}).\end{aligned}$$

Repeat the above inequality, we have

$$\theta \leqslant \xi_n - \eta_n \leqslant (L+N)^n S_N^n(\xi_0).$$

By the regularity of K_C, we gain that

$$\| \xi_n - \eta_n \|_C \leqslant N_0(L+N)^n \| S_N^n \| \| \xi_0 \|_C . \tag{3.19}$$

Since $0 < L + N < \lambda_1 + N - b$, there exists a constant $\varepsilon > 0$ such that $L + N + \varepsilon < \lambda_1 + N - b$. According to the spectral radius of Gelfand formula

$$\lim_{n\to\infty} \sqrt[n]{\| S_N^n \|} = r(S_N) \leqslant \| (I - P_N\mathfrak{B})^{-1} \| \cdot \| P_N \| < \frac{1}{\lambda_1 + N - b},$$

there is a integer $N_1 > 0$ such that $\| S_N^n \| < \frac{1}{(L+N+\varepsilon)^n}$ when $n \geqslant N_1$. Therefore, by (3.19), when $n \geqslant N_1$

$$\| \xi_n - \eta_n \|_C \leqslant N_0 \| \xi_0 \|_C \cdot \Big(\frac{L+N}{L+N+\varepsilon}\Big)^n \to 0 \ (n \to \infty). \tag{3.20}$$

Similar to theorem of nested interval, it follows from (3.18) and (3.20) that there is a unique $y^* \in \bigcap_{n=1}^{\infty} [\eta_n, \xi_n]$ such that

$$\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \xi_n = u^*.$$

So, $y^* = S_N \circ F(y^*)$, as $n \to \infty$ in (3.16). It follows from the definitions of S_N and F that there is one unique ω-periodic mild solution $y^* \in D$ to Eq. (2.4).

Next, we prove that Eq. (2.4) has just one positive ω-periodic mild solution in $C_\omega(\mathbb{R}, X)$. Let y_1, y_2 both be positive ω-periodic mild solutions of Eq. (2.4). Set $R_1 = N_0 \max\{\| y_1 \|_C, \| y_2 \|_C\} + 1$ and let a sufficiently large $N = N(R_1)$ be the corresponding constant in hypothesis (H3)$'$. Let S_N and F be the operators defined in the above proof, then the $S_N \circ F$ is increasing sequentially in the ordered interval $[\theta, y_i](i = 1, 2)$. Repeat the previous argument, the initial entry ξ_0 is replaced by u_i in the iterative equation (3.15). Further, for $S_N \circ F(y_i) = y_i$, so $\xi_n = y_i$. From (3.20), we get that $\| y_i - \eta_n \|_C \to 0 (n \to \infty)$. Therefore,

$$\lim_{n\to\infty} \eta_n = y_1 = y_2.$$

Namely, there is only one positive ω-periodic mild solution $y^* \in C_\omega(\mathbb{R}, X)$ to Eq. (2.4). □

4. Main results

In this section, we use the above abstract results to study the existence and uniqueness of time ω-periodic solution to the parabolic delayed evolution equation (1.2). Now, we give the following definition.

Definition 4.1. If a function $\eta \in C^{2,1}(\Omega \times \mathbb{R})$ which is ω periodic in t, and satisfies

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + A(x, D)\eta \leqslant g(x, t, \eta(x, t), \eta(x, t-\tau)), \ x \in \Omega, \ t \in \mathbb{R}, \\ \eta|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

then η is said to be an lower ω-periodic solution of Eq. (1.2). If the inequality of Eq. (1.2) is inverted, then it is said to be an upper ω-periodic solution of Eq. (1.2).

Theorem 4.1. *Let $g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be a local Hölder-continuous function whose periodic is ω in t. Suppose that g satisfies hypotheses:*

(F1) there exist constant $0 < a < \lambda_1$ and function $b_0 \in C^1_\omega(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ with $b_0(x, \ t) \geqslant 0$ such that

$$g(x, \ t, \ v, \ w) \leqslant av + b_0(x, \ t), \ g(x, \ t, \ -v, \ -w) \geqslant -av - b_0(x, \ t), \ x \in \overline{\Omega}, t \in \mathbb{R},$$

for almost all $v, \ w \geqslant 0$ with $v, \ w \in \mathbb{R}$;

(F2) there is a constant $N > 0$ such that

$$g(x, \ t, \ v_2, \ w_2) - g(x, \ t, \ v_1, \ w_1) \geqslant -N(v_2 - v_1), \ x \in \overline{\Omega}, t \in \mathbb{R},$$

for all $v_1 \leqslant v_2, \ w_1 \leqslant w_2$ with $v_i, \ w_i \in \mathbb{R}(i = 1, 2)$;

then, Eq. (1.2) has least one time ω-periodic solution $\widetilde{u} \in C^{2,1}_\omega(\overline{\Omega} \times \mathbb{R})$.

Proof. From the previous discussion, the time ω-periodic problem to Eq. (1.2) can be replaced by the ω-periodic problem to Eq. (2.4). By (F1), Eq. (1.2) has lower ω-periodic solution $\eta_0 \in C^{2,1}_\omega(\Omega \times \mathbb{R})$ and upper ω-periodic solution $\xi_0 \in C^{2,1}_\omega(\Omega \times \mathbb{R})$. Thus, it is Eq. (2.4) that also has lower and upper ω-periodic solutions. By means of the hypothesis (F2), we is easy to get that the hypothesis (H3) hold. Hence, it follows from Theorem 3.1 that Eq. (2.4) has ω-periodic mild solutions $\widetilde{u} \in C_\omega(\mathbb{R}, X)$. Since the semigroup $T(t)(t \geqslant 0)$ generated by $-A$ is exponentially stable and analytic, obviously, we can be easy to prove that the semigroup $T_N(t) = T(t)e^{-Nt}, \ t \geqslant 0$ generated by $-(A + NI)$ is exponentially stable and analytic. Then, by means of the analyticity to $T_N(t)(t \geqslant 0)$ and the regularization method applied in [1], we can prove that $\widetilde{y} \in C^{2,1}_\omega(\overline{\Omega} \times \mathbb{R})$ is time ω-periodic solutions of Eq. (1.2). □

Theorem 4.2. *Let all the conditions of the theorem 4.1 hold, and g satisfy hypotheses*

(F3) there is a sufficiently small constant $L_1 > 0$ such that

$$v_2 - v_1 \geqslant L_1(w_2 - w_1), \ x \in \overline{\Omega}, \ t \in \mathbb{R},$$

for all $v_1 \leqslant v_2, \ w_1 \leqslant w_2$ with $v_i \ w_i \in \mathbb{R}(i = 1, 2)$;

(F4) there are constants $L_2, \ L_3 > 0$ satisfying $L_2 + \frac{L_3}{L_1} < \lambda_1$ such that

$$g(x, \ t, \ v_2, \ w_2) - g(x, \ t, \ v_1, \ w_1) \leqslant L_2(v_2 - v_1) + L_3(w_2 - w_1), \ x \in \overline{\Omega}, t \in \mathbb{R},$$

for all $v_1 \leqslant v_2,\ w_1 \leqslant w_2$ *with* $v_i\ w_i \in \mathbb{R}(i=1,2)$;

then, Eq. (1.2) has just one time ω*-periodic solution* $y^* \in C_\omega^{2,1}(\overline{\Omega}\times\mathbb{R})$.

Proof. Theorem 4.1 yields the existence of ω-periodic solutions $\widetilde{y} \in C_\omega^{2,1}(\overline{\Omega}\times\mathbb{R})$ to (1.2). Since it follows from (F3)-(F4) that the hypotheses (H4)-(H5) hold. So, by Theorem 3.2, Eq. (2.4) has only one ω-periodic mild solution $y^* \in C_\omega(\mathbb{R}, X)$, which means $y^* \in C_\omega^{2,1}(\overline{\Omega}\times\mathbb{R})$ is a unique time ω-periodic solution of Eq. (1.2). □

Theorem 4.3. *Let* $g : \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ *be a local Hölder-continuous function whose periodic is* ω *in* t *and* $g(x,\ t,\ 0,\ 0) \geqslant 0,\ (x,\ t,\ v,\ w) \in (\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$. *Suppose that* g *satisfies hypotheses:*

(F2)′ for each $R > 0$, *there is a constant* $N = N(R) > 0$ *such that*

$$g(x,\ t,\ v_2,\ w_2) - g(x,\ t,\ v_1,\ w_1) \geqslant -N(v_2 - v_1),\ x \in \overline{\Omega}, t \in \mathbb{R},$$

for all $0 \leqslant v_1 \leqslant v_2,\ 0 \leqslant w_1 \leqslant w_2$ *with* $|\ v_i\ |,\ |\ w_i\ | \leqslant R(i=1,2)$;

(F4)′ there exist constant $L < \lambda_1$ *and* $0 < b < \lambda_1 - L$ *such that*

$$g(x,\ t,\ v_2,\ w_2) - g(x,\ t,\ v_1,\ w_1) \leqslant L(v_2 - v_1) - b(w_2 - w_1),\ x \in \overline{\Omega},\ t \in \mathbb{R},$$

for all for $0 \leqslant v_1 \leqslant v_2,\ 0 \leqslant w_1 \leqslant w_2$;

then there exists only one positive time ω*-periodic solution* $y^* \in C_\omega^{2,1}(\overline{\Omega}\times\mathbb{R})$ *to Eq. (1.2).*

Proof. We clearly know that the time ω-periodic problem of delayed parabolic evolution Eq. (1.2) can be replaced by the ω-periodic problem of abstract evolution Eq. (2.4). By means of the hypotheses (F2)′ and (F4)′, we easily show that the hypotheses (H3)′ and (H4)′ hold. Hence, it follows from Theorem 3.3 that Eq. (2.4) has just one positive ω-periodic mild solutions y^* which can be got by the iterative sequences defined by (3.17) in $[\theta, \xi_0]$. Since the semigroup $T(t)(t \geqslant 0)$ generated by $-A$ is exponentially stable and analytic. Obviously, we can be easy to prove that $-(A+NI)$ can generate an exponentially stable analytic semigroup $T_N(t)(t \geqslant 0)$ of the form $T_N(t) = T(t)e^{-Nt},\ t \geqslant 0$. Then, by the analyticity to $T_N(t)(t \geqslant 0)$ and the regularization method applied in [1], we get that $y^* \in C_\omega^{2,1}(\overline{\Omega}\times\mathbb{R})$ is a unique positive time ω-periodic solutions of Eq. (1.2). □

5. Example

Example 5.1. Let us consider the parabolic delayed evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t}y(x,t) - \dfrac{\partial^2}{\partial x^2}y(x,t) = \dfrac{a\sin^2 t}{3e}y(x,t) + \dfrac{b\cos^2 t}{6e}y(x,t-\pi) + x\sin^2 t, \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,\ t) \in (0,\pi)\times\mathbb{R}, \\ y(0,t) = y(\pi,t) = 0,\ t \in \mathbb{R} \end{cases} \tag{5.1}$$

where constants $a, b \in (0,1)$, nonlinear term $g(x,t,y(x,t),y(x,t-\pi)) = \frac{a\sin^2 t}{3e}y(x,t) + \frac{b\cos^2 t}{6e}y(x, t-\pi) + x\sin^2 t$ is Hölder-continuous function whose periodic is 2π in t. Assume that the work space is $X = L^2(0,\pi)$ with the L^2-norm $\|\cdot\|$. Let $P = \{y \in X | y(x) \geqslant 0, x \in (0,\pi)\}$. Obviously,

X is an ordered Banach space and positive cone P also is a normal regeneration cone. Denote linear operator $A : \mathcal{D}(A) \subset X \to X$ by given

$$\mathcal{D}(A) := H^2(0,\pi) \cap H_0^1(0,\pi), \quad Ay = -\frac{\partial^2 y}{\partial x^2}.$$

By [1, 26], it can be known that the semigroup $T(t)(t \geqslant 0)$ generated by $-A$ is exponentially stable and analytic, and A has a discrete spectrum with eigenvalues of the form n^2, $n \in \mathbb{N}$. It follows from the maximum principle of the parabolic type that $T(t)(t \geqslant 0)$ is positive, In the meantime, operator A has compact resolvent in $L^2(0,\pi)$, which means that $T(t)(t \geqslant 0)$ is compact. Thus, $\| T(t) \| \leqslant e^{-\lambda_1 t} = e^{-t}$, $t \geqslant 0$, then $M_0 = 1$, $\nu_0 = -\lambda_1 = -1$.

We easily verify that g satisfies (F1) and (F2), according to Theorem 4.1, Eq. (5.1) has time 2π-periodic solution $y \in C_{2\pi}^{2,1}([0,\pi] \times \mathbb{R})$.

Example 5.2. We consider the parabolic delayed evolution equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = \dfrac{a \sin^2 t}{3e} y(x,t) + \dfrac{b \cos^2 t}{5e} y(x,t-\pi) + \mid x \mid \sin^2 t, \ \text{in } \Omega \times \mathbb{R}, \\ y|_{\partial\Omega} = 0, \end{cases} \tag{5.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $\Delta y = \frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_2^2} + \cdots + \frac{\partial^2 y}{\partial x_N^2}$ is Laplace operator, constants $a, b \in (0,1)$, nonlinear term $g(x,t,y(x,t),y(x,t-\pi)) = \frac{a \sin^2 t}{3e} y(x,t) + \frac{b \cos^2 t}{5e} y(x,t-\pi) + \mid x \mid \sin^2 t$ is Hölder-continuous function which is 2π-periodic in t. Obviously, $g(x,\ t,\ 0,\ 0) \geqslant 0$. We choose the work space $X = L^2(\Omega)$ equipped with the L^2-norm $\| \cdot \|$. Let $P = \{y \in X | y(x) \geqslant 0, x \in \Omega\}$, so X is an ordered Banach space and positive cone P is a normal regeneration cone. Define linear operator $A : \mathcal{D}(A) \subset X \to X$ by given

$$\mathcal{D}(A) := H^2(\Omega) \cap H_0^1(\Omega), \quad Ay = -\Delta y.$$

By [1, 26], we easily see that the semigroup $T(t)(t \geqslant 0)$ generated by $-A$ is exponentially stable and analytic, and A has a discrete spectrum with eigenvalues of the form n^2, $n \in \mathbb{N}$. It follows from the maximum principle of the parabolic type that $T(t)(t \geqslant 0)$ is positive, In the meantime, operator A has compact resolvent in $L^2(\Omega)$, which means that $T(t)(t \geqslant 0)$ is compact. Thus, the growth exponent of the semigroup $T(t)(t \geqslant 0)$ satisfies $\nu_0 = -\lambda_1 = -1$. Namely, $\| T(t) \| \leqslant e^{-\lambda_1 t} = e^{-t}$, $t \geqslant 0$.

We easily verify that g satisfies (F2)$'$ and (F4)$'$. By Theorem 4.3, Eq. (5.1) has unique positive time 2π-periodic solution $y \in C_{2\pi}^{2,1}(\overline{\Omega} \times \mathbb{R})$.

6. Conclusion

Parabolic partial differential equations have been rapidly developed due to their applications in physics, population system, chemistry, medicine and other fields. In particular, the periodic and positive periodic problem to the parabolic evolution equations with finite delay is motivated by numerous applications, including spacecraft dynamics, natural cycle phenomena, and biological predation models. In the article, we remove the assumption that problem studied has an ordered set of upper and lower solutions. By applying theory of operator semigroup, the Schauder fixed point theorem, and monotone iterative method of upper and lower solutions, the existence and uniqueness results of ω-periodic mild solutions and positive mild solutions to (2.4) are obtained when the nonlinear term f satisfies some ordered conditions that are easy to verify. Furthermore, we extend the abstract results for (2.4) to time periodic problem for (1.2), which improves and generalizes the recent results of this issue. Thus, the paper is valuable. However, the approach to (1.2) is rather single, so we need to find more ways to research periodic problems in depth. Due to needs for more practice application, we also consider asymptotic behavior of periodic solution for neutral delayed evolution equations.

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