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STATIONARY DISTRIBUTION OF A STOCHASTIC ENTERPRISE CLUSTER MODEL WITH HIGH-ORDER PERTURBATION

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Abstract The survival and development of enterprise clusters are closely related to the sustainable development of the entire economy, so research on them has attracted much attention. In this article, we study the dynamic properties of a stochastic enterprise cluster model with high-order perturbation. First, we study stochastically ultimate boundedness of the system. Then, some sufficient condition for the existence of the stationary distribution in the system are obtained. We also discussed the destructiveness and permanence of the system. Finally, three numerical examples are applied to verify the obtained results.

Keywords Cluster model, stationary distribution, stochastic, dynamics.

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1. Introduction

Enterprise cluster refers to the concentration of similar enterprises in a fixed area, forming a large economic output and having a significant economic impact on the regional economy [4]. Many scholars follow with interest its complex behavior including administering process, running mechanism, risk and efficiency, evolvement cycle and so on. The classic competition and cooperation system of two enterprises can be expressed by

$$\begin{aligned} \frac{dv_1(t)}{dt} &= a_1 v_1(t)\left(1 - \frac{v_1(t)}{\alpha} - \frac{\beta_1(v_2(t)-c_2)^2}{\alpha}\right), \\ \frac{dv_2(t)}{dt} &= a_2 v_2(t)\left(1 - \frac{v_2(t)}{\alpha} + \frac{\beta_2(v_1(t)-c_1)^2}{\alpha}\right), \end{aligned} \tag{1.1}$$

where v_1 and v_2 denote the output of enterprises V_1 and V_2, respectively. a_1 and a_2 represent the intrinsic growth rate, α is the carrying capacity of market, β_1 and β_2 denote the competitive coefficients, c_1 and c_2 denote the initial production of enterprises V_1 and V_2, respectively.

In recent years, there has been growing interest among scholars in studying enterprise clusters using concepts and principles from ecological theory and dynamic system theory. Xu and Saho [15] studied a enterprise cluster model with impulsive terms by using Lyapunov functional method and coincidence degree theory, and obtained existence and global attractivity of periodic solution. A enterprises cluster model with feedback controls on time scale was investigated based on the comparison theorems on time scales, some results belonging to asymptotically stable almost periodic solution were obtained [16]. For enterprise cluster models with

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delays, see [3,7,11,12]. For the dynamical behavior of fractional-order enterprise cluster models, see [13,14].

Enterprise clusters are located in complex social and economic environments, and are inevitably subject to various random factors. Therefore, introducing a random disturbance term in system (1.1) can more accurately characterize the dynamic mechanism of system (1.1). We introduce random terms into system (1.1) using different methods. Replace the parameters a_1 and a_2 by

$$a_1 = a_1 + (\omega_{11} + \omega_{12}v_1(t))\dot{B}_1(t),\ \ a_2 = a_2 + (\omega_{21} + \omega_{22}v_2(t))\dot{B}_2(t),$$

where $\dot{B}_i(t)$ is the white noise, $\omega_{ij} > 0$ denotes the intensity of the white noise, $i, j = 1, 2$. The stochastic version corresponding to system (1.1) takes the following form

$$\begin{aligned} dv_1(t) &= a_1 v_1(t)\left(1 - \frac{v_1(t)}{\alpha} - \frac{\beta_1(v_2(t) - c_2)^2}{\alpha}\right)dt + v_1(t)(\omega_{11} + \omega_{12}v_1(t))dB_1(t), \\ dv_2(t) &= a_2 v_2(t)\left(1 - \frac{v_2(t)}{\alpha} + \frac{\beta_2(v_1(t) - c_1)^2}{\alpha}\right)dt + v_2(t)(\omega_{21} + \omega_{22}v_2(t))dB_2(t). \end{aligned} \tag{1.2}$$

The random perturbations in system (1.2) depend on square of v_1 and v_2, which are high-order perturbations. The recent research about the stochastic systems with high-order perturbations, see [5,6,8,10].

To the authors' knowledge, there are no results for the enterprise cluster model with high-order perturbations. Using random analysis theory and Lyapunove functional method, we obtained stationary distribution and some dynamic properties of system (1.2). The main innovations of this paper are given as follows:

(1) There are no results for a enterprise cluster model with high-order perturbations. The research model presented in this article can accurately depict the survival and development status of enterprise clusters, thus the results of this article have important practical significance for the development of enterprise clusters.

(2) The existence of an ergodic stationary distribution can help us formulate reasonable policies and promote the healthy development of enterprises.

(3) The research methodology presented in this article can assist us in studying other types of stochastic systems with high-order perturbations.

The output of a company is limited, in the whole paper we have the following assumption for system (1.2): $v_1 \vee v_2 \leq v^*$, where v^* is a positive constant.

Let $\mathbb{R}^2_+ = \{(v_1, v_2) \in \mathbb{R}^2 : v_1, v_2 > 0\}$ and $(\Theta, F, \{F_t\}_{t\geq 0}, P)$ be a complete probability set with a $\sigma-$field filtration $\{F_t\}_{t\geq 0}$ satisfying the standard conditions. For a function $\chi(\tau)$ on $[0, \infty)$, denote $\chi^+ = \sup_{\tau\geq 0} \chi(\tau)$, $\chi^- = \inf_{\tau\geq 0} \chi(\tau)$.

From the standard proof in [9], we have the following lemma:

Lemma 1.1. *For any initial $(v_1(0), v_2(0)) \in \mathbb{R}^2_+$, system (1.2) has unique positive solution $(v_1(t), v_2(t))$ on $t \geq 0$, and the solution will stay on $\mathbb{R}^2_+$ with probability 1.*

The remainder of the paper is organized as follows. In Section 2, we study stochastically ultimate boundedness of system (1.2). In Section 3, Ergodicity and stationary distribution of system (1.2) are obtained. In Section 4, we discuss the cases of enterprise cluster extinction. In Section 5, we obtained some sufficient condition for guaranteeing permanence in the mean. Three numerical simulations supporting our theoretical results are give in Section 6. Section 7 concludes this paper.

2. Stochastically ultimate boundedness

Since the property of ultimate boundedness indicates the long-term existence of enterprise clusters, we will obtain the stochastically ultimate boundedness of system (1.2). We first give the following definition:

Definition 2.1. System (1.2) is said to be stochastically ultimately bounded if for any $\varrho \in (0,1)$ and initial data $(v_1(0), v_2(0)) \in \mathbb{R}^2_+$, there is a positive constant $M(\varrho)$ such that the solution $v(t) = (v_1(t), v_2(t))$ of system (1.2) satisfies the following property:

$$\lim_{t\to\infty} \sup P(|v(t)| \leq M) \geq 1 - \varrho.$$

Theorem 2.1. *For any given initial value $(v_1(0), v_2(0)) \in \mathbb{R}^2_+$. Then, system (1.2) is stochastically ultimately bounded.*

Proof. Let

$$V(v_1, v_2) = v_1^\theta + v_2^\theta,\ v_1, v_2 \in \mathbb{R}_+,\ \theta \in (0,1].$$

Applying the Itô formula to system (1.2) leads to

$$dV(v_1, v_2) = \mathcal{L}V(v_1, v_2)dt + \theta v_1^\theta(\omega_{11} + \omega_{12}v_1(t))dB_1(t) + \theta v_2^\theta(\omega_{21} + \omega_{22}v_1(t))dB_2(t), \tag{2.1}$$

where

$$\begin{aligned}
\mathcal{L}V(v_1, v_2) &= a_1\theta v_1^\theta\left(1 - \frac{v_1(t)}{\alpha} - \frac{\beta_1(v_2(t) - c_2)^2}{\alpha}\right) + 0.5\theta(\theta-1)(\omega_{11} + \omega_{12}v_1(t))^2 v_1^\theta \\
&\quad + a_2\theta v_2^\theta\left(1 - \frac{v_2(t)}{\alpha} + \frac{\beta_2(v_1(t) - c_2)^2}{\alpha}\right) + 0.5\theta(\theta-1)(\omega_{21} + \omega_{22}v_2(t))^2 v_2^\theta \\
&\leq \theta a_1 v_1^\theta - \frac{\theta a_1}{\alpha}v_1^{1+\theta} + \theta a_2 v_2^\theta - \frac{\theta a_2}{\alpha}v_2^{1+\theta} + a_2 v^* \frac{\beta_2(v^* + c_2)^2}{\alpha} \\
&\leq M,
\end{aligned}$$

where M is a positive constant. Substituting this into (2.1) gives

$$dV(v_1, v_2) \leq Mdt + \theta v_1^\theta(\omega_{11} + \omega_{12}v_1(t))dB_1(t) + \theta v_2^\theta(\omega_{21} + \omega_{22}v_1(t))dB_2(t). \tag{2.2}$$

By (2.2) we have

$$d\left[e^t V(v_1, v_2)\right] \leq Me^t dt + e^t\theta v_1^\theta(\omega_{11} + \omega_{12}v_1(t))dB_1(t) + e^t\theta v_2^\theta(\omega_{21} + \omega_{22}v_1(t))dB_2(t).$$

Thus,

$$e^t\mathbb{E}V(v_1, v_2) \leq e^t V(v_1(0), v_2(0)) + Me^t - M$$

which implies that

$$\lim_{t\to\infty} \sup \mathbb{E}V(v_1(t), v_2(t)) \leq M.$$

We also have

$$|v|^2 \leq 2\Big(\max\{v_1, v_2\}\Big)^2$$

and

$$|v|^{\theta} \leq 2^{\frac{\theta}{2}} \bigg(\max\{v_1, v_2\} \bigg)^{\theta} \leq 2^{\frac{\theta}{2}} V(v_1(t), v_2(t)).$$

Thus,

$$\lim_{t\to\infty} \sup \mathbb{E}|v|^{\theta} \leq 2^{\frac{\theta}{2}} M := \widetilde{M}.$$

For any $\varrho > 0$. let $a = \frac{\widetilde{M}}{\varrho}$. By Chebyshev's inequality, we get

$$P(|v| > a) \leq \frac{\mathbb{E}|v|}{a} \leq \frac{\widetilde{M}}{a} = \varrho$$

which results in

$$P(|v| \leq a) \geq 1 - \varrho.$$

The proof is completed. □

3. Ergodic stationary distribution

Let $Y(t)$ be a homogeneous Markov process in $\mathbb{E}^d$, where $\mathbb{E}^d$ is a $d-$dimensional Euclidean space. Consider the following stochastic differential system:

$$dY(t) = m(Y)dt + \sum_{r=1}^{k} h_r(Y)dB_r.$$

The diffusion matrix is defined by

$$G(y) = g_{ij}(y) = \sum_{r=1}^{k} h_r^i(y)h_r^j(y).$$

Lemma 3.1. [1] *The Markov process $Y(t)$ has a unique ergodic stationary distribution $\nu(\cdot)$ if there exists a bounded domain $\Lambda \subset \mathbb{E}^d$ with regular boundary Γ such that*
(i) there is a positive constant N such that

$$\sum_{i,j=1}^{d} g_{ij}(y)\eta_i\eta_j \geq N|\eta|^2, \ y \in \Lambda, \ \eta \in \mathbb{E}^d;$$

(ii) there is a nonnegative C^2-function V such that such that LV is negative for any $\mathbb{E}^d \setminus \Lambda$. Then,

$$P\bigg\{ \lim_{t\to\infty} \int_0^t h(Y(t))dt = \int_{\mathbb{E}^d} h(y)\nu(dy) \bigg\} = 1,$$

where $y \in \mathbb{E}^d$, h is a integrable function with respect to the measure ν.

Define a parameter

$$H_0 = \frac{\alpha}{2\beta_1 c_2^2} + \frac{\alpha}{2\beta_2 c_1^2}.$$

Theorem 3.1. *Assume that $H_0 > 1$. Then, for any initial value $(v_1(0), v_2(0)) \in \mathbb{R}^+$, system (1.2) admits a unique stationary distribution $\nu(\cdot)$ which has ergodic property.*

Proof. From Lemma 1.1, system (1.2) has a unique positive solution $(v_1(t), v_2(t))$ on $t \geq 0$. We first verify the condition (i) in Lemma 3.1. The diffusion matrix of (1.2) is

$$\begin{aligned}\sum_{i,j=1}^{2} g_{ij}(v_1, v_2)\eta_i\eta_j &= ((\omega_{11}v_1 + \omega_{12}v_1^2)\eta_1, (\omega_{21}v_2 + \omega_{22}v_2^2)\eta_2)\begin{pmatrix}(\omega_{11}v_1 + \omega_{12}v_1^2)\eta_1 \\ (\omega_{21}v_2 + \omega_{22}v_2^2)\eta_2\end{pmatrix}\\ &= (\omega_{11}v_1 + \omega_{12}v_1^2)^2\eta_1^2 + (\omega_{21}v_2 + \omega_{22}v_2^2)^2\eta_2^2\\ &\geq N||\eta||^2,\end{aligned}$$

where $N = \min_{(v_1,v_2)\in\bar{\Lambda}_k}\{(\omega_{11}v_1 + \omega_{12}v_1^2)^2\eta_1^2 + (\omega_{21}v_2 + \omega_{22}v_2^2)^2\eta_2^2\}$, $(v_1, v_2) \in \bar{\Lambda}_k \subset \mathbb{R}_+^2, \eta = (\eta_1, \eta_2) \in \mathbb{R}_+^2$, $\bar{\Lambda}_k = [\frac{1}{k}, k] \times [\frac{1}{k}, k]$, $k > 0$.

To prove the condition (ii) in Lemma 3.1, define a C^2-function $V_2(v_1, v_2) : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$\begin{aligned}V_2(v_1, v_2) &= N\left[- k_1 \ln v_1 - k_2 \ln v_2 + \frac{k_1(\omega_{11} + \omega_{12}v_1)^{\gamma_1}}{\gamma_1(1 - \gamma_1)\omega_{11}^{\gamma_1}}\right]\\ &\quad + v_1^{\gamma_2} + v_2^{\gamma_2}\\ &= NV_1(v_1, v_2) + v_1^{\gamma_2} + v_2^{\gamma_2},\end{aligned}$$

where

$$V_1(v_1, v_2) = -k_1 \ln v_1 - k_2 \ln v_2 + \frac{k_1(\omega_{11} + \omega_{12}v_1)^{\gamma_1}}{\gamma_1(1 - \gamma_1)\omega_{11}^{\gamma_1}},$$

k_1 and k_2 are positive constants to be determined later, $\gamma_1, \gamma_2 \in (0, 1)$, $N > 0$ is a sufficiently large number such that

$$f_1^+ + f_2^+ - 2N(R_0 - 1)\lambda + \frac{k_1\omega_{11}^2}{2} + \frac{k_2\omega_{21}^2}{2} \leq -2,$$

where λ is defined by (3.2). It is easy to see that

$$\lim_{k\to+\infty} \inf V_2(v_1, v_2) = +\infty \text{ for } v_1, v_2 \in \mathbb{R}_+^2 \backslash \Lambda_k.$$

Since $V_2(v_1, v_2)$ is a continuous, $V_2(v_1, v_2)$ has a minimum point (v_1^0, v_2^0). Define a C^2-function $V_2(v_1, v_2) : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$V(v_1, v_2) = V_2(v_1, v_2) - V_2(v_1^0, v_2^0).$$

Making use of the Itô's formula to $V_1(v_1, v_2)$, we get

$$\begin{aligned}LV_1 =& - k_1a_1\left(1 - \frac{v_1(t)}{\alpha} - \frac{\beta_1(v_2(t) - c_2)^2}{\alpha}\right) + \frac{k_1}{2}(\omega_{11} + \omega_{12}v_1)^2\\ &+ \frac{k_1(\omega_{11}v_1 + \omega_{12}v_1^2)^{\gamma_1-1}\omega_{12}}{(1 - \gamma_1)\omega_{11}^{\gamma_1}}a_1v_1\left(1 - \frac{v_1(t)}{\alpha} - \frac{\beta_1(v_2(t) - c_2)^2}{\alpha}\right)\\ &+ \frac{k_1(\omega_{11}v_1 + \omega_{12}v_1^2)^{\gamma_1}\omega_{12}^2v_1^2}{2\omega_{11}^{\gamma_1}}\\ &- k_2a_2\left(1 - \frac{v_2(t)}{\alpha} + \frac{\beta_2(v_1(t) - c_1)^2}{\alpha}\right) + \frac{k_2}{2}(\omega_{21} + \omega_{22}v_2)^2\\ \leq& - k_1a_1 + \frac{k_1a_1v_1}{\alpha} + \frac{k_1a_1\beta_1c_2^2}{\alpha} + \frac{k_1a_1\beta_1v_2^2}{\alpha} + \frac{k_1}{2}(\omega_{11} + \omega_{12}v_1)^2\end{aligned}$$

$$\begin{aligned}&+\frac{k_1(\omega_{11}v_1+\omega_{12}v_1^2)^{\gamma_1-1}\omega_{12}}{(1-\gamma_1)\omega_{11}^{\gamma_1}}a_1v_1\\&-k_2a_2+\frac{k_2a_2v_2}{\alpha}+\frac{k_2a_2\beta_2c_1^2}{\alpha}+\frac{k_2a_2\beta_2v_1^2}{\alpha}+\frac{k_2}{2}(\omega_{21}+\omega_{22}v_2)^2.\end{aligned}\tag{3.1}$$

Let

$$\frac{k_1a_1\beta_1c_2^2}{\alpha}=\frac{k_2a_2\beta_2c_1^2}{\alpha}=\lambda.\tag{3.2}$$

Then,

$$k_1=\frac{\lambda\alpha}{a_1\beta_1c_2^2},\ k_2=\frac{\lambda\alpha}{a_2\beta_2c_1^2}.$$

Substituting these expressions into (3.1) leads to

$$\begin{aligned}LV_1\le&-\left(\frac{\alpha}{\beta_1c_2^2}+\frac{\alpha}{\beta_2c_1^2}-2\right)\lambda\\&+\frac{k_1a_1v_1}{\alpha}+\frac{k_1a_1\beta_1v_2^2}{\alpha}+\frac{k_1}{2}(\omega_{11}+\omega_{12}v_1)^2\\&+\frac{k_1(\omega_{11}v_1+\omega_{12}v_1^2)^{\gamma_1-1}\omega_{12}}{(1-\gamma_1)\omega_{11}^{\gamma_1}}a_1v_1\\&+\frac{k_2a_2v_2}{\alpha}+\frac{k_2a_2\beta_2v_1^2}{\alpha}+\frac{k_2}{2}(\omega_{21}+\omega_{22}v_2)^2\\=&-2(H_0-1)\lambda+\frac{k_1a_1v_1}{\alpha}+\frac{k_1a_1\beta_1v_2^2}{\alpha}+\frac{k_1}{2}(\omega_{11}+\omega_{12}v_1)^2\\&+\frac{k_1(\omega_{11}v_1+\omega_{12}v_1^2)^{\gamma_1-1}\omega_{12}}{(1-\gamma_1)\omega_{11}^{\gamma_1}}a_1v_1\\&+\frac{k_2a_2v_2}{\alpha}+\frac{k_2a_2\beta_2v_1^2}{\alpha}+\frac{k_2}{2}(\omega_{21}+\omega_{22}v_2)^2.\end{aligned}\tag{3.3}$$

On the other hand, we obtain

$$\begin{aligned}L\big[v_1^{\gamma_2}+v_2^{\gamma_2}\big]=&\gamma_2v_1^{\gamma_2}a_1\left(1-\frac{v_1(t)}{\alpha}-\frac{\beta_1(v_2(t)-c_2)^2}{\alpha}\right)\\&+\frac{1}{2}\gamma_2(\gamma_2-1)v_1^{\gamma_2}(\omega_{11}+\omega_{12}v_1)^2\\&+\gamma_2v_2^{\gamma_2}a_2\left(1-\frac{v_2(t)}{\alpha}+\frac{\beta_2(v_1(t)-c_1)^2}{\alpha}\right)\\&+\frac{1}{2}\gamma_2(\gamma_2-1)v_2^{\gamma_2}(\omega_{21}+\omega_{22}v_2)^2\\\le&\gamma_2v_1^{\gamma_2}a_1+\gamma_2v_2^{\gamma_2}a_2+\gamma_2v_2^{\gamma_2}a_2\frac{\beta_2(v^*+c_1)^2}{\alpha}.\end{aligned}\tag{3.4}$$

From (3.1), (3.3) and (3.4), we have

$$\begin{aligned}LV\le&N\bigg[-2(H_0-1)\lambda+\frac{k_1a_1v_1}{\alpha}+\frac{k_1a_1\beta_1v_2^2}{\alpha}+\frac{k_1}{2}(\omega_{11}+\omega_{12}v_1)^2\\&+\frac{k_1(\omega_{11}v_1+\omega_{12}v_1^2)^{\gamma_1-1}\omega_{12}}{(1-\gamma_1)\omega_{11}^{\gamma_1}}a_1v_1\end{aligned}$$

$$
\begin{aligned}
&+\frac{k_2a_2v_2}{\alpha}+\frac{k_2a_2\beta_2v_1^2}{\alpha}+\frac{k_2}{2}(\omega_{21}+\omega_{22}v_2)^2\Bigg]+f_1(v_1)+f_2(v_2)\\
:=&F(v_1,v_2),
\end{aligned}
$$

where

$$
f_1(v_1)=\gamma_2v_1^{\gamma_2}a_1,\quad f_2(v_2)=\gamma_2v_2^{\gamma_2}a_2+\gamma_2v_2^{\gamma_2}a_2\frac{\beta_2(v^*+c_1)^2}{\alpha}.
$$

For $v_1, v_2 \to 0^+$, in view of $f_1^+ + f_2^+ - 2N(H_0-1)\lambda + \frac{k_1\omega_{11}^2}{2} + \frac{k_2\omega_{21}^2}{2} \le -2$, we have

$$
F(v_1,v_2)\le -2.
$$

It is easy to see that

$$
F(v_1,v_2)\le F(+\infty,v_2)\to -\infty \text{ as } v_1\to+\infty
$$

and

$$
F(v_1,v_2)\le F(v_1,+\infty)\to -\infty \text{ as } v_2\to+\infty.
$$

Therefore, there exists a sufficiently large k such that

$$
LV(v_1,v_2)\le -1 \text{ for any } (v_1,v_2)\in\mathbb{R}^2\setminus\Lambda_k.
$$

The condition (2) of Lemma 3.1 holds. Based on Lemma 3.1, system (1.2) is ergodic and has a unique stationary distribution. □

4. Extinction

In this section, we consider the the extinction of the the output v_1 (or v_2). Define two parameters

$$
H_1=\frac{2a_1}{w_{11}^2}+\frac{4a_1\beta_1c_2v^*}{\alpha w_{11}^2} \text{ and } \widetilde{H}_1=\frac{2a_2}{\omega_{21}}+\frac{2\beta_2(v^*+c_1)^2}{\alpha\omega_{21}}.
$$

Theorem 4.1. *Assume $H_1<1$. Let $(v_1(t),v_2(t))$ be the positive solution of (1.2) with initial value $(v_1(0),v_2(0))\in\mathbb{R}_+^2$. Then,*

$$
\lim_{t\to\infty} v_1(t)=0 \ \text{ almost surely.}
$$

Furthermore, assume $\widetilde{H}_1<1$, then

$$
\lim_{t\to\infty} v_2(t)=0 \ \text{ almost surely.}
$$

Proof. Applying the Itô's formula to $\ln v_1(t)$ yields

$$
\begin{aligned}
d\ln v_1(t)=&\left[a_1-\frac{a_1v_1(t)}{\alpha}-\frac{a_1\beta_1(v_2(t)-c_2)^2}{\alpha}-\frac{1}{2}(\omega_{11}+\omega_{12}v_1(t))^2\right]dt\\
&+(\omega_{11}+\omega_{12}v_1(t))dB_1(t)\\
\le&\left(a_1+\frac{2a_1\beta_1c_2v^*}{\alpha}-\frac{1}{2}\omega_{11}^2\right)dt+(\omega_{11}+\omega_{12}v_1(t))dB_1(t).
\end{aligned}
\tag{4.1}
$$

Integrating the both sides of (4.1) from 0 to t, we obtain

$$\ln v_1(t) \le \left(a_1 - \frac{1}{2}\omega_{11}^2 + \frac{2a_1\beta_1 c_2 v^*}{\alpha}\right)t + \Xi(t) + \ln v_1(0), \tag{4.2}$$

where

$$\Xi(t) = (\omega_{11} + \omega_{12}v_1(t))dB_1(t). \tag{4.3}$$

$\Xi(t)$ is a continuous martingale with $\Xi(0) = 0$. Since the quadratic variation

$$\langle \Xi(t), \Xi(t)\rangle = \int_0^t (\omega_{11} + \omega_{12}v_1(s))^2 ds \le (\omega_{11} + \omega_{12}v^*)^2 t$$

and

$$\lim_{t\to\infty}\sup \frac{\langle \Xi(t), \Xi(t)\rangle}{t} \le (\omega_{11} + \omega_{12}v^*)^2 < \infty.$$

Thus,

$$\lim_{t\to\infty} \frac{\langle \Xi(t)}{t} = 0 \text{ almost surely.} \tag{4.4}$$

Dividing both sides of (4.2) and taking the limit superior, in view of (4.4) and $H_1 < 1$, we have

$$\begin{aligned}\lim_{t\to\infty}\sup \frac{\ln v_1(t)}{t} &\le a_1 - \frac{1}{2}\omega_{11}^2 + \frac{2a_1\beta_1 c_2 v^*}{\alpha}\\ &= \frac{1}{2}\omega_{11}^2\left(H_1 - 1\right)\\ &< 0,\end{aligned}$$

which means $\lim_{t\to\infty} v_1(t) = 0$ almost surely. Similar to the above proof, we also have

$$\ln v_2(t) \le \left(a_2 + \frac{\beta_2(v^* + c_1)^2}{\alpha} - \frac{\omega_{21}^2}{2}\right)t + \widetilde{\Xi}(t) + \ln v_2(0), \tag{4.5}$$

where

$$\widetilde{\Xi}(t) = (\omega_{21} + \omega_{22}v_2(t))dB_2(t). \tag{4.6}$$

In view of (4.5), (4.6) and $\widetilde{H}_1 < 1$, we have

$$\lim_{t\to\infty}\sup \frac{\ln v_2(t)}{t} \le \frac{\omega_{21}^2}{2}\left(\widetilde{H}_1 - 1\right) < 0,$$

which means $\lim_{t\to\infty} v_2(t) = 0$ almost surely. □

5. Permanence

Define two parameters

$$H_2 = a_1\left[\frac{a_1\beta_1 c_2^2}{\alpha} + \frac{a_1\beta_1(v^*)^2}{\alpha} + \frac{1}{2}(\omega_{11} + \omega_{12}v^*)^2\right]^{-1}$$

and

$$\widetilde{H}_2 = \frac{2a_2}{(\omega_{21} + \omega_{22}v^*)^2}.$$

Theorem 5.1. *Assume $H_2 > 1$ and $\widetilde{H}_2 > 1$. The positive solution $(v_1(t), v_2(t))$ of (1.2) with initial value $(v_1(0), v_2(0)) \in \mathbb{R}^2_+$ is permanent in the mean, i.e., v_1 and v_2 satisfy the following properties:*

$$\lim_{t\to\infty} \inf\langle v_1(t)\rangle \geq \frac{\alpha}{H_2}(H_2 - 1)$$

and

$$\lim_{t\to\infty} \inf\langle v_2(t)\rangle \geq \frac{\alpha}{\widetilde{H}_2}(\widetilde{H}_2 - 1).$$

Proof. By (4.1), we obtain

$$\frac{\ln v_1(t) - \ln v_1(0)}{t} \geq a_1 - \frac{a_1\beta_1 c_2^2}{\alpha} - \frac{a_1\beta_1(v^*)^2}{\alpha} - \frac{1}{2}(\omega_{11} + \omega_{12}v^*)^2 - \frac{a_1}{\alpha}\langle v_1(t)\rangle + \frac{\Xi(t)}{t},$$

where $\Xi(t)$ is defined by (4.3). Thus,

$$\langle v_1(t)\rangle \geq \frac{\alpha}{a_1}\left[a_1 - \frac{a_1\beta_1 c_2^2}{\alpha} - \frac{a_1\beta_1(v^*)^2}{\alpha} - \frac{1}{2}(\omega_{11} + \omega_{12}v^*)^2 - \frac{\ln v_1(t) - \ln v_1(0)}{t} + \frac{\Xi(t)}{t}\right].$$

If $0 < v_1(t) < 1$, we have

$$\langle v_1(t)\rangle\rangle \geq \frac{\alpha}{a_1}\left[a_1 - \frac{a_1\beta_1 c_2^2}{\alpha} - \frac{a_1\beta_1(v^*)^2}{\alpha} - \frac{1}{2}(\omega_{11} + \omega_{12}v^*)^2 + \frac{\ln v_1(0)}{t} + \frac{\Xi(t)}{t}\right]. \tag{5.1}$$

Using $\lim_{t\to\infty} \frac{\ln v_1(0)}{t} = 0$ and $\lim_{t\to\infty} \frac{\Xi(t)}{t} = 0$, and taking the inferior limit of both sides of (5.1), we have

$$\begin{aligned}\lim_{t\to\infty} \inf\langle v_1(t)\rangle &\geq \frac{\alpha}{a_1}\left[a_1 - \frac{a_1\beta_1 c_2^2}{\alpha} - \frac{a_1\beta_1(v^*)^2}{\alpha} - \frac{1}{2}(\omega_{11} + \omega_{12}v^*)^2\right] \\ &= \frac{a_1\beta_1 c_2^2 + a_1\beta_1(v^*)^2 + \frac{\alpha}{2}(\omega_{11} + \omega_{12}v^*)^2}{a_1}(H_2 - 1) \\ &> 0.\end{aligned}$$

If $v_1(t) \geq 1$, we have

$$\langle v_1(t)\rangle \geq \frac{\alpha}{a_1}\left[a_1 - \frac{a_1\beta_1 c_2^2}{\alpha} - \frac{a_1\beta_1(v^*)^2}{\alpha} - \frac{1}{2}(\omega_{11} + \omega_{12}v^*)^2 - \frac{\ln v_1(t)}{t} + \frac{\Xi(t)}{t}\right]. \tag{5.2}$$

Using $\lim_{t\to\infty} \frac{\ln v_1(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\Xi(t)}{t} = 0$, and taking the inferior limit of both sides of (5.2), we have

$$\begin{aligned}\lim_{t\to\infty} \inf\langle v_1(t)\rangle &\geq \frac{\alpha}{a_1}\left[a_1 - \frac{a_1\beta_1 c_2^2}{\alpha} - \frac{a_1\beta_1(v^*)^2}{\alpha} - \frac{1}{2}(\omega_{11} + \omega_{12}v^*)^2\right] \\ &= \frac{a_1\beta_1 c_2^2 + a_1\beta_1(v^*)^2 + \frac{\alpha}{2}(\omega_{11} + \omega_{12}v^*)^2)}{a_1}(H_2 - 1) \\ &> 0.\end{aligned}$$

Similar to the above proof, if $0 < v_2(t) < 1$, we have

$$\langle v_2(t)\rangle \geq \frac{\alpha}{a_2}\left[a_2 - \frac{1}{2}(\omega_{21} + \omega_{22}v^*)^2 + \frac{\ln v_2(0)}{t} + \frac{\widetilde{\Xi}(t)}{t}\right], \tag{5.3}$$

where $\widetilde{\Xi}(t)$ is defined by (4.6). Using $\lim_{t\to\infty}\frac{\ln v_2(0)}{t}=0$ and $\lim_{t\to\infty}\frac{\widetilde{\Xi}(t)}{t}=0$, and taking the inferior limit of both sides of (5.3), we have

$$\begin{aligned}\lim_{t\to\infty}\inf\langle v_2(t)\rangle &\geq \frac{\alpha}{a_2}\left[a_2-\frac{1}{2}(\omega_{21}+\omega_{22}v^*)^2\right]\\ &=\frac{\alpha}{2a_2}(\omega_{21}+\omega_{22}v^*)^2(\widetilde{H}_2-1)\\ &>0.\end{aligned}$$

If $v_2(t)\geq 1$, we have

$$\langle v_2(t)\rangle \geq \frac{\alpha}{a_2}\left[a_2-\frac{1}{2}(\omega_{21}+\omega_{22}v^*)^2-\frac{\ln v_2(t)}{t}+\frac{\widetilde{\Xi}(t)}{t}\right]. \tag{5.4}$$

Using $\lim_{t\to\infty}\frac{\ln v_2(t)}{t}=0$ and $\lim_{t\to\infty}\frac{\widetilde{\Xi}(t)}{t}=0$, and taking the inferior limit of both sides of (5.4), we have

$$\begin{aligned}\lim_{t\to\infty}\inf\langle v_2(t)\rangle &\geq \frac{\alpha}{a_2}\left[a_2--\frac{1}{2}(\omega_{21}+\omega_{22}v^*)^2\right]\\ &=\frac{\alpha}{2a_2}(\omega_{21}+\omega_{22}v^*)^2(\widetilde{H}_2-1)\\ &>0.\end{aligned}$$

The proof is completed. □

6. Numerical examples

In this section, we present three examples which support our results. Based on Milstein's method [2], system (1.2) can be discretized into the following system:

$$\begin{aligned}(v_1)_{k+1}&=(v_1)_k+a_1(v_1)_k\left(1-\frac{(v_1)_k}{\alpha}-\frac{\beta_1((v_2)_k-c_2)^2}{\alpha}\right)\Delta t+(v_1)_k(\omega_{11}+\omega_{12}(v_1)_k)\sqrt{\Delta t}w_1,\\ (v_2)_{k+1}&=(v_2)_k+a_2(v_2)_k\left(1-\frac{(v_2)_k}{\alpha}-\frac{\beta_2((v_1)_k-c_1)^2}{\alpha}\right)\Delta t++(v_2)_k(\omega_{21}+\omega_{22}(v_2)_k)\sqrt{\Delta t}w_2,\end{aligned}$$

where $\Delta t>0$ is the increment, w_1 and w_2 are the Gaussian random variables.

Example 6.1. To illustrate the ergodicity and stationary distribution of system (1.2), choose the following coefficients:

$$\begin{aligned}&a_1=0.25,\ a_2=0.32,\ \alpha=2.5,\ \beta_1=0.3,\ \beta_2=0.2,\ c_1=1,\ c_2=1,\\ &\omega_{11}=0.022,\ \omega_{12}=0.022,\ \omega_{21}=0.035,\ \omega_{22}=0.042.\end{aligned}$$

By computation, we get

$$H_0=\frac{\alpha}{2\beta_1c_2^2}+\frac{\alpha}{2\beta_2c_1^2}\approx 10.41>1.$$

Hence, all conditions of Theorem 3.1 hold and system (1.2) has a unique ergodic stationary distribution. From Figures 1 and 2, we observe that system (1.2) has a unique ergodic stationary distribution.

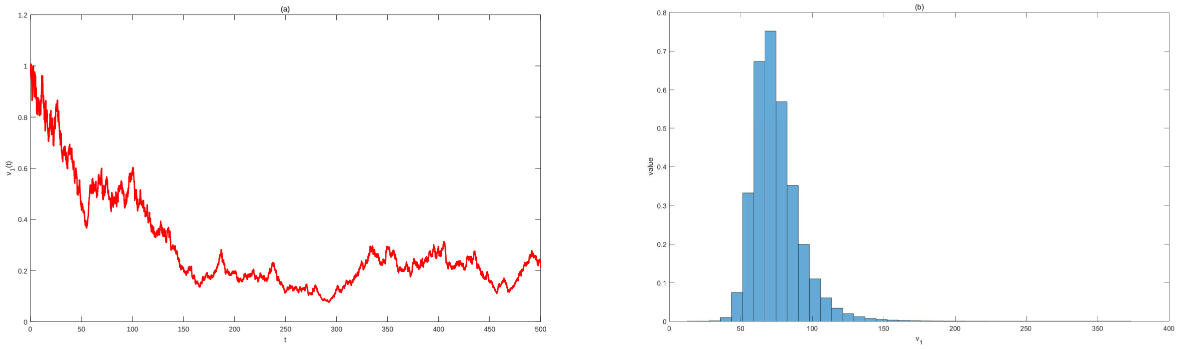

Figure 1. Figure 1(a) is the stochastic path of v_1, Figure 1(b) is the frequency histogram of v_1, using parameters of Example 6.1.

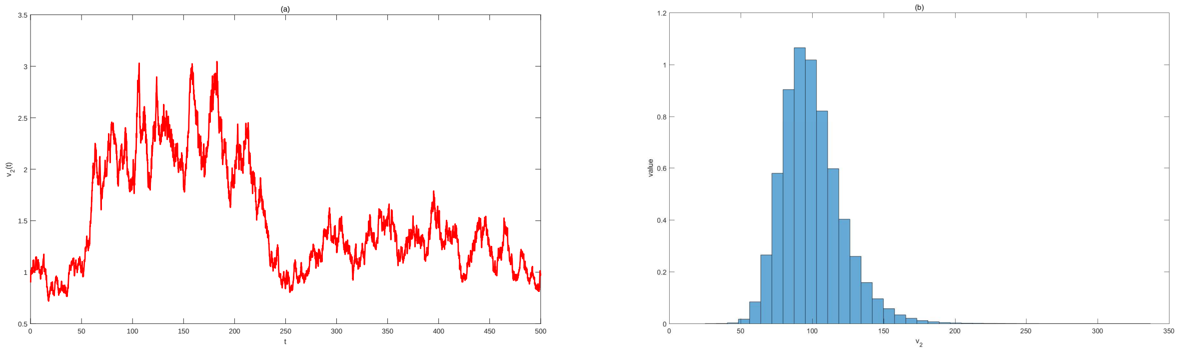

Figure 2. Figure 2(a) is the stochastic path of v_2, Figure 2(b) is the frequency histogram of v_2, using parameters of Example 6.1.

Example 6.2. To illustrate the extinct of enterprise clusters v_1 and v_2 in system (1.2), taking the coefficients as follows:

$$a_1 = 0.62,\ a_2 = 0.42,\ \alpha = 0.5,\ \beta_1 = 1.2,\ \beta_2 = 1.5,\ c_1 = 0.1,\ c_2 = 0.1,$$
$$\omega_{11} = 16.05,\ \omega_{12} = 6.05,\ \omega_{21} = 18.03,\ \omega_{22} = 8.03.$$

Choosing $v^* = 2.5$, we have

$$H_1 = \frac{2a_1}{w_{11}^2} + \frac{4a_1\beta_1 c_2 v^*}{\alpha w_{11}^2} \approx 0.06 < 1$$

and

$$\widetilde{H}_1 = \frac{2a_2}{\omega_{21}} + \frac{2\beta_2(v^* + c_1)^2}{\alpha\omega_{21}} \approx 0.127 < 1.$$

Hence, all conditions of Theorem 4.1 hold and the solution of system (1.2) is extinct.

From Figure 3, we observe that the solution of system (1.2) becomes zero, which means that the enterprise clusters will become extinct.

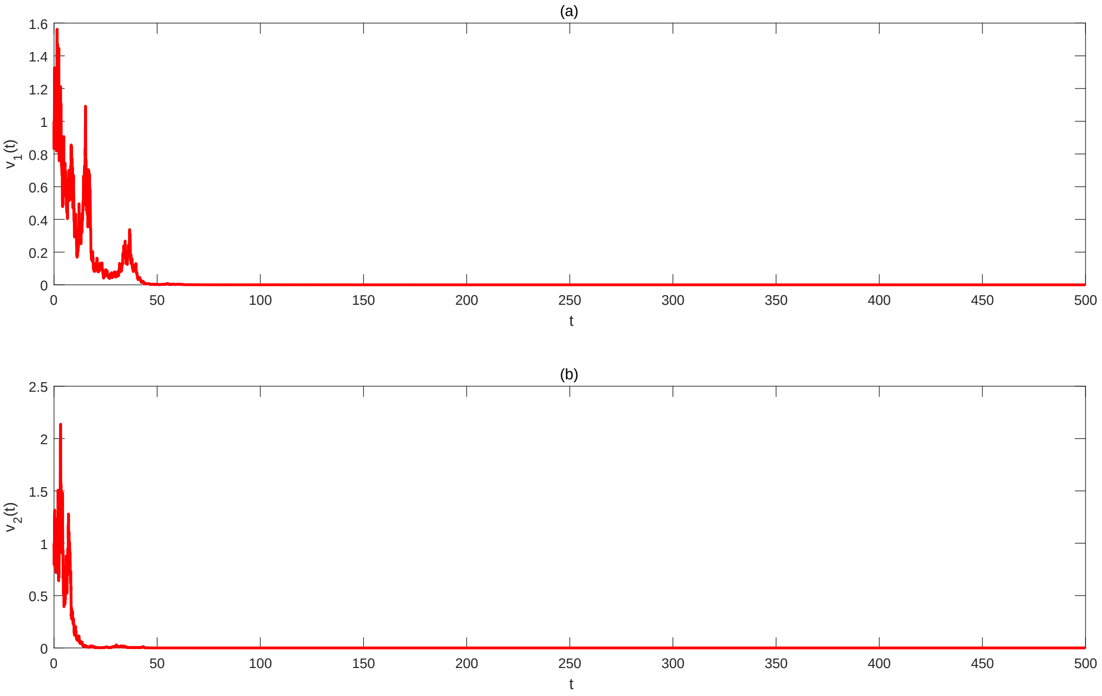

Figure 3. The solution of system (1.2) is extinct by using parameters of Example 6.2.

Example 6.3. To illustrate the results of Theorem 5.1, take the following coefficients:

$$a_1 = 1.2,\ a_2 = 1.4,\ \alpha = 10,\ \beta_1 = 0.1,\ \beta_2 = 1.2,\ c_1 = 1.4,\ c_2 = 1.5,$$
$$\omega_{11} = 0.1,\ \omega_{12} = 0.2,\ \omega_{21} = 0.1,\ \omega_{22} = 0.2.$$

Choosing $v^* = 2$, we have

$$H_2 = a_1\left[\frac{a_1\beta_1 c_2^2}{\alpha} + \frac{a_1\beta_1(v^*)^2}{\alpha} + \frac{1}{2}(\omega_{11} + \omega_{12}v^*)^2\right]^{-1} \approx 14.63 > 1$$

and

$$\widetilde{H}_2 = \frac{2a_2}{(\omega_{21} + \omega_{22}v^*)^2} = 9.6 > 1.$$

Thus,

$$\lim_{t\to\infty} \inf\langle v_1(t)\rangle \geq \frac{a_1\beta_1 c_2^2 + a_1\beta_1(v^*)^2 + \frac{\alpha}{2}(\omega_{11} + \omega_{12}v^*)^2}{a_1}(H_2 - 1) \approx 0.47$$

and

$$\lim_{t\to\infty} \inf\langle v_2(t)\rangle \geq \frac{\alpha}{2a_2}(\omega_{21} + \omega_{22}v^*)^2(\widetilde{H}_2 - 1) \approx 7.67.$$

Hence, all conditions of Theorem 5.1 hold and the solution of system (1.2) is permanent. From Figure 4, we observe that the solution of system (1.2) is permanent.

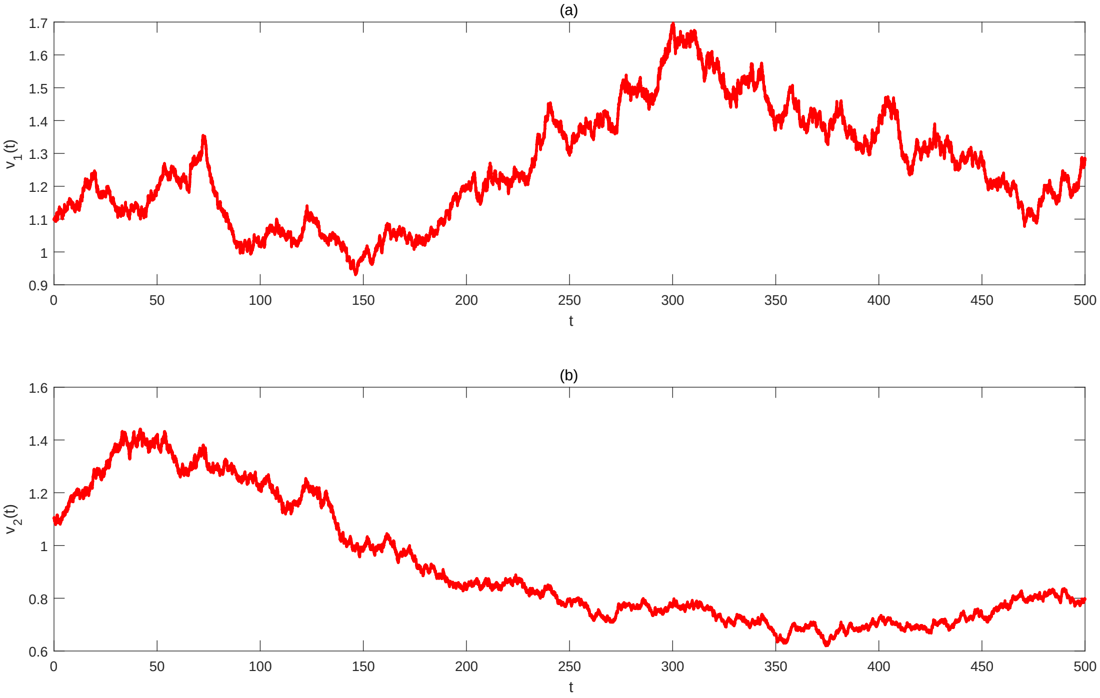

Figure 4. The solution of system (1.2) is permanent by using parameters of Example 6.3.

7. Conclusion

In this paper, we introduce a a stochastic enterprise cluster model with high-order perturbation. On the one hand, from the perspective of the development of enterprise clusters, there exists a relationship of competition and cooperation among enterprises. On the other hand, enterprise clusters in external environment is inevitably affected by stochastic perturbation. Therefore, it is of great significance to study environmental noise in investigating the interaction among enterprise clusters. It is worth noting that the disturbance studied in this article belongs to high-order disturbances, and there is currently not much research work in this area. Systems with high-order disturbances are better able to characterize the survival environment of enterprise clusters. This article considers the long-time dynamical behavior of a stochastic enterprise cluster model with high-order perturbation. We first studied the stochastically ultimate boundedness of the system. Then, we obtained sufficient conditions for the existence of the stationary distribution and Ergodicity of the system and the extinction of enterprise clusters, and further discussed the permanence in the mean. Finally, the theoretical results were verified using three numerical examples. Up to now, there is no paper to study the dynamical behavior of the stochastic enterprise cluster model with high-order perturbation, so this study fills the gap mentioned above and expands the scope of research in the aforementioned fields. Some other interesting problems on stochastic enterprise cluster model deserve further consideration, in the future, we will study stochastic enterprise cluster model with Lévy jumps. Also, we will investigate dynamical behavior and optimal control problems of the enterprise cluster model with impulsive terms, which certainly has value for further exploration.

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Ethical approval. This article does not contain any studies with human participants or animals performed by any of the authors.

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