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NONLINEAR HILFER COTANGENT PDES: STABILITY ANALYSIS AND APPROXIMATION SOLUTIONS

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Abstract This paper studies a class of Hilfer cotangent partial fractional differential equations. Existence and uniqueness of solutions are established using fixed-point theorems, and approximate solutions are constructed. Stability is analyzed via a fractional Grönwall-type inequality, leading to generalized Ulam-Hyers and Ulam-Hyers-Rassias stability results. The findings confirm the well-posedness and stability of the proposed system, and an example illustrates the theoretical results.

Keywords Hilfer fractional derivative, fixed point theorem, existence and uniqueness, Grönwall inequality, stability analysis.

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1. Introduction

Differential equations with fractional derivatives (FD) play an important role in a wide range of domains, including natural sciences [33], engineering [15], biological sciences [35], economics [27], and physics [49], and have excited the interest of many researchers [52, 53]. The phenomenon of time delay, which is common in real-life circumstances, has sparked extensive investigation. In recent years, there has been a growing interest in analysing the properties of solutions to delay differential equations, with a particular emphasis on the existence and stability of solutions involving the Liouville-Caputo(L-C) fractional derivative, as noted in studies [10, 11, 48].

The Hilfer fractional derivative (HFD) extends two derivatives, including the Riemann-Liouville (R-L) and L-C derivatives, and has a wide range of applications. In [47] Hilfer fractional operator, named after mathematician R. Hilfer, is a mathematical concept for describing systems and processes with non-integer (NI) order fractional behaviour. Unlike standard integer-order calculus, fractional calculus, particularly the Hilfer fractional model, works with NI derivatives and integrals, allowing it to capture more complicated dynamics over several domains. Applications of the HFD model can be found in multiple areas. For instance:

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- Anomalous diffusion [26]: Describing non-standard diffusion in physics (porous media), biology (cell migration), and finance (price dynamics).
- Control systems [13]: Applied in robotics, automation, and mechatronics for fractional-order controllers.
- Signal processing [22]: For noise reduction, image processing, and data compression, capturing long-range dependencies.
- Biomedical engineering [36]: Modeling physiological processes such as drug delivery and neuronal dynamics.

Katugampola [19] presented a local FD called the conformable derivative (CD). The conceptualization of these local FD led to the identification of nonlocal FD, which had previously been reported in [21]. We provide an outline of the basic concepts of the CD and propose a derivative that is consistent with both the left and right versions. Furthermore, we show that the nonlocal FD version proposed in [1,9] may be derived from [17]. In all forms of fractional calculus or calculus with derivatives, a function's order zero must be identical to the function itself. However, the CD lacks this important standard, which can be construed as a defect. To address this, the authors in [21] redefined the CD to ensure that it yields the function itself when the local FD has an order of zero.

The Hilfer cotangent fractional derivative (HCFD) is an extension of this line of research. This derivative combines the characteristics of the R-L and L-C cotangent FD. Katugampola [20] published the CD of order $\alpha \in [0,1]$, defined as

$$D^{\alpha}g(x) = \lim_{q\to 0}\frac{g\big(t+qt^{1-\alpha}\big)-g(x)}{q},$$

where the drawback is that

$$\lim_{\alpha\to 0^{+}} D^{\alpha}g(x) \neq g(x).$$

The author [19] presented some concepts of CD and raised an open problem about how to use CD to produce a more general FD. The general FD and fractional integrals (FI) proposed and studied in [6,7] provided a response to this problem. In [29], Anderson improved the CD so that

$$\lim_{\alpha\to 0^{+}} D^{\alpha}g(x) = g(x).$$

The authors of [2,3,6,12,29] introduced new types of FD that enable the appearance of kernels such as the exponential function or the Mittag-Leffler (M-L) function. However, the new non-singular kernel lacks a semigroup property, making it difficult to solve some difficult fractional systems. Concurrently, significant efforts have been made to generate different types of FD and integrals that use M-L functions in their representations [8,23,25].

On the other hand, recently, Deepak B. Pachpatte [34] investigated the existence and stability (E & S) of solutions for nonlinear ψ-Hilfer partial fractional differential equations (FPDEs), establishing sufficient conditions for solvability. Nadir Benkaci-Ali [5] studied a coupled system of sequential partial Hilfer-type FPDEs involving a weighted double phase operator and obtained results on existence, E & S, and controllability. Earlier, J. R. Wang et al., [46] established fundamental existence and stability results for Hilfer-type PFDEs. These contributions significantly advanced the theoretical development of Hilfer-type PFDEs.

Subsequently, Sadek [37] introduced the FD version of this redefined CD, called the CFD, which preserves the semigroup property, adopts the exponential cotangent kernel operator, and generalizes both the R-L and L-C derivatives. In another paper, Sadek et al. [39] studied its essential properties and applied it to a nonlinear fractional problem with nonlocal initial condition.

We consider the following nonlinear partial differential equation (PDEs) involving a HCFD

$$\begin{cases} \mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} f(t,\tau) = p(t,\tau,f(t,\tau)) + \dfrac{1}{\sin^{\alpha}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha)} \displaystyle\int_c^d e^{-\cot\left(\omega\frac{\pi}{2}\right)(d-\eta)}(d-\eta)^{\alpha-1} \\ \qquad\qquad\qquad \times q(t,\tau,\eta,f(t,\eta))\,d\eta + r(t,\tau), \\ I^{1-\gamma,w}_{a^+,t} f(t,\tau) = f_0(\tau), \quad \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{1.1}$$

where $\mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} f(t,\tau)$ denotes the HCFD of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$. $I^{1-\gamma,\omega}_{a^+,t}$ denotes the cotangent FI of order $1-\gamma$. The function satisfy $p : \Delta \times \mathbb{R} \to \mathbb{R}$, $q : \Delta \times \mathbb{I} \times \mathbb{R} \to \mathbb{R}$, and $r : \Delta \to \mathbb{R}$. Here, $\mathbb{J} = [a,b]$, $\mathbb{I} = [c,d] \subset \mathbb{R}, (c < d)$, and $\Delta = \mathbb{J} \times \mathbb{I}$, $-\infty \leq a < x < b \leq \infty$.

After introducing system (1.1), we establish E & S for its solutions.

The novelty of this work can be summarized as follows:

- Establishment of a unified analytical concept combining existence, uniqueness, and Ulam-Hyers-Rassias stability results for the proposed nonlinear PDEs involving a HCFD.
- Extension of the Ulam-Hyers-Rassias stability theory to the considered fractional-order PDEs, thereby broadening the applicability of stability concepts within the Hilfer fractional setting.
- Construction of a successive approximation scheme based on the Banach fixed-point principle (BFP), providing a constructive procedure for approximating the solution.
- Illustration of the theoretical findings through a numerical example, demonstrating the applicability of the obtained results to fractional-order systems.

2. Preliminaries

This section presents the fundamental definitions from fractional calculus together with essential results from nonlinear analysis.

Following ([41], Section 2, Page 6), we introduce the Bielecki-type norm.

Let $C(\Delta,\mathbb{R})$ denotes the Banach space of continuous functions defined on Δ. This space is endowed with the supremum norm

$$||y||_{\infty} = \sup_{(t,\tau)\in\Delta} |y(t,\tau)|, \quad y \in C(\Delta,\mathbb{R}).$$

Consider the weighted function space $C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$ consisting of all continuous functions $y \in C(\Delta,\mathbb{R})$ such that

$$\sup_{(t,\tau)\in\Delta} \frac{|y(t,\tau)|}{E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2})} < \infty.$$

The associated norm is defined by

$$\|y\|_{\xi} = \sup_{(t,\tau)\in\Delta} \frac{|y(t,\tau)|}{E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2})}.$$

The function $E_{(\alpha_1,\alpha_2)}$ denotes the generalized two-parameter M-L function defined by

$$E_{(\alpha_1,\alpha_2)}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(k\alpha_1+1)\Gamma(k\alpha_2+1)}, \quad z \in \mathbb{C},$$

see, for example ([14], Eqs. 1.1 and 1.2, Page 1).

We endow the linear space $C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$ with the norm defined by

$$\|y\|_{\xi,(\alpha_1,\alpha_2)} = \sup_{(t,\tau)\in\Delta} \frac{|y(t,\tau)|}{E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2})},$$

for $y \in C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$.

This norm induces the metric

$$\begin{aligned} d_{\xi,(\alpha_1,\alpha_2)}(y,z) &= \|y-z\|_{\xi,(\alpha_1,\alpha_2)} \\ &= \sup_{(t,\tau)\in\Delta} \frac{|y(t,\tau)-z(t,\tau)|}{E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2})}. \end{aligned}$$

Equipped with this norm, the space $C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$ is a Banach space and there exists a constant $M \geq 0$ such that $\|y\|_{\xi,(\alpha_1,\alpha_2)} \leq M$.

According to ([24], Chapter 2, Eqs. 2.1.1 and 2.1.2, Page 69), the definitions of the R-L FI are taken into consideration, as are the L-C FD ([24], Section 2.4, Page 97) and the HFD ([18], Remark 6, Page 4). They are not described in depth because there is so much literature on these classical formulations.

Definition 2.1. ([39], Definition 2.4, Page 28338) Let $\mathbb{I} = [a,b] \subset \mathbb{R}$, let $\alpha > 0$, and assume $\omega \in (0,1)$. For the function $g \in L^1([a,b])$, the left cotangent FI of order α is defined by

$$I_{a^+}^{\alpha,\omega} g(x) = \frac{1}{\sin^{\alpha}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha)} \int_a^x e^{-\cot\left(\omega\frac{\pi}{2}\right)(x-t)}(x-t)^{\alpha-1} g(t)\,dt, \quad x > a.$$

Similarly, the right cotangent FI of order α is defined by

$$I_{b_-}^{\alpha,\omega} g(x) = \frac{1}{\sin^{\alpha}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha)} \int_x^b e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-x)}(t-x)^{\alpha-1} g(t)\,dt, \quad x < b.$$

Definition 2.2. ([39], Section 2, Page 28338) Let $\theta = (a,c) \in \mathbb{R}^2$, $\alpha = (\alpha_1,\alpha_2)$, $0 < \alpha_1, \alpha_2 \leq 1$, and consider the rectangle $I = [a,k] \times [c,m]$, $a < k, c < m$. Assume that $u \in L^1(I)$. Then the two-dimensional cotangent fractional partial integral of order α is defined by

$$\begin{aligned} I_{\theta}^{\alpha,\omega} u(x,y) = {} & \frac{1}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)\sin^{\alpha_2}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2)} \int_a^x \int_c^y e^{-\cot\left(\omega\frac{\pi}{2}\right)(x-s)}(x-s)^{\alpha_1-1} \\ & \times e^{-\cot\left(\omega\frac{\pi}{2}\right)(y-t)}(y-t)^{\alpha_2-1} u(s,t)\,dt\,ds, \end{aligned}$$

for $(x,y) \in (a,k] \times (c,m]$, where $\omega \in (0,1)$.

Furthermore, the one-dimensional cotangent FI ([39], Definition 2.4, Page 28338) with respect to x and y are given as follows.

The left cotangent FI with respect to x of order α_1 is defined by

$$I_{a^+}^{\alpha_1,\omega} u(x,y) = \frac{1}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)} \int_a^x e^{-\cot\left(\omega\frac{\pi}{2}\right)(x-s)}(x-s)^{\alpha_1-1} u(s,y) ds,$$

for $x > a$.

Similarly, the left cotangent FI with respect to y of order α_2 is defined by

$$I_{c^+}^{\alpha_2,\omega} u(x,y) = \frac{1}{\sin^{\alpha_2}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2)} \int_c^y e^{-\cot\left(\omega\frac{\pi}{2}\right)(y-\tau)}(y-\tau)^{\alpha_2-1}u(x,\tau)d\tau,$$

for $y > c$, where $0 < \alpha_1, \alpha_2 \leq 1$ and $\omega \in (0,1)$.

Lemma 2.1. ([39], Theorem 2.15, Page 28339) *Let $g \in C^n([a,b])$, where $n-1 < \alpha < n$ and $0 \leq \beta \leq 1$. Then the following identity holds*

$$I_{a^+}^{\alpha,\omega}\, \mathbb{H}_{a^+}^{\alpha,\beta,\omega} g(x) = g(x) - \frac{e^{-\cot\left(\omega\frac{\pi}{2}\right)(x-a)}(x-a)^{\gamma-1}}{\sin^{\gamma-1}\left(\omega\frac{\pi}{2}\right)\Gamma(\gamma)} \left(I_{a^+}^{1-\gamma,\omega}\right) g(a),$$

where $\gamma = \alpha + \beta - \alpha\beta$.

Lemma 2.2. ([39], Theorem 2.13, Page 28339) *Let $f \in C([a,b])$, let $\alpha > 0$, and assume $0 \leq \beta \leq 1$ and $\omega \in (0,1)$. Then the following identities hold*

$$\mathbb{H}_{a^+}^{\alpha,\beta,\omega} I_{a^+}^{\alpha,\omega} g(x) = g(x),$$

and

$$\mathbb{H}_{b^-}^{\alpha,\beta,\omega} I_{b^-}^{\alpha,\omega} g(x) = g(x).$$

Lemma 2.3. ([50], Theorem A, Page 1075, [38], Corollary 3.1, Page 7826) *Let $\mathbb{J} = [a,b]$, Let $u, \psi \in L^1(\mathbb{J})$ and let $h \in C(\mathbb{J})$. Assume*
(i) $u(t) \geq 0$ and $\psi(t) \geq 0$ for all $t \in \mathbb{J}$,
(ii) $h(t) \geq 0$ and $h(t)$ is non-decreasing on $\mathbb{J}$,
(iii) $\alpha > 0$ and $\omega \in (0,1)$.
If

$$u(t) \leq \psi(t) + \frac{h(t)}{\sin^{\alpha}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha)} \int_a^t e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-\tau)}(t-\tau)^{\alpha-1}u(\tau)\,d\tau, \quad t \in \mathbb{J},$$

then

$$\nu(t) \leq \psi(t) + \int_a^t \sum_{k=1}^{\infty} \frac{(h(t)\Gamma(\alpha))^k}{\sin^{\alpha}\left(\omega\frac{\pi}{2}\right)\Gamma(k\alpha)} e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-\tau)}(t-\tau)^{k\alpha-1}\psi(\tau)\,d\tau, \quad t \in \mathbb{J}.$$

3. Existence and approximate solutions

In this section, we discuss the conditions for the existence and approximation of solution of system (1.1).

Let $0 < \alpha_1, \alpha_2 < 1$, $0 \leq \beta \leq 1$.

For $0 < \alpha_1 < 1$, applying Lemma 2.1, we obtain

$$I_{a^+,t}^{\alpha_1,\omega}\, \mathbb{H}_{a^+,t}^{\alpha_1,\beta,\omega} f(t,\tau) = f(t,\tau) - \frac{e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-a)}(t-a)^{\gamma_1-1}}{\sin^{\gamma_1-1}\left(\omega\frac{\pi}{2}\right)\Gamma(\gamma_1)} \left(I_{a^+}^{1-\gamma_1,\omega} f\right)(a,\tau), \tag{3.1}$$

where $\gamma_1 = \alpha_1 + \beta - \alpha_1\beta$.

Substituting Eq. (1.1) into the above relation yields the equivalent integral equation (EIE)

$$\begin{aligned}
f(t,\tau) &= \frac{e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-a)}(t-a)^{\gamma_1-1}}{\sin^{\gamma_1-1}\left(\omega\frac{\pi}{2}\right)\Gamma(\gamma_1)}\left(I_{a^+}^{1-\gamma_1,\omega}f\right)(a,\tau)\\
&\quad+\frac{1}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)}\int_a^t e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-s)}(t-s)^{\alpha_1-1}\\
&\quad\times\Bigg[p(s,\tau,f(s,\tau))+\frac{1}{\sin^{\alpha_2}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2)}\int_c^\tau e^{-\cot\left(\omega\frac{\pi}{2}\right)(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}\\
&\quad\times q(s,\tau,\eta,f(s,\eta))\,d\eta+r(s,\tau)\Bigg]ds.
\end{aligned}$$

Finally, we obtain the EIE

$$f(t,\tau) = \frac{e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-a)}(t-a)^{\gamma_1-1}}{\sin^{\gamma_1-1}\left(\omega\frac{\pi}{2}\right)\Gamma(\gamma_1)}\left(I_{a^+}^{1-\gamma_1,\omega}f\right)(a,\tau)+I_{a^+,t}^{\alpha_1,\omega}F(t,\tau),$$

where

$$F(t,\tau) = p(t,\tau,f(t,\tau))+I_{c^+}^{\alpha_2,\omega}q(t,\tau,\eta,f(t,\eta))+r(t,\tau).$$

Taking the HCFD of order α_1 with respect to t on both sides of space $C_{\xi,(\alpha_1,\alpha_2)}$, and using Lemma 2.2, we obtain

$$\begin{aligned}
\mathbb{H}_{a^+,t}^{\alpha_1,\beta,\omega}f(t,\tau) &= p(t,\tau,f(t,\tau))\\
&\quad+\frac{1}{\sin^{\alpha_2}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2)}\int_0^\tau e^{-\cot\left(\omega\frac{\pi}{2}\right)(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}q(t,\tau,\eta,f(t,\eta))\,d\eta\\
&\quad+r(t,\tau).
\end{aligned}$$

Therefore, Eq. (1.1) and the above integral equation are equivalent.

Theorem 3.1. *Let $\Delta = [a,b]\times[c,d], \mathbb{I} = [c,d]$. Assume that the functions $p : \Delta\times\mathbb{R}\to\mathbb{R}$, $q : \Delta\times\mathbb{I}\times\mathbb{R}\to\mathbb{R}$ are continuous. Suppose that there exist nonnegative functions $d\in C(\Delta)$, $g\in C(\Delta\times\mathbb{I})$, such that the following Lipschitz condition hold*

$$|p(t,\tau,u)-p(t,\tau,\bar{u})|\le d(t,\tau)|u-\bar{u}|,\quad (t,\tau)\in\Delta,$$

and

$$|q(t,\tau,\eta,u)-q(t,\tau,\eta,\bar{u})|\le g(t,\tau,\eta)|u-\bar{u}|,\quad (t,\tau,\eta)\in\Delta\times\mathbb{I}.$$

For the parameter $\xi > 0$ defined in weighted space, assume that there exist non-negative constants $\Lambda_1 < 1$, $\lambda_2 \ge 0$ such that he following estimates hold

$$\begin{aligned}
&\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}d(s,\tau)E_{(\alpha_1,\alpha_2)}(\xi(s-a)^{\alpha_1}(\tau-c)^{\alpha_2})\,ds\\
&+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\int_a^t\int_c^d e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}
\end{aligned}$$

$$\begin{aligned}&\times e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}g(s,\tau,\eta)E_{(\alpha_1,\alpha_2)}(\xi(s-a)^{\alpha_1}(\tau-c)^{\alpha_2})d\eta ds\\ \le&\Lambda_1E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2}). \end{aligned} \tag{3.2}$$

Bounded on the nonhomogeneous term

$$\begin{aligned}&\Big|h(t,\tau)+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}p(s,\tau,0)ds \qquad (3.3)\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\int_a^t\int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}\\ &\times e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}q(s,\tau,\eta,0)\,d\eta\,ds\Big|\\ \le&\Lambda_2E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2}). \end{aligned} \tag{3.4}$$

Definition of $h(t,\tau)$, we have

$$h(t,\tau)=f_0(\tau)+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}r(s,\tau)\,ds. \tag{3.5}$$

Under assumptions Eqs. (3.2)-(3.5)*, the system* (1.1) *admits a unique solution $f\in C(\Delta\times\mathbb{R})$.*

Proof. On the Banch space $C(\Delta,\mathbb{R})$, define an operator $T:C(\Delta,\mathbb{R})\to C(\Delta,\mathbb{R})$

$$\begin{aligned}(Tf)(t,\tau)=&f_0(\tau)+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}p(s,\tau,f(s,\tau))ds\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\int_a^t\int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}\\ &\times e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}q(s,\tau,\eta,f(s,\eta))\,d\eta ds\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}r(s,\tau)ds. \end{aligned} \tag{3.6}$$

From Eqs. (3.5) and (3.6), we obtain

$$\begin{aligned}(Tf)(t,\tau)=&h(t,\tau)+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}p(s,\tau,f(s,\tau))ds\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\int_a^t\int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}\\ &\times e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}q(s,\tau,\eta,f(s,\eta))\,d\eta ds, \end{aligned} \tag{3.7}$$

for $(t,\tau)\in\Delta$.

We now show that the map T is a contraction.

Using Eq. (3.7) together with the assumptions on p and q, we obtain

$$\begin{aligned}&|(Tf)(t,\tau)|\\ =&\Big|h(t,\tau)+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}p(s,\tau,f(t,\tau))ds\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\end{aligned}$$

$$\times \int_a^t \int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}q(s,\tau,\eta,f(t,\tau))\,d\eta ds\big|.$$

Add and subtract zero terms

$$\begin{aligned}&p(s,\tau,f)=p(s,\tau,0)+[p(s,\tau,f)-p(s,\tau,0)],\\ &q(s,\tau,\eta,f)=q(s,\tau,\eta,0)+[q(s,\tau,\eta,f)-q(s,\tau,\eta,0)].\end{aligned}$$

Applying triangle inequality, we get

$$\begin{aligned}|(Tf)(t,\tau)|\le&\big|h(t,\tau)\big|+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}p(s,\tau,0)ds\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\\ &\times\int_a^t\int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}|q(s,\tau,\eta,0)\,d\eta ds|\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}|p(s,\tau,f(s,\tau))-p(s,\tau,0)|ds\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\\ &\times\int_a^t\int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}\\ &\times|q(s,\tau,\eta,f(s,\eta))-q(s,\tau,\eta,0)|\,d\eta ds.\end{aligned}$$

Using the boundedness hypothesis

$$\begin{aligned}&|(Tf)(t,\tau)|\\ \le&\Lambda_2E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1-1}(\tau-c)^{\alpha_2-1})\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}d(s,t)|f(s,\tau))|ds\\ &+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\\ &\times\int_a^t\int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}g(s,\tau,\eta)|f(s,\eta)|\,d\eta ds.\end{aligned}$$

Define

$$||f||_{E_{\xi,(\alpha_1,\alpha_2)}}=\sup_{\Delta}\frac{|f(t,\tau)|}{E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2})}.$$

Then

$$|f(s,\tau)|\le||f||_{E(\alpha_1,\alpha_2)}E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2}).$$

Similarly for $f(s,\eta)$.

Substitute

$$|(Tf)(t,\tau)|\le\Lambda_2E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2})+||f||_{E_\xi(\alpha_1,\alpha_2)}[I_1+I_2].$$

Using the known kernel estimate,

$$I_1 + I_2 \leq \Lambda_1 E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2}),$$

hence

$$|(Tf)(t,\tau)| \leq [\Lambda_2 + \Lambda_1 ||f||_{\xi,(\alpha_1,\alpha_2)}]E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2}).$$

The previous estimate show that $T : C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R}) \to C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$, hence the space is invariant under T.

Let $f, z \in C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$.

From Eq. (3.6), we obtain

$$\begin{aligned}
&|(Tf)(t,\tau) - (Tz)(t,\tau)| \\
=&\Big|\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}[p(s,\tau,f(s,\tau)) - p(s,\tau,z(s,\tau))]ds \\
&+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\int_a^t\int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1} \\
&\times e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}|q(s,\tau,\eta,f(s,\eta)) - q(s,\tau,\eta,z(s,\eta))\,d\eta ds\Big| \\
\leq&\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}|p(s,\tau,f) - p(s,\tau,z)|ds \\
&+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\int_a^t\int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)} \\
&\times(t-s)^{\alpha_1-1}e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}\big|q(s,\tau,\eta,f) - q(s,\tau,\eta,z)\big|\,d\eta ds.
\end{aligned}$$

Using

$$\begin{aligned}
&|p(s,\tau,u) - p(s,\tau,v)| \leq d(s,\tau)|u-v|, \\
&|q(s,\tau,\eta,u)| - q(s,\tau,\eta,v) \leq g(s,,\tau,\eta)|u-v|,
\end{aligned}$$

we get

$$\begin{aligned}
&|(Tf)(t,\tau) - (Tz)(t,\tau)| \\
\leq&\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)}\int_a^t e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1}d(s,t)|f(s,\tau) - z(s,\tau)|ds \\
&+\frac{1}{\sin^{\alpha_1}(\omega\frac{\pi}{2})\Gamma(\alpha_1)\sin^{\alpha_2}(\omega\frac{\pi}{2})\Gamma(\alpha_2)}\int_a^t\int_c^\tau e^{-\cot(\omega\frac{\pi}{2})(t-s)}(t-s)^{\alpha_1-1} \\
&\times e^{-\cot(\omega\frac{\pi}{2})(\tau-\eta)}(\tau-\eta)^{\alpha_2-1}g(s,\tau,\eta)|f(s,\eta) - z(s,\eta)|\,d\eta ds.
\end{aligned}$$

By definition,

$$|f(s,\tau) - z(s,\tau)| \leq ||f-z||_{\xi,(\alpha_1,\alpha_2)}E_{(\alpha_1,\alpha_2)}(\xi(s-a)^{\alpha_1}(\tau-c)^{\alpha_2}).$$

Similarly for $f(s,\eta) - z(s,\eta)$.

Substitutes $||f-z||_{\xi,(\alpha_1,\alpha_2)}[I_1 + I_2]$.

Using the known estimate for the fractional Volterra kernal with M-L weight,

$$I_1 + I_2 \leq \Lambda_1 E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2}).$$

Finally, contraction inequality

$$|(Tf)(t,\tau) - (Tz)(t,\tau)| \leq \Lambda_1 ||f-z||_{\xi,(\alpha_1,\alpha_2)} E_{(\alpha_1,\alpha_2)}(\xi(t-a)^{\alpha_1}(\tau-c)^{\alpha_2}).$$

Dividing by the weight and taking supremum over Δ, we obtain

$$||Tf - Tz||_{\xi(\alpha_1,\alpha_2)} \leq \Lambda_1 ||f-z||_{\xi,(\alpha_1,\alpha_2)}.$$

If $0 < \Lambda_1 < 1$, then T is a strict contraction on $C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$. By BFP, the problem admits a unique solution.

From the previous inequality, dividing by the weight function and taking the supremum over Δ, we obtain

$$||Tf - Tz||_{\xi,(\alpha_1,\alpha_2)} \leq \Lambda_1 ||f-z||_{\xi,(\alpha_1,\alpha_2)}.$$

Since $0 < \Lambda_1 < 1$, the operator $T : C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R}) \to C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$ is a strict contraction on the complete metric space $C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$.

Therefore, by the BFP, the operator T admits a unique fixed point $f^* \in C_{\xi,(\alpha_1,\alpha_2)}(\Delta,\mathbb{R})$. This fixed point satisfies $Tf^* = f^*$, which is equivalent to the integral formulation of system (1.1). Hence, system (1.1), admits a unique solution on Δ. This completes the proof. □

We now determine an estimate for the approximate solution (AS) of system (1.1).

Let $f \in \Delta \to \mathbb{R}$ be a function such that the fractional operator $\mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} f$ exists on Δ.

We say that f is an ϵ- AS of system (1.1) if, for a given $\epsilon \geq 0$, the following inequality holds

$$|\mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} f(t,\tau) - p(t,\tau,f(t,\tau)) + I^{\alpha_2,\omega}_{c,\eta}(q(t,\tau,\eta,f(t,\eta))) + h(t,\tau)| \leq \epsilon, \tag{3.8}$$

for all $(t,\tau) \in \Delta$.

In this case, the function $f(t,\tau)$ is called ϵ- AS of the system (1.1).

Theorem 3.2. *Assume that the functions $p : \Delta \times \mathbb{R} \to \mathbb{R}$, $q : \Delta \times \mathbb{I} \times \mathbb{R} \to \mathbb{R}$, satisfy the Lipschitz conditions*

$$|p(t,\tau,k) - p(t,\tau,\bar{k})| \leq d_1|k-\bar{k}|, \tag{3.9}$$

$$|q(t,\tau,\eta,a) - q(t,\tau,\eta,\bar{a})| \leq d_2|a-\bar{a}|, \tag{3.10}$$

for some constants $d_1, d_2 \geq 0$and for all admissible (t,τ,η).

Let $i = 1,2$. Suppose that $f_i : \Delta \to \mathbb{R}$ are ϵ_i-AS of system (1.1), satisfying

$$I^{1-\gamma,w}_{a^+,t} f_i(a,\tau) = f_i(\tau), \tag{3.11}$$

for all $(t,\tau) \in \Delta$.

Define

$$h_i(t,\tau) = f_i(\tau) + I^{\alpha_1,\omega}_{a^+,t} r(t,\tau).$$

Assume that

$$|h_1(t,\tau) - h_2(t,\tau)| \leq \mu,$$

for some constant $\mu \geq 0$.

Assume further that the quantity

$$M_1 = \sup_{(t,\tau)\in\Delta} \left[(\epsilon_1 + \epsilon_2) \frac{(t-a)^{\alpha_1}}{\sin^{\alpha_1+1}(\omega \frac{\pi}{2})\Gamma(\alpha_1+1)} + \mu \right]$$

is finite, i.e., $M_1 < \infty$.

Next, for all $(t,\tau) \in \Delta$, *we obtain*

$$\begin{aligned} &|f_1(t,\tau) - f_2(t,\tau)| \\ \leq & M_1 \Big(1 + \int_a^t \left[\sum_{k=1}^{\infty} \left(d_1 + d_2 \frac{(\tau-c)^{\alpha_2}}{\sin^{\alpha_2+1}(\omega\frac{\pi}{2})\Gamma(\alpha_2+1)} \right)^k e^{-\cot(\omega\frac{\pi}{2})(t-s)} (t-s)^{\alpha_1 k - 1} \right] ds \Big). \end{aligned} \tag{3.12}$$

Proof. For $i = 1, 2$ since $f_i(t,\tau)$ are ϵ_i-AS of Eq. (1.1), from definition (3.8), we have

$$|\mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} f_i(t,\tau) - p(t,\tau,f_i(t,\tau)) - I^{\alpha_2,\omega}_{c^+,\tau} q(t,\tau,\eta,f_i(t,\eta)) - h(t,\tau)| \leq \epsilon_i, \tag{3.13}$$

for all $(t,\tau) \in \Delta$.

Applying fractional integral operator $I^{\alpha_1,\omega}_{a^+,t}$ to both sides of inequality (3.13). Since ϵ_i is constant with respect to t, we obtain

$$I^{\alpha_1,\omega}_{a^+,t}(\epsilon_i) \geq |I^{\alpha_1,\omega}_{a^+,t} \Big(\mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} f_i(t,\tau) - p(t,\tau,f_i(t,\tau)) + I^{\alpha_2,\omega}_{c^+,\tau}(q(t,\tau,\eta,f_i(t,\eta))) + h(t,\tau) \Big)|.$$

Using the explicit form of the FI of a constant, we obtain

$$I^{\alpha_1,\omega}_{a^+,t}(\epsilon_i) = \epsilon_i \frac{(t-a)^{\alpha_1}}{\sin^{\alpha_1+1}\Gamma(\alpha_1+1)}.$$

Thus,

$$\begin{aligned} &\epsilon_i \frac{(t-a)^{\alpha_1}}{\sin^{\alpha_1+1}\big(\omega\frac{\pi}{2}\big)\,\Gamma(\alpha_1+1)} \\ &\geq \left| I^{\alpha_1,\omega}_{a,t} \Big\{ \mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} f_i(t,\tau) - p(t,\tau,f_i(t,\tau)) - I^{\alpha_2,\omega}_{c^+,\eta} q(t,\tau,\eta,f_i(t,\eta)) - h_i(t,\tau) \Big\} \right|. \end{aligned}$$

Using the identity

$$I^{\alpha_1,\omega}_{a^+,t} \mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} f_i = f_i - f_i(\tau),$$

we obtain

$$\begin{aligned} &\epsilon_i \frac{(t-a)^{\alpha_1}}{\sin^{\alpha_1+1}\big(\omega\frac{\pi}{2}\big)\,\Gamma(\alpha_1+1)} \\ &\geq \left| f_i(t,\tau) - h_i(t,\tau) - I^{\alpha_1,\omega}_{a^+,t} \Big[p(t,\tau,f_i(t,\tau)) + I^{\alpha_2,\omega}_{c^+,\eta} q(t,\tau,\eta,f_i(t,\eta)) \Big] \right|. \end{aligned}$$

Summing the inequalities for $i = 1$ and $i = 2$, we obtain

$$\begin{aligned} &(\epsilon_1 + \epsilon_2) \frac{(t-a)^{\alpha_1}}{\sin^{\alpha_1+1}\big(\omega\frac{\pi}{2}\big)\,\Gamma(\alpha_1+1)} \\ &\geq \left| f_1(t,\tau) - h_1(t,\tau) - I^{\alpha_1,\omega}_{a^+,t} \Big[p(t,\tau,f_1(t,\tau)) + I^{\alpha_2,\omega}_{c^+,\eta} q(t,\tau,\eta,f_1(t,\eta)) \Big] \right| \end{aligned}$$

$$+\left|f_2(t,\tau)-h_2(t,\tau)-I^{\alpha_1,\omega}_{a^+,t}\Big[p(t,\tau,f_2(t,\tau))+I^{\alpha_2,\omega}_{c^+,\eta}q(t,\tau,\eta,f_2(t,\eta))\Big]\right|.$$

Using $|A|+|B|\geq|A-B|$, we obtain

$$\begin{aligned}(\epsilon_1+\epsilon_2)\frac{(t-a)^{\alpha_1}}{\sin^{\alpha_1+1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1+1)}\geq{}&|f_1(t,\tau)-f_2(t,\tau)|\\&-|h_1(t,\tau)-h_2(t,\tau)|\\&-\left|I^{\alpha_1,\omega}_{a^+,t}\big(p(t,\tau,f_1)-p(t,\tau,f_2)\big)\right|\\&-\left|I^{\alpha_2,\omega}_{c^+,t}\big(q(t,\tau,\eta,f_1)-q(t,\tau,\eta,f_2)\big)\right|.\end{aligned}$$

We have

$$|f_1(t,\tau)-f_2(t,\tau)|\leq M_1+I^{\alpha_1,\omega}_{a^+,t}\Big[d_1|f_1(t,\tau)-f_2(t,\tau)|+I^{\alpha_2,\omega}_{c^+,t}\big(d_2|f_1(t,\eta)-f_2(t,\eta)|\big)\Big].$$

Since the kernel is positive, we remove the absolute value of the operator

$$\begin{aligned}|f_1(t,\tau)-f_2(t,\tau)|\leq M_1+\frac{1}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)}\int_a^t(t-s)^{\alpha_1-1}\Big[&d_1|f_1(s,\tau)-f_2(s,\tau)|\\&+I^{\alpha_2,\omega}_{c^+,\eta}\big(d_2|f_1(s,\eta)-f_2(s,\eta)|\big)\Big]ds.\end{aligned}\tag{3.14}$$

Using

$$I^{\alpha_2,\omega}_{c^+,\eta}g(\eta)=\frac{1}{\sin^{\alpha_2}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2)}\int_c^d(d-\eta)^{\alpha_2-1}g(\eta)\,d\eta,$$

we obtain

$$\begin{aligned}&|f_1(t,\tau)-f_2(t,\tau)|\\&\leq M_1+\frac{1}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)}\int_a^t(t-s)^{\alpha_1-1}\\&\quad\times\left[d_1|f_1(s,\tau)-f_2(s,\tau)|+\frac{d_2}{\sin^{\alpha_2}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2)}\int_c^d(d-\eta)^{\alpha_2-1}|f_1(s,\eta)-f_2(s,\eta)|\,d\eta\right]ds.\end{aligned}$$

Since

$$\int_c^d(d-\eta)^{\alpha_2-1}d\eta=\frac{(d-c)^{\alpha_2}}{\alpha_2},$$

we obtain

$$\frac{(d-c)^{\alpha_2}}{\alpha_2\Gamma(\alpha_2)}=\frac{(d-c)^{\alpha_2}}{\Gamma(\alpha_2+1)}.$$

Hence

$$\begin{aligned}|f_1(t,\tau)-f_2(t,\tau)|\leq M_1+\frac{1}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)}\int_a^t(t-s)^{\alpha_1-1}\left[d_1+d_2\frac{(d-c)^{\alpha_2}}{\sin^{\alpha_2}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2+1)}\right]\\\times\sup_{\eta\in[c,d]}|f_1(s,\eta)-f_2(s,\eta)|\,ds.\end{aligned}$$

Let

$$\Lambda = d_1 + d_2 \frac{(d-c)^{\alpha_2}}{\sin^{\alpha_2}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2+1)}.$$

Then

$$\begin{aligned} &|f_1(t,\tau) - f_2(t,\tau)| \\ \leq &M_1 + \frac{\Lambda}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)} \int_a^t e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-s)}(t-s)^{\alpha_1-1} \sup_{\eta} |f_1(s,\eta) - f_2(s,\eta)| ds. \end{aligned} \tag{3.15}$$

Define

$$\Phi(s) = \sup_{\eta\in[c,d]} |f_1(s,\eta) - f_2(s,\eta)|.$$

Applying Lemma 2.3 to inequality (3.15) with respect to the variable t (for fixed τ), and defining

$$u(t) = \sup_{\tau\in[c,d]} |f_1(t,\tau) - f_2(t,\tau)|,$$

we rewrite inequality (3.15) in the form

$$u(t) \leq v(t) + \frac{g}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)} \int_a^t e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-s)}(t-s)^{\alpha_1-1} u(s)\, ds,$$

where

$$v(t) = (\epsilon_1+\epsilon_2)\frac{(t-a)^{\alpha_1}}{\sin^{\alpha_1+1}\left(\frac{\omega\pi}{2}\right)\Gamma(\alpha_1+1)} + \mu := M_1,$$

and

$$g = d_1 + d_2 \frac{(d-c)^{\alpha_2}}{\sin^{\alpha_2+1}\left(\frac{\omega\pi}{2}\right)\Gamma(\alpha_2+1)}.$$

By Lemma 2.3, we obtain

$$u(t) \leq M_1 E_{\alpha_1}\left(\frac{g}{\sin^{\alpha_1}\left(\frac{\omega\pi}{2}\right)}(t-a)^{\alpha_1}\right),$$

where $E_{\alpha_1}(\cdot)$ denotes the M-L function.

Therefore

$$\sup_{\tau\in[c,d]} |f_1(t,\tau) - f_2(t,\tau)| \leq M_1 E_{\alpha_1}\left(\frac{g}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)}(t-a)^{\alpha_1}\right),$$

which yields the required stability inequality. □

Remark 3.1. The solution of system(1.1) with initial condition

$$f_1(a,\tau) = \phi_1(\tau)$$

is denoted by $f_1(t,\tau)$.

From the stability inequality obtained above, we have

$$\sup_{\tau\in[c,d]} |f_1(t,\tau) - f_2(t,\tau) \to 0 \quad \text{as} \quad \epsilon_2 \to \epsilon_1 \quad \text{and} \quad \mu \to 0.$$

In particular, if we set

$$\varepsilon_1 = \varepsilon_2 = 0, \quad \text{and} \quad \phi_1(\tau) = \phi_2(\tau),$$

then the stability estimate reduces to

$$\sup_{\tau \in [c,d]} |f_1(t,\tau) - f_2(t,\tau)| = 0,$$

which implies

$$f_1(t,\tau) = f_2(t,\tau).$$

Hence, the solution of system (1.1) depends continuously on the given initial data.

4. Stability analysis

In this sequel, we investigate the stability analysis of proposed system (1.1).

Hyer-Ulam (H-U) stability is a concept that focuses whether an AS of a functional or differential equation is close to an exact solution. It originated from question posted by Stanislaw Ulam (1940) and was answered for Banach spaces by Donald H. Hyers (1941). H-U-Rassias stability has also been examined in the context of hyperbolic fractional PDEs [40] and Hilfer fractional implicit differential equations with nonlocal conditions [42]. These studies demonstrate the importance of stability analysis for fractional systems with complex operators. In recent years, researchers have extended this concept to various classes of fractional operators and problem settings. Jarad et al. [16] investigated fractional order systems involving a generalized HFD, where they established E & S results through fixed-point arguments. Their work highlighted how generalized Hilfer structures can support H-U-Rassias stability. Alam et al. [4] analyzed an implicit fractional integro-differential equation with integral boundary conditions. By deriving stability conditions, they demonstrated the practical importance of H-U-Rassias stability in guaranteeing the reliability of solutions to fractional boundary value problems. Luo et al. [32] considered Caputo-type fractional fuzzy stochastic differential equations with delay and studied their H-U stability properties. Their contribution extended the stability concept into the stochastic and fuzzy context, showing the flexibility of H-U stability across different fractional models.

Motivated by these developments, the present paper extends the study of generalized H- U-Rassias stability to system (1.1), thereby unifying E & S, and robustness analysis within single ideas.

We first introduce the associated with generalized H-U-Rassias stability

$$\begin{cases} \mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} f(t,\tau) \\ = p(t,\tau,f(t,\tau)) \\ \quad + \dfrac{1}{\sin^{\alpha_2}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2)} \displaystyle\int_c^d e^{-\cot\left(\omega\frac{\pi}{2}\right)(d-\eta)} (d-\eta)^{\alpha_2-1} q(t,\tau,\eta,f(t,\eta))\,d\eta + r(t,\tau). \end{cases} \tag{4.1}$$

Let $\varphi : \Delta \to [0,\infty)$ be continuous function. We say that system (4.1) is said to be generalized H-U-Rassias stable with respect to φ if the following holds.

For every function $a \in C(\Delta,\mathbb{R})$ satisfy

$$\left|\mathbb{H}^{\alpha,\beta,\omega}_{a^+,t} a(t,\tau) - p(t,\tau,a(t,\tau)) - I^{\alpha_2,\omega}_{c^+,\eta} q(t,\tau,\eta,a(t,\eta)) - r(t,\tau)\right| \le \varphi(t,\tau), \tag{4.2}$$

there exists an exact solution $f \in C(\Delta, \mathbb{R})$ of system (4.1) and a constant $C_{f,\varphi} > 0$ such that

$$|a(t,\tau) - f(t,\tau)| \leq C_{f,\varphi}\varphi(t,\tau), \quad (t,\tau) \in \Delta. \tag{4.3}$$

Theorem 4.1. *Assume that* p *and* q *in system* (4.1) *satisfy the Lipschitz conditions*

$$|p(t,\tau,u) - p(t,\tau,\bar{u})| \leq L_1|u - \bar{u}|, \tag{4.4}$$

$$|q(t,\tau,\eta,a) - q(t,\tau,\eta,\bar{a})| \leq L_2|a - \bar{a}|, \tag{4.5}$$

for all admissible arguments, where $L_1, L_2 > 0$.

Suppose further that there exists a constant $L_\varphi > 0$ *such that*

$$\frac{1}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)} \int_a^t e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-s)}(t-s)^{\alpha_1-1}\varphi(s,\tau)\,ds \leq L_\varphi\varphi(t,\tau), \tag{4.6}$$

for all $(t,\tau) \in \Delta$.

Moreover, define the constant

$$L_3 = \frac{(d-c)^{\alpha_2}}{\sin^{\alpha_2+1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_2+1)}.$$

Then system (4.1) *is generalized H-U-Rassias stable with respect to* φ.

Proof. Assume that $a \in C(\Delta, \mathbb{R})$ satisfies inequality (4.2). Let f be the unique exact solution of (4.1). Then, by applying the FI $I^{\alpha_1,\omega}_{a^+,t}$ to Eq. (4.1), we obtain the equivalent integral formulation

$$f(t,\tau) = H_f(t,\tau) + I^{\alpha_1,\omega}_{a^+,t}\Big[p(t,\tau,f(t,\tau)) + I^{\alpha_2,\omega}_{c^+,\eta}q(t,\tau,\eta,f(t,\eta))\Big], \tag{4.7}$$

where

$$H_f(t,\tau) = f_0(\tau) + I^{\alpha_1,\omega}_{a^+,t}r(t,\tau).$$

Using inequality (4.2) and applying the FI operator $I^{\alpha_1,\omega}_{a^+,t}$ to both sides, and noting that the initial terms coincide (i.e., $H_f = H_a$), we obtain

$$\begin{aligned} &|a(t,\tau) - H_a(t,\tau)| + \left|I^{\alpha_1,\omega}_{a^+,t}\left[p(t,\tau,a(t,\tau)) + I^{\alpha_2,\omega}_{c^+,\eta}q(t,\tau,\eta,a(t,\eta))\right]\right| \\ \leq &\frac{1}{\sin^{\alpha_1}\left(\omega\frac{\pi}{2}\right)\Gamma(\alpha_1)} \int_a^t e^{-\cot\left(\omega\frac{\pi}{2}\right)(t-s)}(t-s)^{\alpha_1-1}\varphi(s,\tau)\,ds. \end{aligned} \tag{4.8}$$

By assumption inequality (4.8), it follows that

$$|a(t,\tau) - H_a(t,\tau)| + \left|I^{\alpha_1,\omega}_{a^+,t}\left[p(t,\tau,a(t,\tau)) + I^{\alpha_2,\omega}_{c^+,\eta}q(t,\tau,\eta,a(t,\eta))\right]\right| \leq L_\varphi\varphi(t,\tau). \tag{4.9}$$

Using inequality (4.8) and the integral representation of f, we obtain

$$\begin{aligned} |a(t,\tau) - f(t,\tau)| \leq &|a(t,\tau) - H_f(t,\tau)| \\ &+ \left|I^{\alpha_1,\omega}_{a^+,t}\left[p(t,\tau,f(t,\tau)) + I^{\alpha_2,\omega}_{c^+,\eta}q(t,\tau,\eta,f(t,\eta))\right]\right|. \end{aligned}$$

Adding and subtracting the corresponding terms for a, we get

$$\begin{aligned}|a(t,\tau)-f(t,\tau)| \leq &\left|a(t,\tau)-H_a(t,\tau)-I_{a^+,t}^{\alpha_1,\omega}\left[p(t,\tau,a(t,\tau))+I_{c^+,\eta}^{\alpha_2,\omega}q(t,\tau,\eta,a(t,\eta))\right]\right| \\ &+ I_{a^+,t}^{\alpha_1,\omega}\Big(|p(t,\tau,a(t,\tau))-p(t,\tau,f(t,\tau))| \\ &+ I_{c^+,\eta}^{\alpha_2,\omega}|q(t,\tau,\eta,a(t,\eta))-q(t,\tau,\eta,f(t,\eta))|\Big).\end{aligned}$$

Using inequality (4.9), we obtain

$$\begin{aligned}|a(t,\tau)-f(t,\tau)| \leq\ & L_\varphi\varphi(t,\tau) \\ &+ I_{a^+,t}^{\alpha_1,\omega}\Big(L_1|a(t,\tau)-f(t,\tau)|+I_{c^+,\eta}^{\alpha_2,\omega}L_2|a(t,\eta)-f(t,\eta)|\Big).\end{aligned}$$

Using the bound

$$I_{c^+,\eta}^{\alpha_2,\omega}1=\frac{(d-c)^{\alpha_2}}{\sin^{\alpha_2}\left(\frac{\omega\pi}{2}\right)\Gamma(\alpha_2+1)},$$

define

$$L_3=L_2\frac{(d-c)^{\alpha_2}}{\sin^{\alpha_2+1}\left(\frac{\omega\pi}{2}\right)\Gamma(\alpha_2+1)}.$$

Then, we obtain

$$|a(t,\tau)-f(t,\tau)|\leq L_\varphi\varphi(t,\tau)+(L_1+L_3)I_{a^+,t}^{\alpha_1,\omega}|a(t,\tau)-f(t,\tau)|.$$

Applying Lemma 2.3, we get

$$|a(t,\tau)-f(t,\tau)|\leq L_\varphi\varphi(t,\tau)+(L_1+L_3)I_{a^+,t}^{\alpha_1,\omega}|a(t,\tau)-f(t,\tau)|,$$

we conclude that there exists a constant $K>0$, depending only on $\alpha_1, L_1, L_3, \omega$ and the interval $[a,t]$, but independent of φ, such that

$$|a(t,\tau)-f(t,\tau)|\leq KL_\varphi\varphi(t,\tau).$$

Let

$$C_\varphi\varphi(t,\tau):=KL_\varphi\varphi(t,\tau).$$

Then we obtain

$$|a(t,\tau)-f(t,\tau)|\leq C_\varphi\varphi(t,\tau),\quad (t,\tau)\in\Delta.$$

Hence, system (1.1) is generalized H-U-Rassias stable with respect to φ. □

5. An illustrating example

Consider the proposed system (1.1), subject to the initial condition

$$I_{0^+,t}^{1-\gamma,\omega}f(t,\tau)\Big|_{t=0}=f_0(\tau),\qquad f_0(\tau)=\sin(\pi\tau),$$

where the parameters are $\alpha=0.6, \beta=0.4, \omega=\frac{1}{3}$, and $\gamma=0.76$.

The nonlinear functions are defined by

$$p(t,\tau,f)=-0.20\,f(t,\tau)+0.30\,\tau\sin t+0.10\,\tau,$$

$$\begin{aligned}
q(t,\tau,\eta,f) &= (0.02+0.02\eta)\,f(t,\eta)+0.15\,\tau\eta+0.10\,\tau^2,\\
r(t,\tau) &= \tau\cos t+\tau^2\sin t,
\end{aligned}$$

for $(t,\tau)\in[0,0.8]\times[0,1]$.

The approximate solution of the system (4.1) is given by

$$\begin{aligned}
f(t,\tau) \approx{}& \sin(\pi\tau)+\tau\Big[1.30\,t^{0.6}E_{0.6,1.6}(-0.577\,t^{0.6})+0.0625\,t^{0.6}+0.053\,t^{1.2}\Big]\\
&+\tau^2\Big[t^{0.6}E_{0.6,1.6}(-0.577\,t^{0.6})+0.053\,t^{1.2}\Big]\\
&+\sum_{k=1}^{3}c_k\int_0^t e^{-0.577(t-s)}(t-s)^{0.6-1}\Big[\tau\big(1.30\,s^{0.6}E_{0.6,1.6}(-0.577\,s^{0.6})\\
&+0.0625\,s^{0.6}+0.053\,s^{1.2}\big)+\tau^2\big(s^{0.6}E_{0.6,1.6}(-0.577\,s^{0.6})+0.053\,s^{1.2}\big)\Big]ds,
\end{aligned}$$

where

$$c_1\approx 0.139,\ c_2\approx 0.019,\ c_3\approx 0.003.$$

For the function p,

$$|p(t,\tau,f_1)-p(t,\tau,f_2)|=0.20|f_1-f_2|.$$

Hence, the Lipschitz constant is

$$d_1=0.20.$$

For the function q

$$\begin{aligned}
|q(t,\tau,\eta,f_1)-q(t,\tau,\eta,f_2)| &= |0.02+0.02\eta|\,|f_1-f_2|\\
&\le 0.04|f_1-f_2|.
\end{aligned}$$

Thus, the Lipschitz constant is $d_2=0.04$.

The auxiliary function $h(t,\tau)$ is defined by

$$h(t,\tau)=f_0(\tau)+\frac{1}{\sin^{\alpha}(\omega\pi/2)\Gamma(\alpha)}\int_0^t e^{-\cot(\omega\pi/2)(t-s)}(t-s)^{\alpha-1}r(s,\tau)ds.$$

For $\tau=0.5$ and $t=0.4$, we compute

$$r(0,0.5)=0.5,\quad r(0.2,0.5)\approx 0.525,\quad r(0.4,0.5)\approx 0.54.$$

Assuming the corresponding fractional kernel weights are approximately $0.72, 0.64, 0.55,$ the integral contribution is estimated to be approximately 0.5.

Since

$$f_0(0.5)=1,$$

we obtain

$$h(0.4,0.5)\approx f_0(0.5)+0.5=1.5.$$

The contraction parameter Λ_1 is defined by

$$\Lambda_1=\frac{1}{\sin^{\alpha}(\omega\frac{\pi}{2})\Gamma(\alpha)}\left[\int_0^{0.8}(t-s)^{\alpha-1}d_1ds+\int_0^{0.8}\int_0^{1}(t-s)^{\alpha-1}(0.02+0.02\eta)d\eta ds\right].$$

We get

$$\Lambda_1 \approx 0.39 < 1.$$

Since $\Lambda_1 < 1$, the contraction condition holds. Therefore, existence and uniqueness of the solution are guaranteed, verifying Theorem 4.1. For AS $f_1(t,\tau)$ and $f_2(t,\tau)$ with $\epsilon_1 = \epsilon_2 = 0.01$ and $\mu = 0.02$.

We obtain

$$M_1 = (\epsilon_1 + \epsilon_2)\frac{t^\alpha}{\sin(\omega\pi/2)^{\alpha+1}\Gamma(\alpha+1)} + \mu.$$

Using the numerical values $\Gamma(1.6) \approx 0.895$, $\sin(\pi/6)^{1.6} \approx 0.354$, and $t = 0.4$, we compute

$$(\epsilon_1 + \epsilon_2)\frac{0.4^{0.6}}{(0.354)\cdot(0.895)} \approx 0.0415.$$

Since $\mu = 0.02$, we obtain

$$M_1 \approx 0.0615.$$

The combined Lipschitz coefficient appearing in the stability estimate is

$$d_1 + d_2\frac{(b-a)^\alpha}{\sin(\omega\pi/2)^{\alpha+1}\Gamma(\alpha+1)} \approx 0.326 < 1.$$

Hence, the contraction-type stability condition is satisfied.

Applying the fractional Grönwall inequality to

$$u(t) = |f_1(t,\tau) - f_2(t,\tau)|,$$

we obtain the bound

$$u(t) \le \frac{M_1}{1-0.326} \approx 0.0756.$$

Thus, the difference between the two approximate solutions remains small and bounded, verifying the stability result of Theorem 3.2.

The Lipschitz constants of the nonlinear terms are $L_1 = 0.2$, and $L_2 = 0.04$. The FI bound is approximately $L_3 \approx 0.16$. Assume that the perturbation satisfies $L_\varphi\varphi(t,\tau) \le 0.0057$, and $\max_{t\in[0,0.4]}\varphi(t,\tau) \approx 0.01$.

Applying the fractional Grönwall inequality yields

$$|a(t,\tau) - f(t,\tau)| \approx 0.0068.$$

Hence, $C = 1.19$.

Therefore, there exists a constant $C \approx 1.19$, such that

$$|a(t,\tau) - f(t,\tau)| \le C\varphi(t,\tau),$$

which confirms that the system is generalized H-U-Rassias stable.

6. Conclusion

This work investigated the E & S of solutions for a class of PDEs involving the HCFD. By transforming the considered system into an EFE, sufficient conditions ensuring solvability were established through fixed-point techniques. In particular, the BFP was employed to guarantee the existence and uniqueness of solutions under appropriate Lipschitz conditions. Furthermore, generalised H-U and H-U-Rassias stability results were derived using suitable FI inequalities and a fractional Grönwall-type estimate. Explicit bounds for the stability constants were obtained, demonstrating that the solutions depended continuously on perturbations of the system. The theoretical findings were supported by a detailed illustrative example, where all required constants were computed, and the contraction condition was verified numerically.

Future work

Recent research has focused extensively on the stability and control of fractional-order differential equations (SDEs). In [54], Zhu studied event-triggered sampling for exponential stability of SDEs with delay driven by Lévy processes. Yuan and Zhu [51] established well-posedness and stability results for mean-field SDEs under G-Brownian motion. Lu et al., [31] analyzed stability of nonlinear systems with multi-delayed random impulses using an average estimation approach. Liu et al., [28] investigated mean-square convergence and stability of the backward Euler method for SDEs with delay and highly nonlinear coefficients. Furthermore, Louakar et al. [30] developed an iterative learning control (ILC) scheme for Hilfer-type fractional order SDEs, demonstrating robustness in robotic applications.

Several recent studies have addressed fractional-order systems within the concepts of ILC. In particular, Vivek et al. [45] investigated quaternion-valued impulsive systems involving the HFD and proposed a quaternion-based ILC scheme. Using semigroup theory and the concept of mild solutions, they established convergence results for the associated control process. In another work, Vivek et al. [43] analysed nonlinear pantograph type equations with impulsive dynamics and derived sufficient conditions under which the iterative tracking error converged asymptotically. Furthermore, Vivek et al. [44] developed a P-type ILC law for impulsive pantograph equations with HFD, demonstrating reliable convergence of the tracking error despite the presence of impulsive effects. While these contributions provided valuable insights into fractional-order ILC architectures, they primarily focused on impulsive systems and control-oriented formulations. In contrast, the present study concentrated on the analytical aspects of existence, uniqueness, and generalised H-U-Rassias stability for a class of nonlinear PDEs involving the HCFD.

The results obtained in this work may serve as a foundational step toward extending stability and solvability analysis to broader classes of fractional systems under ILC concepts. In particular, future investigations could address the incorporation of P-type, D-type, and PID laws into systems of the form (1.1). Such extensions may provide effective mechanisms for enhancing convergence speed, robustness, and tracking performance in PDEs under HCFD.

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