Journal of Applied Analysis and Computation
Volume 16, Number 5, October 2026, 2438–2457

Website:http://www.jaac-online.com
DOI:10.11948/20240447

NON-HOMOGENEOUS INITIAL-BOUNDARY VALUE PROBLEM FOR COUPLED HIROTA EQUATION ON THE HALF LINE

Huifeng Wang[1] and Ju Wu[2,†]

Abstract We study the initial-boundary value problem of the coupled Hirota equation on the right half line with nonhomogeneous data. It is shown that the initial-boundary value problem is local well-posed. The main idea of the proof for the local well-posedness is to derive an explicit solution formula, which is obtained by applying the Fourier and Laplace transforms, and then obtain a priori estimates using the restricted norm method. Additionally, we obtain the smoothing results that the nonlinearities of the coupled Hirota equation on the half line are smoother than the initial data.

Keywords Coupled Hirota equation, initial-boundary value problem, well-posedness.

MSC(2010) 35Q55, 35Q53, 35G61.

1. Introduction

We are concerned with the following initial-boundary value problem (IBVP) of the coupled Hirota equation posed on the right half line:

$$\begin{cases} iu_t + \alpha_1 \partial_x^2 u + i\beta \partial_x^3 u + i\gamma \left\{ (2|u|^2 + |v|^2) u_x + u\bar{v}v_x \right\} + \delta \left(|u|^2 + |v|^2 \right) u = 0, \\ iv_t + \alpha_2 \partial_x^2 v + i\beta \partial_x^3 v + i\gamma \left\{ (|u|^2 + 2|v|^2) v_x + \bar{u}vu_x \right\} + \delta \left(|u|^2 + |v|^2 \right) v = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}^+, \\ u(0,t) = f(t), \quad v(0,t) = g(t), \qquad t \in \mathbb{R}^+, \end{cases} \tag{1.1}$$

where $\alpha_j (j=1,2) > 0, \beta > 0,\ \gamma,\ \delta$ are given real constants and u, v are complex valued functions. The data $(u_0, v_0, f, g) \in H_x^s(\mathbb{R}^+) \times H_x^s(\mathbb{R}^+) \times H_t^{\frac{s+1}{3}}(\mathbb{R}^+) \times H_t^{\frac{s+1}{3}}(\mathbb{R}^+)$ with the additional conditions $u_0(0) = f(0),\ v_0(0) = g(0)$ for $\frac{1}{2} < s < 3$. The given compatibility conditions are to ensure that the solutions are continuous space-time functions with $\frac{1}{2} < s < 3$.

It is well known that the nonlinear Schrödinger (NLS) equation models slowly varying amplitude electromagnetic waves in a nonlinear medium. But the NLS equation becomes insufficient when pulses lengths become comparable to the wavelength. Adding some terms and the generated pulse propagation is called the higher-order nonlinear Schrödinger (HNLS) equation, which contains higher-order dispersive effect, self-steepening and inelastic Raman scattering terms [2, 29, 34]. The generalization of HNLS equation to a coupled Hirota equation was first proposed by Tasgal and Potasek [34], who eliminated the inelastic Raman scattering effect and explained the co-propagation of two ultrashort pulses with the effects of third order dispersion

†The corresponding author.
[1]School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing 100021, China
[2]School of Mathematics and Statistics, Henan University, Kaifeng 475004, China
Email: wanghuifeng_math@126.com(H. Wang), wujun@henu.edu.cn(J. Wu)

and self-steepening. In view of physical scenarios, for IBVP with the half line, the semi-infinite domain is practical as it avoids the need to consider downstream boundaries, and solutions to the coupled Hirota equations approximate waves moving in the positive x direction.

There are many references to the Cauchy problem for the Hirota equation or the coupled Hirota equation in the literature. We refer the readers to see [8, 16, 22, 23, 25, 33] for the well-posedness theory of Hirota equation. In particular, [16] obtained local well posedness for $s = -\frac{1}{4}$ only for a more simple case, where one of the nonlinear terms does not appear. Later in [25] and [33], Laurey and Staffilani obtained the local well-posedness in H^s for $s > \frac{3}{4}$ and $s \geq \frac{1}{4}$, respectively. In addition, Laurey also proved global well-posedness in H^1 and $H^s(s \geq 2)$. By the Fourier restricted norm method (also named $X^{s,b}$ method), Huo and Guo [22] also obtained the local well-posedness in H^s for $s \geq \frac{1}{4}$ and global well-posedness in H^s for $1 \leq s \leq 2$. To the best of our knowledge, the most recent result is [23], which established the local well-posedness of Cauchy problem with the nonlinear terms at the endpoint $s = -\frac{1}{4}$ in H^s and the global well-posedness in H^s for $s \geq 0$. There is only one result for well-posedness research of the coupled Hirota equation: Huo and Jia [24] used Fourier restricted norm method ($X^{s,b}$ method) to prove the local well-posedness in $H^s \times H^s$ for $s \geq \frac{1}{4}$.

Concerning the initial-boundary value problem, it has undergone in-depth research depending on different tools. Relying on Duhamel forcing operator and $X^{s,b}$ method, Colliander, Kenig [10] and Holmer [21] studied the Korteweg–de Vries(KdV) equation on the right half-line, left half-line and line segment with low regularity requirement. Holmer [20] also investigated the NLS equation depending on the same tools. Bona et al. [3–5] introduced the Laplace transform method to study the different dispersive equations. The IBVP of Hirota equation posed on the half line was considered by Guo and Wu [17] using the Laplace transform and $X^{s,b}$ method to investigate the local well-posedness in H^s for $s > \frac{1}{2}$ and global well-posedness in H^1. In addition, we also obtained that the nonlinearities are smoother than the initial data. Recently, Guo and Wu [18] also studied IBVP of the Hirota equation posed on a finite interval. The classical multiplier method, Lions-Magenes's interpolation and Tartar's interpolation theorem are used to establish the local well-posedness in $C(0, T; H^s) \cap L^2(0, T; H^{s+1})$ for $s \geq 0$, and the local solution is extended to global one by a priori bound. We also refer to [1, 9, 11–13, 15, 19, 26–28, 30–32] for more IBVP related to a higher-order equation or coupled system. Specifically, we note that in the work [1], Alkın, Mantzavinos and Özsarı investigated the initial-boundary value problem for the higher-order nonlinear Schrödinger equation on the half-line based on the Fokas unified transform and Strichartz estimates, where the nonlinearity in the equation is of power type. The aforementioned method can be extended to derivative nonlinearities, but it requires refining multilinear estimates and regularity conditions to compensate for the additional complexity induced by the derivatives.

In this paper we aim to study the IBVP (1.1) by using suitable tools, which include the Laplace transform and $X^{s,b}$ method, initiated in [3, 10]. The $X^{s,b}$ method was proposed by Bourgain [6, 7], and the application of this method to IBVP was first presented in [10]. Their idea was to use a boundary forcing operator that could convert IBVP to an initial value problem, and the theory of initial value problem succeeds to the IBVP. The introduction of the Laplace transform was another step forward in the study of IBVP, and the idea was first proposed by Bona et al. [3], who used it to obtain a series of results [3–5]. We are now prepared to study the IBVP of the coupled Hirota equation using the tools provided, which include the Laplace transform and $X^{s,b}$ method. In order to give our main theorem, we start with a definition.

Definition 1.1. We say that (1.1) is locally well-posed in $H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+)$ if for any $(u_0, v_0, f,$

$g) \in H_x^s(\mathbb{R}^+) \times H_x^s(\mathbb{R}^+) \times H_t^{\frac{s+1}{3}}(\mathbb{R}^+) \times H_t^{\frac{s+1}{3}}(\mathbb{R}^+)$, with the additional compatibility condition $u_0(0) = f(0), v_0(0) = g(0)$ for $s > \frac{1}{2}$, the equation $\Phi(u, v) = (u, v)$, where Φ is defined in (3.3), has a unique solution in

$$\left\{X_1^{s,b} \cap C_t^0 H_x^s \cap C_x^0 H_t^{\frac{s+1}{3}}\right\} \times \left\{X_2^{s,b} \cap C_t^0 H_x^s \cap C_x^0 H_t^{\frac{s+1}{3}}\right\}$$

for any $b < \frac{1}{2}$ and some sufficiently small

$$T := T\left(\|u_0\|_{H^s(\mathbb{R}^+)}, \|f\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}, \|v_0\|_{H^s(\mathbb{R}^+)}, \|g\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}\right).$$

Furthermore, the solution depends continuously on the initial and boundary data.

More precisely, we state our main theorem as follows.

Theorem 1.1. *For any $\frac{1}{2} < s < \frac{7}{2}, s \neq \frac{1}{2} + n, n \in \mathbb{N}$. Then equations* (1.1) *is locally well-posed in $H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+)$. Moreover, for $a < \min\{2s-1, \frac{1}{2}, \frac{7}{2} - s\}$, the solution of* (1.1) *satisfies*

$$u(x,t) - W_0^t(u_0, f) \in C_t^0 H_x^{s+a}([0,T] \times \mathbb{R}^+),$$
$$v(x,t) - W_0^t(v_0, g) \in C_t^0 H_x^{s+a}([0,T] \times \mathbb{R}^+),$$

where W_0^t is the solution of the corresponding linear problem (3.1) *below.*

The proof of the above theorem relies on an equivalent integral system, and we intend to find a priori estimates of it. Note that the equivalent integral system, which includes the group operator, Duhamel term and boundary operator, is established by the Fourier transform, Laplace transform, and the superposition of solution operators of linear free equations with nonlinearity by Duhamel principle. Furthermore, the restricted norm method can be used to achieve the contraction argument. In addition, it is not clear that the different extensions of initial data affect the uniqueness of solutions on $\mathbb{R}^+$. The uniqueness issue will be resolved in Section 5.2 below.

The rest of this paper is organized as follows. Section 2 contains some notations, the function spaces and the given equivalent integral equations. In section 4, we present some linear and nonlinear estimates, which are used to complete the fixed point theory. Section 5 states the proof of Theorem 1.1.

2. Preliminaries

2.1. Notations and function spaces

We define the one dimension Fourier transform as

$$\mathcal{F}f = \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

The inhomogeneous L^2-based Sobolev space $H^s = H^s(\mathbb{R})$ are defined by the norm

$$\|f\|_{H^s} = \|f\|_{H^s(\mathbb{R})} = \left\|\langle\xi\rangle^s \widehat{f}(\xi)\right\|_{L^2_\xi},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. Sobolev space $H^s(\mathbb{R}^+)$ on the half line for $s > -\frac{1}{2}$ are defined by the norm

$$\|g\|_{H^s(\mathbb{R}^+)} = \inf\{\|\widetilde{g}\|_{H^s(\mathbb{R})} : \widetilde{g}\chi_{(0,\infty)} = g\},$$

where χ is the characteristic function. Note that we need to restrict $s > -\frac{1}{2}$ since multiplication with characteristic function is not defined for H^s distribution when $s \le -\frac{1}{2}$.

We define the Hirota propagator as

$$W_{\mathbb{R}}^j g(x,t) = e^{t\left(i\alpha_j\partial_x^2 - \beta\partial_x^3\right)} g(x) = \mathcal{F}^{-1}\left(e^{-it\phi_j(\cdot)}\widehat{g}(\cdot)\right)(x), \quad j = 1, 2,$$

where $g \in L^2(\mathbb{R})$ and $\phi_j(\xi) = \alpha_j\xi^2 - \beta\xi^3$. For $s, b \in \mathbb{R}$, the restricted norm spaces $X_j^{s,b}(\mathbb{R}\times\mathbb{R})$ corresponding to the Hirota flow are defined by the norm

$$\|u\|_{X_j^{s,b}} = \left(\iint_{\xi,\tau} \langle\xi\rangle^{2s}\langle\tau+\phi_j(\xi)\rangle^{2b}|\widehat{u}(\xi,\tau)|^2\,d\xi\,d\tau\right)^{\frac{1}{2}}.$$

We define a smooth compactly supported function $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta(t) = 1$ on $[-1,1]$, *supp* $\eta \subset [-2,2]$, and $\eta_T(t) = \eta(tT^{-1})$. Let ρ be a smooth function supported on $[-1,\infty)$, and $\rho(x) = 1$ for $x > 0$. For any function f, we set the notation

$$D_0[f(x,t)] = f(0,t).$$

We define the notations

$$f(\xi,\tau) = \langle\xi\rangle^s\langle\tau+\phi_j(\xi)\rangle^b|\widehat{u}(\xi,\tau)|,$$
$$\sigma = \tau + \phi_j(\xi), \quad j = 1, 2.$$

Denote the notation $\int_*$ as the integral

$$\int_{\xi_0-\xi_1+\xi_2-\xi_3=0;\tau_0-\tau_1+\tau_2-\tau_3=0} d\tau_1 d\tau_2 d\tau_3 d\xi_1 d\xi_2 d\xi_3.$$

Throughout this paper, the notation $a \lesssim (\gtrsim) b$ is defined by $a \le (\ge) Cb$ for any constant C. The notation $a\pm$ indicates $a \pm \epsilon$, where ϵ is an arbitrarily small constant.

3. Formulation of the problem

To obtain a solution representation of (1.1), we need to construct an integral equation on $\mathbb{R}$ in this section, which is equivalent to the IBVP (1.1). We state the following lemma of extensions of $H^s(\mathbb{R}^+)$ functions. Note that any H^s extension $\widetilde{u}_0$ is continuous on $\mathbb{R}$ and $u_0(0)$ is well defined for $u_0 \in H^s(\mathbb{R}^+)(s > \frac{1}{2})$. For the sake of convenience, we will use notations α, $W_{\mathbb{R}}$, $X^{s,b}$ to denote the previous notations α_j, $W_{\mathbb{R}}^j$, $X_j^{s,b}$.

Lemma 3.1. ([14, Lemma 2.1]) *Let $h \in H^s(\mathbb{R}^+)$ for some $-\frac{1}{2} < s < \frac{3}{2}$.*

(1) *If* $-\dfrac{1}{2} < s < \dfrac{1}{2}$, *then* $\|\chi_{(0,\infty)}h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.

(2) *If* $\dfrac{1}{2} < s < \dfrac{3}{2}$, *and* $h(0) = 0$, *then* $\|\chi_{(0,\infty)}h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.

To construct the integral solution of (1.1), we consider the linear initial-boundary value problem of single Hirota equation on $\mathbb{R}^+$:

$$\begin{cases} i\zeta_t + \alpha\zeta_{xx} + i\beta\zeta_{xxx} = 0, \quad (x,t) \in (0,+\infty) \times (0,\infty), \\ \zeta(x,0) = \zeta_0(x), \\ \zeta(0,t) = h(t), \end{cases} \tag{3.1}$$

where $\alpha = \alpha_1$ or α_2, $\zeta = u$ or v, $h(t) = f(t)$ or $g(t)$, $\zeta_0(x) \in H^s(\mathbb{R}^+)$, $h(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$ with additional compatibility condition $\zeta_0(0) = h(0)$ for $s > \frac{1}{2}$. Using $\zeta_0 = h = 0$ with the method of odd extension, we have the uniqueness of solution of (3.1). In the following expression, we decompose the solution operator into two parts: One of which is a boundary operator, which contains zero initial data, and the propagator on the whole real line. Relying on the idea of construction of solution in [4], we can write the solution as

$$W_0^t(\zeta_0, h) = W_0^t(0, h-p) + W_{\mathbb{R}}(t)\widetilde{\zeta}_0,$$

where $\widetilde{\zeta}_0$ is an extension of the function ζ_0 on the full line $\mathbb{R}$, and $\widetilde{\zeta}_0$ satisfies $\|\widetilde{\zeta}_0\|_{H^s(\mathbb{R})} \lesssim \|\zeta_0\|_{H^s(\mathbb{R}^+)}$,

$$p(t) = \eta(t)D_0\left[W_{\mathbb{R}}(t)\widetilde{\zeta}_0\right] = \eta(t)\left[W_{\mathbb{R}}(t)\widetilde{\zeta}_0\right]\Big|_{x=0},$$

which is well-defined and is in $H^{\frac{s+1}{3}}(\mathbb{R}^+)$.

To obtain the boundary operator $W_0^t(0,f)$, we consider the following linear boundary value problem:

$$\begin{cases} i\zeta_t + \alpha\zeta_{xx} + i\beta\zeta_{xxx} = 0, \\ \zeta(x,0) = 0, \\ \zeta(0,t) = h(t), \end{cases} \tag{3.2}$$

where $h(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$. Moreover, we have the compatibility condition $h(0) = 0$. Following [17] we can write solution formula as

$$\begin{aligned} \zeta(x,t) &= \frac{1}{2\pi i}\int_I e^{-i(\alpha\mu^2-\beta\mu^3)t}e^{\omega(\mu)x}(3\beta\mu^2 - 2\alpha\mu)\widetilde{h}(\beta\mu^3 - \alpha\mu^2)\,d\mu \\ &= \frac{1}{2\pi}\int_I e^{-i(\alpha\mu^2-\beta\mu^3)t}e^{\omega(\mu)x}(3\beta\mu^2 - 2\alpha\mu)\widehat{h}(\beta\mu^3 - \alpha\mu^2)\,d\mu, \end{aligned}$$

where

$$\omega = \frac{-i\mu + \frac{i\alpha}{\beta} - \sqrt{3\mu^2 - \frac{2\alpha}{\beta}\mu - \frac{\alpha^2}{\beta^2}}}{2},$$

$I = (\alpha/\beta, +\infty)$ and $\widehat{h}(\xi) = \mathcal{F}\left(\chi_I h\right)(\xi)$. For $x \geq 0$, we can write W_{0j}^t as

$$\begin{aligned} \zeta_j(x,t) = &\frac{1}{2\pi}\int_{\mathbb{R}} \chi_I e^{-i(\alpha\mu^2-\beta\mu^3)t+\omega(\mu)x}(3\beta\mu^2 - 2\alpha_j\mu)\rho\left(x\sqrt{3\mu^2 - \frac{2\alpha_j}{\beta}\mu - \frac{\alpha_j^2}{\beta^2}}/2\right) \\ &\times \widehat{h}(\beta\mu^3 - \alpha_j\mu^2)\,d\mu. \end{aligned}$$

We added the cut-off function ρ in solution so that the integral converge for all x.

We set a system of integral equations $\Phi(u,v)$ equivalent to (1.1) on $[0,T]$,

$$\begin{cases} \Phi u = \eta_T(t)W_{\mathbb{R}}(t)\widetilde{u}_0 + i\eta_T(t)\displaystyle\int_0^t W_{\mathbb{R}}(t-t')G_1(u)\,dt' + \eta_T(t)W_0^t(0, f-p_1-q_1)(t), \\ \Phi v = \eta_T(t)W_{\mathbb{R}}(t)\widetilde{v}_0 + i\eta_T(t)\displaystyle\int_0^t W_{\mathbb{R}}(t-t')G_2(u)\,dt' + \eta_T(t)W_0^t(0, g-p_2-q_2)(t), \end{cases} \tag{3.3}$$

where $\widetilde{u}_0$ and $\widetilde{v}_0$ are the H^s extensions of u_0 and v_0 to $\mathbb{R}$, respectively,

$$G_1(u) = i\gamma\left\{(2|u|^2+|v|^2)u_x + u\bar{v}v_x\right\} + \delta\left(|u|^2+|v|^2\right)u, \tag{3.4}$$

$$G_2(u) = i\gamma\left\{(|u|^2+2|v|^2)v_x + \bar{u}vu_x\right\} + \delta\left(|u|^2+|v|^2\right)v, \tag{3.5}$$

$$p_1(t) = \eta(t)D_0\left(W_{\mathbb{R}}(t)\widetilde{u}_0\right), \quad q_1(t) = \eta(t)D_0\left(\int_0^t W_{\mathbb{R}}(t-t')G_1(u)\,dt'\right), \tag{3.6}$$

$$p_2(t) = \eta(t)D_0\left(W_{\mathbb{R}}(t)\widetilde{v}_0\right), \quad q_2(t) = \eta(t)D_0\left(\int_0^t W_{\mathbb{R}}(t-t')G_2(u)\,dt'\right). \tag{3.7}$$

For sufficiently small T, we will prove that the integral system (3.3) has a unique solution in the Banach space $X^{s,b}\cap C_t^0H_x^s\cap C_x^0H_t^{\frac{s+1}{3}}$ on $\mathbb{R}\times\mathbb{R}$. It is obvious that the restriction of u to $\mathbb{R}^+\times[0,T]$ satisfies (1.1) in the distribution sense by the definition of boundary operator, and the smooth solution of (3.3) satisfies (1.1) in classical sense.

It is well known that the embedding $X^{s,b}\subset C_t^0H_x^s$ for $b>\frac{1}{2}$. To prove the contraction argument, we also recall the following estimates from [17]. For any s and b, we have

$$\|\eta(t)W_{\mathbb{R}}g\|_{X^{s,b}} \lesssim \|g\|_{H^s}. \tag{3.8}$$

For $s\in\mathbb{R},\ 0\le b_1<\frac{1}{2}$ and $0\le b_2<1-b_1$, we have

$$\left\|\eta(t)\int_0^t W_{\mathbb{R}}(t-t')F(t')dt'\right\|_{X^{s,b_2}} \lesssim \|F\|_{X^{s,-b_1}}. \tag{3.9}$$

For sufficiently small T and $-\frac{1}{2}<b_1<b_2<\frac{1}{2}$, we have

$$\|\eta(t/T)F\|_{X^{s,b_1}} \lesssim T^{b_2-b_1}\|F\|_{X^{s,b_2}}. \tag{3.10}$$

Finally, the following inequality will be used throughout this paper. The proof of the lemma can be found in [14].

Lemma 3.2. *If $\beta\ge\gamma\ge0$ and $\beta+\gamma>1$, then we have*

$$\int\frac{1}{\langle x-a\rangle^{\beta}\langle x-b\rangle^{\gamma}}\mathrm{d}x \lesssim \langle a-b\rangle^{-\gamma}\varphi_\beta(a-b),$$

where

$$\varphi_\beta(c) = \begin{cases} 1 & \beta>1, \\ \log(1+\langle c\rangle) & \beta=1, \\ \langle c\rangle^{1-\beta} & \beta<1. \end{cases}$$

4. A priori estimates

4.1. Linear estimates

We start with the corresponding Kato smoothing estimates from [17] for the Hirota group of the equivalent integral system.

Lemma 4.1. ([17, Lemma 3.1]) *For fixed $s \geq 0$ and any $\zeta \in H^s(\mathbb{R})$, we have $\eta(t)W_{\mathbb{R}}\zeta \in C_x^0 H_t^{\frac{s+1}{3}}(\mathbb{R}\times\mathbb{R})$, and*

$$\left\|\eta(t)W_{\mathbb{R}}^j\zeta\right\|_{L_x^\infty H_t^{\frac{s+1}{3}}} \lesssim_{\alpha_j,\beta} \|\zeta\|_{H^s(\mathbb{R})},$$

where $j = 1, 2$.

The following Lemma 4.2 and 4.3 demonstrate that the Hirota boundary operator belongs to $X^{s,b} \cap C_t^0 H_x^s \cap C_x^0 H_t^{\frac{s+1}{3}}$.

Lemma 4.2. ([17, Lemma 3.2]) *Let $s \geq 0, b \leq \frac{1}{2}$. Then for any compactly supported smooth function η and h satisfying $\chi_{(0,\infty)}h \in H^{\frac{s+1}{3}}(\mathbb{R})$, we have*

$$\left\|\eta(t)W_{0j}^t(0,h)\right\|_{X^{s,b}} \lesssim_{\alpha_j,\beta} \left\|\chi_{(0,\infty)}h\right\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})},$$

where $j = 1, 2$.

Lemma 4.3. ([17, Lemma 3.3]) *For any $s \geq 0$, and h such that $\chi_{(0,\infty)}h \in H^{\frac{s+1}{3}}(\mathbb{R})$, we have*

$$W_{0j}^t(0,h) \in C_t^0 H_x^s(\mathbb{R}\times\mathbb{R})$$

and

$$\eta(t)W_{0j}^t(0,h) \in C_x^0 H_t^{\frac{s+1}{3}}(\mathbb{R}\times\mathbb{R}),$$

for $j = 1, 2$.

Lemma 4.4. *For fixed $b < \frac{1}{2}$ and any smooth compactly supported function $\eta(t)$, we have*

$$\left\|\eta(t)\int_0^t W_{\mathbb{R}}^j(t-t')F\,dt'\right\|_{C_x^0 H_t^{\frac{s+1}{3}}(\mathbb{R}\times\mathbb{R})}$$
$$\lesssim_{\alpha_j,\beta} \begin{cases} \|F\|_{X_j^{s,-b}}, & \text{for } 0 \leq s \leq \frac{1}{2}, \\ \|F\|_{X_j^{s,-b}} + \|F\|_{X_j^{\frac{1}{2},\frac{2s-1-6b}{6}}}, & \text{for } s > \frac{1}{2}, \end{cases}$$

where $j = 1, 2$.

Proof. Relying on the fact that the $X^{s,b}$ norm is independent of space translation, it suffices to prove the bound for $\eta D_0\left(\int_0^t W_{\mathbb{R}}^j(t-t')F\,dt'\right)$. At $x = 0$, we have

$$D_0\left(\int_0^t W_{\mathbb{R}}^j(t-t')F\,dt'\right) = \int_{\mathbb{R}}\int_0^t e^{-i(t-t')\phi_j(\xi)}F(\widehat{\xi},t')\,dt'd\xi,$$

where $\phi_j(\xi) = \alpha_j\xi^2 - \beta\xi^3$. Using the facts that

$$F(\widehat{\xi},t') = \int_{\mathbb{R}} e^{it'\lambda}\widehat{F}(\xi,\lambda)\,d\lambda,$$

and

$$\int_0^t e^{it'(\lambda+\phi_j(\xi))}dt' = \frac{e^{it(\lambda+\phi_j(\xi))}-1}{i(\lambda+\phi_j(\xi))},$$

we obtain

$$D_0\left(\int_0^t W_{\mathbb{R}}^j(t-t')F\,dt'\right) = \int_{\mathbb{R}^2} \frac{e^{it\lambda}-e^{-it\phi_j(\xi)}}{i(\lambda+\phi_j(\xi))}\widehat{F}(\xi,\lambda)\,d\lambda d\xi.$$

Using a smooth cut-off function ψ in $[-1,1]$ and taking $\psi^c = 1-\psi$ we write

$$\begin{aligned}\eta(t)D_0\left(\int_0^t W_{\mathbb{R}}^j(t-t')F\,dt'\right) =& \eta(t)\int_{\mathbb{R}^2} \frac{e^{it\lambda}-e^{-it\phi_j(\xi)}}{i(\lambda+\phi_j(\xi))}\psi(\lambda+\phi_j(\xi))\widehat{F}(\xi,\lambda)\,d\lambda d\xi\\ &+\eta(t)\int_{\mathbb{R}^2}\frac{e^{it\lambda}}{i(\lambda+\phi_j(\xi))}\psi^c(\lambda+\phi_j(\xi))\widehat{F}(\xi,\lambda)\,d\lambda d\xi\\ &-\eta(t)\int_{\mathbb{R}^2}\frac{e^{-it\phi_j(\xi)}}{i(\lambda+\phi_j(\xi))}\psi^c(\lambda+\phi_j(\xi))\widehat{F}(\xi,\lambda)\,d\lambda d\xi\\ :=& \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Using Taylor expansion we have

$$\frac{e^{it\lambda}-e^{-it\phi_j(\xi)}}{i(\lambda+\phi_j(\xi))} = ie^{it\lambda}\sum_{k=1}^{\infty}\frac{(-it)^k}{k!}(\lambda+\phi_j(\xi))^{k-1}.$$

The facts that $\|uv\|_{H^s} \lesssim \|u\|_{H^1}\|v\|_{H^s}$, $\|t^k\eta\|_{H^1} \lesssim k$ and $\sum_{k=1}^{\infty}\frac{1}{(k-1)!}$ convergence give

$$\begin{aligned}\|\mathrm{I}\|_{H_t^{\frac{s+1}{3}}} &\lesssim \sum_{k=1}^{\infty}\frac{\|\eta(t)t^k\|_{H^1}}{k!}\left\|\int_{\mathbb{R}^2} e^{it\lambda}(\lambda+\phi_j)^{k-1}\psi(\lambda+\phi_j)\widehat{F}(\xi,\lambda)\,d\lambda d\xi\right\|_{H_t^{\frac{s+1}{3}}}\\ &\lesssim \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left\|\langle\lambda\rangle^{\frac{s+1}{3}}\int_{\mathbb{R}}(\lambda+\phi_j)^{k-1}\psi(\lambda+\phi_j)\widehat{F}(\xi,\lambda)\,d\xi\right\|_{L^2_\lambda}\\ &\lesssim \left\|\langle\lambda\rangle^{\frac{s+1}{3}}\int_{\mathbb{R}}\psi(\lambda+\phi_j)\left|\widehat{F}(\xi,\lambda)\right|\,d\xi\right\|_{L^2_\lambda}.\end{aligned}$$

By Cauchy-Schwarz inequality in ξ, we have

$$\begin{aligned}&\left(\int\langle\lambda\rangle^{\frac{2s+2}{3}}\left(\int\limits_{|\lambda+\phi|<1}\langle\xi\rangle^{-2s}\,d\xi\right)\left(\int\limits_{|\lambda+\phi|<1}\langle\xi\rangle^{2s}\left|\widehat{F}(\xi,\lambda)\right|^2\,d\xi\right)d\lambda\right)^{\frac12}\\ \lesssim&\sup_\lambda\left(\langle\lambda\rangle^{\frac{2s+2}{3}}\int\limits_{|\lambda+\phi|<1}\langle\xi\rangle^{-2s}\,d\xi\right)^{\frac12}\|F\|_{X^{s,-b}},\end{aligned}$$

which can be bounded by $\|F\|_{X_j^{s,-b}}$ since

$$\langle\lambda\rangle^{\frac{2s+2}{3}}\int\langle\xi\rangle^{-2s}\,d\xi \lesssim \begin{cases}1, & \text{if } |\lambda|\le 2,\\ \langle\lambda\rangle^{\frac{2s+2}{3}}\int\langle z\rangle^{-\frac{2s}{3}-\frac23}\,dz, & \text{if } |\lambda|>2,\end{cases}$$

where the latter bound from the change of variable $z=\xi^3$.

For the third term III, if $|\xi| \le \left|\frac{3\alpha}{2\beta}\right|$, we have the bound that

$$\begin{aligned}
\|\mathrm{III}\|_{H_t^{\frac{s+1}{3}}} &\lesssim \int_{\mathbb{R}} \int_{|\xi|\le\left|\frac{3\alpha}{2\beta}\right|} \frac{\left\|\eta e^{it\phi_j}\right\|_{H_t^{\frac{s+1}{3}}}}{|\lambda+\phi_j|} \psi^c(\lambda+\phi_j)|\widehat{F}(\xi,\lambda)|\, d\xi d\lambda \\
&\lesssim \int_{\mathbb{R}^2} \frac{\chi_{[-\frac{9\alpha^3}{8\beta^2},\frac{45\alpha^3}{8\beta^2}]}(\phi_j)}{\langle\lambda+\phi_j\rangle} |\widehat{F}(\xi,\lambda)|\, d\xi d\lambda \\
&\lesssim \|F\|_{X_j^{s,-b}} \left\| \frac{\chi_{[-\frac{9\alpha^3}{8\beta^2},\frac{45\alpha^3}{8\beta^2}]}(\phi_j)}{\langle\lambda+\phi_j\rangle^{1-b}} \right\|_{L^2_{\xi,\lambda}} \\
&\lesssim \|F\|_{X_j^{s,-b}} .
\end{aligned}$$

We estimate III for $|\xi| \ge \left|\frac{3\alpha}{2\beta}\right|$, and setting $z=\xi^3$. Then we have

$$\begin{aligned}
&\left\| \eta(t) \int_{\mathbb{R}^2} \frac{e^{it\phi_j(\xi)}}{i(\lambda+\phi_j(\xi))} \psi^c(\lambda+\phi_j(\xi)) \widehat{F}(\xi,\lambda)\, d\lambda d\xi \right\|_{H_t^{\frac{s+1}{3}}} \\
\lesssim &\left\| \langle z\rangle^{\frac{s+1}{3}} \int_{\mathbb{R}} \frac{|\widehat{F}(\xi(z),\lambda)|}{\langle\lambda+\phi_j\rangle} \frac{1}{|3\xi^2(z)|}\, d\lambda \right\|_{L^2_{|z|\ge 27\alpha^3/8\beta^2}} .
\end{aligned}$$

By Cauchy-Schwarz inequality in λ, this is bounded by

$$\left\| \langle z\rangle^{\frac{s+1}{3}} \frac{|\widehat{F}(\xi(z),\lambda)|}{\langle\lambda+\phi_j\rangle^b} \right\|_{L^2_z L^2_\lambda} .$$

Changing variables back to ξ, we have the bound $\|F\|_{X_j^{s,-b}}$ for $b<\frac{1}{2}$.

For the second term II, noticing that $\langle\lambda\rangle \lesssim \langle\lambda+\phi_j\rangle + |\xi|^3$ we have

$$\begin{aligned}
\|\mathrm{II}\|_{H_t^{\frac{s+1}{3}}} &\lesssim \|\eta(t)\|_{H^1} \left\| \langle\lambda\rangle^{\frac{s+1}{3}} \int_{\mathbb{R}} \frac{1}{\lambda+\phi_j} \psi^c(\lambda+\phi_j)\widehat{F}(\xi,\lambda)\, d\xi \right\|_{L^2_\lambda} \\
&\lesssim \left\| \langle\lambda\rangle^{\frac{s+1}{3}} \int_{\mathbb{R}} \frac{1}{\langle\lambda+\phi_j\rangle} \left|\widehat{F}(\xi,\lambda)\right| d\xi \right\|_{L^2_\lambda} \\
&\lesssim \left\| \int_{\mathbb{R}} \langle\lambda+\phi_j\rangle^{\frac{2s-1-6b}{6}} \left|\widehat{F}(\xi,\lambda)\right| d\xi \right\|_{L^2_\lambda} + \left\| \int_{\mathbb{R}} \frac{|\xi|^{s+1}}{\langle\lambda+\phi_j\rangle} \left|\widehat{F}(\xi,\lambda)\right| d\xi \right\|_{L^2_\lambda} .
\end{aligned}$$

For $s>\frac{1}{2}$, by Cauchy-Schwarz inequality in ξ, the second summand is bounded by

$$\begin{aligned}
&\left(\int \left(\int \frac{|\xi|^2}{\langle\lambda+\phi_j\rangle^{2-2b}}\, d\xi \right) \left(\int \frac{|\xi|^{2s}}{\langle\lambda+\phi_j\rangle^{2b}} \left|\widehat{F}(\xi,\lambda)\right|^2 d\xi \right) d\lambda \right)^{\frac{1}{2}} \\
\lesssim &\sup_\lambda \left(\int \frac{|\xi|^2}{\langle\lambda+\phi_j\rangle^{2-2b}}\, d\xi \right)^{\frac{1}{2}} \|F\|_{X_j^{s,-b}} \\
\lesssim &\|F\|_{X_j^{s,-b}} ,
\end{aligned}$$

which holds provided that

$$\sup_{\lambda}\left(\int \frac{|\xi|^2}{\langle \lambda+\phi_j\rangle^{2-2b}}\,d\xi\right)$$

is finite for $b<\frac{1}{2}$. This completes the proof for the case $s>\frac{1}{2}$. For $s\le\frac{1}{2}$, by Cauchy-Schwarz inequality in ξ, we bound

$$\left\|\langle\lambda\rangle^{\frac{s+1}{3}}\int_{\mathbb{R}}\frac{1}{\langle\lambda+\phi_j\rangle}\left|\widehat{F}(\xi,\lambda)\right|\,d\xi\right\|_{L^2_\lambda}$$

by

$$\begin{aligned}&\left(\int\langle\lambda\rangle^{\frac{2s+2}{3}}\left(\int\frac{1}{\langle\lambda+\phi_j\rangle^{2-2b}\langle\xi\rangle^{2s}}\,d\xi\right)\left(\int\frac{\langle\xi\rangle^{2s}}{\langle\lambda+\phi_j\rangle^{2b}}\left|\widehat{F}(\xi,\lambda)\right|^2\,d\xi\right)d\lambda\right)^{\frac{1}{2}}\\&\lesssim\sup_{\lambda}\left(\langle\lambda\rangle^{\frac{2s+2}{3}}\int\frac{1}{\langle\lambda+\phi_j\rangle^{2-2b}\langle\xi\rangle^{2s}}\,d\xi\right)^{\frac{1}{2}}\|F\|_{X_j^{s,-b}}\\&\lesssim\|F\|_{X_j^{s,-b}}.\end{aligned}$$

To obtain the last inequality, consider $|\phi_j|\ll 1$ and $|\phi_j|\gtrsim 1$ separately. Setting $z=\xi^3$ and applying Lemma 3.2 we can bound the supremum that

$$\sup_{\lambda}\left(\langle\lambda\rangle^{\frac{2s+2}{3}}\int\frac{1}{\langle\lambda+\phi_j\rangle^{2-2b}\langle z\rangle^{\frac{2s}{3}+\frac{2}{3}}}\,dz\right)\lesssim\sup_{\lambda}\langle\lambda\rangle^{\frac{2s}{3}+\frac{2}{3}-\min\left\{2-2b,\frac{2s}{3}+\frac{2}{3}\right\}}<\infty,$$

which holds for $b<\frac{1}{2},\ s\le\frac{1}{2}$. For the latter case, by setting $z=\xi^3$ and applying Lemma 3.2, the supremum is bounded by $\sup_\lambda\langle\lambda\rangle^{\frac{2s}{3}+\frac{2}{3}-2+2b}<\infty$, which holds for $b<\frac{1}{2},\ s\le\frac{1}{2}$. □

4.2. Nonlinear estimates

In this section we establish nonlinear estimates of (3.3) in following lemmas. Recall the notations

$$\begin{aligned}&f_\nu(\xi_j,\tau_j)=\langle\xi_j\rangle^s\langle\tau_j+\phi_\nu(\xi_j)\rangle^b|\widehat{u}(\xi_j,\tau_j)|,\\&g(\xi_0,\tau_0)=\langle\xi_0\rangle^{-(s+a)}\langle\tau_0+\phi(\xi_0)\rangle^b|\widehat{\overline{\varphi}}(\xi_0,\tau_0)|,\ \sigma_j^\nu=\tau_j+\phi_\nu(\xi_j)=\tau_j+\beta\xi_j^3+\alpha_\nu\xi_j^2,\\&\phi=\phi_1\text{ or }\phi_2,\ \sigma_0=\sigma_0^1\text{ or }\sigma_0^2,\quad\sigma_j=\sigma_j^1\text{ or }\sigma_j^2,\quad f(\xi_j,\tau_j)=f_1(\xi_j,\tau_j)\text{ or }f_2(\xi_j,\tau_j),\\&\mathcal{F}F_l^0=\frac{g(\xi_0,\tau_0)}{\langle\sigma_0^\nu\rangle^l},\quad\mathcal{F}F_l^j=\frac{f(\xi_j,\tau_j)}{\langle\sigma_j^\nu\rangle^l},\quad j=1,2,3,\quad\nu=1,2,\end{aligned}$$

and define functions $u_1,u_2,u_3=u$ or v. We also recall the Kato smoothing inequality and Maximal function inequality. Then we have

$$\left\|D_xF_{\frac{1}{2}+}\right\|_{L_x^\infty L_t^2}\lesssim\|f\|_{L^2_{\xi,\tau}},\tag{4.1}$$

$$\left\|D_x^{-\frac{1}{4}}F_{\frac{1}{2}+}\right\|_{L_x^4L_t^\infty}\lesssim\|f\|_{L^2_{\xi,\tau}}.\tag{4.2}$$

Interpolation (4.1) and (4.2) with Plancherel identity we obtain

$$\left\|D_x^{1-}F_{\frac{1}{2}-}\right\|_{L_x^{\infty-}L_t^2}\lesssim\|f\|_{L^2_{\xi,\tau}},\tag{4.3}$$

$$\left\| D_x^{-\frac{1}{4}+} F_{\frac{1}{2}-} \right\|_{L_x^{4-} L_t^{\infty-}} \lesssim \|f\|_{L^2_{\xi,\tau}}. \tag{4.4}$$

By Sobolev embedding we have

$$\left\| D_x^{-\frac{1}{2}+\frac{1}{p}-} F_{\frac{1}{2}-\frac{1}{p}+} \right\|_{L_x^p L_t^p} \lesssim \|f\|_{L^2_{\xi,\tau}}, \quad 2 \le p < \infty. \tag{4.5}$$

Strichartz estimate is given by

$$\left\| F_{\frac{1}{2}+} \right\|_{L_x^6 L_t^6} \lesssim \|f\|_{L^2_{\xi,\tau}}. \tag{4.6}$$

Interpolation (4.6) with Plancherel identity, we have

$$\left\| F_{\frac{3}{4}-\frac{3}{2p}+} \right\|_{L_x^p L_t^p} \lesssim \|f\|_{L^2_{\xi,\tau}}, \quad 2 < p < 6. \tag{4.7}$$

Interpolation (4.6) with (4.5), we have

$$\left\| D_x^{-\frac{1}{2}+\frac{3}{p}-} F_{\frac{1}{2}-} \right\|_{L_x^p L_t^p} \lesssim \|f\|_{L^2_{\xi,\tau}}, \quad 6 < p < \infty. \tag{4.8}$$

By (4.7), (4.8) and Hölder inequality, for $0 \le c \le \frac{1}{2}$, we have

$$\left\| \left(\frac{f(\xi,\tau)}{\langle \sigma \rangle^{c+}} \right)^{\vee} \left[\left(\frac{f(\xi,\tau)}{\langle \xi \rangle^{\frac{1}{2}-c+} \langle \sigma \rangle^{\frac{1}{2}-}} \right)^{\vee} \right]^2 \right\|_{L^{2+}_{x,t}} \lesssim \|f\|^3_{L^2_{\xi,\tau}}. \tag{4.9}$$

By (4.3), (4.4) and Hölder inequality, we have

$$\left\| \left(\frac{\langle \xi \rangle^{1-} f(\xi,\tau)}{\langle \sigma \rangle^{\frac{1}{2}-}} \right)^{\vee} \left[\left(\frac{f(\xi,\tau)}{\langle \xi \rangle^{\frac{1}{4}-} \langle \sigma \rangle^{\frac{1}{2}-}} \right)^{\vee} \right]^2 \right\|_{L^{2-}_{x,t}} \lesssim \|f\|^3_{L^2_{\xi,\tau}}. \tag{4.10}$$

Lemma 4.5. *For $s > 1/4$ and $a < \min\{2s - \frac{1}{2}, 1\}$, there exists $\epsilon > 0$ such that $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, we have*

$$\|u_1 \overline{u}_2 u_3\|_{X_j^{s+a,-b}} \lesssim \|u_1\|_{X^{s,b}_{j_1}} \|u_2\|_{X^{s,b}_{j_2}} \|u_3\|_{X^{s,b}_{j_3}}, \tag{4.11}$$

where $j = 1, 2,\ j_m = 1, 2 (m = 1, 2, 3)$.

Proof. Writing the Fourier transform of $u_1 \bar{u}_2 u_3$ as a convolution, we obtain

$$\widehat{u_1 \overline{u}_2 u_3}(\xi,\tau) = \int_* \widehat{u}_1(\xi_1,\tau_1) \widehat{\overline{u}}_2(\xi_2,\tau_2) \widehat{u}_3(\xi_3,\tau_3).$$

It suffices to prove that

$$\left| \iint (u_1 \overline{u}_2 u_3) \bar{g}\, dx\, dt \right| \lesssim \|u_1\|_{X^{s,b}_{j_1}} \|u_2\|_{X^{s,b}_{j_2}} \|u_3\|_{X^{s,b}_{j_3}} \|g\|_{X_j^{-(s+a),b}},$$

for $j_m = 1, 2$ $(m = 1, 2, 3)$, $j = 1, 2$ and $g \in X_j^{-(s+a),b}$. By duality, the desired result is equivalent to prove that

$$\left| \int_* M \left(\frac{\langle \xi_1 \rangle^{1-} f(\xi_1,\tau_1) f(\xi_2,\tau_2) f(\xi_3,\tau_3)}{\langle \xi_2 \rangle^{\frac{1}{4}-} \langle \xi_3 \rangle^{\frac{1}{4}-} \langle \sigma_1 \rangle^{b} \langle \sigma_2 \rangle^{b} \langle \sigma_3 \rangle^{b}} \cdot \frac{g(\xi_0,\tau_0)}{\langle \sigma_0 \rangle^{b}} \right) \right| \lesssim \|f\|^3_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}},$$

where

$$M(\xi_1,\xi_2,\xi,\tau_1,\tau_2,\tau)=\sup\frac{\langle\xi_2\rangle^{\frac{1}{4}-}\langle\xi_3\rangle^{\frac{1}{4}-}\langle\xi_0\rangle^{s+a}}{\langle\xi_1\rangle^{1-}\prod_{j=1}^{3}\langle\xi_j\rangle^{s}}.$$

By Young inequality and Hölder inequality, we have the bound that

$$\left\|\left(\iint_{\xi_1,\xi_2,\tau_1,\tau_2}M^2\right)^{1/2}\left(\iint_{\xi_1,\xi_2,\tau_1,\tau_2}\left(\frac{\langle\xi_1\rangle^{1-}f(\xi_1,\tau_1)f(\xi_2,\tau_2)f(\xi_3,\tau_3)}{\langle\xi_2\rangle^{\frac{1}{4}-}\langle\xi_3\rangle^{\frac{1}{4}-}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}\langle\sigma_3\rangle^{b}}\cdot\frac{g(\xi_0,\tau_0)}{\langle\sigma_0\rangle^{b}}\right)^2\right)^{1/2}\right\|_{L^2_{\xi,\tau}}$$

$$\lesssim\sup_{\xi,\tau}\left(\iint_{\xi_1,\xi_2,\tau_1,\tau_2}M^2\right)\left\|\iint_{\xi_1,\xi_2,\tau_1,\tau_2}\left(\frac{\langle\xi_1\rangle^{1-}f(\xi_1,\tau_1)f(\xi_2,\tau_2)f(\xi_3,\tau_3)}{\langle\xi_2\rangle^{\frac{1}{4}-}\langle\xi_3\rangle^{\frac{1}{4}-}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}\langle\sigma_3\rangle^{b}}\cdot\frac{g(\xi_0,\tau_0)}{\langle\sigma_0\rangle^{b}}\right)^2\right\|_{L^2_{\xi_1,\xi_2,\tau_1,\tau_2}}$$

$$\lesssim\left(\sup_{\xi,\tau}\|M\|_{L^2}\right)\|f\|^3_{L^2_{\xi,\tau}}\|g\|_{L^2_{\xi,\tau}}.$$

By symmetry, we consider the case $|\xi_1|\geq|\xi_2|\geq|\xi_3|$, which implies that $|\xi_1|\gtrsim|\xi_0|$. It is easy to obtain that the supremum is finite if $a<\min\{2s-\frac{1}{2},1\}$. By the Plancherel identity, (4.10) and (4.7), we can achieve the bound $\|f\|^3_{L^2}\|g\|_{L^2}$. This completes the proof of (4.11). □

Lemma 4.6. *For $s>\frac{1}{2}$ and $0<a<2s-1$, there exists $\epsilon>0$ such that $\frac{1}{2}-\epsilon<b<\frac{1}{2}$, we have*

$$\|\partial_x(u_1\bar{u}_2u_3)\|_{X_j^{s+a,-b}}\lesssim\|u_1\|_{X^{s,b}_{j_1}}\|u_2\|_{X^{s,b}_{j_2}}\|u_3\|_{X^{s,b}_{j_3}},\tag{4.12}$$

where $j=1,2,\ j_m=1,2(m=1,2,3)$.

Proof. For (4.12), as in Lemma 4.5, by duality it suffices to show that

$$\left|\iint(u_1\bar{u}_2u_3)_x\overline{\varphi}\,dx\,dt\right|\lesssim\|u_1\|_{X^{s,b}_{j_1}}\|u_2\|_{X^{s,b}_{j_2}}\|u_3\|_{X^{s,b}_{j_3}}\|g\|_{X_j^{-(s+a),b}},$$

where $j_m=1$ or $2(m=1,2,3),\ j=1,2$ and $g\in X_j^{-(s+a),b}$. The desired result is equivalent to prove that

$$I:=\int_*\frac{\langle\xi_0\rangle^{s+a}|\xi_0|g(\xi_0,\tau_0)\prod_{j=1}^{3}f(\xi_j,\tau_j)}{\prod_{j=1}^{3}\langle\xi_j\rangle^{s}\prod_{j=0}^{3}\langle\sigma_j\rangle^{b}}\lesssim\|f\|^3_{L^2}\|g\|_{L^2}.$$

We briefly outline the proof, which is similar to Lemma 3.7 in [17]. We split the domain of integral in following several cases. Consider

$$\begin{aligned}&|\xi_0-\xi_3|\leq2k,\ |\xi_0-\xi_1|\leq2k\text{ or }|\xi_0-\xi_2|\leq2k,\\&|\xi_0-\xi_3|\leq2k,\ |\xi_0-\xi_1|\geq2k\text{ and }|\xi_0-\xi_2|\geq2k,\\&|\xi_0-\xi_3|\geq2k,\ |\xi_0-\xi_1|\leq2k\text{ or }|\xi_0-\xi_2|\leq2k,\end{aligned}$$

where $k=\max\{1,|\frac{2\alpha_1}{3\beta}|,|\frac{2\alpha_2}{3\beta}|\}$. Invoking $\langle\xi_1\rangle\langle\xi_0-\xi_1+\xi_2\rangle\sim\langle\xi_0\rangle\langle\xi_2\rangle$, we have

$$\begin{aligned}I&\lesssim\|F_b^0\|_{L^2_xL^2_t}\|F_b^1\|_{L^6_xL^6_t}\|F_b^2\|_{L^6_xL^6_t}\|F_b^3\|_{L^6_xL^6_t}\\&\lesssim\|f\|^3_{L^2_{\xi,\tau}}\|g\|_{L^2_{\xi,\tau}},\end{aligned}$$

for $a<2s-1$. In the case $|\xi_0-\xi_3|\geq 2k,\ |\xi_0-\xi_1|\geq 2k$ and $|\xi_0-\xi_2|\geq 2k$, we have

$$\max\{|\sigma_0|,|\sigma_1|,|\sigma_2|,|\sigma_3|\}\gtrsim|\xi_0-\xi_1||\xi_0-\xi_2||\xi_0-\xi_3|.$$

Relying on (4.1) and (4.2), we obtain

$$\min\left(\frac{\langle\xi_i\rangle^{\frac14}\langle\xi_j\rangle^{\frac14}}{\langle\xi_k\rangle},\frac{\langle\xi_i\rangle^{\frac14}\langle\xi_k\rangle^{\frac14}}{\langle\xi_j\rangle},\frac{\langle\xi_j\rangle^{\frac14}\langle\xi_k\rangle^{\frac14}}{\langle\xi_i\rangle}\right)\lesssim\frac{\langle\xi_i\rangle^{\frac14}\langle\xi_j\rangle^{\frac14}\langle\xi_k\rangle^{\frac14}}{\langle\xi_i\rangle^{\frac54}+\langle\xi_j\rangle^{\frac54}+\langle\xi_k\rangle^{\frac54}}.$$

Restricting the variables $\xi_0,\cdots,\xi_3$ by the size as $|\xi_{\min}|\leq|\xi_{\mathrm{mid}}|\leq|\xi_{\max 1}|\approx|\xi_{\max}|$, we have

$$\max_{0\leq i,j,k\leq 3}\frac{\langle\xi_i\rangle^{\frac14}\langle\xi_j\rangle^{\frac14}\langle\xi_k\rangle^{\frac14}}{\langle\xi_i\rangle^{\frac54}+\langle\xi_j\rangle^{\frac54}+\langle\xi_k\rangle^{\frac54}}\approx\frac{\langle\xi_{\mathrm{mid}}\rangle^{\frac14}}{\langle\xi_{\max}\rangle^{\frac34}}.$$

It remains to prove that

$$\sup_{\xi_0-\xi_1+\xi_2-\xi_3=0}\frac{\langle\xi_0\rangle^{s+a}\langle\xi_0\rangle\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}}{\langle(\xi_0-\xi_1)(\xi_0-\xi_2)(\xi_0-\xi_3)\rangle^{\frac12-}}\frac{\langle\xi_{\mathrm{mid}}\rangle^{\frac14}}{\langle\xi_{\max}\rangle^{\frac34}}\lesssim 1.$$

At this stage, by symmetry, it suffices to show that, for $|\xi_1|\leq|\xi_3|$,

$$\frac{\langle\xi_0\rangle^{s+a}\langle\xi_0\rangle\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}}{\langle(\xi_0-\xi_1)(\xi_0-\xi_2)(\xi_0-\xi_3)\rangle^{\frac12-}}\frac{\langle\xi_{\mathrm{mid}}\rangle^{\frac14}}{\langle\xi_{\max}\rangle^{\frac34}}\lesssim 1. \tag{4.13}$$

Case 1. $|\xi_1|\leq|\xi_3|\lesssim|\xi_2|\approx|\xi_0|$.

$$\begin{aligned}(4.13)&\lesssim\frac{\langle\xi_0\rangle^{a+\frac14}}{\langle\xi_1\rangle^{s}\langle\xi_0-\xi_1\rangle^{\frac12-}\langle\xi_0-\xi_2\rangle^{\frac12-}\langle\xi_0-\xi_3\rangle^{\frac12-}\langle\xi_3\rangle^{s-\frac14}}\\&\lesssim\begin{cases}\langle\xi_0\rangle^{a-\frac54},\ \text{if } s>\dfrac34,\\ \langle\xi_0\rangle^{-\frac12+a-s-\min\{s-\frac12,\frac34-s\}+},\ \text{if } \dfrac12<s<\dfrac34,\end{cases}\end{aligned}$$

which is finite for $a<\min\{2s,\frac54\}$.

Case 2. $|\xi_1|\leq|\xi_3|\approx|\xi_0|,\ |\xi_3|\gg|\xi_2|$.

$$(4.13)\lesssim\frac{\langle\xi_0\rangle^{a+\frac14}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}(\langle\xi_1\rangle^{\frac14}+\langle\xi_2\rangle^{\frac14})}{\langle\xi_0\rangle^{\frac32-}}\lesssim\langle\xi_0\rangle^{a-\frac54+}\langle\xi_1\rangle^{-s+\frac14-}\lesssim 1,$$

which holds for $a<\frac54$.

Case 3. $|\xi_1|\leq|\xi_3|\approx|\xi_2|,\ |\xi_3|\gg|\xi_0|$.

$$(4.13)\lesssim\frac{\langle\xi_0\rangle^{s+a+1}\langle\xi_3\rangle^{-\frac34-2s+}(\langle\xi_0\rangle^{\frac14}+\langle\xi_1\rangle^{\frac14})}{\langle\xi_1\rangle^{s}\langle\xi_0-\xi_1\rangle^{\frac12-}\langle\xi_0-\xi_2\rangle^{\frac12-}\langle\xi_0-\xi_3\rangle^{\frac12-}}\lesssim\langle\xi_0\rangle^{s+a-\frac14+}\langle\xi_3\rangle^{-\frac34-2s+}\lesssim 1,$$

which holds for $a<s+1$.

Case 4. $|\xi_0|,|\xi_2|\ll|\xi_1|\approx|\xi_3|$.

$$(4.13)\lesssim\langle\xi_0\rangle^{s+a+1}\langle\xi_2\rangle^{-s-\frac12+}\langle\xi_3\rangle^{-2s-\frac74+}(\langle\xi_0\rangle^{\frac14}+\langle\xi_2\rangle^{\frac14})\lesssim\langle\xi_0\rangle^{a-2+}\langle\xi_2\rangle^{-2s+1+}\lesssim 1,$$

which holds for $a<2$. This completes the proof of (4.12). □

Lemma 4.7. *For* $0 < s < \frac{7}{2}$ *and* $a < \min\{2s, \frac{1}{2}, \frac{7}{2} - s\}$, *there exists* $\epsilon > 0$ *such that* $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, *we have*

$$\|u_1 \bar{u}_2 u_3\|_{X_j^{\frac{1}{2}, \frac{2s+2a-1-6b}{6}}} \lesssim \|u_1\|_{X_{j_1}^{s,b}} \|u_2\|_{X_{j_2}^{s,b}} \|u_3\|_{X_{j_3}^{s,b}}, \tag{4.14}$$

where $j = 1, 2,\ j_m = 1, 2 (m = 1, 2, 3)$.

Proof. As in Lemma 4.5, by duality it suffices to show that

$$I := \int_* \frac{\langle \xi_0 \rangle^{\frac{1}{2}} \langle \tau_0 + \phi(\xi_0) \rangle^{\frac{2s+2a-1-6b}{6}} g(\xi_0, \tau_0) \prod_{j=1}^3 f(\xi_j, \tau_j)}{\prod_{j=1}^3 \langle \xi_j \rangle^s \langle \tau_j + \phi(\xi_j) \rangle^b} \lesssim \|f\|_{L^2}^3 \|g\|_{L^2}.$$

We divide the analysis of the integrals by considering the sign of $s + a$. We begin with the case $2 \le s + a < \frac{7}{2}$. Using the fact that

$$\langle \tau_0 + \phi(\xi_0) \rangle \lesssim \langle \xi_{\max} \rangle^3 \max_{j=1,\dots,3} \langle \tau_j + \phi(\xi_j) \rangle,$$

and setting $\max_{j=1,\dots,3} \langle \tau_j + \phi(\xi_j) \rangle = \langle \tau_3 + \phi(\xi_3) \rangle$, we have

$$I \lesssim \|f\|_{L^2}^6 \|g\|_{L^2}^2 \sup_{\xi_0} \int_* \frac{\langle \xi_0 \rangle \langle \xi_{\max} \rangle^{2s+2a-1-6b}}{\prod_{j=1}^3 \langle \xi_j \rangle^{2s}}.$$

The supremum is bounded by $\sup \langle \xi_0 \rangle \langle \xi_{max} \rangle^{2a-1-6b}$, which is finite for $a \le 3b$.

By symmetry, we can restrict a case: $|\xi_1| \ge |\xi_2| \ge |\xi_3|$, which implies that $|\xi_1| \gtrsim |\xi_0|$. We distinguish two cases of $s + a$: $s + a < \frac{3}{2}$ and $s + a \ge \frac{3}{2}$. For case $\frac{1}{2} < s + a < \frac{3}{2}$, in view of (4.9) with $c = \frac{2-(s+a)}{3} \in (\frac{1}{6}, \frac{1}{2})$ for $g(\xi_0, \tau_0), f(\xi_2, \tau_2), f(\xi_3, \tau_3)$ and (4.8) for $f(\xi_1, \tau_1)$. For case $\frac{3}{2} \le s + a < 2$, in view of (4.9) with $c = \frac{2-(s+a)}{3} \in (0, \frac{1}{6})$ for $g(\xi_0, \tau_0), f(\xi_2, \tau_2), f(\xi_3, \tau_3)$ and (4.8) for $f(\xi_1, \tau_1)$. In the above two cases, we estimate supremum by

$$\sup \langle \xi_1 \rangle^{\frac{1}{2} - c+} \langle \xi_2 \rangle^{\frac{1}{2} - c+} \langle \xi_3 \rangle^{\frac{1}{2} - c+} \frac{\langle \xi_0 \rangle^{\frac{1}{2}}}{\prod_{j=1}^3 \langle \xi_j \rangle^s},$$

which is finite for $a < \min\{2s, \frac{1}{2}\}$. This completes the proof of (4.14). □

Lemma 4.8. *For* $s > 0$ *and* $a < \min\{2s + \frac{1}{2}, \frac{5}{4}, \frac{7}{2} - s\}$, *there exists* $\epsilon > 0$ *such that* $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, *we have*

$$\|\partial_x (u_1 \bar{u}_2 u_3)\|_{X_j^{\frac{1}{2}, \frac{2s+2a-1-6b}{6}}} \lesssim \|u_1\|_{X_{j_1}^{s,b}} \|u_2\|_{X_{j_2}^{s,b}} \|u_3\|_{X_{j_3}^{s,b}}, \tag{4.15}$$

where $j = 1, 2,\ j_m = 1, 2 (m = 1, 2, 3)$.

Proof. For (4.15), as in Lemma 4.6, by duality and letting $b = \frac{1}{2}-$, it suffices to show that

$$I := \int_* \frac{\langle \xi_0 \rangle^{\frac{1}{2}} |\xi_0| g(\xi_0, \tau_0) \prod_{j=1}^3 f(\xi_j, \tau_j)}{\langle \sigma_0 \rangle^{\frac{2-(s+a)}{3}-} \prod_{j=1}^3 \langle \xi_j \rangle^s \langle \sigma_j \rangle^{\frac{1}{2}-}} \lesssim \|f\|_{L^2}^3 \|g\|_{L^2}.$$

We begin with the case $2 \le s + a < \frac{7}{2}$. Restricting the variables ξ_1, ξ_2, ξ_3 by the size as $|\xi_{\min}| \le |\xi_{\text{mid}}| \le |\xi_{\max}|$, we have

$$\langle \sigma_0 \rangle \lesssim \max\{\langle \xi_{\max} \rangle^3, \langle \sigma_1 \rangle, \langle \sigma_2 \rangle, \langle \sigma_3 \rangle\}.$$

If the maximum is one of $\langle\sigma_j\rangle$, without loss of generality, we write that the maximum is $\langle\sigma_3\rangle$. Then we have

$$\langle\sigma_0\rangle^{\frac{2-(s+a)}{3}-}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}-}\gtrsim\langle\sigma_1\rangle^{\frac{1}{2}-}\langle\sigma_2\rangle^{\frac{1}{2}-}\langle\sigma_3\rangle^{\frac{1}{2}+\frac{2-(s+a)}{3}-}$$
$$\gtrsim\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\xi_{\max}\rangle^{\frac{7}{2}-(s+a)-}.$$

Invoking the Cauchy-Schwarz inequality, it suffices to show that

$$\sup_{\xi_0}\int_{\xi_0-\xi_1+\xi_2-\xi_3=0}\frac{\langle\xi_0\rangle^3}{\langle\xi_{\max}\rangle^{7-2(s+a)-}\prod_{j=1}^{3}\langle\xi_j\rangle^{2s}}\lesssim 1.$$

Using Lemma 3.2 and $|\xi_{\max}|\gtrsim|\xi_0|$, we obtain the desired result provided that $a<2$.

If the maximum is $\langle\xi_{\max}\rangle^3$, we have

$$\langle\sigma_0\rangle^{\frac{2-(s+a)}{3}-}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}-}\gtrsim\langle\xi_{\max}\rangle^{2-(s+a)-}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}+}.$$

Thus, by (4.1) and (4.2), it suffices to prove that

$$\langle\xi_0\rangle^{\frac{3}{2}}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}\langle\xi_{\max}\rangle^{s+a-2+}\frac{\langle\xi_{\mathrm{mid}}\rangle^{\frac{1}{4}}}{\langle\xi_{\max}\rangle^{\frac{3}{4}}}\lesssim\frac{\langle\xi_{\max}\rangle^{s+a-\frac{5}{4}+}\langle\xi_{\mathrm{mid}}\rangle^{\frac{1}{4}}}{\prod_{j=1}^{3}\langle\xi_j\rangle^{s}}\lesssim 1,$$

provided that $a<\frac{5}{4}$.

It remains to consider the case $s+a<2$. It is easy to check that

$$\langle\sigma_0\rangle^{\frac{2-(s+a)}{3}-}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}-}\gtrsim\langle(\xi_0-\xi_1)(\xi_0-\xi_3)\rangle^{\frac{2-(s+a)}{3}-}\frac{\prod_{j=0}^{3}\langle\sigma_j\rangle^{\frac{1}{2}+}}{\max_{j=0,\ldots,3}\langle\sigma_j\rangle^{\frac{1}{2}+}}.$$

The cases

$$|\xi_0-\xi_3|\le 2k,\ |\xi_0-\xi_1|\le 2k \text{ or } |\xi_0-\xi_2|\le 2k,$$
$$|\xi_0-\xi_3|\le 2k,\ |\xi_0-\xi_1|\ge 2k \text{ and } |\xi_0-\xi_2|\ge 2k,$$
$$|\xi_0-\xi_3|\ge 2k,\ |\xi_0-\xi_1|\le 2k \text{ or } |\xi_0-\xi_2|\le 2k,$$

are immediate for $s>\frac{1}{2}$. Now we consider the case: $|\xi_0-\xi_3|\ge 2k,\ |\xi_0-\xi_1|\ge 2k$ and $|\xi_0-\xi_2|\ge 2k$.

Using (4.1) and (4.2), and taking the size of $\xi_{\max}$ and ξ_{mid} as in Lemma 4.6, we need to bound that

$$\sup_{\xi_0-\xi_1+\xi_2-\xi_3=0}\frac{\langle\xi_0\rangle^{\frac{3}{2}}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}}{\langle(\xi_0-\xi_1)(\xi_0-\xi_2)(\xi_0-\xi_3)\rangle^{\frac{2-(s+a)}{3}-}}\frac{\langle\xi_{\mathrm{mid}}\rangle^{\frac{1}{4}}}{\langle\xi_{\max}\rangle^{\frac{3}{4}}}\lesssim 1.$$

By symmetry, when $\xi_0-\xi_1+\xi_2-\xi_3=0$ and $|\xi_1|\le|\xi_3|$, it suffices to show that

$$\frac{\langle\xi_0\rangle^{\frac{3}{2}}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}}{\langle(\xi_0-\xi_1)(\xi_0-\xi_2)(\xi_0-\xi_3)\rangle^{\frac{2-(s+a)}{3}-}}\frac{\langle\xi_{\mathrm{mid}}\rangle^{\frac{1}{4}}}{\langle\xi_{\max}\rangle^{\frac{3}{4}}}\lesssim 1. \tag{4.16}$$

Concerning the four cases: $|\xi_1|\le|\xi_3|\lesssim|\xi_2|\approx|\xi_0|$; $|\xi_1|\le|\xi_3|\approx|\xi_0|,\ |\xi_3|\gg|\xi_2|$; $|\xi_1|\le|\xi_3|\approx|\xi_2|,\ |\xi_3|\gg|\xi_0|$; $|\xi_0|,|\xi_2|\ll|\xi_1|\approx|\xi_3|$, we uniformly bound (4.16) by constant for $a<\min\{2s+\frac{1}{2},\frac{5}{4}\}$. This completes the proof of (4.15). □

5. Proof of Theorem 1.1

5.1. Existence

In this section, we will prove that the map Φ defined in (3.3) has a fixed point in $X_1^{s,b} \times X_2^{s,b}$ by the a priori estimates. There exists an extension $(\widetilde{u}_0, \widetilde{v}_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ of (u_0, v_0) such that $\|\widetilde{u}_0\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R}^+)}$, $\|\widetilde{v}_0\|_{H^s(\mathbb{R})} \lesssim \|v_0\|_{H^s(\mathbb{R}^+)}$.

Recall that the map $\Phi(u, v)$,

$$\begin{cases} \Phi u = \eta_T(t) W_{\mathbb{R}}^1(t)\widetilde{u}_0 + i\eta_T(t)\displaystyle\int_0^t W_{\mathbb{R}}^1(t-t')G_1(u)\,dt' + \eta_T(t)W_{01}^t(0, f - p_1 - q_1)(t), \\ \Phi v = \eta_T(t) W_{\mathbb{R}}^2(t)\widetilde{v}_0 + i\eta_T(t)\displaystyle\int_0^t W_{\mathbb{R}}^2(t-t')G_2(u)\,dt' + \eta_T(t)W_{02}^t(0, g - p_2 - q_2)(t), \end{cases} \tag{5.1}$$

where $G_j(u)$, p_j and q_j for $j = 1, 2$ are defined in (3.4)–(3.7). To show that the map $\Phi \in X_1^{s,b} \times X_2^{s,b}$, we use (3.8) to obtain

$$\begin{aligned} \left\|\eta(t/T)W_{\mathbb{R}}^1(t)\widetilde{u}_0\right\|_{X_1^{s,b}} &\lesssim \|\widetilde{u}_0\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R}^+)}, \\ \left\|\eta(t/T)W_{\mathbb{R}}^2(t)\widetilde{v}_0\right\|_{X_2^{s,b}} &\lesssim \|\widetilde{v}_0\|_{H^s(\mathbb{R})} \lesssim \|v_0\|_{H^s(\mathbb{R}^+)}. \end{aligned}$$

By (3.9), (3.10), (4.11) and (4.12), we have

$$\begin{aligned} &\left\|\eta(t/T)\int_0^t W_{\mathbb{R}}^j(t-t')G_j(u)\,dt'\right\|_{X_j^{s,b}} \\ \lesssim& T^{\frac{1}{2}-b-}\left\|G_j(u)\right\|_{X_j^{s,-b}} \\ \lesssim& T^{\frac{1}{2}-b-}(\|u\|_{X_1^{s,b}}^3 + \|v\|_{X_2^{s,b}}^3 + \|u\|_{X_1^{s,b}}^2\|v\|_{X_2^{s,b}} + \|u\|_{X_1^{s,b}}\|v\|_{X_2^{s,b}}^2). \end{aligned} \tag{5.2}$$

We apply Lemma 3.1 and Lemma 4.2 to obtain

$$\begin{aligned} \left\|\eta(t/T)W_{01}^t(0, f - p_1 - q_1)(t)\right\|_{X_1^{s,b}} &\lesssim \left\|\chi_{(0,\infty)}(f - p_1 - q_1)\right\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \\ &\lesssim \|f - p_1\|_{H_t^{\frac{s+1}{3}}(\mathbb{R}^+)} + \|q_1\|_{H_t^{\frac{s+1}{3}}(\mathbb{R}^+)}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \left\|\eta(t/T)W_{02}^t(0, g - p_2 - q_2)(t)\right\|_{X_2^{s,b}} &\lesssim \left\|\chi_{(0,\infty)}(g - p_2 - q_2)\right\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \\ &\lesssim \|g - p_2\|_{H_t^{\frac{s+1}{3}}(\mathbb{R}^+)} + \|q_2\|_{H_t^{\frac{s+1}{3}}(\mathbb{R}^+)}. \end{aligned} \tag{5.4}$$

Using Lemma 4.1, we have

$$\|p_1\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \lesssim \|\widetilde{u}_0\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R}^+)}, \tag{5.5}$$

$$\|p_2\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \lesssim \|\widetilde{v}_0\|_{H^s(\mathbb{R})} \lesssim \|v_0\|_{H^s(\mathbb{R}^+)}. \tag{5.6}$$

Moreover, by Lemma 4.4, (3.10) and Lemma 4.5, 4.6, 4.7, 4.8, we have

$$\|q_1\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \lesssim T^{\frac{1}{2}-b-}(\|u\|_{X_1^{s,b}}^3 + \|v\|_{X_2^{s,b}}^3 + \|u\|_{X_1^{s,b}}^2\|v\|_{X_2^{s,b}} + \|u\|_{X_1^{s,b}}\|v\|_{X_2^{s,b}}^2), \tag{5.7}$$

$$\|q_2\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \lesssim T^{\frac{1}{2}-b-}(\|u\|^3_{X_1^{s,b}} + \|v\|^3_{X_2^{s,b}} + \|u\|^2_{X_1^{s,b}}\|v\|_{X_2^{s,b}} + \|u\|_{X_1^{s,b}}\|v\|^2_{X_2^{s,b}}). \tag{5.8}$$

Combining these estimates, we obtain

$$\begin{aligned}\|\Phi(u,v)\|_{X^{s,b}} \lesssim &\|u_0\|_{H^s(\mathbb{R}^+)} + \|v_0\|_{H^s(\mathbb{R}^+)} + \|f\|_{H_t^{\frac{s+1}{3}}(\mathbb{R}^+)} + \|g\|_{H_t^{\frac{s+1}{3}}(\mathbb{R}^+)} \\ &+ T^{\frac{1}{2}-b-}(\|u\|^3_{X_1^{s,b}} + \|v\|^3_{X_2^{s,b}} + \|u\|^2_{X_1^{s,b}}\|v\|_{X_2^{s,b}} + \|u\|_{X_1^{s,b}}\|v\|^2_{X_2^{s,b}}).\end{aligned}$$

For $\delta > 0$ sufficiently small, we define a Banach space B, where

$$B = \{U = (u,v) : \ \|U\|_{X_1^{s,b}\times X_2^{s,b}} < \delta\}$$

such that $\Phi(\cdot) : B \to B$. Furthermore, combining with similar estimates for the difference $\Phi(u_1,v_1) - \Phi(u_2,v_2)$, we obtain the existence of a fixed point (u,v) of Φ in $X_1^{s,b} \times X_2^{s,b}$ for T sufficiently small, where T only depends on $\|u_0\|_{H^s(\mathbb{R}^+)}, \|v_0\|_{H^s(\mathbb{R}^+)}, \|f\|_{H_t^{\frac{s+1}{3}}(\mathbb{R}^+)}, \|g\|_{H_t^{\frac{s+1}{3}}(\mathbb{R}^+)}$.

We prove that $u, v \in C_t^0 H_x^s([0,T]\times\mathbb{R})$ is obtained by the a priori estimates. The Hirota group operator is continuous in H^s. For the boundary term W_0^t, it is continuous by Lemma 4.3, (5.7) and (5.8). The continuity of the Duhamel term $\int_0^t W_{\mathbb{R}}(t-t')G_j(u)\,dt'$ follows from (3.9), (4.11), (4.12) and the embedding $X^{s,b} \subset C_t^0 H_x^s$ for $b > \frac{1}{2}$. In addition, the fact that $u \in C_x^0 H_t^{\frac{s+1}{3}}(\mathbb{R}\times[0,T])$ follows from Lemma 4.1, Lemma 4.3 and Lemma 4.4. Relying on the above contraction mapping argument and the a priori estimates, we obtain the continuous dependence on the initial and boundary data. However, it is not clear whether the different extensions of initial data affect the uniqueness of solutions on $\mathbb{R}^+$. The uniqueness issue will be resolved in Section 5.2 below. Until then, we prove that the nonlinearities of the solution are smoother than the initial data by the fixed point argument and the a priori estimates. Recall that

$$\begin{aligned} u - W_{01}^t(u_0,f) &= i\eta(t/T)\int_0^t W_{\mathbb{R}}^1(t-t')G_1(u)\,dt' - \eta(t/T)W_{01}^t(0,q_1)(t), \\ v - W_{02}^t(v_0,g) &= i\eta(t/T)\int_0^t W_{\mathbb{R}}^2(t-t')G_2(u)\,dt' - \eta(t/T)W_{02}^t(0,q_2)(t).\end{aligned}$$

The fact that the first term is in $C_t^0 H_x^{s+a}$ by embedding $X^{s+a,\frac{1}{2}+} \subset C_t^0 H_x^{s+a}$ and (5.2). By Lemma 4.3, (5.7) and (5.8), the second term also in $C_t^0 H_x^{s+a}$. Thus, we have

$$\begin{aligned}\left\|u - W_{01}^t(u_0,f)\right\|_{C_t^0 H_x^{s+a}} &\lesssim \|G_1(u)\|_{X_1^{s+a,-b}} + \|q_1\|_{H^{\frac{s+a+1}{3}}} \\ &\lesssim \|u\|^3_{X_1^{s,b}} + \|v\|^3_{X_2^{s,b}} + \|u\|^2_{X_1^{s,b}}\|v\|_{X_2^{s,b}} + \|u\|_{X_1^{s,b}}\|v\|^2_{X_2^{s,b}},\end{aligned}$$

which implies that $u(t) - W_{01}^t(u_0,f)$ belongs to $C_t^0 H_x^{s+a}$. The analogous statement for $v(t) - W_{02}^t(v_0,g)$ is proved similarly.

5.2. Uniqueness

In this section, we discuss the uniqueness argument for equations (1.1). The solution we proved above is a unique fixed point of (5.1), and we still need to show that different extensions of initial data produce the same solution on $\mathbb{R}^+$. Now we consider two solutions (u_1,v_1), (u_2,v_2) of the IBVP (1.1). The idea is to apply a limit argument, and we start with an extension argument in [14]:

Lemma 5.1. *Fix $s \geq 0$ and $k \geq s$. Let $u_0 \in H^s(\mathbb{R}^+), f \in H^k(\mathbb{R}^+)$, and let $\widetilde{u}_0$ be an H^s extension of u_0 to $\mathbb{R}$. Then there is an H^k extension $\widetilde{f}$ of f to $\mathbb{R}$ such that*

$$\|\widetilde{u}_0 - \widetilde{f}\|_{H^s(\mathbb{R})} \lesssim \|u_0 - f\|_{H^s(\mathbb{R}^+)}.$$

By the standard argument, one can check that the solution (u, v) is unique in $C_t^0 H_x^s \times C_t^0 H_x^s$ with $s \geq 2$. Now we check the uniqueness for $s \in (\frac{1}{4}, 2), s \neq \frac{1}{2}, \frac{3}{2}$. Define

$$Z^s = H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+),$$

and we consider the data $(u_0, v_0, f, g) \in Z^s$. Take a sequence $(u_{0n}, v_{0n}) \subset H^2(\mathbb{R}^+) \times H^2(\mathbb{R}^+)$ converging to (u_0, v_0) in $H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+)$. Also take sequence $(f_n, g_n) \subset H^1(\mathbb{R}^+)$ converging to $(f, g) \in H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+)$. Given two extensions $(u_0^e, v_0^e), (\widetilde{u}_0^e, \widetilde{v}_0^e)$ of (u_0, v_0) in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, we have the corresponding solutions $(u, v), (\widetilde{u}, \widetilde{v})$ by the contraction argument. By Lemma 5.1, let sequences $(u_{0n}^e, v_{0n}^e), (\widetilde{u}_{0n}^e, \widetilde{v}_{0n}^e) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ be extensions of (u_{0n}, v_{0n}) converging to $(u_0^e, v_0^e), (\widetilde{u}_0^e, \widetilde{v}_0^e)$ in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. It is clear that using two extensions $(u_{0n}^e, v_{0n}^e), (\widetilde{u}_{0n}^e, \widetilde{v}_{0n}^e)$ and Theorem 1.1, we can construct two sequences of solutions $(u_n, v_n), (\widetilde{u}_n, \widetilde{v}_n)$. By the uniqueness of $H^2(\mathbb{R}^+)$ solutions, the corresponding solutions sequences $(u_n, v_n) = (\widetilde{u}_n, \widetilde{v}_n)$ on $(x, t) \in \mathbb{R}^+ \times [0, T]$. By fixed point argument, (u_n, v_n) and $(\widetilde{u}_n, \widetilde{v}_n)$ converge in $H^s \times H^s$ to (u, v) and $(\widetilde{u}, \widetilde{v})$ respectively. Then we obtain $(u, v) = (\widetilde{u}, \widetilde{v})$ on $(x, t) \in \mathbb{R}^+ \times [0, T]$. The uniqueness for $\frac{1}{4} < s < 2$ can be obtained by iteration.

Acknowledgements

J. Wu is supported by Natural Science Foundation of Henan under Grant No. 262300422609 and the China Postdoctoral Science Foundation under Grant No. 2023M733692.

References

[1] A. Alkın, D. Mantzavinos and T. Özsarı, *Local well-posedness of the higher-order nonlinear Schrödinger equation on the half-line: Single-boundary condition case*, Stud. Appl. Math., 2024, 152, 203–248.

[2] S. G. Bindu, A. Mahalingam and K. Porsezian, *Dark soliton solutions of the coupled Hirota equation in nonlinear fiber*, Phys. Lett. A, 2001, 286, 321–331.

[3] J. L. Bona, S. M. Sun and B. Y. Zhang, *A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane*, Trans. Amer. Math. Soc., 2001, 354, 427–490.

[4] J. L. Bona, S. M. Sun and B. Y. Zhang, *A nonhomogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finite domain*, Comm. Partial Differential Equations, 2003, 28, 1391–1436.

[5] J. L. Bona, S. M. Sun and B. Y. Zhang, *Nonhomogeneous boundary-value problems for one-dimensional nonlinear Schrödinger equations*, J. Math. Pures Appl., 2018, 109, 1–66.

[6] J. Bourgain, *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part I: Schrödinger equation*, Geom. Funct. Anal., 1993, 3, 107–156.

[7] J. Bourgain, *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part II: The KdV equation*, Geom. Funct. Anal., 1993, 3, 209–262.

[8] X. Carvajal, *Local well-posedness for a higher order nonlinear Schrödinger quation in Sobolev spaces of negative indices*, Electron. J. Differential Equations, 2004, 2004(13), 1–10.

[9] M. Cavalcante and A. J. Corcho, *The initial-boundary value problem for the Schrödinger–Korteweg–de Vries system on the half-line*, Commun. Contemp. Math., 2019, 21(8), 1850066.

[10] J. E. Colliander and C. E. Kenig, *The generalized Korteweg-de Vries equation on the half line*, Comm. Partial Differential Equations, 2002, 27, 2187–2266.

[11] E. Compaan, W. Shin and N. Tzirakis, *Well-posedness for the Schrödinger-KdV system on the half-line*, J. Math. Anal. Appl., 2024, 537, 128313.

[12] E. Compaan and N. Tzirakis, *Low regularity well–posedness for dispersive equations on semi–infinite intervals*, Commun. Pure Appl. Anal., 2023, 22(8), 2481–2500.

[13] M. B. Erdoğan, T. B. Gürel and N. Tzirakis, *The fifth order KP–II equation on the upper half-plane*, Differential Integral Equations, 2020, 33, 555–596.

[14] M. B. Erdoğan and N. Tzirakis, *Regularity properties of the cubic nonlinear Schrödinger equation on the half line*, J. Funct. Aanl., 2016, 271, 2539–2568.

[15] A. V. Faminskii, *Global weak solutions of an initial-boundary value problem on a half-line for the higher order nonlinear Schrödinger equation*, J. Math. Anal. Appl., 2024, 533, 128003.

[16] B. L. Guo and S. B. Tan, *Global smooth solution for nonlinear evolution equation of Hirota type*, Sci. China, 1992, 35A, 1425–1433.

[17] B. L. Guo and J. Wu, *Well-posedness of the initial-boundary value problem for the Hirota equation on the half line*, J. Math. Anal. Appl., 2021, 504, 125571.

[18] B. L. Guo and J. Wu, *Initial-boundary value problem for the Hirota equation posed on a finite interval*, preprint.

[19] A. Himonas and F. C. Yan, *A higher dispersion KdV equation on the half-line*, J. Differential Equations, 2022, 333, 55–102.

[20] J. Holmer, *The initial-boundary value problem for the 1D nonlinear Schrödinger equation on the half-line*, Differential Integral Equations, 2005, 18, 647–668.

[21] J. Holmer, *The initial-boundary value problem for Korteweg-de Vries equation*, Comm. Partial Differential Equations, 2006, 31, 1151–1190.

[22] Z. H. Huo and B. L. Guo, *Well-posedness of the Cauchy problem for the Hirota equation in Sobolev spaces H^s*, Nonlinear Anal., 2005, 60(6), 1093–1110.

[23] Z. H. Huo and Y. L. Jia, *Well-posedness for the Cauchy problem to the Hirota equation in Sobolev spaces of negative indices*, Chin. Ann. Math., 2005, 26B(1), 75–88.

[24] Z. H. Huo and Y. L. Jia, *Well-posedness for the Cauchy problem of coupled Hirota equations with low regularity data*, J. Math. Anal. Appl., 2006, 322, 566–579.

[25] C. Laurey, *The Cauchy problem for a third order nonlinear Schrödinger equation*, Nonlinear Anal., 1997, 29, 121–158.

[26] S. H. Li, M. Chen, X. Yang and B. Y. Zhang, *Lower regularity solutions of the non-homogeneous boundary-value problem for a higher order Boussinesq equation in a quarter plane*, Nonlinear Anal., 2022, 221, 112893.

[27] S. Li, C. Mu and D. Zhou, *Non-homogeneous initial boundary value problems and small data scattering of the Hirota equations posed on the half line*, Nonlinearity, 2024, 37, 115009.

[28] C. Mayo, D. Mantzavinos and T. Özsari, *Well-posedness of the higher-order nonlinear Schrödinger equation on a finite interval*, 2024. arXiv:2406.15579.

[29] D. Mihalache, N. Truta and L.-C. Crasovan, *Painlevé analysis and bright solitary waves of the higher-order nonlinear Schrödinger equation containing third-order dispersion and self-steepening term*, Phys. Rev. E, 1997, 56, 1064.

[30] T. Özsari and K. C. Yilmaz, *Stabilization of higher order Schrödinger equations on a finite interval: Part II*, Evol. Equ. Control Theory, 2022, 11, 1087–1148.

[31] M. Sriskandasingam, S. M. Sun and B. Y. Zhang, *General boundary value problems of a class of fifth order KdV equations on a bounded interval*, Differential Integral Equations, 2024, 37, 817–842.

[32] M. Sriskandasingam, S. M. Sun and B. Y. Zhang, *Non-homogeneous boundary value problems of the Kawahara equation posed on a finite interval*, Nonlinear Anal., 2023, 227, 113158.

[33] G. Staffilani,*On the generalized Korteweg-de Vries-type equations*, Differential Integral Equations, 1997, 10, 777–796.

[34] R. S. Tasgal and M. J. Potasek, *Soliton solutions to coupled higher-order nonlinear Schrödinger equations*, J. Math. Phys., 1992, 33, 1208–1215.

Received October 2024; Accepted March 2026; Available online April 2026.