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MULTIPLICATIVE FRACTIONAL BULLEN-TYPE INEQUALITIES IN THE FRAMEWORK OF *G*-CALCULUS*

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Abstract This paper introduces a multiplicative analogue of the Bullen quadrature rule and develops a suitable notion of convexity tailored to the G-calculus framework. Building on these foundations, we derive a new fractional identity in the multiplicative setting, which serves as a key enabler for establishing Bullen-type inequalities via multiplicative Riemann-Liouville fractional integrals. This work integrate fractional calculus with multiplicative analysis for the study of integral inequalities, thereby proposing a novel pathway within non-Newtonian mathematical systems. Our results advance the theory of generalized calculus and open promising directions for future investigations into multiplicative fractional inequalities.

Keywords G-calculus, multiplicative fractional integrals, multiplicative Bullen rule, GG-convex functions, GA-convex functions.

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1. Introduction

Bullen's inequality has emerged as a pivotal tool across multiple branches of mathematical analysis, especially in the context of numerical integration and approximation techniques. It delivers a crucial bound on the error associated with Bullen's quadrature formula, a widely employed scheme prized for its precision when integrating sufficiently smooth functions. In recent decades, this classical result has been extended through diverse frameworks, including convex function theory, fractional calculus, and parameter-dependent formulations. These generalizations have given rise to a rich family of inequalities of Bullen type, significantly broadening their utility in

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both analytical investigations and computational applications.

Hwang et al. [13] gave the following Bullen-type inequality for differentiable convex functions:

$$\left|\frac{1}{4}\left[\Phi(\mathfrak{r}_1)+2\Phi\left(\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right)+\Phi(\mathfrak{r}_2)\right]-\frac{1}{\mathfrak{r}_2-\mathfrak{r}_1}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\Phi(\mathfrak{z})\,\mathrm{d}\mathfrak{z}\right|\le\frac{\mathfrak{r}_2-\mathfrak{r}_1}{16}\left(\left|\Phi'(\mathfrak{r}_1)\right|+\left|\Phi'(\mathfrak{r}_2)\right|\right),$$

where $|\Phi'|$ is convex on $[\mathfrak{r}_1,\mathfrak{r}_2]$.

In the wake of this development, the inequality has become a focal point for further theoretical exploration, with key advances reported in [19, 21], among other studies.

Fractional calculus generalizes the traditional notions of differentiation and integration to arbitrary real (or complex) orders, offering a powerful mathematical formalism for describing systems endowed with memory effects and hereditary properties. Among its core constructs are the Riemann–Liouville fractional integral operators, which are instrumental in shaping the theoretical landscape of fractional analysis and its diverse applications. For a function $\Phi\in L^1([\mathfrak{r}_1,\mathfrak{r}_2])$ and order $\alpha>0$ by

$$\mathscr{I}^{\alpha}_{\mathfrak{r}_1^+}\Phi(\varsigma)=\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{r}_1}^{\varsigma}(\varsigma-\mathfrak{z})^{\alpha-1}\Phi(\mathfrak{z})\,\mathrm{d}\mathfrak{z},\quad \varsigma>\mathfrak{r}_1$$

and

$$\mathscr{I}^{\alpha}_{\mathfrak{r}_2^-}\Phi(\varsigma)=\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\mathfrak{r}_2}(\mathfrak{z}-\varsigma)^{\alpha-1}\Phi(\mathfrak{z})\,\mathrm{d}\mathfrak{z},\quad \varsigma<\mathfrak{r}_2,$$

respectively.

These operators extend the classical idea of iterated integration and constitute a foundational pillar in fractional calculus, underpinning a wide spectrum of integral inequalities, including fractional variants. In recent years, they have been widely employed to establish refined inequalities of Bullen type, particularly under diverse convexity conditions. Researchers have leveraged Riemann–Liouville fractional integrals to derive parametric, midpoint, and Bullen-type estimates, enriching the theoretical framework. Significant advances in this direction have been made by Du et al. [8], Budak et al. [7], Zhao et al. [24], Sassane et al. [22], and Fahad et al. [9], whose contributions have substantially expanded the scope and depth of the field.

The classical calculus framework, pioneered by Newton and Leibniz, rests fundamentally on additive structures and linear limiting processes. However, in numerous theoretical and applied settings — particularly those characterized by exponential trends, multiplicative evolution, or relative rate changes — this conventional paradigm may prove suboptimal. This limitation has motivated the development of alternative calculi, collectively referred to as non-Newtonian calculi. In these systems, the standard operations of addition and subtraction are supplanted by alternative binary operations, and the notions of differentiation and integration are redefined accordingly. Among the most prominent is multiplicative calculus, in which derivatives and integrals are formulated using multiplication and division rather than addition and subtraction. This renders it especially well-suited for modeling phenomena governed by exponential or proportional dynamics. First formalized by Grossman and Katz in the late 1960s and early 1970s, multiplicative calculus constitutes a self-contained mathematical system, complete with its own definitions of gradient, derivative, integral, and mean, all grounded in a multiplicative arithmetic structure. Within this framework, functions possessing a constant multiplicative derivative serve a role analogous to linear functions in classical calculus. Since its inception, multiplicative calculus has attracted growing scholarly attention and found applications across diverse domains, including biology, economics, signal processing, and numerical analysis. These advances have further spurred extensions into fractional, stochastic, and discrete formulations. For a detailed exposition of the foundations and applications of multiplicative calculus, we refer the reader to the foundational works of Grossman and Katz [12], Bashirov et al. [4], Boruah and Hazarika [5,6], Lakhdari et al. [14,16], and Stanley [23].

We now present the definitions of the elementary operations in multiplicative calculus; comprehensive treatments are available in [10].

Let $\mathbb{R}_\star = \{\mathfrak{z} : \mathfrak{z} = \exp\{z\}, z \in \mathbb{R}\}$. For $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{R}_\star$, the elementary operations within the multiplicative calculus framework are specified by:

- $\mathfrak{z}_1 +_\star \mathfrak{z}_2 = \exp\{\ln(\mathfrak{z}_1) + \ln(\mathfrak{z}_2)\} = \mathfrak{z}_1\mathfrak{z}_2$,
- $\mathfrak{z}_1 -_\star \mathfrak{z}_2 = \exp\{\ln(\mathfrak{z}_1) - \ln(\mathfrak{z}_2)\} = \frac{\mathfrak{z}_1}{\mathfrak{z}_2}$,
- $\mathfrak{z}_1 \cdot_\star \mathfrak{z}_2 = \exp\{\ln(\mathfrak{z}_1).\ln(\mathfrak{z}_2)\} = \mathfrak{z}_1^{\ln(\mathfrak{z}_2)}$,
- $\mathfrak{z}_1 /_\star \mathfrak{z}_2 = \exp\left\{\frac{\ln(\mathfrak{z}_1)}{\ln(\mathfrak{z}_2)}\right\} = \mathfrak{z}_1^{\frac{1}{\ln(\mathfrak{z}_2)}}$, with $\mathfrak{z}_2 \neq 1$.

Remark 1.1. The sets $(\mathbb{R}_\star, +_\star)$ and $(\mathbb{R}_\star, \cdot_\star)$ each form abelian groups. Equipped with the operations $+_\star$ and $\cdot_\star$, the set $\mathbb{R}_\star$ constitutes a field. Within this multiplicative algebraic framework, the additive identity, commonly called the multiplicative zero, is $0_\star = 1$, and the multiplicative identity, often termed the multiplicative unit, is $1_\star = e$.

In [12], Grossman and Katz introduced the multiplicative derivative via the following limit:

$$\frac{\mathrm{d}_\star \Phi(\mathfrak{z})}{\mathrm{d}_\star \mathfrak{z}} = \lim_{\varpi \to 0} \left(\frac{\Phi(\mathfrak{z} + \varpi)}{\Phi(\mathfrak{z})}\right)^{\frac{1}{\varpi}}.$$

This operator plays a pivotal role in quantitative economics and financial modeling [4, 12]. Distinct from the classical derivative, it measures relative change with respect to proportional shifts in the independent variable, and is commonly referred to as the *geometric derivative*. It underpins *geometric calculus*, a formalism inherently confined to functions taking strictly positive values.

Alternatively, Grossman [11] later proposed a distinct multiplicative derivative—termed the *bigeometric derivative*—in which additive increments are supplanted by multiplicative ones. It is formally defined by:

$$\frac{\mathrm{d}_\star \Phi(\mathfrak{z})}{\mathrm{d}_\star \mathfrak{z}} = \lim_{\varpi \to 1} \left(\frac{\Phi(\mathfrak{z}\varpi)}{\Phi(\mathfrak{z})}\right)^{\frac{1}{\ln \varpi}}.$$

To clearly distinguish between these two multiplicative frameworks, the authors of [6] coined the term G-calculus. Henceforth in this paper, we adopt their terminology: The adjective "multiplicative" will exclusively denote constructs and operations within the G-calculus system including the multiplicative derivative and integral.

Bas et al. [2] (2025) formulated a multiplicative counterpart to the classical Riemann-Liouville fractional integrals, given by the following expressions:

Definition 1.1 ([2]). Let $\Phi : \mathrm{D} \subset \mathbb{R}_\star \to \mathbb{R}_\star$. The left and right multiplicative Riemann-Liouville fractional integrals of order $\alpha > 0$ are expressed as:

$$_\star\mathscr{I}^{\alpha}_{\mathfrak{r}_1^+}\Phi(\varsigma) = \exp\left\{\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{r}_1}^{\varsigma} (\ln\varsigma - \ln\mathfrak{z})^{\alpha-1} \frac{\ln\Phi(\mathfrak{z})}{\mathfrak{z}}\mathrm{d}\mathfrak{z}\right\},\ \varsigma > \mathfrak{r}_1, \tag{1.1}$$

and

$$ {}_{\star}\mathscr{I}^{\alpha}_{\mathfrak{r}_2^-}\Phi(\varsigma) = \exp\left\{\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\mathfrak{r}_2}(\ln\mathfrak{z}-\ln\varsigma)^{\alpha-1}\frac{\ln\Phi(\mathfrak{z})}{\mathfrak{z}}\mathrm{d}\mathfrak{z}\right\},\ \varsigma<\mathfrak{r}_2. \tag{1.2} $$

Moreover, Baş et al. [3] established the definition of multiplicative fractional integrals and derivatives for functions of two variables, whereas Lakhdari et al. proposed the concept of multiplicative tempered fractional integrals in [18] and the multiplicative Hadamard and Katugampola operators in [15].

Recently, leveraging the above multiplicative Riemann-Liouville fractional integrals, Lakhdari and Saleh [17] provided multiplicative fractional Hermite–Hadamard inequalities as follows:

Theorem 1.1. *Let $\Phi : [\mathfrak{r}_1,\mathfrak{r}_2] \subset \mathbb{R}_\star \to \mathbb{R}_\star$ be a multiplicative integrable function. If Φ is GG-convex on $[\mathfrak{r}_1,\mathfrak{r}_2]$, then the following multiplicative Riemann-Liouville fractional Hermite-Hadamard inequalities hold*

$$ \Phi\left(\sqrt{\mathfrak{r}_1\mathfrak{r}_2}\right) \le \left[\left({}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}^+}\Phi(\mathfrak{r}_2)\right)\left({}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}^-}\Phi(\mathfrak{r}_1)\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)^\alpha}} \le \sqrt{\Phi(\mathfrak{r}_1)\Phi(\mathfrak{r}_2)}, \tag{1.3} $$

where ${}_{\star}\mathscr{I}^{\alpha}_{\mathfrak{r}_1^+}$ and ${}_{\star}\mathscr{I}^{\alpha}_{\mathfrak{r}_2^-}$ are defined as in (1.1) *and* (1.2)*, respectively.*

In that work, the authors derived midpoint- and trapezium-type inequalities for functions whose first-order multiplicative derivatives are GG-convex in multiplicative absolute value.

$$ \left|\frac{\Phi\left(\sqrt{\mathfrak{r}_1\mathfrak{r}_2}\right)}{\left[\left({}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}^+}\Phi(\mathfrak{r}_2)\right)\left({}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}^-}\Phi(\mathfrak{r}_1)\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)^\alpha}}}\right|_\star \le \left(\left|\Phi^\star(\mathfrak{r}_1)\right|_\star\left|\Phi^\star(\mathfrak{r}_2)\right|_\star\right)^{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{4(\alpha+1)}}, $$

and

$$ \left|\frac{\sqrt{\Phi(\mathfrak{r}_1)\Phi(\mathfrak{r}_2)}}{\left[\left({}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}^+}\Phi(\mathfrak{r}_2)\right)\left({}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}^-}\Phi(\mathfrak{r}_1)\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)^\alpha}}}\right|_\star \le \left(\left|\Phi^\star(\mathfrak{r}_1)\right|_\star\left|\Phi^\star(\mathfrak{r}_2)\right|_\star\right)^{\frac{\alpha(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)}{4(\alpha+1)}}. $$

In this work, we seek to generalize Bullen's inequality within the context of fractional G-calculus. This extension provides a fresh perspective on classical analytic inequalities and opens new avenues for progress in non-Newtonian analysis, numerical approximation, and optimization in multiplicative settings.

The remainder of this paper is structured as follows. Section 2 recalls key notions from G-calculus, providing the essential definitions and theoretical machinery needed for our subsequent developments. We then introduce a multiplicative counterpart of the Bullen quadrature rule and investigate the notion of multiplicative convexity. In Section 3, building upon this groundwork, we first derive a novel integral identity in the multiplicative fractional setting; leveraging this identity, we establish new Bullen-type inequalities for functions whose $\star$-derivative, in $\star$-absolute value, satisfies GG-convexity. Section 4 presents a numerical case study supported by graphical visualization, offering both quantitative and visual validation of the obtained inequalities, and discusses potential applications of our results to special means. Finally, Section 5 concludes the paper by summarizing the principal contributions and outlining promising directions for future research in multiplicative fractional analysis.

2. Preliminaries

2.1. Fundamental principles of multiplicative calculus

This subsection lays out the theoretical groundwork in multiplicative calculus, emphasizing the definitions and properties critical to the development of our results.

Definition 2.1 ([10]). For a function $\phi : \mathbb{R}_\star \to \mathbb{R}$, we define the associated multiplicative function $\Phi : \mathbb{R}_\star \to \mathbb{R}_\star$ as follows:

$$\Phi(\mathfrak{z}) = \exp\{\phi(\ln(\mathfrak{z}))\}, \quad \mathfrak{z} \in \mathbb{R}_\star.$$

Example 2.1. Here are some examples of the multiplicative counterparts of some usual functions.

- The multiplicative counterpart of the function $\phi(\mathfrak{z}) = \mathfrak{z}^k$ is $\Phi(\mathfrak{z}) = \mathfrak{z}^{k_\star} = \exp\left\{(\ln \mathfrak{z})^k\right\}$, $k \geq 0$. In particular, we have $\phi(\mathfrak{z}) = \mathfrak{z} = \Phi(\mathfrak{z})$.
- The exponential and logarithmic functions retain their fundamental role in the multiplicative calculus, behaving identically to their multiplicative analogues. Hence, they are invariant under the transition to the multiplicative framework.
- The multiplicative function associated with $\phi(\mathfrak{z}) = \exp\{-\mathfrak{z}\}$ is $\Phi(\mathfrak{z}) = \exp\{-\mathfrak{z}\}_\star = \exp\left\{\frac{1}{\mathfrak{z}}\right\}$.
- The multiplicative counterpart of the polynomial $Q_n(\mathfrak{z}) = \sum\limits_{i=0}^{n} \mathfrak{c}_i \mathfrak{z}^i$ is the multiplicative polynomial $\mathcal{Q}_n(\mathfrak{z}) = \sum\limits_{i=0}^{n} \mathfrak{c}_{i*} \cdot_\star \mathfrak{z}^{i_\star}$, where $\mathfrak{c}_{i*} = \exp\{\mathfrak{c}_i\}$.

Definition 2.2 ([10]). A function $\Phi : \mathrm{D} \subseteq \mathbb{R}_\star \to \mathbb{R}_\star$ is called multiplicatively continuous at a point $\mathfrak{z}_0 \in \mathrm{D}$ if the function $\exp\left\{\phi(\ln \mathfrak{z})\right\}$ is continuous at $\mathfrak{z}_0$. It is multiplicatively continuous on D if this holds for every point in D, and we write $\Phi \in \mathcal{C}_\star(\mathrm{D})$ in this case.

Definition 2.3 ([10]). Let $\mathrm{D} \subset \mathbb{R}_\star$ and $\phi : \mathrm{D} \to \mathbb{R}_\star$ be a differentiable function. Then, Φ is multiplicative differentiable ($\star$-differentiable), and the multiplicative derivative ($\star$-derivative) of $\Phi_\star$ at $\mathfrak{z} \in \mathrm{D}$ is defined by:

$$\frac{\mathrm{d}_\star \Phi(\mathfrak{z})}{\mathrm{d}_\star \mathfrak{z}} = \Phi^\star(\mathfrak{z}) = \exp\left\{\mathfrak{z}\frac{\Phi'(\mathfrak{z})}{\Phi(\mathfrak{z})}\right\}.$$

Proposition 2.1 ([10]). *Let $\Phi, \Psi : \mathrm{D} \subseteq \mathbb{R}_\star \to \mathbb{R}_\star$ be two multiplicative differentiable functions on* D. *Then, the following properties are valid*

- $(\mathfrak{c} \cdot_\star \Phi)^* (\mathfrak{z}) = \mathfrak{c} \cdot_\star \Phi^* (\mathfrak{z}) = (\Phi^* (\mathfrak{z}))^{\ln(\mathfrak{c})}$, $\mathfrak{c} \in \mathbb{R}_\star$,
- $(\Phi +_\star \Psi)^* (\mathfrak{z}) = \Phi^* (\mathfrak{z}) +_\star \Psi^* (\mathfrak{z}) = \Phi^* (\mathfrak{z}) \Psi^* (\mathfrak{z})$,
- $(\Phi \cdot_\star \Psi)^* (\mathfrak{z}) = [\Phi^* (\mathfrak{z}) \cdot_\star \Psi (\mathfrak{z})] +_\star [\Phi (\mathfrak{z}) \cdot_\star \Psi^\star (\mathfrak{z})] = (\Phi^* (\mathfrak{z}))^{\ln(\Psi(\mathfrak{z}))} (\Psi^* (\mathfrak{z}))^{\ln(\Phi(\mathfrak{z}))}$,
- $(\Phi /_\star \Psi_\star)^* (\mathfrak{z}) = ([\Phi^* (\mathfrak{z}) \cdot_\star \Psi (\mathfrak{z})] -_\star [\Phi (\mathfrak{z}) \cdot_\star \Psi^\star (\mathfrak{z})]) /_\star \Psi^{2_\star} (\mathfrak{z}) = \left(\frac{(\Psi(\mathfrak{z}))^{\Phi^\star(\mathfrak{z})}}{(\Phi(\mathfrak{z}))^{\Psi^\star(\mathfrak{z})}} \right)^{\frac{1}{(\ln(\Psi(\mathfrak{z})))^2}}$,
- $(\Phi \circ \Psi_\star) (\mathfrak{z}) = \Psi^\star(\mathfrak{z}) \cdot_\star \Phi^\star (\Psi(\mathfrak{z})) = (\Phi^\star (\Psi(\mathfrak{z})))^{\ln(\Psi^\star(\mathfrak{z}))}$.

Definition 2.4 ([10]). Assume that $\Phi : \mathrm{D} \subseteq \mathbb{R}_\star \to \mathbb{R}_\star$ is a function such that $\Phi \in \mathcal{C}_\star^n$, $n \in \mathbb{N}$ with $n \geq 2$. The multiplicative differential $\mathrm{d}_\star \mathfrak{z}$ is given by

$$\mathrm{d}_\star \mathfrak{z} = \exp\left\{\mathrm{d}\left(\ln \mathfrak{z}\right)\right\} = \exp\left\{\frac{\mathrm{d}\mathfrak{z}}{\mathfrak{z}}\right\}$$

and

$$\mathrm{d}_\star \Phi(\mathfrak{z}) = \exp\left\{\mathrm{d}\left(\ln \Phi(\mathfrak{z})\right)\right\} = \exp\left\{\frac{\Phi'(\mathfrak{z})}{\Phi(\mathfrak{z})}\mathrm{d}\mathfrak{z}\right\}.$$

Definition 2.5 ([10]). Assume that $\Phi : [\mathfrak{r}_1, \mathfrak{r}_2] \subset \mathbb{R}_\star \to \mathbb{R}_\star$ is a function satisfying $\Phi \in \mathcal{C}_\star\left[\mathfrak{r}_1, \mathfrak{r}_2\right])$. Then, Φ is $\star$-integrable (integrable in the multiplicative sense). Its multiplicative integral is given by:

$$\int_{\star \mathfrak{r}_1}^{\mathfrak{r}_2} \Phi(\mathfrak{z}) \cdot_\star \mathrm{d}_\star \mathfrak{z} = \exp\left\{\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{\ln \Phi\left(\mathfrak{z}\right)}{\mathfrak{z}} \mathrm{d}\mathfrak{z}\right\}.$$

Proposition 2.2 ([10]). *Let $\Phi, \Psi : [\mathfrak{r}_1, \mathfrak{r}_2] \subseteq \mathbb{R}_\star \to \mathbb{R}_\star$ be $\star$-integrable functions on $[\mathfrak{r}_1, \mathfrak{r}_2]$ and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{R}_\star$. Then, the following properties hold:*

- $\int_{\star \mathfrak{r}_1}^{\mathfrak{r}_1} \Phi\left(\mathfrak{z}\right) \cdot_\star \mathrm{d}_\star \mathfrak{z} = 0_\star = 1,$
- $\int_{\star \mathfrak{r}_1}^{\mathfrak{r}_2} \left(\mathfrak{z}_1 \cdot_\star \Phi\left(\mathfrak{z}\right) +_\star \mathfrak{z}_2 \cdot_\star \Psi(\mathfrak{z})\right) \cdot_\star \mathrm{d}_\star \mathfrak{z} = \mathfrak{z}_1 \cdot_\star \int_{\star \mathfrak{r}_1}^{\mathfrak{r}_2} \Phi\left(\mathfrak{z}\right) \cdot_\star \mathrm{d}_\star \mathfrak{z} +_\star \mathfrak{z}_2 \cdot_\star \int_{\star \mathfrak{r}_1}^{\mathfrak{r}_2} \Psi\left(\mathfrak{z}\right) \cdot_\star \mathrm{d}_\star \mathfrak{z},$
- $\int_{\star \mathfrak{r}_1}^{\mathfrak{r}_2} \Phi\left(\mathfrak{z}\right) \cdot_\star \mathrm{d}_\star \mathfrak{z} = \int_{\star \mathfrak{r}_1}^{\mathfrak{c}} \Phi\left(\mathfrak{z}\right) \cdot_\star \mathrm{d}_\star \mathfrak{z} +_\star \int_{\star \mathfrak{c}}^{\mathfrak{r}_2} \Phi\left(\mathfrak{z}\right) \cdot_\star \mathrm{d}_\star \mathfrak{z},$
- $\int_{\star \mathfrak{r}_1}^{\mathfrak{r}_2} \left(\Phi\left(\mathfrak{z}\right)\right)^k \cdot_\star \mathrm{d}_\star \mathfrak{z} = \left(\int_{\star \mathfrak{r}_1}^{\mathfrak{r}_2} \Phi\left(\mathfrak{z}\right) \cdot_\star \mathrm{d}_\star \mathfrak{z}\right)^k.$

Proposition 2.3. *Let $\Phi : \mathrm{D} \subset \mathbb{R}_\star \to \mathbb{R}_\star$ be a $\star$-integrable function, and $\Psi : \mathrm{D} \subset \mathbb{R}_\star \to \mathbb{R}_\star$ be a differentiable function. Then, we have*

$$\int_{\star \mathfrak{r}_1}^{\mathfrak{r}_2} \Phi\left(\Psi(\mathfrak{z})\right) \cdot_\star \mathrm{d}_\star \left(\Psi(\mathfrak{z})\right) = \exp\left\{\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \ln\left(\Phi\left(\Psi(\mathfrak{z})\right)\right) \frac{\Psi'(\mathfrak{z})}{\Psi(\mathfrak{z})} \mathrm{d}\mathfrak{z}\right\}.$$

Proposition 2.4 ([10]). *Let $\Phi : [\mathfrak{r}_1, \mathfrak{r}_2] \to \mathbb{R}_\star$ be a $\star$-differentiable function. Then the following hold:*

If the multiplicative derivative satisfies $\Phi^\star(\mathfrak{z}) \geq 1$ for all $\mathfrak{z} \in [\mathfrak{r}_1, \mathfrak{r}_2]$, then Φ is non-decreasing in the multiplicative sense on the interval ($\star$-increasing).

Conversely, if $\Phi^\star(\mathfrak{r}) \leq 1$ throughout $[\mathfrak{r}_1, \mathfrak{r}_2]$, then Φ is multiplicatively non-increasing over the same domain ($\star$-decreasing).

Proposition 2.5 ([10]). *For $\mathcal{U} \in \mathbb{R}_\star$, the multiplicative absolute value ($\star$-absolute value) is defined as:*

$$|\mathcal{U}|_\star = \begin{cases} \mathcal{U}, & \textit{if } \ \mathcal{U} \geq 1, \\ \dfrac{1}{\mathcal{U}}, & \textit{if } \ 0 < \mathcal{U} < 1. \end{cases}$$

Furthermore, the $\star$-absolute value and the classical absolute value are linked by the following relation:

$$|\ln \mathcal{U}| = \ln |\mathcal{U}|_\star.$$

Remark 2.1 ([10]). For $\mathcal{U}, \mathcal{V} \in \mathbb{R}_\star$, the multiplicative triangle inequality, commonly referred to as the $\star$-triangle inequality, takes the form

$$|\mathcal{U} +_\star \mathcal{V}|_\star \leq |\mathcal{U}|_\star +_\star |\mathcal{V}|_\star.$$

2.2. Multiplicative analogues of the Bullen rule and convexity concepts

In this setting, we define the multiplicative analogue of Bullen's quadrature rule and introduce the concept of multiplicative convexity, which plays a central role in establishing our main inequality results.

Definition 2.6. Let $\Phi : [\mathfrak{r}_1, \mathfrak{r}_2] \subset \mathbb{R}_\star \to \mathbb{R}_\star$ be $\star$-integrable on $[\mathfrak{r}_1, \mathfrak{r}_2]$. We defined the multiplicative Bullen formula, denoted by $\mathcal{B}_\star(\Phi)$ as follows:

$$\begin{aligned}\left(\int_{\star\mathfrak{r}_1}^{\mathfrak{r}_2} \Phi(\mathfrak{z}) \cdot_\star \mathrm{d}_\star\mathfrak{z}\right)^{\frac{1}{\ln(\mathfrak{r}_2)-\ln(\mathfrak{r}_1)}} &\simeq \mathcal{B}_\star(\Phi_\star)\\ &= \exp\left\{\frac{\phi(\ln(\mathfrak{r}_1)) + 2\phi(\ln\sqrt{\mathfrak{r}_1\mathfrak{r}_2}) + \phi(\ln(\mathfrak{r}_2))}{4}\right\}\\ &= \left[\Phi(\mathfrak{r}_1)\left(\Phi(\sqrt{\mathfrak{r}_1\mathfrak{r}_2})\right)^2 \Phi(\mathfrak{r}_2)\right]^{\frac{1}{4}}. \end{aligned} \tag{2.1}$$

Definition 2.7 ([1]). A function $\Phi : \mathbb{R} \to \mathbb{R}_\star$ is said to be GA-convex on $[\mathfrak{r}_1, \mathfrak{r}_2] \subset \mathbb{R}$, if the following inequality

$$\Phi\left(\mathfrak{z}^{\varpi}\mathfrak{y}^{1-\varpi}\right) \leq \varpi\Phi(\mathfrak{z}) + (1-\varpi)\Phi(\mathfrak{y})$$

holds true for all $\mathfrak{z}, \mathfrak{y} \in [\mathfrak{r}_1, \mathfrak{r}_2]$ and all $\varpi \in [0,1]$.

Definition 2.8 ([1]). A function $\Phi : \mathbb{R}_\star \to \mathbb{R}_\star$ is said to be GG-convex on $[\mathfrak{r}_1, \mathfrak{r}_2] \subset \mathbb{R}_\star$, if the following inequality

$$\Phi\left(\mathfrak{z}^{\varpi}\mathfrak{y}^{1-\varpi}\right) \leq (\Phi(\mathfrak{z}))^{\varpi}(\Phi(\mathfrak{y}))^{1-\varpi}$$

holds true for all $\mathfrak{z}, \mathfrak{y} \in [\mathfrak{r}_1, \mathfrak{r}_2]$ and all $\varpi \in [0,1]$.

Remark 2.2 ([1]). A function $\Phi : \mathrm{D} \subset \mathbb{R}_\star \to \mathbb{R}_\star$ is GG-convex on I if and only if $\ln \circ \Phi$ is GA-convex on D.

Definition 2.9 ([17]). A function Φ is deemed to be multiplicatively convex ($\star$-convex) on $[\mathfrak{r}_1, \mathfrak{r}_2] \subset \mathbb{R}_\star$ if the inequality

$$\Phi\left(\varpi_\star \cdot_\star \mathfrak{z} +_\star (1-\varpi)_\star \times \mathfrak{y}\right) \leq \varpi_\star \cdot_\star \Phi(\mathfrak{z}) +_\star (1-\varpi)_\star \cdot_\star \Phi(\mathfrak{y}) \tag{2.2}$$

holds for all $\mathfrak{z}, \mathfrak{y} \in [\mathfrak{r}_1, \mathfrak{r}_2]$ and $\varpi \in [0,1]$, where $\varkappa_\star = \exp\{\varkappa\}$.

Remark 2.3. Rewriting inequality (2.2) using standard operations reveals that $\star$-convexity is equivalent to GG-convexity.

Proposition 2.6 ([17]). *A twice $\star$-differentiable function Φ is $\star$-convex on $[\mathfrak{r}_1, \mathfrak{r}_2] \subset \mathbb{R}_\star$ if and only if $\Phi^{**} \geq 0_\star$, (ie. $\Phi^{**} \geq 1$).*

Proposition 2.7 ([17]). *For $\tau \in (-\infty, 0] \cup [1, \infty)$, the function $\Phi(\mathfrak{z}) = \mathfrak{z}^{\tau_\star}$ is GG-convex on $[1, \infty)$.*

3. Main results

With the necessary theoretical groundwork in place, this section presents the main contributions of the paper: Bullen-type inequalities formulated in terms of multiplicative Riemann-Liouville fractional integrals.

3.1. Multiplicative fractional integral identity

We start by deriving a novel fractional identity within the G-calculus framework; this result will serve as a pivotal instrument in establishing our main outcomes.

Lemma 3.1. *Let $\Phi : [\mathfrak{r}_1, \mathfrak{r}_2] \subset \mathbb{R}_* \to \mathbb{R}_*$ be a $*$-differentiable function with $\Phi^* \in L[\mathfrak{r}_1, \mathfrak{r}_2]$ with $\mathfrak{r}_1 < \mathfrak{r}_2$, then for $\alpha > 0$, the following multiplicative identity holds*

$$\begin{aligned}
&\mathcal{B}_\star(\Phi) -_\star \left({}_\star\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}+}\Phi(\mathfrak{r}_2)\ {}_\star\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}-}\Phi(\mathfrak{r}_1)\right)^{\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)^{\alpha}}} \\
&= \left(\int_{\star 0}^{1} \exp\{2\varpi^{\alpha}-1\} \cdot_\star \Phi^\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right) \cdot_\star \mathrm{d}_\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\right)^{\frac{1}{4}} \\
&\quad -_\star \left(\int_{\star 0}^{1} \exp\{2\varpi^{\alpha}-1\} \cdot_\star \Phi^\star\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right) \cdot_\star \mathrm{d}_\star\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right)\right)^{\frac{1}{4}}, \qquad (3.1)
\end{aligned}$$

where $\mathcal{B}_\star(\Phi)$ is defined as in (2.1).

Proof. Let

$$\mathcal{J} = \mathcal{J}_1 -_\star \mathcal{J}_2, \qquad (3.2)$$

where

$$\mathcal{J}_1 = \left(\int_{\star 0}^{1} \exp\{2\varpi^{\alpha}-1\} \cdot_\star \Phi^\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right) \cdot_\star \mathrm{d}_\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\right)^{\frac{1}{4}}$$

and

$$\mathcal{J}_2 = \left(\int_{\star 0}^{1} \exp\{2\varpi^{\alpha}-1\} \cdot_\star \Phi^\star\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right) \cdot_\star \mathrm{d}_\star\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right)\right)^{\frac{1}{4}}.$$

By making using of Proposition 2.3 along with properties of multiplicative integral and derivative, we obtain for $\mathcal{J}_1$

$$\begin{aligned}
\mathcal{J}_1 &= \left(\int_{\star 0}^{1} \exp\{2\varpi^{\alpha}-1\} \cdot_\star \Phi^\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right) \cdot_\star \mathrm{d}_\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\right)^{\frac{1}{4}} \\
&= \left(\int_{\star 0}^{1} \left(\Phi^\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\right)^{2\varpi^{\alpha}-1} \cdot_\star \mathrm{d}_\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\right)^{\frac{1}{4}} \\
&= \exp\left\{ \frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{8} \int_0^1 \left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right) \frac{\ln\left[\Phi^\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\right]^{2\varpi^{\alpha}-1}}{\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}} \mathrm{d}\varpi \right\} \\
&= \exp\left\{ \frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{8} \int_0^1 (2\varpi^{\alpha}-1)\ln\Phi^\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\mathrm{d}\varpi \right\} \\
&= \exp\left\{ \frac{1}{4}\int_{\mathfrak{r}_1}^{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}} \left[2\left(2\frac{\ln\varkappa-\ln\mathfrak{r}_1}{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}\right)^{\alpha}-1\right] \frac{\ln\Phi^\star(\varkappa)}{\varkappa}\mathrm{d}\varkappa \right\} \\
&= \exp\left\{ \frac{1}{4}\int_{\mathfrak{r}_1}^{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}} \left[2\left(2\frac{\ln\varkappa-\ln\mathfrak{r}_1}{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}\right)^{\alpha}-1\right] (\ln\Phi)'(\varkappa)\,\mathrm{d}\varkappa \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{\frac{1}{4}\left[2\left(2\frac{\ln\varkappa-\ln\mathfrak{r}_1}{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}\right)^{\alpha}-1\right]\ln\Phi\left(\varkappa\right)\Big|_{\mathfrak{r}_1}^{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}}\right.\\
&\quad\left.-\frac{2^{\alpha}\alpha}{2\left(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1\right)^{\alpha}}\int_{\mathfrak{r}_1}^{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}}\left(\ln\varkappa-\ln\mathfrak{r}_1\right)^{\alpha-1}\frac{\ln\Phi(\varkappa)}{\varkappa}\mathrm{d}\varkappa\right\}\\
&= \exp\left\{\frac{1}{4}\ln\Phi(\mathfrak{r}_1)+\frac{1}{4}\ln\Phi\left(\sqrt{\mathfrak{r}_1\mathfrak{r}_2}\right)\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\left(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1\right)^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{r}_1}^{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}}\left(\ln\varkappa-\ln\mathfrak{r}_1\right)^{\alpha-1}\frac{\ln\Phi(\varkappa)}{\varkappa}\mathrm{d}\varkappa\right)\right\}. \qquad (3.3)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\mathscr{J}_2 &= \left(\int_{\star 0}^{1}\exp\left\{2\varpi^{\alpha}-1\right\}\cdot_{\star}\Phi^{\star}\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right)\cdot_{\star}\mathrm{d}_{\star}\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right)\right)^{\frac{1}{4}}\\
&= \exp\left\{\frac{1}{4}\ln\Phi\left(\mathfrak{r}_2\right)+\frac{1}{4}\ln\Phi(\sqrt{\mathfrak{r}_1\mathfrak{r}_2})\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1\right)^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\int_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}}^{\mathfrak{r}_2}\left(\ln\mathfrak{r}_2-\ln\varkappa\right)^{\alpha-1}\frac{\ln\Phi(\varkappa)}{\varkappa}\mathrm{d}\varkappa\right)\right\}. \qquad (3.4)
\end{aligned}
$$

By using the equalities (3.3) and (3.4) in (3.2), we get equality (3.1). This completes the proof. □

3.2. Multiplicative fractional Bullen-type inequalities in G-calculus

Leveraging the identity derived above and exploiting the structural properties of multiplicative convex functions, we establish novel Bullen-type inequalities, effectively extending their classical counterparts into the G-calculus setting.

Theorem 3.1. *Let Φ be as in Lemma 3.1. If $|\Phi^{\star}|_{\star}$ is GG-convex, then the following Bullen-type inequality holds*

$$
\begin{aligned}
&\left|\mathcal{B}_{\star}\left(\Phi\right)-_{\star}\left({}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}+}\Phi\left(\mathfrak{r}_2\right)\,{}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}-}\Phi\left(\mathfrak{r}_1\right)\right)^{\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)^{\alpha}}}\right|_{\star}\\
&\le\left(\left|\Phi^{\star}\left(\mathfrak{r}_1\right)\right|_{\star}\left|\Phi^{\star}\left(\mathfrak{r}_2\right)\right|_{\star}\right)^{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{8(\alpha+1)}\left(\alpha\left(2^{1-\frac{1}{\alpha}}-1\right)+1\right)}.
\end{aligned}
$$

Proof. By using the properties of multiplicative integral and derivative along with the $\star$-absolute value, (3.1) yields

$$
\begin{aligned}
&\left|\mathcal{B}_{\star}\left(\Phi\right)-_{\star}\left({}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}+}\Phi\left(\mathfrak{r}_2\right)\,{}_{\star}\mathscr{I}^{\alpha}_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}-}\Phi\left(\mathfrak{r}_1\right)\right)^{\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)^{\alpha}}}\right|_{\star}\\
&\le\left|\left(\int_{\star 0}^{1}\exp\left\{2\varpi^{\alpha}-1\right\}\cdot_{\star}\Phi^{\star}\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\cdot_{\star}\mathrm{d}_{\star}\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\right)^{\frac{1}{4}}\right.\\
&\quad\left.-_{\star}\left(\int_{\star 0}^{1}\exp\left\{2\varpi^{\alpha}-1\right\}\cdot_{\star}\Phi^{\star}\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right)\cdot_{\star}\mathrm{d}_{\star}\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right)\right)^{\frac{1}{4}}\right|_{\star}
\end{aligned}
$$

$$
\begin{aligned}
&= \left| \exp\left\{ \frac{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}{8} \int_0^1 (2\varpi^\alpha - 1) \ln \Phi^\star \left(\mathfrak{r}_1^{\frac{2-\varpi}{2}} \mathfrak{r}_2^{\frac{\varpi}{2}} \right) \mathrm{d}\varpi \right.\right. \\
&\quad \left.\left. - \frac{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}{8} \int_1^1 \left(2\varpi^\alpha - \tfrac{1}{3}\right) \ln \Phi^\star \left(\mathfrak{r}_1^{\frac{\varpi}{2}} \mathfrak{r}_2^{\frac{2-\varpi}{2}} \right) \mathrm{d}\varpi \right\} \right|_\star \\
&= \exp\left\{ \left| \frac{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}{8} \int_0^1 (2\varpi^\alpha - 1) \left(\ln \Phi^\star \left(\mathfrak{r}_1^{\frac{2-\varpi}{2}} \mathfrak{r}_2^{\frac{\varpi}{2}} \right) - \ln \Phi^\star \left(\mathfrak{r}_1^{\frac{\varpi}{2}} \mathfrak{r}_2^{\frac{2-\varpi}{2}} \right) \right) \mathrm{d}\varpi \right| \right\} \\
&\leq \exp\left\{ \frac{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}{8} \int_0^1 |2\varpi^\alpha - 1| \left(\left| \ln \Phi^\star \left(\mathfrak{r}_1^{\frac{2-\varpi}{2}} \mathfrak{r}_2^{\frac{\varpi}{2}} \right) \right| + \left| \ln \Phi^\star \left(\mathfrak{r}_1^{\frac{\varpi}{2}} \mathfrak{r}_2^{\frac{2-\varpi}{2}} \right) \right| \right) \mathrm{d}\varpi \right\} \\
&= \exp\left\{ \frac{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}{8} \int_0^1 |2\varpi^\alpha - 1| \left(\ln \left| \Phi^\star \left(\mathfrak{r}_1^{\frac{2-\varpi}{2}} \mathfrak{r}_2^{\frac{\varpi}{2}} \right) \right|_\star + \ln \left| \Phi^\star \left(\mathfrak{r}_1^{\frac{\varpi}{2}} \mathfrak{r}_2^{\frac{2-\varpi}{2}} \right) \right|_\star \right) \mathrm{d}\varpi \right\}. \qquad (3.5)
\end{aligned}
$$

By using the fact that $|\Phi^\star|_\star$ is GG-convex, inequality (3.5) gives

$$
\begin{aligned}
&\left| \mathcal{B}_\star(\Phi) -_\star \left({}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}+} \Phi(\mathfrak{r}_2)\, {}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}-} \Phi(\mathfrak{r}_1) \right)^{\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1)^\alpha}} \right|_\star \\
&\leq \exp\left\{ \frac{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}{8} \left(\int_0^1 |2\varpi^\alpha - 1| \left(\begin{array}{l} \ln \left(|\Phi^\star(\mathfrak{r}_1)|_\star^{\frac{2-\varpi}{2}} |\Phi^\star(\mathfrak{r}_2)|_\star^{\frac{\varpi}{2}} \right) \\ + \ln \left(|\Phi^\star(\mathfrak{r}_1)|_\star^{\frac{\varpi}{2}} |\Phi^\star(\mathfrak{r}_2)|_\star^{\frac{2-\varpi}{2}} \right) \end{array} \right) \mathrm{d}\varpi \right) \right\} \\
&= \exp\left\{ \frac{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}{8} \left(\frac{\alpha\left(2^{1-\frac{1}{\alpha}} - 1\right) + 1}{\alpha + 1} \right) \left(\ln |\Phi^\star(\mathfrak{r}_1)|_\star + \ln |\Phi^\star(\mathfrak{r}_2)|_\star \right) \right\} \\
&= \left(|\Phi^\star(\mathfrak{r}_1)|_\star |\Phi^\star(\mathfrak{r}_2)|_\star \right)^{\frac{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}{8(\alpha+1)} \left(\alpha \left(2^{1-\frac{1}{\alpha}} - 1 \right) + 1 \right)}.
\end{aligned}
$$

The proof is completed. □

Corollary 3.1. *In Theorem 3.1, if we attempte to take $\alpha = 1$, then we get the following Bullen-type inequality for multiplicative integrals*

$$
\left| \mathcal{B}_\star(\Phi) -_\star \left(\int_{\star\mathfrak{r}_1}^{\mathfrak{r}_2} \Phi(\mathfrak{z}) \mathrm{d}\mathfrak{z} \right)^{\frac{1}{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}} \right|_\star \leq \left[\Phi^\star(\mathfrak{r}_1)\, \Phi^\star(\mathfrak{r}_2) \right]^{\frac{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}{16}}.
$$

Theorem 3.2. *Let Φ be as in Lemma 3.1. If $(\ln |\Phi^\star|_\star)^\jmath$ is GA-convex for $\jmath > 0$ with $\frac{1}{\imath} + \frac{1}{\jmath} = 1$, then the following Bullen-type inequality holds*

$$
\begin{aligned}
&\left| \mathcal{B}_\star(\Phi) -_\star \left({}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}+} \Phi(\mathfrak{r}_2)\, {}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}-} \Phi(\mathfrak{r}_1) \right)^{\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1)^\alpha}} \right|_\star \\
&\leq \left(|\Phi^\star(\mathfrak{r}_1)|_\star |\Phi^\star(\mathfrak{r}_2)|_\star \right)^{(\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1) \frac{1+3^{\frac{1}{\jmath}}}{2^{3+\frac{2}{\jmath}}} \mathcal{L}(\alpha,\imath)^{\frac{1}{\imath}}}.
\end{aligned}
$$

Here $\mathcal{L}(\alpha,\imath)$ is given by

$$
\mathcal{L}(\alpha,\imath) = \int_0^1 |2\varpi^\alpha - 1|^\imath \, \mathrm{d}\varpi. \qquad (3.6)
$$

Proof. Applying the Hölder inequality then using the GA-convexity of $(\ln|\Phi|_\star)^\jmath$, we obtain from (3.5) that

$$\begin{aligned}
&\left|\mathcal{B}_\star(\Phi) -_\star \left({}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}+}\Phi(\mathfrak{r}_2)\ {}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}-}\Phi(\mathfrak{r}_1)\right)^{\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)^\alpha}}\right|_\star \\
&\leq \exp\left\{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{8}\left(\int_0^1|2\varpi^\alpha-1|^\imath\, d\varpi\right)^{\frac{1}{\imath}}\left[\begin{array}{l}\left(\int_0^1\left(\ln\left|\Phi^\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\right|_\star\right)^\jmath d\varpi\right)^{\frac{1}{\jmath}}\\ +\left(\int_0^1\left(\ln\left|\Phi^\star\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right)\right|_\star\right)^\jmath d\varpi\right)^{\frac{1}{\jmath}}\end{array}\right]\right\} \\
&\leq \exp\left\{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{8}\left(\int_0^1|2\varpi^\alpha-1|^\imath\, d\varpi\right)^{\frac{1}{\imath}}\right. \\
&\quad\left.\times\left[\begin{array}{l}\left(\int_0^1\frac{2-\varpi}{2}(\ln|\Phi^\star(\mathfrak{r}_1)|)^\jmath+\frac{\varpi}{2}(\ln|\Phi(\mathfrak{r}_2)|_\star)^\jmath d\varpi\right)^{\frac{1}{\jmath}}\\ +\left(\int_0^1\frac{\varpi}{2}(\ln|\Phi^\star(\mathfrak{r}_1)|)^\jmath+\frac{2-\varpi}{2}(\ln|\Phi(\mathfrak{r}_2)|_\star)^\jmath d\varpi\right)^{\frac{1}{\jmath}}\end{array}\right]\right\} \\
&= \exp\left\{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{8}\mathcal{L}(\alpha,\imath)^{\frac{1}{\imath}}\left[\begin{array}{l}\left(\frac{3[\ln\Phi^\star(\mathfrak{r}_1)]^\jmath+[\ln\Phi^\star(\mathfrak{r}_2)]^\jmath}{4}\right)^{\frac{1}{\jmath}}\\ +\left(\frac{[\ln\Phi^\star(\mathfrak{r}_1)]^\jmath+3[\ln\Phi^\star(\mathfrak{r}_2)]^\jmath}{4}\right)^{\frac{1}{\jmath}}\end{array}\right]\right\} \\
&\leq \exp\left\{(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)\frac{1+3^{\frac{1}{\jmath}}}{2^{3+\frac{2}{\jmath}}}\mathcal{L}(\alpha,\imath)^{\frac{1}{\imath}}\left[\ln|\Phi^\star(\mathfrak{r}_1)|_\star+\ln|\Phi^\star(\mathfrak{r}_2)|_\star\right]\right\} \\
&= \left(|\Phi^\star(\mathfrak{r}_1)|_\star|\Phi^\star(\mathfrak{r}_2)|_\star\right)^{(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)\frac{1+3^{\frac{1}{\jmath}}}{2^{3+\frac{2}{\jmath}}}\mathcal{L}(\alpha,\imath)^{\frac{1}{\imath}}},
\end{aligned}$$

where we have used (3.6), and the fact that $(P+Q)^\omega \leq P^\omega + Q^\omega$ for $P, Q > 0$ and $0 \leq \omega < 1$. The proof is completed. □

Theorem 3.3. *Let Φ be as in Lemma 3.1. If $(\ln|\Phi^\star|_\star)^\jmath$ is GA-convex for $\jmath > 0$, then the following Bullen-type inequality holds*

$$\begin{aligned}
&\left|\mathcal{B}_\star(\Phi) -_\star \left({}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}+}\Phi(\mathfrak{r}_2)\ {}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}-}\Phi(\mathfrak{r}_1)\right)^{\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)^\alpha}}\right|_\star \\
&\leq \left(|\Phi^\star(\mathfrak{r}_1)|_\star|\Phi^\star(\mathfrak{r}_2)|_\star\right)^{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{4}\left(\frac{\alpha\left(2^{1-\frac{1}{\alpha}}-1\right)+1}{\alpha+1}\right)^{1-\frac{1}{\jmath}}\left(\left(\frac{\alpha 2^{\frac{\alpha-1}{\alpha}}+1-\alpha}{\alpha+1}-\frac{\alpha 2^{\frac{\alpha-2}{\alpha}}+2-\alpha}{4(\alpha+2)}\right)^{\frac{1}{\jmath}}+\left(\frac{\alpha 2^{\frac{\alpha-2}{\alpha}}+2-\alpha}{4(\alpha+2)}\right)^{\frac{1}{\jmath}}\right)}.
\end{aligned}$$

Proof. Applying the power mean inequality then using the GA-convexity of $(\ln|\Phi|_\star)^\jmath$, (3.5) gives

$$\begin{aligned}
&\left|\mathcal{B}_\star(\Phi) -_\star \left({}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}+}\Phi(\mathfrak{r}_2)\ {}_\star\mathscr{I}^\alpha_{\sqrt{\mathfrak{r}_1\mathfrak{r}_2}-}\Phi(\mathfrak{r}_1)\right)^{\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)^\alpha}}\right|_\star \\
&\leq \exp\left\{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{8}\left(\int_0^1|2\varpi^\alpha-1|\, d\varpi\right)^{1-\frac{1}{\jmath}}\right.
\end{aligned}$$

$$\times\left[\begin{array}{l}\left(\int_0^1|2\varpi^\alpha-1|\left(\ln\left|\Phi^\star\left(\mathfrak{r}_1^{\frac{2-\varpi}{2}}\mathfrak{r}_2^{\frac{\varpi}{2}}\right)\right|_\star\right)^\jmath\mathrm{d}\varpi\right)^{\frac{1}{\jmath}}\\+\left(\int_0^1|2\varpi^\alpha-1|\left(\ln\left|\Phi^\star\left(\mathfrak{r}_1^{\frac{\varpi}{2}}\mathfrak{r}_2^{\frac{2-\varpi}{2}}\right)\right|_\star\right)^\jmath\mathrm{d}\varpi\right)^{\frac{1}{\jmath}}\end{array}\right]\Bigg\}$$

$$\leq\exp\left\{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{8}\left(\int_0^1|2\varpi^\alpha-1|\,\mathrm{d}\varpi\right)^{1-\frac{1}{\jmath}}\right.$$

$$\left.\times\left[\begin{array}{l}\left(\int_0^1|2\varpi^\alpha-1|\left(\frac{2-\varpi}{2}\left(\ln\left|\Phi^\star(\mathfrak{r}_1)\right|\right)^\jmath+\frac{\varpi}{2}\left(\ln\left|\Phi(\mathfrak{r}_2)\right|_\star\right)^\jmath\right)\mathrm{d}\varpi\right)^{\frac{1}{\jmath}}\\+\left(\int_0^1|2\varpi^\alpha-1|\left(\frac{\varpi}{2}\left(\ln\left|\Phi^\star(\mathfrak{r}_1)\right|\right)^\jmath+\frac{2-\varpi}{2}\left(\ln\left|\Phi(\mathfrak{r}_2)\right|_\star\right)^\jmath\right)\mathrm{d}\varpi\right)^{\frac{1}{\jmath}}\end{array}\right]\right\}$$

$$=\exp\left\{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{4}\left(\frac{\alpha\left(2^{1-\frac{1}{\alpha}}-1\right)+1}{\alpha+1}\right)^{1-\frac{1}{\jmath}}\left(\ln\left|\Phi^\star(\mathfrak{r}_1)\right|_\star+\ln\left|\Phi^\star(\mathfrak{r}_2)\right|_\star\right)\right.$$

$$\left.\times\left(\left(\frac{\alpha2^{\frac{\alpha-1}{\alpha}}+1-\alpha}{\alpha+1}-\frac{\alpha2^{\frac{\alpha-2}{\alpha}}+2-\alpha}{4(\alpha+2)}\right)^{\frac{1}{\jmath}}+\left(\frac{\alpha2^{\frac{\alpha-2}{\alpha}}+2-\alpha}{4(\alpha+2)}\right)^{\frac{1}{\jmath}}\right)\right\}$$

$$=\left(\left|\Phi^\star(\mathfrak{r}_1)\right|_\star\left|\Phi^\star(\mathfrak{r}_2)\right|_\star\right)^{\frac{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}{4}\left(\frac{\alpha\left(2^{1-\frac{1}{\alpha}}-1\right)+1}{\alpha+1}\right)^{1-\frac{1}{\jmath}}\left(\left(\frac{\alpha2^{\frac{\alpha-1}{\alpha}}+1-\alpha}{\alpha+1}-\frac{\alpha2^{\frac{\alpha-2}{\alpha}}+2-\alpha}{4(\alpha+2)}\right)^{\frac{1}{\jmath}}+\left(\frac{\alpha2^{\frac{\alpha-2}{\alpha}}+2-\alpha}{4(\alpha+2)}\right)^{\frac{1}{\jmath}}\right)},$$

where we have used the fact that $(A+B)^w\leq A^w+B^w$ for $A,B>0$ and $0\leq w<1$. The proof is completed. □

4. Examples and applications

4.1. Graphical and numerical verification

For illustrative purposes, this subsection presents two numerical examples and their graphical representations, offering concrete evidence of the consistency of our results within the multiplicative setting. All graphical representations were carried out using MATLAB R2022b on a computer equipped with an Intel Core i5 processor.

Example 4.1. Let $\Phi\colon[1,2]\to\mathbb{R}_\star$ be the multiplicative function defined by

$$\Phi(\mathfrak{z})=\mathfrak{z}^{3_\star}/_\star\exp\{3\}.$$

Its $\star$-derivative, given by $\Phi^\star(\mathfrak{z})=\mathfrak{z}^{2_\star}$, satisfies the GG-convexity condition on $[1,2]$, as established in Proposition 2.7. This ensures that Φ fulfills the key hypothesis necessary for the applicability of our main theoretical results.

By invoking Theorem 3.1, the following inequality holds:

$$\left|\exp\left((\ln2)^3\left(\frac{5}{48}-\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+4)}-\frac{\alpha}{6(\alpha+3)}\right)\right)\right|_\star\leq\exp\left(\frac{(\ln2)^3}{8(\alpha+1)}\left(\alpha\left(2^{1-\frac{1}{\alpha}}-1\right)+1\right)\right).\quad(4.1)$$

Figure 1 illustrates both sides of inequality (4.1). The plot confirms that, over the full range of $\alpha\in(0,3]$, the right-hand expression consistently dominates the left-hand side, thereby providing numerical and visual confirmation of the theoretical bound.

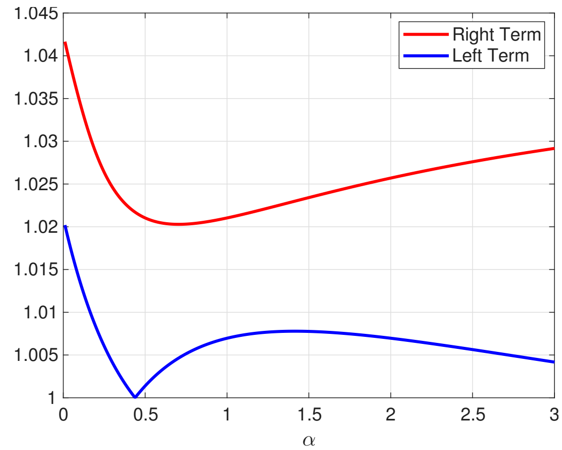

Figure 1. Graphical validation of inequality (4.1).

Example 4.2. Consider the function defined on $[1,2]$ by $\Phi(\mathfrak{z}) = \exp\left\{(\ln\mathfrak{z})^2 + (\ln\mathfrak{z})^3\right\}$ and for which the $\star$-derivative given by $\Phi^\star(\mathfrak{z}) = \exp\left\{2\ln\mathfrak{z} + 3(\ln\mathfrak{z})^2\right\}$ is GG-convex on $[1,2]$. By invoking Theorem 3.1, we obtain

$$\left|\exp\left\{(\ln 2)^2\left[\frac{6+5\ln 2}{16} - \frac{(2+\ln 2)\left(\alpha^2+3\alpha+4\right)+4\ln 2}{8(\alpha+1)(\alpha+2)}\right]\right\}\right|_\star \leq \exp\left(\frac{2(\ln 2)^2+3(\ln 2)^3}{8(\alpha+1)}\left(\alpha\left(2^{1-\frac{1}{\alpha}}-1\right)+1\right)\right). \quad (4.2)$$

A graphical comparison of both members of inequality (4.2) is presented in Figure 2. The curves clearly show that the right-hand side remains greater than the left-hand side throughout the interval $\alpha \in (0,3]$, effectively validating the theoretical bound from a numerical perspective.

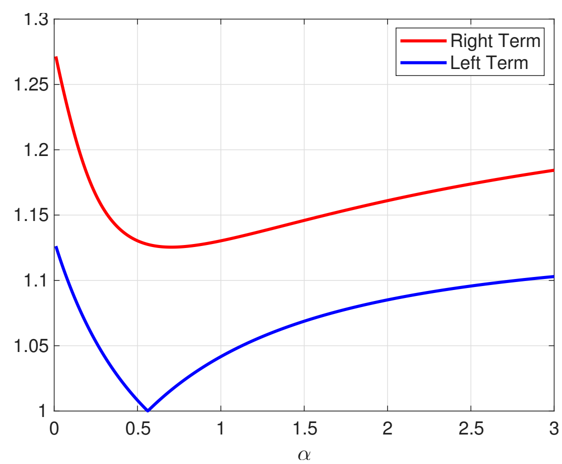

Figure 2. Graphical validation of inequality (4.2).

4.2. Applications to special means

To highlight the practical utility and analytical power of the established results, this subsection develops several concrete applications. We begin by recalling the classical means associated with two positive real numbers $\mathfrak{r}_1$ and $\mathfrak{r}_2$: The arithmetic mean $\mathscr{A}(\mathfrak{r}_1, \mathfrak{r}_2)$, the geometric mean $\mathscr{G}(\mathfrak{r}_1, \mathfrak{r}_2)$, and the logarithmic mean $\mathscr{L}(\mathfrak{r}_1, \mathfrak{r}_2)$, defined respectively as

$$\mathscr{A}(\mathfrak{r}_1, \mathfrak{r}_2) = \frac{\mathfrak{r}_1 + \mathfrak{r}_2}{2},$$
$$\mathscr{G}(\mathfrak{r}_1, \mathfrak{r}_2) = \sqrt{\mathfrak{r}_1 \mathfrak{r}_2}, \ \mathfrak{r}_1, \mathfrak{r}_2 > 0$$

and

$$\mathscr{L}(\mathfrak{r}_1, \mathfrak{r}_2) = \frac{\mathfrak{r}_2 - \mathfrak{r}_1}{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}, \ \mathfrak{r}_1, \mathfrak{r}_2 > 0 \text{ with } \mathfrak{r}_1 \neq \mathfrak{r}_2.$$

Proposition 4.1. *For $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathbb{R}_\star$ such that $1 < \mathfrak{r}_1 < \mathfrak{r}_2$, we have*

$$\left| \frac{\mathscr{A}(\mathfrak{r}_1, \mathfrak{r}_2) + \mathscr{G}^4(\mathfrak{r}_1, \mathfrak{r}_2)}{4} - \mathscr{L}(\mathfrak{r}_1, \mathfrak{r}_2) \right| \leq \frac{1}{16} (\mathfrak{r}_1 + \mathfrak{r}_2)(\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1).$$

Proof. The result follows directly by applying Corollary 3.1 to the function $\Phi(\mathfrak{z}) = \exp\{\mathfrak{z}\}$, where

$$\Phi^*(\mathfrak{z}) = \exp\{\mathfrak{z}\}$$

and

$$\left(\int_{\star \mathfrak{r}_1}^{\mathfrak{r}_2} \Phi(\mathfrak{z}) \cdot_\star d_\star \mathfrak{z} \right)^{\frac{1}{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1}} = \exp\left\{ \frac{1}{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_2} \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} d\mathfrak{z} \right\} = \exp\left\{ \frac{\mathfrak{r}_2 - \mathfrak{r}_1}{\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1} \right\} = \exp\{\mathscr{L}(\mathfrak{r}_1, \mathfrak{r}_2)\}.$$

□

Proposition 4.2. *For $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathbb{R}_\star$ such that $1 < \mathfrak{r}_1 < \mathfrak{r}_2$, we have*

$$\left| \frac{\mathscr{A}(\mathfrak{r}_1, \mathfrak{r}_2) + \mathscr{G}^4(\mathfrak{r}_1, \mathfrak{r}_2)}{4} - \mathscr{L}(\mathfrak{r}_1, \mathfrak{r}_2) \right| \leq \mathscr{A}(\mathfrak{r}_1, \mathfrak{r}_2)(\ln \mathfrak{r}_2 - \ln \mathfrak{r}_1) \left(\frac{8}{9\sqrt{3}} + \frac{1}{6} \right).$$

Proof. This result is derived by invoking Theorem 3.1 with $\alpha = 2$ to the function $\Phi(\mathfrak{z}) = \exp\{\mathfrak{z}\}$. □

5. Conclusion

In conclusion, this study establishes a multiplicative analogue of Bullen's inequality within the framework of multiplicative Riemann-Liouville fractional calculus. By adapting classical convexity to a geometric setting and harnessing the tools of non-Newtonian analysis, we have shown that fundamental integral inequalities can be meaningfully recast in multiplicative terms. These results not only enrich the theoretical landscape of fractional calculus but also provide a springboard for deriving multiplicative counterparts of other classical bounds. Our work contributes to the growing body of non-Newtonian fractional analysis and offers a solid foundation for future theoretical explorations and computational applications in this innovative domain.

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Data availability. This study did not involve the generation or analysis of new datasets; therefore, data sharing is not applicable.

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