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OSCILLATORY SINGULAR SPECIAL FUNCTION TRANSFORM

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Abstract We introduce a new class of integral transforms, called the Oscillatory Singular Special Function Transform (OSSFT), whose kernels combine algebraic singularities, nonlinear oscillatory phases, and special-function components of Mittag–Leffler type. Fundamental properties of the OSSFT are established, including boundedness on weighted Lebesgue spaces, stability, compactness, and smoothing effects. Under suitable symmetry and decay conditions, a Plancherel-type theorem and Heisenberg-type uncertainty inequalities are proved. The compactness of the associated operators further yields spectral discreteness and Sobolev regularity of eigenfunctions.

Keywords Oscillatory integral transforms, singular kernels, special functions, Mittag–Leffler functions, weighted Lebesgue spaces.

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1. Introduction

Integral transforms play a fundamental role in mathematical analysis and its applications, providing powerful tools for solving differential and integral equations, performing spectral analysis, and transferring problems between different functional settings. Classical transforms such as the Fourier and Laplace transforms are cornerstones of harmonic analysis and have been extensively studied, both from theoretical and applied perspectives. Their success has motivated the development of generalized integral transforms designed to address increasingly complex models arising in fractional calculus, anomalous diffusion, nonlocal dynamics, and mathematical physics. Recent developments in harmonic analysis and transform methods, including representation-theoretic approaches and wavelet frameworks, have been investigated in various contexts; see, for example, [1, 2, 5, 7, 8].

In recent years, particular attention has been devoted to integral operators whose kernels involve special functions, singular behavior, or oscillatory structures. Special functions of Mittag–Leffler type, for instance, naturally arise in the theory of fractional differential equations and memory-dependent processes. Integral transforms incorporating such functions have been investigated in connection with generalized fractional operators and nonlocal models. Pal [15] studied generalized integral transforms involving Mittag–Leffler type functions and established connections with fractional calculus operators. Kabra [10] derived integral representations of generalized Mittag–Leffler functions through classical transforms such as the Euler, Laplace, Whittaker, and Mellin transforms, expressed in terms of generalized Wright functions.

More broadly, the theory of special functions with general kernels has been developed to unify and extend classical transform methods. Ata et al. [4] investigated integral representa-

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tions, functional relations, and double Laplace transforms of special functions whose kernels include Mittag–Leffler, Wright, and Fox–Wright functions. Related developments emphasize the versatility of generalized integral transforms and their relevance in applied analysis and fractional models [6, 12, 14, 16].

Parallel to these developments, oscillatory integral operators with singular kernels have been intensively studied in harmonic analysis. Such operators present significant analytical challenges due to the simultaneous presence of algebraic singularities and rapidly oscillating phases. Classical techniques from oscillatory integral theory and real-variable harmonic analysis, as developed for instance in [9, 18], play a central role in understanding their mapping properties. Recent contributions have addressed oscillatory Fourier integrals with algebraic singularities and specialized oscillators, highlighting both analytical and numerical aspects of these operators [3, 11, 13, 17].

Despite this substantial progress, there remains a lack of a unified framework for integral transforms whose kernels simultaneously incorporate algebraic singularities, nonlinear oscillatory phases, and special-function components beyond separable or classical forms. Such kernels naturally arise in models combining nonlocal interactions, fractional dynamics, and oscillatory phenomena, yet their systematic analysis is still limited.

The purpose of this paper is to introduce and study a new class of integral transforms, which we call the Oscillatory Singular Special Function Transform. The defining feature of the Oscillatory Singular Special Function Transform is a kernel that combines an algebraic singularity with a nonlinear oscillatory phase and a Mittag–Leffler type special-function factor. This structure allows the transform to capture both singular and oscillatory effects while retaining sufficient decay to ensure well-posedness on appropriate function spaces.

We develop a comprehensive analytical theory for the Oscillatory Singular Special Function Transform. In particular, we establish boundedness properties on weighted Lebesgue spaces under explicit and interpretable parameter conditions. We further prove stability with respect to admissible kernel perturbations and show that enhanced decay of the special-function component leads to compactness of the associated operators. Under suitable symmetry and decay assumptions, we derive a Plancherel-type theorem and establish Heisenberg-type uncertainty inequalities, demonstrating that the Oscillatory Singular Special Function Transform behaves as a genuine Fourier-type transform in an appropriate functional setting.

Moreover, we show that the Oscillatory Singular Special Function Transform exhibits a quantitative smoothing effect, yielding Sobolev regularity improvement for transformed functions. As a consequence of compactness and smoothing, we obtain a discrete spectral decomposition for the associated normal operator, with eigenfunctions enjoying enhanced regularity. Concrete examples are provided to illustrate the admissibility conditions and to demonstrate the nontrivial nature of the proposed framework.

The results presented in this work place the Oscillatory Singular Special Function Transform at the intersection of oscillatory integral theory, special-function analysis, and operator theory. The framework introduced here opens several directions for future research, including extensions to non-Euclidean settings, anisotropic and multi-parameter kernels, and applications to fractional differential equations, inverse problems, and nonlocal models in mathematical physics.

Nomenclature

- OSSFT: Oscillatory Singular Special Function Transform.
- $L^p(\mathbb{R}^N)$: Lebesgue space of p-integrable functions on $\mathbb{R}^N$.

- $L^p_\alpha(\mathbb{R}^N)$: Weighted Lebesgue space with weight $(1+|x|)^{-\alpha}$.
- $H^s(\mathbb{R}^N)$: Sobolev space of order s on $\mathbb{R}^N$.

2. Preliminaries

Throughout this paper, $\mathbb{R}^N$ denotes the N-dimensional Euclidean space with $N \geq 1$, and integration is taken with respect to the Lebesgue measure unless otherwise specified. Complex-valued functions are considered unless explicitly stated otherwise. The symbol C denotes a generic positive constant whose value may change from line to line.

We begin by fixing the class of special functions appearing in the kernel of the Oscillatory Singular Special Function Transform (OSSFT). Let $\beta, \gamma > 0$ and define the function

$$\mathcal{E}_{\beta,\gamma}(z) := \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\beta m + \gamma)}, \qquad z \in \mathbb{C}.$$

Unlike classical Mittag–Leffler functions, we shall not rely on their known asymptotic theory; instead, we record the following growth property, which will be sufficient for our analysis.

Definition 2.1. A function $\mathcal{E}_{\beta,\gamma}$ is said to be oscillatory-admissible if there exists a constant $C > 0$ such that

$$|\mathcal{E}_{\beta,\gamma}(it)| \leq C(1+|t|)^{-1/\beta} \quad \text{for all } t \in \mathbb{R}.$$

This decay condition ensures integrability of kernels involving $\mathcal{E}_{\beta,\gamma}$ when combined with algebraic singularities.

Next, we introduce the oscillatory–singular kernel structure underlying the OSSFT. Let $\alpha \in (0, N)$ and let $p, q, \mu, \nu > 0$ be fixed parameters. For $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$, define

$$\mathcal{K}(x,\xi) := |x-\xi|^{-\alpha}\, \mathcal{E}_{\beta,\gamma}(i\,|x|^{\mu}|\xi|^{\nu}) \exp\!\left(i\, \frac{|x|^p}{|\xi|^q + 1} \right).$$

The singularity at $x = \xi$ is controlled by α, while oscillation is induced by the nonlinear phase term. The following notion of kernel admissibility will be central in establishing boundedness.

Definition 2.2. The kernel $\mathcal{K}$ is said to be *OSSFT-admissible* if

$$\sup_{\xi \in \mathbb{R}^N} \int_{\mathbb{R}^N} |\mathcal{K}(x,\xi)|\, dx < \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} |\mathcal{K}(x,\xi)|\, d\xi < \infty.$$

To verify admissibility, we split the domain into near and far regions. For fixed ξ, write

$$\int_{\mathbb{R}^N} |\mathcal{K}(x,\xi)|\, dx = \int_{|x-\xi|\leq 1} + \int_{|x-\xi|>1} =: I_1 + I_2.$$

For I_1, using the oscillatory-admissibility of $\mathcal{E}_{\beta,\gamma}$ and polar coordinates,

$$I_1 \leq C \int_{|y|\leq 1} |y|^{-\alpha}\, dy = C \int_0^1 r^{N-\alpha-1}\, dr,$$

which converges since $\alpha < N$.

For I_2, the decay of $\mathcal{E}_{\beta,\gamma}$ yields

$$|\mathcal{K}(x,\xi)| \le C|x-\xi|^{-\alpha}(1+|x|^{\mu}|\xi|^{\nu})^{-1/\beta},$$

and thus

$$I_2 \le C\int_{\mathbb{R}^N} \frac{dx}{|x-\xi|^{\alpha}(1+|x|^{\mu}|\xi|^{\nu})^{1/\beta}},$$

which is finite for $\mu/\beta > N-\alpha$. Hence $\mathcal{K}$ is OSSFT-admissible under explicit parameter constraints.

We now define the function spaces in which the OSSFT will act.

Definition 2.3. Let $1 \le p < \infty$. The space $\mathcal{L}^p_\alpha(\mathbb{R}^N)$ consists of all measurable functions f such that

$$\|f\|_{\mathcal{L}^p_\alpha} := \left(\int_{\mathbb{R}^N} |f(x)|^p(1+|x|)^{-\alpha p}\,dx\right)^{1/p} < \infty.$$

This weighted structure compensates for the singular growth of the kernel and is intrinsic to the OSSFT framework.

Finally, we introduce the integral operator associated with $\mathcal{K}$ in a preliminary form.

Definition 2.4. For $f \in \mathcal{L}^1_\alpha(\mathbb{R}^N)$, define

$$(\mathcal{T}f)(\xi) := \int_{\mathbb{R}^N} \mathcal{K}(x,\xi)f(x)\,dx,$$

whenever the integral converges absolutely.

Using the kernel admissibility and Fubini-type arguments adapted to oscillatory integrals, one verifies that $\mathcal{T}$ is well defined on $\mathcal{L}^1_\alpha(\mathbb{R}^N)$ and extends to larger function spaces. These properties will be sharpened in the next section, where boundedness, continuity, and inversion results for the OSSFT are established.

3. Mapping properties and structural results of the OSSFT

In this section, we establish the principal analytical properties of the Oscillatory Singular Special Function Transform (OSSFT). The results concern boundedness, stability, compactness, duality, and reconstruction properties of the transform. All statements are formulated under explicit parameter constraints that arise naturally from the oscillatory–singular structure of the kernel.

We now establish the fundamental mapping property of the OSSFT, showing that the interplay between the algebraic singularity and the special-function decay yields boundedness on weighted Lebesgue spaces. The following result provides the basic L^p–boundedness of the OSSFT under explicit and sharp parameter constraints.

Theorem 3.1. *Let $\alpha \in (0,N)$ and assume that the kernel $\mathcal{K}$ is OSSFT-admissible. If*

$$\frac{\mu}{\beta} > N-\alpha,$$

then the operator $\mathcal{T}$ extends uniquely to a bounded linear operator

$$\mathcal{T} : \mathcal{L}^p_\alpha(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$$

for every $1 \le p \le \infty$, and there exists a constant $C_p > 0$ such that

$$\|\mathcal{T}f\|_{L^p} \le C_p\|f\|_{\mathcal{L}^p_\alpha}.$$

Proof. Let $f \in \mathcal{L}^p_\alpha(\mathbb{R}^N)$ and recall that

$$(\mathcal{T}f)(\xi) = \int_{\mathbb{R}^N} \mathcal{K}(x,\xi) f(x)\,dx.$$

Using the explicit form of the kernel,

$$\mathcal{K}(x,\xi) = |x-\xi|^{-\alpha}\,\mathcal{E}_{\beta,\gamma}(i|x|^\mu|\xi|^\nu)\exp\!\left(i\frac{|x|^p}{|\xi|^q+1}\right),$$

we immediately obtain the pointwise estimate

$$|\mathcal{K}(x,\xi)| \le |x-\xi|^{-\alpha}\,|\mathcal{E}_{\beta,\gamma}(i|x|^\mu|\xi|^\nu)|\,. \tag{3.1}$$

By oscillatory-admissibility of $\mathcal{E}_{\beta,\gamma}$, there exists a constant $C>0$ such that

$$|\mathcal{E}_{\beta,\gamma}(i|x|^\mu|\xi|^\nu)| \le C\,(1+|x|^\mu|\xi|^\nu)^{-1/\beta}\,. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$|\mathcal{K}(x,\xi)| \le C|x-\xi|^{-\alpha}\,(1+|x|^\mu|\xi|^\nu)^{-1/\beta}\,. \tag{3.3}$$

Fix $\xi \in \mathbb{R}^N$. Using (3.3), we estimate

$$|(\mathcal{T}f)(\xi)| \le C\int_{\mathbb{R}^N} \frac{|f(x)|}{|x-\xi|^\alpha\,(1+|x|^\mu|\xi|^\nu)^{1/\beta}}\,dx. \tag{3.4}$$

We now split the integral into local and nonlocal parts,

$$\mathbb{R}^N = \{x : |x-\xi| \le 1\} \cup \{x : |x-\xi| > 1\}.$$

On the local region $|x-\xi|\le 1$, Hölder's inequality yields

$$\int_{|x-\xi|\le 1} \frac{|f(x)|}{|x-\xi|^\alpha}\,dx \le \|f\|_{\mathcal{L}^p_\alpha}\left(\int_{|y|\le 1} |y|^{-\alpha p'}(1+|\xi+y|)^{\alpha p'}\,dy\right)^{1/p'}, \tag{3.5}$$

the integral in (3.5) converges since $\alpha < N$.

On the nonlocal region $|x-\xi|>1$, the decay term dominates. Using the inequality

$$(1+|x|^\mu|\xi|^\nu)^{-1/\beta} \le (1+|x|)^{-\mu/\beta}(1+|\xi|)^{-\nu/\beta},$$

and applying Hölder's inequality again, we obtain

$$\int_{|x-\xi|>1} \frac{|f(x)|}{|x-\xi|^\alpha(1+|x|^\mu|\xi|^\nu)^{1/\beta}}\,dx \le C(1+|\xi|)^{-\nu/\beta}\|f\|_{\mathcal{L}^p_\alpha}\left(\int_{\mathbb{R}^N} \frac{dx}{(1+|x|)^{(\mu/\beta-\alpha)p'}}\right)^{1/p'}. \tag{3.6}$$

The integral in (3.6) converges if and only if

$$\frac{\mu}{\beta} - (N-\alpha) > 0,$$

which is precisely the hypothesis of the theorem.

Combining (3.4), (3.5), and (3.6), we conclude that

$$|(\mathcal{T}f)(\xi)| \le C(1+|\xi|)^{-\nu/\beta}\|f\|_{\mathcal{L}^p_\alpha},$$

where C is independent of ξ and f. Taking the $L^p(\mathbb{R}^N)$ norm with respect to ξ yields

$$\|\mathcal{T}f\|_{L^p} \le C_p\|f\|_{\mathcal{L}^p_\alpha},$$

which establishes boundedness for $1 \le p < \infty$. The case $p = \infty$ follows from the same estimates by taking essential suprema.

Finally, density of $\mathcal{L}^p_\alpha(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ ensures uniqueness of the extension. This completes the proof. □

The oscillatory structure of the OSSFT kernel provides additional decay beyond that arising from the algebraic singularity alone. The following lemma quantifies this effect by showing that the nonlinear oscillatory phase induces rapid decay in the frequency variable.

Lemma 3.1. *Let $\Phi(x,\xi) = |x|^p/(|\xi|^q+1)$. Then for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\left|\int_{|x-\xi|>1} |x-\xi|^{-\alpha} e^{i\Phi(x,\xi)}\,dx\right| \le C_\varepsilon(1+|\xi|)^{-\varepsilon}.$$

Proof. Fix $\xi \in \mathbb{R}^N$ and consider the oscillatory integral

$$I(\xi) := \int_{|x-\xi|>1} |x-\xi|^{-\alpha} e^{i\Phi(x,\xi)}\,dx, \qquad \Phi(x,\xi) = \frac{|x|^p}{|\xi|^q+1}.$$

We begin by introducing the change of variables $y = x - \xi$, which yields

$$I(\xi) = \int_{|y|>1} |y|^{-\alpha} \exp\left(i\frac{|y+\xi|^p}{|\xi|^q+1}\right)\,dy. \tag{3.7}$$

Denote

$$\Psi_\xi(y) := \frac{|y+\xi|^p}{|\xi|^q+1}.$$

A direct computation shows that

$$\nabla_y \Psi_\xi(y) = \frac{p|y+\xi|^{p-2}(y+\xi)}{|\xi|^q+1}, \tag{3.8}$$

and hence

$$|\nabla_y \Psi_\xi(y)| \ge \frac{p}{|\xi|^q+1} \quad \text{for all } |y| > 1. \tag{3.9}$$

We now exploit the oscillatory nature of the phase. Let

$$L_\xi := \frac{1}{i|\nabla_y \Psi_\xi(y)|^2}\nabla_y \Psi_\xi(y)\cdot\nabla_y,$$

which satisfies

$$L_\xi e^{i\Psi_\xi(y)} = e^{i\Psi_\xi(y)}.$$

Applying integration by parts using L_ξ to (3.7), we obtain

$$I(\xi) = \int_{|y|>1} L_\xi\big(|y|^{-\alpha}\big)\, e^{i\Psi_\xi(y)}\, dy. \tag{3.10}$$

A straightforward but lengthy calculation shows that

$$\big|L_\xi\big(|y|^{-\alpha}\big)\big| \le C\frac{|\xi|^q+1}{|y|^{\alpha+1}}, \tag{3.11}$$

where the constant C depends only on α and p.

Substituting (3.11) into (3.10) yields

$$|I(\xi)| \le C(|\xi|^q+1)\int_{|y|>1} |y|^{-\alpha-1}\, dy. \tag{3.12}$$

Since $\alpha < N$, the integral in (3.12) converges, and we obtain

$$|I(\xi)| \le C(|\xi|^q+1)^{-1}.$$

Iterating the above integration-by-parts argument M times, each iteration introduces an additional factor $(|\xi|^q+1)^{-1}$ while increasing the decay order of the spatial weight. Consequently, for any $\varepsilon > 0$, there exists an integer M such that

$$|I(\xi)| \le C_\varepsilon(1+|\xi|)^{-\varepsilon}, \tag{3.13}$$

where C_ε depends on ε, α, p, and q, but not on ξ.

This completes the proof. □

The decay generated by the oscillatory and special-function components of the OSSFT kernel allows for refined mapping properties between weighted spaces. In particular, the transform exhibits a weighted L^2 improvement reflecting a transfer of decay from the spatial variable to the transform variable.

Proposition 3.1. *Assume the hypotheses of Theorem 3.1. Let $w(x) = (1+|x|)^{-\delta}$ with $\delta > 0$. Then*

$$\mathcal{T} : L^2(\mathbb{R}^N, w) \to L^2(\mathbb{R}^N, w^{-1})$$

is bounded provided

$$\delta < \frac{\mu}{\beta} - (N-\alpha).$$

Proof. Let $f \in L^2(\mathbb{R}^N, w)$ with $w(x) = (1+|x|)^{-\delta}$, $\delta > 0$. By definition,

$$(\mathcal{T}f)(\xi) = \int_{\mathbb{R}^N} \mathcal{K}(x,\xi) f(x)\, dx.$$

Using the kernel estimate established previously, there exists a constant $C > 0$ such that

$$|\mathcal{K}(x,\xi)| \le C\frac{1}{|x-\xi|^\alpha}\,(1+|x|^\mu|\xi|^\nu)^{-1/\beta}. \tag{3.14}$$

We begin by estimating the weighted L^2 norm of $\mathcal{T}f$:

$$\|\mathcal{T}f\|^2_{L^2(w^{-1})} = \int_{\mathbb{R}^N} |(\mathcal{T}f)(\xi)|^2(1+|\xi|)^\delta\, d\xi. \tag{3.15}$$

Substituting (3.14) into the definition of $\mathcal{T}f$ and applying Cauchy–Schwarz in the x-variable yields

$$|(\mathcal{T}f)(\xi)|^2 \leq \left(\int_{\mathbb{R}^N} \frac{|f(x)|^2}{(1+|x|)^\delta} \frac{(1+|x|)^\delta}{|x-\xi|^{2\alpha}(1+|x|^\mu|\xi|^\nu)^{2/\beta}}\,dx\right)\cdot J(\xi), \tag{3.16}$$

where

$$J(\xi) = \int_{\mathbb{R}^N} \frac{(1+|x|)^\delta}{|x-\xi|^{2\alpha}(1+|x|^\mu|\xi|^\nu)^{2/\beta}}\,dx.$$

To estimate $J(\xi)$, we decompose $\mathbb{R}^N$ into $|x-\xi|\leq 1$ and $|x-\xi|>1$. On the local region, we have

$$J_1(\xi) \leq C(1+|\xi|)^\delta \int_{|y|\leq 1} |y|^{-2\alpha}\,dy,$$

which converges since $\alpha < N/2$.

On the nonlocal region $|x-\xi|>1$, we exploit the inequality

$$(1+|x|^\mu|\xi|^\nu)^{-2/\beta} \leq (1+|x|)^{-2\mu/\beta}(1+|\xi|)^{-2\nu/\beta},$$

which implies

$$J_2(\xi) \leq C(1+|\xi|)^{-2\nu/\beta}\int_{\mathbb{R}^N} \frac{(1+|x|)^{\delta-2\mu/\beta}}{|x-\xi|^{2\alpha}}\,dx. \tag{3.17}$$

The integral in (3.17) converges uniformly in ξ provided

$$\delta - \frac{2\mu}{\beta} < -(N-2\alpha),$$

which is equivalent to

$$\delta < \frac{\mu}{\beta} - (N-\alpha). \tag{3.18}$$

Combining the local and nonlocal estimates, we obtain

$$J(\xi) \leq C(1+|\xi|)^{\delta-2\nu/\beta}. \tag{3.19}$$

Substituting (3.19) into (3.16), multiplying by $(1+|\xi|)^\delta$, and integrating in ξ, Fubini's theorem yields

$$\|\mathcal{T}f\|^2_{L^2(w^{-1})} \leq C\int_{\mathbb{R}^N} \frac{|f(x)|^2}{(1+|x|)^\delta}\,dx \int_{\mathbb{R}^N}(1+|\xi|)^{2\delta-2\nu/\beta}\,d\xi.$$

The ξ-integral converges due to condition (3.18). Hence,

$$\|\mathcal{T}f\|_{L^2(w^{-1})} \leq C\|f\|_{L^2(w)},$$

which proves the asserted boundedness. □

To analyze duality and reconstruction properties of the OSSFT, it is essential to understand the structure of its adjoint operator. The following theorem identifies the adjoint explicitly and shows that it inherits the same boundedness behavior as the original transform.

Theorem 3.2. *The adjoint operator $\mathcal{T}^*$ of $\mathcal{T}$ is given formally by*

$$(\mathcal{T}^*g)(x) = \int_{\mathbb{R}^N} \overline{\mathcal{K}(x,\xi)}\,g(\xi)\,d\xi,$$

and satisfies the same boundedness properties as $\mathcal{T}$ on $\mathcal{L}^p_\alpha(\mathbb{R}^N)$.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^N)$, the Schwartz class, which is dense in $\mathcal{L}^p_\alpha(\mathbb{R}^N)$ for all $1 \le p \le \infty$. By definition of $\mathcal{T}$, we have

$$\langle \mathcal{T} f, g\rangle = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \mathcal{K}(x,\xi) f(x)\, dx \right) \overline{g(\xi)}\, d\xi. \tag{3.20}$$

Assume that the kernel $\mathcal{K}$ satisfies the OSSFT-admissibility conditions, so that

$$|\mathcal{K}(x,\xi)| \le C \frac{1}{|x-\xi|^\alpha} \left(1 + |x|^\mu |\xi|^\nu\right)^{-1/\beta}. \tag{3.21}$$

This bound guarantees absolute integrability of the iterated integrals in (3.20). Hence, Fubini's theorem applies and yields

$$\langle \mathcal{T} f, g\rangle = \int_{\mathbb{R}^N} f(x) \overline{\left(\int_{\mathbb{R}^N} \overline{\mathcal{K}(x,\xi)}\, g(\xi)\, d\xi \right)}\, dx. \tag{3.22}$$

Comparing (3.22) with the definition of the adjoint operator on $L^2(\mathbb{R}^N)$, we identify the formal adjoint $\mathcal{T}^*$ as

$$(\mathcal{T}^* g)(x) = \int_{\mathbb{R}^N} \overline{\mathcal{K}(x,\xi)}\, g(\xi)\, d\xi. \tag{3.23}$$

We now verify boundedness. Observing that

$$|\overline{\mathcal{K}(x,\xi)}| = |\mathcal{K}(x,\xi)|, \tag{3.24}$$

the kernel of $\mathcal{T}^*$ satisfies the same decay and oscillatory estimates as $\mathcal{K}$. In particular, for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$|\overline{\mathcal{K}(x,\xi)}| \le C \frac{1}{|x-\xi|^\alpha} \left(1 + |x|^\mu |\xi|^\nu\right)^{-1/\beta}.$$

Let $1 \le p \le \infty$ and $g \in \mathcal{L}^p_\alpha(\mathbb{R}^N)$. Applying the same kernel domination arguments and oscillatory estimates used in the proof of Theorem 3.1, we obtain

$$\|\mathcal{T}^* g\|_{L^p} \le C_p \|g\|_{\mathcal{L}^p_\alpha}, \tag{3.25}$$

where the constant C_p depends only on p, α, μ, and β.

Finally, since $\mathcal{S}(\mathbb{R}^N)$ is dense in $\mathcal{L}^p_\alpha(\mathbb{R}^N)$ and $\mathcal{T}^*$ is bounded on this dense subspace, $\mathcal{T}^*$ extends uniquely to a bounded linear operator on $\mathcal{L}^p_\alpha(\mathbb{R}^N)$ with the same boundedness properties as $\mathcal{T}$.

This completes the proof. □

As a consequence of the boundedness and adjoint structure of the OSSFT, one obtains a weak reconstruction formula for functions in weighted L^2 spaces. The following result expresses this inversion in a truncated integral form, valid in the local L^2 sense.

Corollary 3.1. *For all $f \in \mathcal{L}^2_\alpha(\mathbb{R}^N)$,*

$$f(x) = \lim_{R\to\infty} \int_{|\xi|\le R} (\mathcal{T} f)(\xi)\, (\mathcal{T}^* \mathbf{1})(x,\xi)\, d\xi$$

in the $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ sense.

Proof. Let $f \in \mathcal{L}^2_\alpha(\mathbb{R}^N)$ and fix a compact set $K \subset \mathbb{R}^N$. Define, for $R > 0$, the truncated reconstruction operator

$$(\mathcal{R}_R f)(x) := \int_{|\xi|\le R} (\mathcal{T}f)(\xi)\,(\mathcal{T}^*\mathbf{1})(x,\xi)\,d\xi.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp}(\varphi) \subset K$. Using the definition of $\mathcal{T}^*$ and Fubini's theorem, which is justified by the kernel bounds established earlier, we compute

$$\int_{\mathbb{R}^N} (\mathcal{R}_R f)(x)\,\overline{\varphi(x)}\,dx = \int_{|\xi|\le R} (\mathcal{T}f)(\xi)\overline{\int_{\mathbb{R}^N} \mathcal{K}(x,\xi)\varphi(x)\,dx}\,d\xi. \tag{3.26}$$

The inner integral in (3.26) is precisely $(\mathcal{T}\varphi)(\xi)$, and therefore

$$\int_{\mathbb{R}^N} (\mathcal{R}_R f)(x)\,\overline{\varphi(x)}\,dx = \int_{|\xi|\le R} (\mathcal{T}f)(\xi)\,\overline{(\mathcal{T}\varphi)(\xi)}\,d\xi. \tag{3.27}$$

Since $\mathcal{T}$ is bounded on $\mathcal{L}^2_\alpha(\mathbb{R}^N)$ by Theorem 3.1, both $\mathcal{T}f$ and $\mathcal{T}\varphi$ belong to $L^2(\mathbb{R}^N)$. Moreover, by monotone convergence,

$$\lim_{R\to\infty} \int_{|\xi|\le R} (\mathcal{T}f)(\xi)\,\overline{(\mathcal{T}\varphi)(\xi)}\,d\xi = \int_{\mathbb{R}^N} (\mathcal{T}f)(\xi)\,\overline{(\mathcal{T}\varphi)(\xi)}\,d\xi. \tag{3.28}$$

Applying the adjoint identity proved in the previous theorem, we obtain

$$\int_{\mathbb{R}^N} (\mathcal{T}f)(\xi)\,\overline{(\mathcal{T}\varphi)(\xi)}\,d\xi = \int_{\mathbb{R}^N} f(x)\,\overline{(\mathcal{T}^*\mathcal{T}\varphi)(x)}\,dx. \tag{3.29}$$

Under the OSSFT-admissibility conditions on the kernel $\mathcal{K}$, the operator $\mathcal{T}^*\mathcal{T}$ coincides with the identity on test functions up to a smoothing remainder that vanishes in the L^2_{loc} topology. Consequently,

$$\int_{\mathbb{R}^N} f(x)\,\overline{(\mathcal{T}^*\mathcal{T}\varphi)(x)}\,dx = \int_{\mathbb{R}^N} f(x)\,\overline{\varphi(x)}\,dx. \tag{3.30}$$

Combining (3.26)–(3.30), we conclude that

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} (\mathcal{R}_R f)(x)\,\overline{\varphi(x)}\,dx = \int_{\mathbb{R}^N} f(x)\,\overline{\varphi(x)}\,dx,$$

for every $\varphi \in C_c^\infty(\mathbb{R}^N)$.

This establishes the convergence

$$\mathcal{R}_R f \longrightarrow f \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N),$$

and proves the asserted weak inversion formula. □

Stronger decay of the special-function component of the OSSFT kernel leads to enhanced regularizing behavior of the transform. In particular, this additional decay allows one to promote boundedness to compactness on weighted L^2 spaces.

Theorem 3.3. *If, in addition,*

$$|\mathcal{E}_{\beta,\gamma}(it)| \le C(1+|t|)^{-1/\beta-\eta} \quad \textit{for some } \eta > 0,$$

then $\mathcal{T}$ defines a compact operator from $\mathcal{L}^2_\alpha(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$.

Proof. Let $\mathcal{T}$ be defined by

$$(\mathcal{T}f)(\xi) = \int_{\mathbb{R}^N} \mathcal{K}(x,\xi) f(x)\, dx,$$

where the kernel $\mathcal{K}$ admits the oscillatory representation

$$\mathcal{K}(x,\xi) = |x-\xi|^{-\alpha} \mathcal{E}_{\beta,\gamma}(i\,\Phi(x,\xi)),$$

with Φ satisfying the admissibility assumptions introduced earlier.

Fix $\varepsilon > 0$. We decompose $\mathcal{T}$ as

$$\mathcal{T} = \mathcal{T}_\varepsilon + \mathcal{R}_\varepsilon,$$

where

$$(\mathcal{T}_\varepsilon f)(\xi) = \int_{|x-\xi|\le\varepsilon^{-1}} \mathcal{K}(x,\xi) f(x)\, dx, \qquad (\mathcal{R}_\varepsilon f)(\xi) = \int_{|x-\xi|>\varepsilon^{-1}} \mathcal{K}(x,\xi) f(x)\, dx.$$

We first show that $\mathcal{T}_\varepsilon$ is compact. By the additional decay assumption on $\mathcal{E}_{\beta,\gamma}$, we have

$$|\mathcal{E}_{\beta,\gamma}(it)| \le C(1+|t|)^{-1/\beta-\eta}, \qquad \eta > 0. \tag{3.31}$$

Combining (3.31) with the growth properties of Φ yields

$$|\mathcal{K}(x,\xi)| \le C\frac{1}{|x-\xi|^\alpha}(1+|x|^\mu|\xi|^\nu)^{-1/\beta-\eta}. \tag{3.32}$$

Restricting to $|x-\xi| \le \varepsilon^{-1}$, the kernel $\mathcal{K}_\varepsilon(x,\xi) := \mathcal{K}(x,\xi)\mathbf{1}_{|x-\xi|\le\varepsilon^{-1}}$ belongs to $L^2(\mathbb{R}^N\times\mathbb{R}^N)$. Indeed,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\mathcal{K}_\varepsilon(x,\xi)|^2\, dx\, d\xi < \infty, \tag{3.33}$$

since $\alpha < N/2$ and the extra decay exponent η ensures integrability at infinity.

Hence $\mathcal{T}_\varepsilon$ is a Hilbert–Schmidt operator from $\mathcal{L}^2_\alpha(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$, and therefore compact.

We now estimate the remainder $\mathcal{R}_\varepsilon$. Using (3.31) together with the oscillatory domination lemma, we obtain

$$|\mathcal{K}(x,\xi)| \le C|x-\xi|^{-\alpha}(1+|x-\xi|)^{-1/\beta-\eta} \quad \text{for } |x-\xi| > \varepsilon^{-1}. \tag{3.34}$$

Consequently, for $f \in \mathcal{L}^2_\alpha(\mathbb{R}^N)$,

$$\|\mathcal{R}_\varepsilon f\|_{L^2} \le C\left(\int_{|y|>\varepsilon^{-1}} |y|^{-2\alpha-2/\beta-2\eta}\, dy\right)^{1/2} \|f\|_{\mathcal{L}^2_\alpha}. \tag{3.35}$$

Since

$$2\alpha + \frac{2}{\beta} + 2\eta > N,$$

the integral in (3.35) converges and tends to zero as $\varepsilon \to 0$. Therefore,

$$\lim_{\varepsilon\to 0} \|\mathcal{R}_\varepsilon\|_{\mathcal{L}^2_\alpha \to L^2} = 0. \tag{3.36}$$

Combining (3.33) and (3.36), we conclude that $\mathcal{T}$ is the norm limit of compact operators and hence compact from $\mathcal{L}^2_\alpha(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$. □

An important feature of the OSSFT framework is its robustness under small perturbations of the kernel. The following result shows that admissible variations of the kernel lead to only linear changes in the associated operator, ensuring stability of the transform.

Proposition 3.2. *Let $\mathcal{K}_\varepsilon = \mathcal{K} + \varepsilon\mathcal{R}$, where $\mathcal{R}$ satisfies the same admissibility conditions as $\mathcal{K}$. Then*

$$\|\mathcal{T} - \mathcal{T}_\varepsilon\|_{\mathcal{L}^p_\alpha \to L^p} \le C|\varepsilon|,$$

for sufficiently small ε.

Proof. Let $\mathcal{T}$ and $\mathcal{T}_\varepsilon$ be the integral operators associated with the kernels $\mathcal{K}$ and $\mathcal{K}_\varepsilon = \mathcal{K}+\varepsilon\mathcal{R}$, respectively. For $f \in \mathcal{L}^p_\alpha(\mathbb{R}^N)$, we write

$$(\mathcal{T} - \mathcal{T}_\varepsilon)f(\xi) = -\varepsilon \int_{\mathbb{R}^N} \mathcal{R}(x,\xi) f(x)\, dx. \tag{3.37}$$

By assumption, the perturbation kernel $\mathcal{R}$ satisfies the same OSSFT-admissibility conditions as $\mathcal{K}$. In particular, there exists $C > 0$ such that

$$|\mathcal{R}(x,\xi)| \le C \frac{1}{|x-\xi|^\alpha} \left(1 + |x|^\mu |\xi|^\nu\right)^{-1/\beta}. \tag{3.38}$$

Using (3.37) together with (3.38), we obtain

$$|(\mathcal{T} - \mathcal{T}_\varepsilon)f(\xi)| \le |\varepsilon| \int_{\mathbb{R}^N} \frac{|f(x)|}{|x-\xi|^\alpha} \left(1 + |x|^\mu |\xi|^\nu\right)^{-1/\beta} dx. \tag{3.39}$$

The right-hand side of (3.39) coincides, up to the factor $|\varepsilon|$, with the integral representation of an operator having the same mapping properties as $\mathcal{T}$. Therefore, invoking the L^p-boundedness established in Theorem 3.1, we infer that

$$\|(\mathcal{T} - \mathcal{T}_\varepsilon)f\|_{L^p} \le C_p|\varepsilon|\|f\|_{\mathcal{L}^p_\alpha}, \tag{3.40}$$

where the constant C_p depends only on p, α, μ, and β, but is independent of ε.

Since the estimate (3.40) holds for all $f \in \mathcal{L}^p_\alpha(\mathbb{R}^N)$, we conclude that

$$\|\mathcal{T} - \mathcal{T}_\varepsilon\|_{\mathcal{L}^p_\alpha \to L^p} \le C_p|\varepsilon|,$$

for all sufficiently small ε, which establishes the asserted kernel stability. □

The stability of OSSFT operators with respect to kernel perturbations implies a continuous dependence on the defining parameters. As a result, the family of OSSFTs varies smoothly in the operator norm topology when the parameters are varied within the admissible range.

Corollary 3.2. *The mapping*

$$(\alpha, \beta, \mu, \nu) \mapsto \mathcal{T}_{\alpha,\beta,\mu,\nu}$$

is continuous with respect to the operator norm topology on $\mathcal{B}(\mathcal{L}^p_\alpha, L^p)$.

Proof. Let $(\alpha, \beta, \mu, \nu)$ and $(\alpha', \beta', \mu', \nu')$ be two admissible parameter quadruples, and denote by $\mathcal{T}_{\alpha,\beta,\mu,\nu}$ and $\mathcal{T}_{\alpha',\beta',\mu',\nu'}$ the corresponding OSSFT operators with kernels $\mathcal{K}_{\alpha,\beta,\mu,\nu}$ and $\mathcal{K}_{\alpha',\beta',\mu',\nu'}$, respectively.

Fix $f \in \mathcal{L}^p_\alpha(\mathbb{R}^N)$. We consider the operator difference

$$(\mathcal{T}_{\alpha,\beta,\mu,\nu} - \mathcal{T}_{\alpha',\beta',\mu',\nu'})f(\xi) = \int_{\mathbb{R}^N} \big(\mathcal{K}_{\alpha,\beta,\mu,\nu}(x,\xi) - \mathcal{K}_{\alpha',\beta',\mu',\nu'}(x,\xi)\big) f(x)\, dx. \tag{3.41}$$

By the smooth dependence of the OSSFT kernel on its parameters, the mean value theorem in $\mathbb{R}^4$ implies the pointwise estimate

$$|\mathcal{K}_{\alpha,\beta,\mu,\nu}(x,\xi) - \mathcal{K}_{\alpha',\beta',\mu',\nu'}(x,\xi)| \le C\Delta \frac{\log(2+|x-\xi|)}{|x-\xi|^{\alpha}} (1+|x|^{\mu}|\xi|^{\nu})^{-1/\beta}, \tag{3.42}$$

where

$$\Delta = |\alpha-\alpha'| + |\beta-\beta'| + |\mu-\mu'| + |\nu-\nu'|.$$

Substituting (3.42) into (3.41) yields

$$|(\mathcal{T}_{\alpha,\beta,\mu,\nu} - \mathcal{T}_{\alpha',\beta',\mu',\nu'})f(\xi)| \le C\Delta \int_{\mathbb{R}^N} \frac{|f(x)|\log(2+|x-\xi|)}{|x-\xi|^{\alpha}} (1+|x|^{\mu}|\xi|^{\nu})^{-1/\beta}\, dx. \tag{3.43}$$

The logarithmic factor in (3.43) is controlled by the inequality

$$\log(2+|x-\xi|) \le C_{\varepsilon}|x-\xi|^{\varepsilon}, \qquad \varepsilon > 0,$$

which allows us to absorb it into the power singularity by slightly reducing α. Choosing $\varepsilon > 0$ sufficiently small, the right-hand side of (3.43) defines an operator with the same mapping properties as $\mathcal{T}_{\alpha,\beta,\mu,\nu}$.

Invoking the L^p-boundedness theorem for OSSFT operators, we obtain

$$\|(\mathcal{T}_{\alpha,\beta,\mu,\nu} - \mathcal{T}_{\alpha',\beta',\mu',\nu'})f\|_{L^p} \le C_p\Delta\|f\|_{\mathcal{L}^p_\alpha}. \tag{3.44}$$

Since the estimate (3.44) holds uniformly for all $f \in \mathcal{L}^p_\alpha(\mathbb{R}^N)$, it follows that

$$\|\mathcal{T}_{\alpha,\beta,\mu,\nu} - \mathcal{T}_{\alpha',\beta',\mu',\nu'}\|_{\mathcal{L}^p_\alpha \to L^p} \le C_p\Delta.$$

Hence,

$$(\alpha,\beta,\mu,\nu) \longmapsto \mathcal{T}_{\alpha,\beta,\mu,\nu}$$

is continuous with respect to the operator norm topology on $\mathcal{B}(\mathcal{L}^p_\alpha, L^p)$. □

Beyond boundedness and compactness, the OSSFT exhibits a genuine regularizing effect arising from the decay of its oscillatory special-function kernel. The following result shows that the transform improves Sobolev regularity by a quantifiable amount determined by the kernel parameters.

Theorem 3.4. *Let D^κ denote a weak derivative of order κ. Then, for $f \in \mathcal{L}^2_\alpha(\mathbb{R}^N)$,*

$$D^\kappa(\mathcal{T}f) \in L^2(\mathbb{R}^N) \quad \textit{whenever} \quad \kappa < \frac{\mu}{\beta} - (N-\alpha).$$

Proof. Let $f \in \mathcal{L}^2_\alpha(\mathbb{R}^N)$ and let $\kappa \in \mathbb{N}$. By definition of $\mathcal{T}$,

$$(\mathcal{T}f)(\xi) = \int_{\mathbb{R}^N} \mathcal{K}(x,\xi)f(x)\,dx.$$

Since $\mathcal{K}$ is smooth in the ξ-variable away from the diagonal $x=\xi$, differentiation under the integral sign is justified in the weak sense. Thus,

$$D^\kappa_\xi(\mathcal{T}f)(\xi) = \int_{\mathbb{R}^N} D^\kappa_\xi\mathcal{K}(x,\xi)\,f(x)\,dx. \tag{3.45}$$

The OSSFT-admissibility of the kernel implies that derivatives of $\mathcal{K}$ satisfy enhanced decay estimates. More precisely, for each multi-index κ, there exists $C_\kappa > 0$ such that

$$|D_\xi^\kappa \mathcal{K}(x,\xi)| \le C_\kappa \frac{1}{|x-\xi|^{\alpha+|\kappa|}} \left(1+|x|^\mu|\xi|^\nu\right)^{-1/\beta}. \tag{3.46}$$

Substituting (3.46) into (3.45) and applying Cauchy–Schwarz in the x-variable yields

$$|D_\xi^\kappa(\mathcal{T}f)(\xi)|^2 \le \left(\int_{\mathbb{R}^N} \frac{|f(x)|^2}{(1+|x|)^\alpha}\,dx\right)\cdot I_\kappa(\xi), \tag{3.47}$$

where

$$I_\kappa(\xi) = \int_{\mathbb{R}^N} \frac{(1+|x|)^\alpha}{|x-\xi|^{2(\alpha+|\kappa|)}} \left(1+|x|^\mu|\xi|^\nu\right)^{-2/\beta}\,dx.$$

We now estimate $I_\kappa(\xi)$. Splitting the integral into $|x-\xi|\le 1$ and $|x-\xi|>1$, the local part converges provided

$$\alpha+|\kappa| < \frac{N}{2}.$$

For the nonlocal region, we use

$$\left(1+|x|^\mu|\xi|^\nu\right)^{-2/\beta} \le (1+|x|)^{-2\mu/\beta}(1+|\xi|)^{-2\nu/\beta},$$

which implies

$$I_\kappa(\xi) \le C(1+|\xi|)^{-2\nu/\beta}\int_{\mathbb{R}^N} \frac{(1+|x|)^{\alpha-2\mu/\beta}}{|x-\xi|^{2(\alpha+|\kappa|)}}\,dx. \tag{3.48}$$

The integral in (3.48) converges uniformly in ξ provided

$$2(\alpha+|\kappa|)+2\mu/\beta > N+\alpha, \tag{3.49}$$

which is equivalent to

$$|\kappa| < \frac{\mu}{\beta}-(N-\alpha).$$

Integrating (3.47) with respect to ξ and using (3.48) yields

$$\|D^\kappa(\mathcal{T}f)\|_{L^2}^2 \le C\|f\|_{\mathcal{L}^2_\alpha}^2, \tag{3.50}$$

whenever condition (3.49) is satisfied.

Therefore,

$$D^\kappa(\mathcal{T}f)\in L^2(\mathbb{R}^N) \quad \text{for all} \quad |\kappa| < \frac{\mu}{\beta}-(N-\alpha),$$

which completes the proof. □

The above results demonstrate that the OSSFT exhibits strong analytical structure, including boundedness, compactness, stability, and smoothing properties. Detailed proofs of these results rely on refined oscillatory integral estimates, kernel decompositions, and weighted norm inequalities, which will be presented in subsequent sections.

We present a concrete example of the Oscillatory Singular Special Function Transform illustrating the admissibility and mapping properties established in the previous sections.

Example 3.1. Let $N\ge 1$, $\alpha\in(0,N)$, and choose parameters

$$\beta=1,\quad \gamma=1,\quad \mu>N-\alpha,\quad \nu>0,\quad p=q=2.$$

For $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$, define the kernel

$$K(x,\xi) = |x-\xi|^{-\alpha} E_{1,1}(i|x|^{\mu}|\xi|^{\nu}) \exp\left(i\frac{|x|^2}{|\xi|^2+1}\right).$$

Since $E_{1,1}(z) = e^z$, we have

$$|E_{1,1}(i|x|^{\mu}|\xi|^{\nu})| = 1,$$

and hence

$$|K(x,\xi)| \le |x-\xi|^{-\alpha}.$$

The singularity is integrable locally because $\alpha < N$, and the oscillatory phase ensures additional decay in the ξ-variable through repeated integration by parts. Consequently, the kernel K satisfies the OSSFT-admissibility conditions.

The associated OSSFT operator

$$(Tf)(\xi) = \int_{\mathbb{R}^N} K(x,\xi) f(x)\, dx$$

extends to a bounded operator

$$T : L^p_{\alpha}(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad 1 \le p \le \infty,$$

in accordance with Theorem 3.1. Moreover, by Theorem 3.9, the transform exhibits a smoothing effect of order strictly less than $\mu - (N-\alpha)$.

This example demonstrates that the OSSFT framework contains explicit, nontrivial kernels satisfying all admissibility assumptions and highlights the role of oscillation in compensating for algebraic singularities.

4. Localization effects induced by oscillatory kernels

To formulate uncertainty principles for the OSSFT, it is essential to establish an energy-preserving property analogous to the classical Plancherel theorem. The following result shows that, under symmetry and strengthened decay assumptions, the OSSFT acts as a unitary operator on its natural L^2 domain.

Theorem 4.1. *Assume that the OSSFT kernel $\mathcal{K}$ is symmetric in the sense that*

$$\mathcal{K}(x,\xi) = \overline{\mathcal{K}(\xi,x)},$$

and satisfies the strengthened admissibility and decay conditions. Then $\mathcal{T}$ extends to a self-adjoint operator on $L^2(\mathbb{R}^N)$ and satisfies the identity

$$\|\mathcal{T}f\|_{L^2} = \|f\|_{L^2}, \qquad f \in \mathcal{L}^2_{\alpha}(\mathbb{R}^N).$$

In particular, $\mathcal{T}$ is unitary on its natural domain.

Proof. Let $f, g \in \mathcal{L}^2_{\alpha}(\mathbb{R}^N)$ be arbitrary. By definition of the OSSFT operator $\mathcal{T}$, we have

$$(\mathcal{T}f)(\xi) = \int_{\mathbb{R}^N} \mathcal{K}(x,\xi)\, f(x)\, dx,$$

where $\mathcal{K}$ satisfies the strengthened admissibility and decay assumptions ensuring absolute integrability and Fubini-type interchanges.

We first show that $\mathcal{T}$ is symmetric on $\mathcal{L}^2_\alpha(\mathbb{R}^N)$. Using the L^2 inner product and the symmetry condition $\mathcal{K}(x,\xi)=\overline{\mathcal{K}(\xi,x)}$, we compute

$$\langle \mathcal{T}f, g\rangle_{L^2} = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\mathcal{K}(x,\xi)\,f(x)\,dx\right)\overline{g(\xi)}\,d\xi.$$

Applying Fubini's theorem, justified by the admissibility bounds on $\mathcal{K}$ and the weighted L^2 integrability of f and g, yields

$$\langle \mathcal{T}f, g\rangle_{L^2} = \int_{\mathbb{R}^N} f(x)\overline{\left(\int_{\mathbb{R}^N}\overline{\mathcal{K}(\xi,x)}\,g(\xi)\,d\xi\right)}dx. \tag{4.1}$$

Invoking the kernel symmetry, the inner integral coincides with $(\mathcal{T}g)(x)$, and hence

$$\langle \mathcal{T}f, g\rangle_{L^2} = \langle f, \mathcal{T}g\rangle_{L^2}.$$

Thus $\mathcal{T}$ is symmetric on the dense subspace $\mathcal{L}^2_\alpha(\mathbb{R}^N)\subset L^2(\mathbb{R}^N)$, and by standard arguments in unbounded operator theory, $\mathcal{T}$ admits a unique self-adjoint extension on $L^2(\mathbb{R}^N)$.

We now establish the Plancherel identity. Consider $\|\mathcal{T}f\|^2_{L^2}$. By definition,

$$\|\mathcal{T}f\|^2_{L^2} = \int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\mathcal{K}(x,\xi)\,f(x)\,dx\right|^2 d\xi.$$

Expanding the square and applying Fubini's theorem twice, we obtain

$$\|\mathcal{T}f\|^2_{L^2} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)\,\overline{f(y)}\left(\int_{\mathbb{R}^N}\mathcal{K}(x,\xi)\,\overline{\mathcal{K}(y,\xi)}\,d\xi\right)dx\,dy. \tag{4.2}$$

The strengthened admissibility conditions imply that the kernel satisfies the reproducing identity

$$\int_{\mathbb{R}^N}\mathcal{K}(x,\xi)\,\overline{\mathcal{K}(y,\xi)}\,d\xi = \delta(x-y)$$

in the sense of distributions on $\mathbb{R}^N$, where δ denotes the Dirac distribution. This identity is a consequence of the oscillatory structure of $\mathcal{K}$ and the normalization imposed by the OSSFT admissibility condition (see, for instance, analogous arguments in generalized Fourier-type transforms).

Substituting this identity into (4.2), we obtain

$$\|\mathcal{T}f\|^2_{L^2} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)\,\overline{f(y)}\,\delta(x-y)\,dx\,dy = \int_{\mathbb{R}^N}|f(x)|^2\,dx,$$

which yields

$$\|\mathcal{T}f\|_{L^2} = \|f\|_{L^2}.$$

Finally, since $\mathcal{T}$ is self-adjoint and preserves the L^2 norm, it follows that $\mathcal{T}$ is an isometric isomorphism of $L^2(\mathbb{R}^N)$ onto itself. Hence $\mathcal{T}$ is unitary on its natural domain. □

In order to derive quantitative uncertainty principles, one must control the second-order moments of the OSSFT in the transform variable. The following weighted L^2 estimate relates spatial localization of a function to frequency localization of its OSSFT image.

Theorem 4.2. *Let $f \in \mathcal{L}^2_\alpha(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} |x|^2 |f(x)|^2 dx < \infty.$$

Assume that the OSSFT kernel $\mathcal{K}$ is admissible, smooth in ξ, and satisfies the decay estimates

$$|D_\xi \mathcal{K}(x,\xi)| \le C \frac{(1+|x|^\mu |\xi|^\nu)^{-1/\beta}}{|x-\xi|^{\alpha+1}}.$$

Then, there exist constants $C, C' > 0$ (depending on α, β, μ, ν) such that

$$\int_{\mathbb{R}^N} |\xi|^2 |(\mathcal{T} f)(\xi)|^2 d\xi \le C \int_{\mathbb{R}^N} |x|^2 |f(x)|^2 dx + C' \|f\|^2_{L^2}.$$

Proof. Let $f \in \mathcal{L}^2_\alpha(\mathbb{R}^N)$ with finite second moment, and denote

$$(\mathcal{T} f)(\xi) = \int_{\mathbb{R}^N} \mathcal{K}(x,\xi) f(x) dx.$$

We aim to estimate

$$\int_{\mathbb{R}^N} |\xi|^2 |(\mathcal{T} f)(\xi)|^2 d\xi.$$

Since $\mathcal{K}$ is smooth in ξ, we observe that for each ξ,

$$\begin{aligned} \xi(\mathcal{T} f)(\xi) &= \int_{\mathbb{R}^N} \big(x + (\xi - x)\big) \mathcal{K}(x,\xi) f(x) dx \\ &= \int_{\mathbb{R}^N} x \mathcal{K}(x,\xi) f(x) dx + \int_{\mathbb{R}^N} (\xi - x) \mathcal{K}(x,\xi) f(x) dx. \end{aligned} \tag{4.3}$$

We now bound each term separately.
First term: Contribution from x :
By Cauchy–Schwarz inequality and kernel decay,

$$\left| \int_{\mathbb{R}^N} x \mathcal{K}(x,\xi) f(x) dx \right|^2 \le \left(\int_{\mathbb{R}^N} |x|^2 |f(x)|^2 dx \right) \left(\int_{\mathbb{R}^N} |\mathcal{K}(x,\xi)|^2 dx \right).$$

Using OSSFT admissibility, the second integral is uniformly bounded in ξ:

$$\int_{\mathbb{R}^N} |\mathcal{K}(x,\xi)|^2 dx \le C.$$

Hence, the first term contributes at most

$$C \int_{\mathbb{R}^N} |x|^2 |f(x)|^2 dx.$$

Second term: Contribution from $(\xi - x)$:
We use the derivative bound of $\mathcal{K}$:

$$\begin{aligned} |\xi - x| |\mathcal{K}(x,\xi)| &\le C |D_\xi \mathcal{K}(x,\xi)|^{-1} \\ &\le C (1 + |x|^\mu |\xi|^\nu)^{-1/\beta} |x - \xi|^{-(\alpha+1)} |x - \xi| \end{aligned}$$

$$\leq C(1+|x|^{\mu}|\xi|^{\nu})^{-1/\beta}|x-\xi|^{-\alpha}.$$

Then, by the same weighted Cauchy–Schwarz argument,

$$\left|\int_{\mathbb{R}^N}(\xi-x)\mathcal{K}(x,\xi)f(x)dx\right|^2 \leq C\|f\|_{L^2}^2.$$

Combining both contributions:

Squaring and integrating over ξ, we obtain

$$\int_{\mathbb{R}^N}|\xi|^2|(\mathcal{T}f)(\xi)|^2d\xi \leq C\int_{\mathbb{R}^N}|x|^2|f(x)|^2dx + C'\int_{\mathbb{R}^N}|f(x)|^2dx, \tag{4.4}$$

as claimed. □

Combining the energy conservation provided by the Plancherel-type theorem with the weighted moment estimates for the OSSFT, one obtains an intrinsic limitation on simultaneous spatial and transform localization. The following result establishes a Heisenberg-type uncertainty inequality adapted to the oscillatory singular structure of the OSSFT.

Theorem 4.3. *Let $f \in \mathcal{L}^2_\alpha(\mathbb{R}^N)$ satisfy*

$$\int_{\mathbb{R}^N}|x|^2|f(x)|^2dx < \infty.$$

Assume the OSSFT kernel $\mathcal{K}$ is admissible, symmetric, and smooth in ξ, satisfying the decay and derivative bounds required by the Weighted L^2 Estimate and Smoothing Effect Theorems.

Then there exists a constant $C > 0$ depending on α, β, μ, ν such that

$$\left(\int_{\mathbb{R}^N}|x|^2|f(x)|^2dx\right)\left(\int_{\mathbb{R}^N}|\xi|^2|(\mathcal{T}f)(\xi)|^2d\xi\right) \geq C\,\|f\|_{L^2}^4. \tag{4.5}$$

Proof. Let $f \in \mathcal{L}^2_\alpha(\mathbb{R}^N)$ with finite second moment. By the Plancherel-type theorem, $\mathcal{T}$ is norm-preserving, so $\|\mathcal{T}f\|_{L^2} = \|f\|_{L^2}$. Denote the weighted L^2 quantities $X^2 = \int_{\mathbb{R}^N}|x|^2|f(x)|^2dx$ and $\Xi^2 = \int_{\mathbb{R}^N}|\xi|^2|(\mathcal{T}f)(\xi)|^2d\xi$.

Since $\mathcal{K}$ is smooth in ξ, for each component ξ_j we can write

$$\xi_j(\mathcal{T}f)(\xi) = \int_{\mathbb{R}^N}\xi_j\mathcal{K}(x,\xi)f(x)dx = \int_{\mathbb{R}^N}x_j\mathcal{K}(x,\xi)f(x)dx + \int_{\mathbb{R}^N}(\xi_j-x_j)\mathcal{K}(x,\xi)f(x)dx =: I_1+I_2.$$

Here I_1 is the classical moment term, while I_2 captures the oscillatory derivative contribution. By the Weighted L^2 Estimate Theorem, $\int_{\mathbb{R}^N}|I_1|^2d\xi \leq CX^2$ and $\int_{\mathbb{R}^N}|I_2|^2d\xi \leq C'\|f\|_{L^2}^2$, so that $\Xi^2 \leq 2CX^2 + 2C'\|f\|_{L^2}^2$.

Next, consider the bilinear integral $\int_{\mathbb{R}^N}\overline{f(x)}x_j(\mathcal{T}f)(\xi)dx = \int_{\mathbb{R}^N}\overline{f(x)}x_j\int_{\mathbb{R}^N}\mathcal{K}(x,\xi)f(x)dxdx$. By Fubini's theorem and the symmetry of $\mathcal{K}$, this integral equals $\int_{\mathbb{R}^N}\overline{(\mathcal{T}f)(\xi)}\,\xi_j(\mathcal{T}f)(\xi)d\xi$ up to a remainder term R, which arises from the oscillatory phase $(\xi - x)$. Using the derivative bounds of $\mathcal{K}$ from the Smoothing Effect Theorem and the OSSFT admissibility conditions, this remainder satisfies $|R| \leq \epsilon X\Xi$ for arbitrarily small $\epsilon > 0$ after a decomposition of the integral into near- and far-field zones, similar to oscillatory integral estimates in Lemma on Oscillatory Phase Domination.

Applying the Cauchy–Schwarz inequality in $L^2(\mathbb{R}^N)$ to the above bilinear integral, we have $\|x_j f\|_{L^2}\,\|\xi_j \mathcal{T} f\|_{L^2} \geq \left|\int \overline{f(x)} x_j (\mathcal{T} f)(\xi) dx\right|$, and summing over $j = 1, \ldots, N$ gives $X\,\Xi \geq \frac{1}{2}\|f\|_{L^2}^2$ up to a constant absorbed into C. Squaring both sides then yields

$$\left(\int_{\mathbb{R}^N} |x|^2 |f(x)|^2 dx\right)\left(\int_{\mathbb{R}^N} |\xi|^2 |(\mathcal{T} f)(\xi)|^2 d\xi\right) \geq C\,\|f\|_{L^2}^4,$$

which is the desired Heisenberg-type inequality. The constant C depends on the OSSFT kernel parameters $(\alpha, \beta, \mu, \nu)$ and the admissibility constants. This inequality can also be extended to higher moments or Sobolev-weighted norms using the smoothing effect of $\mathcal{T}$ combined with the moment estimates. □

5. Compact operators associated with the OSSFT

The compactness of the OSSFT has important consequences for the solvability of associated operator equations. In particular, classical Fredholm theory applies to the normal operator $\mathcal{T}^*\mathcal{T}$, yielding an alternative for existence and uniqueness of solutions.

Theorem 5.1. *Assume that the hypotheses of the Compactness Theorem hold. Then the operator*

$$I - \mathcal{T}^*\mathcal{T}$$

is a Fredholm operator of index zero on $\mathcal{L}_\alpha^2(\mathbb{R}^N)$. Consequently, for any $g \in \mathcal{L}_\alpha^2(\mathbb{R}^N)$, either the equation

$$f - \mathcal{T}^*\mathcal{T} f = g$$

admits a unique solution, or the corresponding homogeneous equation has a finite-dimensional space of nontrivial solutions.

Proof. Let $\mathcal{T} : \mathcal{L}_\alpha^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be the OSSFT operator. By the hypotheses of the Compactness Theorem, $\mathcal{T}$ is a compact operator from $\mathcal{L}_\alpha^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$. Consequently, its adjoint $\mathcal{T}^* : L^2(\mathbb{R}^N) \to \mathcal{L}_\alpha^2(\mathbb{R}^N)$ is also compact.

Consider the composition

$$\mathcal{T}^*\mathcal{T} : \mathcal{L}_\alpha^2(\mathbb{R}^N) \to \mathcal{L}_\alpha^2(\mathbb{R}^N),$$

which is compact because the composition of a bounded operator with a compact operator is compact.

Define

$$\mathcal{F} := I - \mathcal{T}^*\mathcal{T}. \tag{5.1}$$

Since $\mathcal{T}^*\mathcal{T}$ is compact, $\mathcal{F}$ is a Fredholm operator of index zero by the standard property of compact perturbations of the identity in Hilbert spaces as can be seen in [12]. Specifically, for any compact operator K on a Hilbert space H, $I - K$ is Fredholm with index zero, and the kernel and cokernel are finite-dimensional.

Let $g \in \mathcal{L}_\alpha^2(\mathbb{R}^N)$ be given. We consider the equation

$$\mathcal{F} f = f - \mathcal{T}^*\mathcal{T} f = g.$$

By Fredholm theory, either the homogeneous equation

$$f - \mathcal{T}^*\mathcal{T} f = 0$$

has only the trivial solution, in which case $\mathcal{F}$ is invertible and the inhomogeneous equation has a unique solution for every g, or the homogeneous equation has a finite-dimensional kernel, in which case the range of $\mathcal{F}$ is closed, has finite codimension equal to the dimension of the kernel, and for the inhomogeneous equation to be solvable, g must lie in the range. The general solution is then unique up to addition of an element of the kernel, giving a finite-dimensional space of nontrivial solutions.

Finally, the finite-dimensionality of the kernel of $\mathcal{F}$ follows directly from the compactness of $\mathcal{T}^*\mathcal{T}$, because any compact operator on a Hilbert space has discrete spectrum accumulating at zero, so 1 can be an eigenvalue only with finite multiplicity. This establishes the Fredholm alternative for $\mathcal{F}$. □

The combination of compactness and smoothing properties yields a refined spectral structure for the normal OSSFT operator. In particular, the spectrum becomes discrete and the associated eigenfunctions inherit quantitative Sobolev regularity from the kernel decay.

Theorem 5.2. *Under the assumptions of the Compactness and Smoothing Effect Theorems, the spectrum of $\mathcal{T}^*\mathcal{T}$ consists of a sequence of nonnegative eigenvalues*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \lambda_n \to 0,$$

and the corresponding eigenfunctions belong to $H^\kappa(\mathbb{R}^N)$ for every

$$\kappa < \frac{\mu}{\beta} - (N - \alpha).$$

Proof. Let $\mathcal{T} : \mathcal{L}^2_\alpha(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be the OSSFT operator, and assume the hypotheses of the Compactness and Smoothing Effect Theorems.

By the Compactness Theorem, $\mathcal{T}$ is compact, hence the composition

$$\mathcal{A} := \mathcal{T}^*\mathcal{T} : \mathcal{L}^2_\alpha(\mathbb{R}^N) \to \mathcal{L}^2_\alpha(\mathbb{R}^N)$$

is compact, self-adjoint, and positive semi-definite. Compact, self-adjoint operators on a Hilbert space admit a discrete spectrum with only possible accumulation point at zero (see [12]). Thus there exists a sequence of eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \lambda_n \to 0,$$

and corresponding eigenfunctions $\{\phi_n\}$ forming an orthonormal set in $L^2(\mathbb{R}^N)$.

Let ϕ_n satisfy

$$\mathcal{T}^*\mathcal{T}\phi_n = \lambda_n \phi_n.$$

To prove regularity, we invoke the Smoothing Effect Theorem. For each ϕ_n, we have

$$\mathcal{T}\phi_n \in L^2(\mathbb{R}^N), \qquad D^\kappa(\mathcal{T}\phi_n) \in L^2(\mathbb{R}^N), \quad \kappa < \frac{\mu}{\beta} - (N - \alpha).$$

Since $\mathcal{T}^*$ maps L^2 to $\mathcal{L}^2_\alpha$ boundedly (as in the Adjoint Theorem), we have

$$D^\kappa(\mathcal{T}^*\mathcal{T}\phi_n) = \mathcal{T}^* D^\kappa(\mathcal{T}\phi_n) \in \mathcal{L}^2_\alpha(\mathbb{R}^N),$$

and using the eigenvalue relation $\mathcal{T}^*\mathcal{T}\phi_n = \lambda_n \phi_n$, we conclude

$$\phi_n \in H^\kappa(\mathbb{R}^N), \quad \forall \kappa < \frac{\mu}{\beta} - (N - \alpha),$$

where H^κ denotes the standard Sobolev space.

Thus the eigenfunctions inherit the full smoothing gain allowed by the OSSFT kernel decay and the $\mathcal{L}^2_\alpha$-weighted norms.

Finally, the positivity of $\mathcal{A}$ ensures all eigenvalues $\lambda_n \geq 0$, completing the spectral characterization. The combination of compactness and smoothing yields discrete, decaying spectrum with regular eigenfunctions, as asserted. □

6. Conclusion

In this paper, we introduced the Oscillatory Singular Special Function Transform (OSSFT), a new class of integral transforms whose kernels simultaneously incorporate algebraic singularities, nonlinear oscillatory phases, and special-function components of Mittag–Leffler type. This combination leads to a flexible analytical framework that extends several classical and generalized integral transforms arising in harmonic analysis and fractional calculus.

A detailed investigation of the OSSFT revealed a rich operator-theoretic structure. We established boundedness of the transform on weighted Lebesgue spaces under explicit parameter constraints that reflect a precise balance between singularity order and oscillatory decay. These mapping properties demonstrate that the oscillatory and special-function components effectively compensate for the algebraic singularity of the kernel.

Furthermore, we showed that enhanced decay of the special-function factor yields compactness of the associated OSSFT operators. This compactness result plays a central role in the subsequent spectral analysis, leading to the discreteness of the spectrum of the normal operator and to quantitative Sobolev regularity of the corresponding eigenfunctions. These results highlight a genuine regularizing effect induced by the OSSFT, which goes beyond mere boundedness.

Under suitable symmetry and strengthened admissibility assumptions, we proved a Plancherel-type identity, establishing that the OSSFT acts as an energy-preserving transform on its natural L^2 domain. This property allows the OSSFT to be viewed as a Fourier-type transform adapted to oscillatory singular kernels. Building on this framework, we derived Heisenberg-type uncertainty inequalities that quantify the intrinsic limitation on simultaneous spatial and transform localization, thereby confirming that the OSSFT exhibits localization behavior analogous to classical harmonic analysis transforms.

The stability of the OSSFT with respect to admissible kernel perturbations further demonstrates the robustness of the proposed framework. Small variations in the defining parameters lead to controlled changes in the associated operators, ensuring continuous dependence in the operator norm topology.

Overall, the analytical results obtained in this work show that the OSSFT provides a coherent and powerful extension of oscillatory integral theory, combining singular analysis, special-function techniques, and operator-theoretic methods. The framework developed here opens several avenues for future research, including extensions to non-Euclidean settings, anisotropic kernels, and multi-parameter transforms, as well as applications to fractional differential equations, inverse problems, and nonlocal models arising in mathematical physics.

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References

[1] I. Ahmad, *Representation-theoretic framework for the linear canonical transform in quantum harmonic analysis*, Afrika Matematika, 2025, 36, 158.

[2] I. Ahmad and N. Ahmad, *Some study on uncertainty principle*, International Journal of Modern Mathematical Sciences, 2015, 13(3), 198–209.

[3] J. P. Antoine and R. Murenzi, *Two-dimensional directional wavelets and the scale-angle representation*, Signal Processing, 1996, 52, 259–281.

[4] E. Ata and I. O. Kıymaz, *New generalized Mellin transform and applications to partial and fractional differential equations*, International Journal of Mathematics and Computer in Engineering, 2023, 1(1), 45–66.

[5] E. Fadhal, A. G. Abdul, N. S. Alharthi, A. Khan, D. Fathima and A. E. Elamin, *On the analysis and deeper properties of the fractional complex physical models pertaining to nonsingular kernels*, Scientific Reports, 2024, 14(1), 22182.

[6] D. Fathima, A. H. Ganie and S. A. Lone, *Some new structures of orthonormal wavelets in* $H_s(\mathbb{R})$, Journal of Applied Mathematics and Informatics, 2025, 43(2), 287–307.

[7] A. H. Ganie, G. A. Basendwah, B. Sultan, N. M. Aloraini and I. L. Popa, *Trace theorems on Herz-Morrey spaces with applications to Sobolev type inequalities*, Boundary Value Problems, 2025, 133, 1–11.

[8] A. H. Ganie, M. Houas and M. M. AlBaidani, *Solvability and Mittag–Leffler–Ulam stability for fractional Duffing problem with three sequential fractional derivatives*, Mathematical Methods in the Applied Sciences, 2024, 47(4), 1807–1822.

[9] M. Goli, A. Ansari and R. Ghaderi, *Integral transforms and probability distributions associated with Fox–Wright type functions*, Mathematics and Computers in Simulation, 2023, 203, 803–825.

[10] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, 2014.

[11] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, 1983.

[12] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1995.

[13] M. Khan and G. H. Abdul, *On the characteristic polynomial and energy of Hermitian quasi-Laplacian matrix of mixed graphs*, Asian-European Journal of Mathematics, 2023, 16(7), 2350116.

[14] H. Kim, *Various approaches for generalized integral transforms*, Symmetry, 2024, 16, 576. DOI: 10.3390/sym16050576.

[15] A. Pal, *Generalized integral transform and fractional calculus operators involving a generalized Mittag–Leffler type function*, Computational Science and Mathematics Forum, 2023, 7, 42.

[16] J. Saba, P. A. Naik, M. Farman, M. U. Saleem and G. H. Abdul, *Stability and complex dynamical analysis of COVID-19 epidemic model with non-singular kernel of Mittag-Leffler law*, Journal of Applied Mathematics and Computing, 2024, 70(4), 3441–3476.

[17] A. Shahid, A. Ganie and N. Alharthi, *A new parameterized iterative scheme for nonlinear equations: Convergence and basin of attraction analysis*, Journal of Applied Mathematics and Computing, 2025, 71, 7929–7958.

[18] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, 1993.

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