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GENERALIZED SOR-LIKE ITERATION METHODS FOR ABSOLUTE VALUE EQUATIONS ASSOCIATED WITH CIRCULAR CONES*

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Abstract In this paper, we study the absolute value equations associated with circular cones (referred to as CCAVE). The circular cone is a type of non-symmetric cone that generalizes the second-order cone. By equivalently reformulating CCAVE as a two-by-two block nonlinear equation, we propose a class of generalized SOR-like iteration methods for solving CCAVE. Useful properties of the circular cone are explored, and sufficient conditions for ensuring the convergence of the generalized SOR-like iteration methods are analyzed. Numerical experiments demonstrate the effectiveness of the proposed methods in solving CCAVE.

Keywords Circular cone, absolute value equation, matrix splitting, convergence.

MSC(2010) 65F10, 65H10.

1. Introduction

Consider iteration solutions of the following absolute value equations associated with circular cones (CCAVE for short):

$$Ax + B|x| = b, \tag{1.1}$$

or equivalently,

$$F(x) := Ax + B|x| - b = 0, \tag{1.2}$$

where $A, B \in \mathbb{R}^{n\times n}$ are given large and sparse real matrices, $b \in \mathbb{R}^n$ is a given real vector, $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_r^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ is a variable with $r, n_1, n_2, \ldots, n_r \geq 1$ and $n_1 + n_2 + \cdots + n_r = n$, and

$$|x| := (|x_1|^{\mathrm{T}}, |x_2|^{\mathrm{T}}, \ldots, |x_r|^{\mathrm{T}})^{\mathrm{T}}$$

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with the absolute value $|x_i|$ determined by the corresponding circular cone in $\mathbb{R}^{n_i}$ for $i = 1, 2, \ldots, r$. The Cartesian product $\mathcal{L}$ of circular cones $\mathcal{L}_{\theta_i}^{n_i}$ is denoted as

$$\mathcal{L} := \mathcal{L}_{\theta_1}^{n_1} \times \mathcal{L}_{\theta_2}^{n_2} \times \cdots \times \mathcal{L}_{\theta_r}^{n_r}, \tag{1.3}$$

where $\mathcal{L}_{\theta_i}^{n_i} (i = 1, 2, \ldots, r)$ denotes a circular cone in $\mathbb{R}^{n_i}$ defined by

$$\begin{aligned}\mathcal{L}_{\theta_i}^{n_i} :=& \{x_i = (x_{i1}, x_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} : \|x_i\| \cos\theta_i \leq x_{i1}\} \\ =& \{x_i = (x_{i1}, x_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} : \|x_{i2}\| \leq x_{i1} \tan\theta_i\},\end{aligned}$$

where $\|\cdot\|$ denotes the usual Euclidean norm of a vector. The circular cone [5] is a pointed closed convex cone having hyper-spherical sections orthogonal to its axis of revolution about which the cone is invariant to rotation. When $\theta_i = \dfrac{\pi}{4}$, the circular cone $\mathcal{L}_{\theta_i}^{n_i}$ reduces to the second-order cone $\mathcal{K}^{n_i}$, i.e.,

$$\mathcal{K}^{n_i} := \{x_i = (x_{i1}, x_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} : \|x_{i2}\| \leq x_{i1}\},$$

which is also called Lorentz cone or ice-cream cone.

The circular cone is a very generalization of the second-order cone which is a powerful tool to handle optimization problems, and the important relations between circular cone $\mathcal{L}_\theta^n$ and second-order cone $\mathcal{K}^n$ are discovered:

$$\mathcal{L}_\theta^n = G^{-1}\mathcal{K}^n \quad \text{and} \quad \mathcal{K}^n = G\mathcal{L}_\theta^n \quad \text{with} \quad G = \begin{pmatrix} \tan\theta & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

In other words, for any vectors $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, there have

$$x \in \mathcal{L}_\theta^n \Longleftrightarrow Gx \in \mathcal{K}^n \quad \text{and} \quad y \in (\mathcal{L}_\theta^n)^* \Longleftrightarrow G^{-1}y \in \mathcal{K}^n.$$

This simple and basic relation helps us to study CCAVE throughout some properties of the second-order cone based on Jordan algebra.

When $\theta_1 = \theta_2 = \cdots = \theta_r = \theta$, CCAVE (1.1) reduces to the problem studied in [20]. When $\theta_i = \dfrac{\pi}{4}$ for all i, CCAVE (1.1) reduces to the second-order cone absolute value equation (SOCAVE for short), which was studied in [8]. When $n_i = 1$ for all i, it is easy to see that CCAVE (1.1) reduces to the standard absolute value equation (AVE for short), i.e.,

$$Ax + B|x| = b, \quad \text{where} \quad |x| := (|x_1|, |x_2|, \ldots, |x_n|)^{\mathrm{T}} \in \mathbb{R}^n, \tag{1.4}$$

which was first introduced by Rohn [27] and is capable to formulate many optimization problems [15, 16, 25].

Henceforth, CCAVE (1.1) constitutes a generalization of the previously studied SOCAVE and AVE. In recent years, the problem of solving AVE (1.4) has garnered significant attention and has been extensively investigated in the literature see for example ([1, 3, 4, 6, 7, 10–14, 17–19, 21–24, 26, 28–33]) and references therein. A wide variety of iterative methods for solving AVE (1.4) have been developed. Most of these methods are based on the Newton algorithm, as AVE (1.4) is a weakly nonlinear equation.

In [8], Hu, Huang and Zhang proposed a generalized Newton method for solving SOCAVE and it is shown to be globally linearly and locally quadratically convergent under suitable assumptions. After that, Miao, Yang and Hu [20] extended the generalized Newton method for solving CCAVE (1.1) with $\theta_1 = \theta_2 = \cdots = \theta_r = \theta$.

In this paper, by reformulating CCAVE (1.1) as a two-by-two block nonlinear equation, we propose a novel class of generalized SOR-like iteration methods for solving it, which is based on a splitting of the two-by-two block coefficient matrix. The aforementioned facts and new observations (Propositions 2.1 and 2.2) on the circular cone attract our attention to give novel convergence analysis technique for the proposed iteration method. We prove that the proposed iteration methods will converge to the solution of CCAVE (1.1) under suitable choices of the involved splitting matrix and parameter. In addition, we also use some test examples to show that the generalized SOR-like iteration method is feasible and effective in computing.

The outline of this paper is as follows. In Section 2, we present some basic concepts and background materials about the circular cone as well as some new observations. In addition, we will present some auxiliary results about the nonnegative matrix. In Section 3, we establish a novel class of matrix splitting iteration methods for solving CCAVE (1.1) and also give the convergence of the proposed iteration methods. Numerical results about the new iteration methods are shown and discussed in Section 4. Finally, in Section 5, we end the paper by some concluding remarks.

At the end of this section, we present some notations, which will be used throughout this paper. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real $m \times n$ matrices, $A \geq B$ ($A > B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) holds for all $1 \leq i \leq m$ and $1 \leq j \leq n$. A matrix $A \in \mathbb{R}^{m\times n}$ is said to be nonnegative (positive) if its entries satisfy $a_{ij} \geq 0$ ($a_{ij} > 0$) for all $1 \leq i \leq m$ and $1 \leq j \leq n$. For the matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|$ denotes the spectral norm defined by $\|A\| := \max\{\|Ax\| : x \in \mathbb{R}^n, \ \|x\| = 1\}$, where $\|\cdot\|$ is Euclidean norm. Without loss of generality, we also denote $(x_1, x_2, \ldots, x_r) = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_r^{\mathrm{T}})^{\mathrm{T}}$.

2. Preliminaries

In this section, we briefly review some basic concepts and background materials about the circular cone. In addition, we will give some new observations on the circular cone, which will be extensively used in following theoretical analysis.

For any vector $x \in \mathbb{R}^n$, let x_+ denote the projection of x onto the circular cone $\mathcal{L}_\theta^n$ and x_- be the projection of $-x$ onto the dual cone $(\mathcal{L}_\theta^n)^*$. With these notations, for any vector $x \in \mathbb{R}^n$, it can be verified that

$$x = x_+ - x_-.$$

Moreover, due to the special structure of the circular cone $\mathcal{L}_\theta^n$, the explicit formula of the projection of $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ onto $\mathcal{L}_\theta^n$ as follows:

$$x_+ = \begin{cases} x \text{ if } x \in \mathcal{L}_\theta^n, \\ 0 \text{ if } x \in -(\mathcal{L}_\theta^n)^*, \\ u \text{ otherwise,} \end{cases} \tag{2.1}$$

where

$$u = \frac{x_1 + \|x_2\| \tan\theta}{1 + \tan^2\theta} \begin{pmatrix} 1 \\ \dfrac{x_2}{\|x_2\|} \end{pmatrix}.$$

Likewise, we can obtain the expression of x_- as follows:

$$x_- = \begin{cases} 0 & \text{if } x \in \mathcal{L}_\theta^n, \\ -x & \text{if } x \in -(\mathcal{L}_\theta^n)^*, \\ v & \text{otherwise,} \end{cases} \tag{2.2}$$

where

$$v = \frac{x_1 - \|x_2\| \cot\theta}{1+\cot^2\theta} \begin{pmatrix} -1 \\ \frac{x_2}{\|x_2\|} \end{pmatrix}.$$

From the expressions (2.1) and (2.2) for x_+ and x_-, it can be shown that $\langle x_+, x_- \rangle = 0$ for any $x \in \mathbb{R}^n$.

In [34], Zhou and Chen gave the following spectral decomposition of x with respect to the circular cone.

Theorem 2.1. [34] *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, then x can be decomposed as*

$$x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}, \tag{2.3}$$

where λ_1, λ_2 and $u^{(1)}$, $u^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\lambda_1 = x_1 - \|x_2\| \cot\theta, \quad \lambda_2 = x_1 + \|x_2\| \tan\theta, \tag{2.4}$$

$$u^{(1)} = \frac{1}{1+\cot^2\theta} \begin{pmatrix} 1 \\ -(\cot\theta)\widetilde{u} \end{pmatrix}, \quad u^{(2)} = \frac{1}{1+\tan^2\theta} \begin{pmatrix} 1 \\ (\tan\theta)\widetilde{u} \end{pmatrix}, \tag{2.5}$$

with $\widetilde{u} = \frac{x_2}{\|x_2\|}$ if $x_2 \neq 0$, and any vector in $\mathbb{R}^{n-1}$ satisfying $\|\widetilde{u}\| = 1$ if $x_2 = 0$.

Moreover, in [34], the expression of x_+ has the following form:

$$x_+ = (\lambda_1)_+ u^{(1)} + (\lambda_2)_+ u^{(2)}, \tag{2.6}$$

where $\lambda_+ = \max\{0, \lambda\}$ for $\lambda \in \mathbb{R}$, λ_i and $u^{(i)}$ for $i = 1, 2$ are given as in Theorem 2.1.

Utilizing a similar technique, Miao, Yang and Hu [20] generalize this to the expression for x_- as follows:

$$x_- = (\lambda_1)_- u^{(1)} + (\lambda_2)_- u^{(2)}, \tag{2.7}$$

where $\lambda_- = \max\{0, -\lambda\}$ for $\lambda \in \mathbb{R}$, λ_i and $u^{(i)}$ for $i = 1, 2$ are given as in Theorem 2.1.

For any vector $x \in \mathbb{R}^n$, let us define the absolute value of x with respect to the circular cone as

$$|x| := x_+ + x_-.$$

Then, we have

$$\begin{aligned} |x| &= \big[(\lambda_1)_+ + (\lambda_1)_-\big] u^{(1)} + \big[(\lambda_2)_+ + (\lambda_2)_-\big] u^{(2)} \\ &= |\lambda_1| u^{(1)} + |\lambda_2| u^{(2)}, \end{aligned}$$

that is

$$|x| = \begin{cases} \begin{pmatrix} |x_1| \\ 0 \end{pmatrix}, & \text{if } x_2 = 0, \\ \begin{pmatrix} \dfrac{|x_1 - \|x_2\| \cot\theta|}{1+\cot^2\theta} + \dfrac{|x_1 + \|x_2\| \tan\theta|}{1+\tan^2\theta} \\ \Big[\dfrac{|x_1 + \|x_2\| \tan\theta|}{1+\tan^2\theta}\tan\theta - \dfrac{|x_1 - \|x_2\| \cot\theta|}{1+\cot^2\theta}\cot\theta\Big]\dfrac{x_2}{\|x_2\|}, \end{pmatrix}, & \text{if } x_2 \neq 0. \end{cases} \tag{2.8}$$

Based on these facts, we can get the following results.

Proposition 2.1. [9] *For any vectors x and y in $\mathbb{R}^n$, let $|x|$ and $|y|$ be defined as in (2.8), it holds*

$$\| \, |x| - |y| \, \| \leq \|x - y\|.$$

From Proposition 2.1, we can obtain the following result.

Proposition 2.2. *For any vectors $x = (x_1, x_2, \cdots, x_r)$ and $y = (y_1, y_2, \cdots, y_r)$ in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ with $n_1 + n_2 + \cdots + n_r = n$, let*

$$|x| = (|x_1|, |x_2|, \ldots, |x_r|), \quad |y| = (|y_1|, |y_2|, \ldots, |y_r|)$$

with $|x_i|, |y_i| \in \mathbb{R}^{n_i}$ defined as (2.8), it holds

$$\| \, |x| - |y| \, \| \leq \|x - y\|.$$

About the nonnegative matrix, we have the following results.

Lemma 2.1. [2] *For any vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, the following results hold:*

(1) *If $0 \leq x \leq y$, then $\|x\| \leq \|y\|$;*

(2) *if $x \leq y$ and P is a nonnegative matrix, then $Px \leq Py$.*

Lemma 2.2. [2] *For any matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, if $0 \leq A \leq B$, then $\|A\| \leq \|B\|$.*

3. Generalized SOR-like iteration methods

In this section, we introduce a class of matrix splitting iteration methods for solving CCAVE (1.1). Let $y = B|x|$, then CCAVE (1.1) is equivalent to

$$\begin{cases} Ax + y = b, \\ -B|x| + y = 0, \end{cases}$$

that is

$$\mathcal{A}\mathbf{z} := \begin{pmatrix} A & I \\ -BG & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} := \mathbf{b}, \tag{3.1}$$

where $Gx := |x|$. If $A = M - N$ is a splitting of the matrix A and

$$\mathcal{A} = \mathcal{D} - \mathcal{L} - \mathcal{U},$$

where

$$\mathcal{D} = \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & 0 \\ BG & 0 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} N & -I \\ 0 & 0 \end{pmatrix},$$

then we can obtain the following fixed point equation

$$(\mathcal{D} - \omega\mathcal{L})\mathbf{z} = [(1-\omega)\mathcal{D} + \omega\mathcal{U}]\mathbf{z} + \omega\mathbf{b},$$

where the parameter $\omega > 0$. That is

$$\begin{pmatrix} M & 0 \\ -\omega BG & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1-\omega)M + \omega N & -\omega I \\ 0 & (1-\omega)I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \omega b \\ 0 \end{pmatrix}. \tag{3.2}$$

Based on the fixed point equation (3.2), we can establish the following matrix splitting iteration methods for solving CCAVE (1.1), called the generalized SOR-like iteration method.

Method 3.1 (Generalized SOR-like iteration method for CCAVE (1.1))**.**

Step 1. Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Given the initial vectors $x^{(0)} \in \mathbb{R}^n$ and $y^{(0)} \in \mathbb{R}^n$. Choose $0 < \varepsilon \ll 1$. Set $k := 0$.

Step 2. Solve the linear system

$$Mz^{(k)} = Nx^{(k)} - y^{(k)} + b$$

for $z^{(k)}$. Update

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega z^{(k)}.$$

Step 3. For $x^{(k+1)} = (x_1^{(k+1)}, x_2^{(k+1)}, \cdots, x_r^{(k+1)}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$, compute

$$|x^{(k+1)}| = (|x_1^{(k+1)}|, |x_2^{(k+1)}|, \cdots, |x_r^{(k+1)}|)$$

by using (2.8). If $\|b - Ax^{(k+1)} - B|x^{(k+1)}|\,\| \le \varepsilon$, break; else, go to Step 4.

Step 4. Set

$$y^{(k+1)} = (1-\omega)y^{(k)} + \omega B|x^{(k+1)}|.$$

Method 3.1 provides a general framework of the generalized SOR-like iteration method for solving CCAVE (1.1). It can yield a series of the iteration methods with suitable choices of the splitting of the matrix A and the iteration parameter ω. In specific, we let $A = D - L - U$, where D, $-L$, and $-U$ are the diagonal, the strictly lower-triangular, and the strictly upper-triangular matrices of the matrix A. Then

(a) when $M = A$ and $N = 0$, Method 3.1 gives GSOR iteration method

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} - \omega A^{-1}(y^{(k)} - b), \\ y^{(k+1)} = (1-\omega)y^{(k)} + \omega B|x^{(k+1)}|; \end{cases}$$

(b) when $M = D$ and $N = L + U$, Method 3.1 gives GSOR-Jacobi iteration method

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega D^{-1}\big[(L+U)x^{(k)} - y^{(k)} + b\big], \\ y^{(k+1)} = (1-\omega)y^{(k)} + \omega B|x^{(k+1)}|; \end{cases}$$

(c) when $M = D - L$ and $N = U$, Method 3.1 gives GSOR-Seidel iteration method

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega(D-L)^{-1}(Ux^{(k)} - y^{(k)} + b), \\ y^{(k+1)} = (1-\omega)y^{(k)} + \omega B|x^{(k+1)}|; \end{cases}$$

(d) when $M = \frac{1}{\alpha}D - L$ and $N = (\frac{1}{\alpha} - 1)D + U$, Method 3.1 gives GSOR-SOR iteration method

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega(\frac{1}{\alpha}D-L)^{-1}\big[\big((\frac{1}{\alpha}-1)D+U\big)x^{(k)} - y^{(k)} + b\big], \\ y^{(k+1)} = (1-\omega)y^{(k)} + \omega B|x^{(k+1)}|; \end{cases}$$

(e) when $M = \frac{1}{\alpha}(D - \beta L)$ and $N = \frac{1}{\alpha}[(1-\alpha)D + (\alpha-\beta)L + \alpha U]$, Method 3.1 gives GSOR-AOR iteration method

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega(D-\beta L)^{-1}\big[\big((1-\alpha)D+(\alpha-\beta)L+\alpha U\big)x^{(k)} - \alpha y^{(k)} + \alpha b\big], \\ y^{(k+1)} = (1-\omega)y^{(k)} + \omega B|x^{(k+1)}|; \end{cases}$$

(f) when $M = \alpha I + H$ and $N = \alpha I - S$ with $H = \frac{1}{2}(A + A^\top)$ and $S = \frac{1}{2}(A - A^\top)$, Method 3.1 gives GSOR-HSS iteration method

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega(\alpha I+H)^{-1}\big[(\alpha I - S)x^{(k)} - y^{(k)} + b\big], \\ y^{(k+1)} = (1-\omega)y^{(k)} + \omega B|x^{(k+1)}|. \end{cases}$$

Assume x^* is the solution of CCAVE (1.1), or equivalently, the vector pair (x^*, y^*) satisfies the following equations

$$\begin{cases} Ax^* + y^* = b, \\ -B|x^*| + y^* = 0, \end{cases}$$

that is

$$\begin{cases} x^* = (1-\omega)x^* + \omega M^{-1}(Nx^* - y^* + b), \\ y^* = (1-\omega)y^* + \omega B|x^*|. \end{cases} \tag{3.3}$$

Let the vector pair $(x^{(k)}, y^{(k)})$ be generated by Method 3.1. Define the iteration errors

$$e_k^x = x^* - x^{(k)}, \quad e_k^y = y^* - y^{(k)}.$$

Then we can obtain the convergence results for Method 3.1 as follows.

Theorem 3.1. *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Denote*

$$\xi = \|M^{-1}\|, \quad \eta = \|M^{-1}N\|, \quad \zeta = \|B\|, \quad \textit{and}$$
$$T = \begin{pmatrix} |1-\omega| + \omega\eta & \omega\xi \\ \omega|1-\omega|\zeta + \omega^2\eta\zeta & |1-\omega| + \omega^2\xi\zeta \end{pmatrix}. \tag{3.4}$$

If $\|T\| < 1$, *then the following inequality holds*

$$\|E_{k+1}\| < \|E_k\|, \quad k = 0, 1, 2, \ldots, \tag{3.5}$$

where

$$E_{k+1} = \begin{pmatrix} \|e^x_{k+1}\| \\ \|e^y_{k+1}\| \end{pmatrix}.$$

Proof. From Method 3.1 and (3.3), we have

$$e^x_{k+1} = (1-\omega)e^x_k + \omega M^{-1}(Ne^x_k - e^y_k), \tag{3.6}$$

$$e^y_{k+1} = (1-\omega)e^y_k + \omega B(|x^*| - |x^{(k+1)}|). \tag{3.7}$$

From (3.6), we can obtain

$$\begin{aligned} \|e^x_{k+1}\| \le & |1-\omega| \cdot \|e^x_k\| + \omega(\|M^{-1}N\| \cdot \|e^x_k\| + \|M^{-1}\| \cdot \|e^y_k\|) \\ := & |1-\omega| \cdot \|e^x_k\| + \omega(\eta\|e^x_k\| + \xi\|e^y_k\|) \\ = & (|1-\omega| + \omega\eta)\|e^x_k\| + \omega\xi\|e^y_k\|. \end{aligned} \tag{3.8}$$

From (3.7) and Proposition 2.2, we have

$$\begin{aligned} \|e^y_{k+1}\| & \le |1-\omega| \cdot \|e^y_k\| + \omega\|B\| \cdot \| \, |x^*| - |x^{(k+1)}| \, \| \\ & \le |1-\omega| \cdot \|e^y_k\| + \omega\zeta\|x^* - x^{(k+1)}\| \\ & = |1-\omega| \cdot \|e^y_k\| + \omega\zeta\|e^x_{k+1}\|. \end{aligned} \tag{3.9}$$

Consequently, based on (3.8) and (3.9), the following inequalities are derived:

$$\begin{cases} -\omega\zeta\|e^x_{k+1}\| + \|e^y_{k+1}\| \le |1-\omega|\|e^x_{k+1}\|, \\ \|e^x_{k+1}\| \le (|1-\omega| + \omega\eta)\|e^x_k\| + \omega\xi\|e^y_k\|. \end{cases}$$

Subsequently, it follows that

$$\begin{pmatrix} -\omega\zeta & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \|e^x_{k+1}\| \\ \|e^y_{k+1}\| \end{pmatrix} \le \begin{pmatrix} 0 & |1-\omega| \\ |1-\omega| + \omega\eta & \omega\xi \end{pmatrix} \begin{pmatrix} \|e^x_k\| \\ \|e^y_k\| \end{pmatrix}. \tag{3.10}$$

Let

$$P = \begin{pmatrix} 0 & 1 \\ 1 & \omega\zeta \end{pmatrix} \ge 0.$$

Multiplying (3.10) from left by the nonnegative matrix P and according to Lemma 2.1, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & \omega\zeta \end{pmatrix} \begin{pmatrix} -\omega\zeta & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \|e^x_{k+1}\| \\ \|e^y_{k+1}\| \end{pmatrix} \le \begin{pmatrix} 0 & 1 \\ 1 & \omega\zeta \end{pmatrix} \begin{pmatrix} 0 & |1-\omega| \\ |1-\omega| + \omega\eta & \omega\xi \end{pmatrix} \begin{pmatrix} \|e^x_k\| \\ \|e^y_k\| \end{pmatrix},$$

$$\begin{pmatrix} \|e^x_{k+1}\| \\ \|e^y_{k+1}\| \end{pmatrix} \le \begin{pmatrix} |1-\omega| + \omega\eta & \omega\xi \\ \omega|1-\omega|\zeta + \omega^2\eta\zeta & |1-\omega| + \omega^2\xi\zeta \end{pmatrix} \begin{pmatrix} \|e^x_k\| \\ \|e^y_k\| \end{pmatrix}. \tag{3.11}$$

Denote

$$E_{k+1} = \begin{pmatrix} \|e^x_{k+1}\| \\ \|e^y_{k+1}\| \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} |1-\omega| + \omega\eta & \omega\xi \\ \omega|1-\omega|\zeta + \omega^2\eta\zeta & |1-\omega| + \omega^2\xi\zeta \end{pmatrix} \geq 0.$$

According to (3.11), we have

$$\|E_{k+1}\| \leq \|TE_k\| \leq \|T\| \cdot \|E_k\|.$$

If $\|T\| < 1$, then we have

$$\|E_{k+1}\| < \|E_k\|.$$

This completes the proof. □

Theorem 3.2. *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Denote*

$$\begin{aligned} &\xi = \|M^{-1}\|, \quad \zeta = \|B\|, \\ &\eta = \|M^{-1}N\|, \quad \delta = \omega^2\xi\zeta, \\ &\varphi = |1-\omega| + \omega\eta, \\ &\psi = |1-\omega| + \omega^2\xi\zeta. \end{aligned}$$

If

$$2\varphi^2 + \delta^2 + \psi^2 < \min\{1 + \varphi^2(1-\omega)^2, 2\}, \tag{3.12}$$

then the following inequality holds

$$|||(e^x_{k+1}, e^y_{k+1})||| < |||(e^x_k, e^y_k)|||, \quad k = 0, 1, 2, \ldots. \tag{3.13}$$

Here, the norm is defined by

$$|||(e^x, e^y)||| := \sqrt{\|e^x\|^2 + (\omega\zeta)^{-2}\|e^y\|^2}.$$

Proof. From the proof of Theorem 3.1, we have

$$\begin{pmatrix} \|e^x_{k+1}\| \\ \|e^y_{k+1}\| \end{pmatrix} \leq \begin{pmatrix} |1-\omega| + \omega\eta & \omega\xi \\ \omega|1-\omega|\zeta + \omega^2\eta\zeta & |1-\omega| + \omega^2\xi\zeta \end{pmatrix} \begin{pmatrix} \|e^x_k\| \\ \|e^y_k\| \end{pmatrix}. \tag{3.14}$$

Denote

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & \omega^{-1}\zeta^{-1} \end{pmatrix} \geq 0.$$

Multiplying (3.14) from left by the nonnegative matrix Q, we can get

$$\begin{pmatrix} \|e^x_{k+1}\| \\ \omega^{-1}\zeta^{-1}\|e^y_{k+1}\| \end{pmatrix} \leq \begin{pmatrix} |1-\omega| + \omega\eta & \omega^2\xi\zeta \\ |1-\omega| + \omega\eta & |1-\omega| + \omega^2\xi\zeta \end{pmatrix} \begin{pmatrix} \|e^x_k\| \\ \omega^{-1}\zeta^{-1}\|e^y_k\| \end{pmatrix}.$$

Hence, we have

$$|||(e^x_{k+1}, e^y_{k+1})||| \leq \|\widehat{T}\| \cdot |||(e^x_k, e^y_k)|||,$$

where

$$\widehat{T} = \begin{pmatrix} |1-\omega| + \omega\eta & \omega^2\xi\zeta \\ |1-\omega| + \omega\eta & |1-\omega| + \omega^2\xi\zeta \end{pmatrix} \geq 0.$$

Now, we consider the choice of the splitting of the matrix A and the iteration parameter ω such that $\|\widehat{T}\| < 1$, thus the inequality (3.13) holds. Denote

$$\widehat{T} = \begin{pmatrix} \varphi & \delta \\ \varphi & \psi \end{pmatrix}.$$

Since

$$\widehat{T}^\top \widehat{T} = \begin{pmatrix} 2\varphi^2 & \varphi(\delta+\psi) \\ \varphi(\delta+\psi) & \delta^2+\psi^2 \end{pmatrix},$$

then we have

$$\operatorname{tr}(\widehat{T}^\top \widehat{T}) = 2\varphi^2 + \delta^2 + \psi^2 \tag{3.15}$$

and

$$\det(\widehat{T}^\top \widehat{T}) = \varphi^2(1-\omega)^2. \tag{3.16}$$

Assume λ is the eigenvalues of the matrix $\widehat{T}^\top \widehat{T}$ with $\lambda \geq 0$, thus λ will satisfy

$$\lambda^2 - \operatorname{tr}(\widehat{T}^\top \widehat{T})\lambda + \det(\widehat{T}^\top \widehat{T}) = 0.$$

Hence, we have the following relations

$$\lambda_1 + \lambda_2 = \operatorname{tr}(\widehat{T}^\top \widehat{T}), \quad \lambda_1\lambda_2 = \det(\widehat{T}^\top \widehat{T}). \tag{3.17}$$

If

$$0 \leq 2\varphi^2 + \delta^2 + \psi^2 < \min\{1 + \varphi^2(1-\omega)^2, 2\},$$

from (3.15), (3.16) and (3.17), we have

$$0 \leq \lambda_1 + \lambda_2 < 2 \quad \text{and} \quad \lambda_1 + \lambda_2 < 1 + \lambda_1\lambda_2,$$

that is

$$0 \leq \lambda_1 + \lambda_2 < 2 \quad \text{and} \quad (\lambda_1 - 1)(\lambda_2 - 1) > 0.$$

Note that $\lambda_1\lambda_2 \geq 0$, then we can get

$$0 \leq \lambda_1 < 1 \quad \text{and} \quad 0 \leq \lambda_2 < 1.$$

Hence

$$\|\widehat{T}\| = \sqrt{\max\{\lambda_1, \lambda_2\}} < 1.$$

This completes the proof. □

It is easy to see that if the conditions of Theorem 3.2 are satisfied, then we have

$$0 \leq |||(e_k^x, e_k^y)||| \leq \|\widehat{T}\| \cdot |||(e_{k-1}^x, e_{k-1}^y)||| \leq \cdots \leq \|\widehat{T}\|^k \cdot |||(e_0^x, e_0^y)|||.$$

Since $\|\widehat{T}\| < 1$, we have $\lim_{k\to\infty} |||(e_k^x, e_k^y)||| = 0$. Using the definition of the norm $|||\cdot|||$ gives $\lim_{k\to\infty} \|e_k^x\| = 0$ and $\lim_{k\to\infty} \|e_k^y\| = 0$. Therefore, the sequence $\{x^{(k)}\}$ generated by Method 3.1 will converge to the solution of CCAVE (1.1).

4. Numerical experiments

In this section, we use some test problems to examine the effectiveness of the generalized SOR-like iteration method 3.1. All test problems are started from the initial zero vector, are terminated if the current iterations satisfy

$$\mathrm{ERR} := \|b - Ax^{(k)} - B|x^{(k)}|\,\| \leq 10^{-6}$$

or if the number of the prescribed iteration steps $k_{\max} = 1000$ is exceeded, and are performed under Matlab R2011b on a personal computer with 3.60GHz central processing unit (Intel(R) Core(TM) i7-7700), 32.0GB memory and Windows 10 operating system. In addition, 'IT' denotes the number of iteration steps and 'CPU' denotes the elapsed CPU time in seconds. The symbol '—' in the tables of numerical results means that the methods do not converge within the preset maximum number of iterations. And all linear systems are solved by direct methods.

Example 4.1. Consider CCAVE (1.1) associated with

$$\mathcal{L} := \mathcal{L}_{\theta_1}^{l^2} \times \mathcal{L}_{\theta_2}^{l^2} \times \cdots \times \mathcal{L}_{\theta_l}^{l^2} \quad \text{and} \quad \theta_i = \frac{\pi}{2} + (-1)^i \frac{\pi}{3}, \quad i = 1, 2, \ldots, l,$$
$$A = T_x \otimes I \otimes I + I \otimes T_y \otimes I + I \otimes I \otimes T_z \in \mathbb{R}^{l^3 \times l^3}.$$

Here, $\otimes$ denotes the Kronecker product and T_x, T_y, T_z are tridiagonal matrices given by

$$T_x = \mathrm{tridiag}(t_2, t_1, t_3), \quad T_y = \mathrm{tridiag}(t_2, 0, t_3) \quad \text{and} \quad T_z = \mathrm{tridiag}(t_3, 0, t_2)$$

with $t_1 = 5 + 7r$, $t_2 = -1 - r$, and $t_3 = -r$. Let $r = qh/2$ with $h = 1/(l + 1)$ and $q = 10$.

Furthermore, let $B = \frac{1}{2} I$, $x^* = (-1, 1, -1, \ldots, (-1)^{l^3})^{\mathrm{T}} \in \mathbb{R}^{l^3}$, and $b := Ax^* + B|x^*|$.

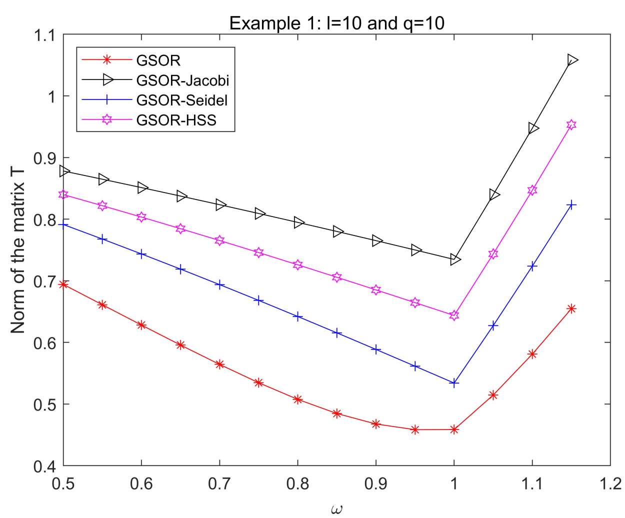

Figure 1. Norm of the matrix T with the parameter ω for Example 4.1.

For Example 4.1, we utilize GSOR, GSOR-Jacobi, GSOR-Seidel and GSOR-HSS iteration methods for solving it and quickly approximate the exact solution x^*. The norm of the matrix T defined in (3.4) and the iteration numbers with the parameter ω for GSOR, GSOR-Jacobi,

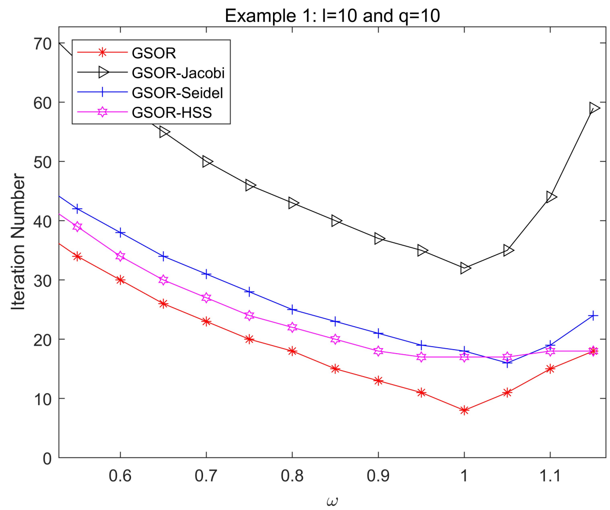

Figure 2. Iteration numbers with the parameter ω for Example 4.1.

Table 1. Numerical results of Example 4.1 and $q = 10$.

Method	l	10	20	30	40	50
GSOR	ω	1.00	1.00	1.00	1.00	1.00
	IT	8	9	10	10	10
	CPU	0.0372	0.5528	4.5849	21.6193	114.1439
	ERR	9.7928e-07	6.5642e-07	1.9212-08	3.6481e-08	5.8042e-07
GSOR-Jacobi	ω	1.00	1.00	1.00	1.00	1.00
	IT	32	34	35	35	35
	CPU	0.0035	0.0099	0.0371	0.0965	0.2450
	ERR	8.8788e-07	8.9338e-07	6.5315e-07	7.3379e-07	8.1499e-07
GSOR-Seidle	ω	1.05	1.00	1.00	0.95	0.95
	IT	16	18	18	19	19
	CPU	0.0008	0.0052	0.199	0.0536	0.1161
	ERR	6.3365e-07	6.4019e-07	6.9944e-07	9.0062e-07	9.2700e-07
GSOR-HSS ($\alpha = 1$)	ω	1.00	1.00	1.00	0.90	0.95
	IT	17	20	21	22	20
	CPU	0.0245	0.4022	2.5304	11.1034	70.2414
	ERR	7.9522e-07	5.8675e-07	7.0669e-07	8.1115e-07	9.4459e-07

GSOR-Seidel and GSOR-HSS ($\alpha = 1$) iteration methods to solve Example 4.1 are plotted on Figures 1 and 2, respectively.

For different values of l, the optimal parameter ω was selected. Table 1 lists the number of iterations, CPU time and residuals for each method. The numerical results show that the number of iterations of the GSOR method is significantly lower than that of other methods, and the GSOR-Seidel iterative method always has the shortest computing time, demonstrating higher computational efficiency.

Example 4.2. Consider CCAVE (1.1) associated with

$$\mathcal{L} := \mathcal{L}_{\theta_1}^{l^2} \times \mathcal{L}_{\theta_2}^{l^2} \times \cdots \times \mathcal{L}_{\theta_l}^{l^2} \quad \text{and} \quad \theta_i = \frac{\pi}{4} + (-1)^i \frac{\pi}{8}, \quad i = 1, 2, \ldots, l,$$

$$A = T_x \otimes I \otimes I + I \otimes T_y \otimes I + I \otimes I \otimes T_z \in \mathbb{R}^{l^3 \times l^3}.$$

Here $\otimes$ denotes the Kronecker product and T_x, T_y, T_z are tridiagonal matrices given by

$$T_x = \text{tridiag}(t_2, t_1, t_3), \quad T_y = \text{tridiag}(t_2, 0, t_3) \quad \text{and} \quad T_z = \text{tridiag}(t_2, 0, t_3).$$

Let $r = qh/2$ with $h = 1/(l+1)$. And we consider the following cases:

Case (1)**.** $t_1 = 8$, $t_2 = -1 - r$, $t_3 = -1 + r$.

Case (2)**.** $t_1 = 6 + 6r$, $t_2 = -1 - 2r$, $t_3 = -1$.

Let $B = \dfrac{1}{10}I$, $x^* = (-1, 2, -3, 4, -5, \ldots, (-1)^{l^3} l^3)^{\mathrm{T}} \in \mathbb{R}^{l^3}$ and $b := Ax^* + B|x^*|$.

For Example 4.2, we compare GSOR, GSOR-Jacobi, GSOR-Seidel and GSOR-HSS ($\alpha = 1$) iteration methods. Figures 3 and 4 depicts the norm of the matrix T and iteration numbers with the parameter ω of these iteration methods. From those subfigures, we can see that the iteration methods may converge to the solution when $\|T\| \geq 1$.

In Tables 2-5, for different size l and parameter q, we list the iteration steps, the CPU time, and the residual norms with respect to the generalized SOR-like iteration methods for Example 4.2. The quasi-optimal parameters are also listed in the corresponding tables, which are chosen as the best ones experimentally. These numerical results show that all tested methods can quickly compute a satisfactory approximation solution. Moreover, we can see that GSOR-Seidel iteration method shows much higher computing efficiency than other the iteration methods.

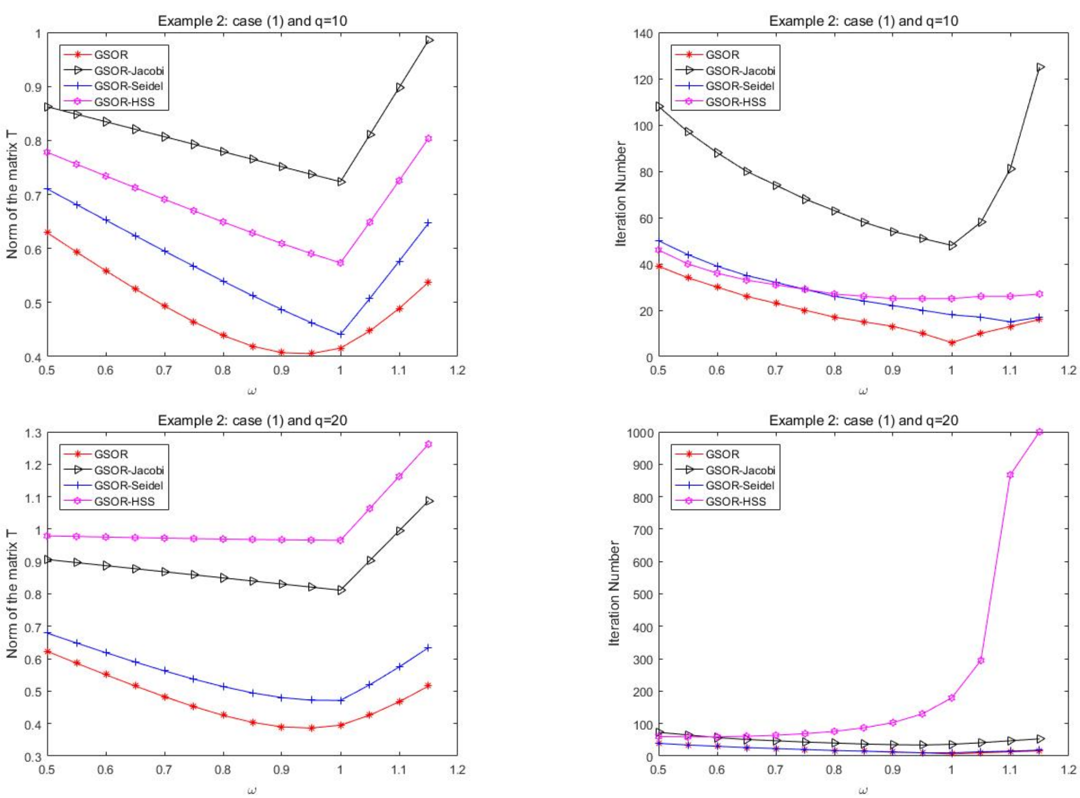

Figure 3. Norm of the matrix T (left) and iteration numbers (right) with the parameter ω of the iteration methods for Example 4.2 with Case (1) and $l = 8$.

Table 2. Numerical results of Example 4.2 with Case (1) and $q = 10$.

Method	l	8	16	24	32	40
	ω	1.00	1.00	1.00	1.00	1.00
	IT	6	7	8	8	8
GSOR	CPU	0.0156	0.2748	1.5079	7.2162	21.8959
	ERR	6.0069e-07	2.2314e-07	2.5548e-08	9.3294e-08	2.5482e-07
	ω	1.00	1.00	1.00	1.00	1.00
	IT	48	71	78	82	86
GSOR-Jacobi	CPU	0.0153	0.0656	0.1924	0.4268	1.5391
	ERR	6.0036e-07	7.4529e-07	8.6402e-07	9.6271e-07	7.5043e-07
	ω	1.10	1.15	1.15	1.15	1.15
	IT	15	26	31	35	37
GSOR-Seidle	CPU	0.0050	0.0245	0.0754	0.1889	0.9474
	ERR	6.3849e-07	4.7942e-07	8.3206e-07	5.7052e-07	8.0330e-07
	ω	1.00	1.15	1.15	1.15	1.15
GSOR-HSS	IT	25	17	17	18	18
($\alpha = 1$)	CPU	0.0411	0.2018	0.9698	4.6729	13.9632
	ERR	7.5173e-07	4.7562e-07	3.5270e-07	2.7587e-07	6.5505e-07

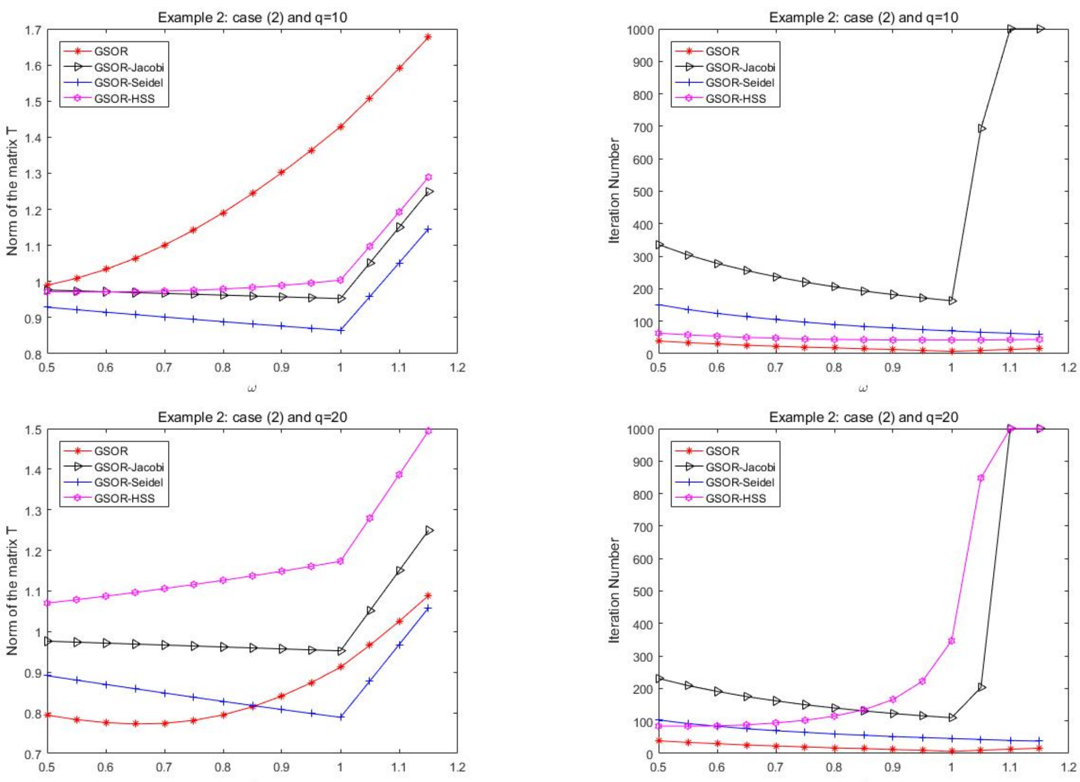

Figure 4. Norm of the matrix T (left) and iteration numbers (right) with the parameter ω of the iteration methods for Example 4.2 with Case (2) and $l = 8$.

Table 3. Numerical results of Example 4.2 with Case (1) and $q = 20$.

Method	l	8	16	24	32	40
GSOR	ω	1.00	1.00	1.00	1.00	1.00
	IT	6	7	8	8	8
	CPU	0.0198	0.2815	1.5066	7.2162	21.5379
	ERR	5.4210e-07	2.1982e-07	2.6324e-08	9.3294e-08	2.6080e-07
GSOR-Jacobi	ω	0.95	1.00	1.00	1.00	1.00
	IT	34	66	78	82	87
	CPU	0.0122	0.0588	0.1903	0.4268	1.5473
	ERR	9.8932e-07	9.9887e-07	9.2203e-07	9.6271e-07	7.6217e-07
GSOR-Seidle	ω	1.00	1.10	1.15	1.15	1.15
	IT	10	18	24	35	33
	CPU	0.0035	0.0170	0.0602	0.1889	1.1391
	ERR	5.9835e-07	4.9983e-07	8.8095e-07	5.7052e-07	4.8416e-07
GSOR-HSS ($\alpha = 1$)	ω	0.55	1.00	1.05	1.15	1.15
	IT	59	29	22	18	19
	CPU	0.1055	0.3429	1.2637	4.6729	19.0462
	ERR	8.7818e-07	7.9624e-07	8.4064e-07	2.7587e-07	3.0678e-07

Table 4. Numerical results of Example 4.2 with Case (2) and $q = 10$.

Method	l	8	16	24	32	40
GSOR	ω	1.00	1.00	1.00	1.00	1.00
	IT	7	9	11	13	15
	CPU	0.0490	0.3859	2.1109	11.7558	41.8952
	ERR	1.4766e-07	5.7752e-07	6.3320e-07	5.2020e-07	4.1148e-07
GSOR-Jacobi	ω	1.00	1.00	—	—	—
	IT	163	522	—	—	—
	CPU	0.0652	0.4894	—	—	—
	ERR	9.4991e-07	9.8877e-07	—	—	—
GSOR-Seidle	ω	1.35	1.50	1.55	1.55	1.55
	IT	49	152	312	541	836
	CPU	0.0182	0.1482	0.7957	2.9903	8.9076
	ERR	8.3114e-07	9.4683e-07	9.8972e-07	9.8405e-07	9.8290e-07
GSOR-HSS ($\alpha = 1$)	ω	1.00	1.40	1.60	1.70	1.70
	IT	42	34	84	148	238
	CPU	0.0763	0.4212	4.8953	37.2411	173.1226
	ERR	6.1121e-07	6.6188e-07	8.2151e-07	8.9313e-07	9.7361e-07

5. Concluding remarks

We have introduced a class of generalized SOR-like iteration methods for solving the absolute value equation associated with circular cones. This approach is derived by equivalently reformulating the CCAVE as a two-by-two block nonlinear equation. Furthermore, we have made several novel observations regarding circular cones. Leveraging these insights and a newly developed convergence analysis technique, we have established sufficient conditions for the convergence of

Table 5. Numerical results of Example 4.2 with Case (2) and $q = 20$.

Method	l	8	16	24	32	40
GSOR	ω	1.00	1.00	1.00	1.00	1.00
	IT	6	9	10	12	13
	CPU	0.0181	0.3462	1.9594	10.4786	34.6034
	ERR	8.6458e-07	3.9429e-08	4.8689e-07	1.6076e-07	3.5629e-07
GSOR-Jacobi	ω	1.00	1.00	1.00	1.00	—
	IT	110	304	585	953	—
	CPU	0.0402	0.2809	1.4531	4.9564	—
	ERR	8.2288e-07	9.9003e-07	9.5589e-07	9.6887e-07	7.6217e-07
GSOR-Seidle	ω	1.30	1.45	1.50	1.55	1.55
	IT	32	86	167	269	404
	CPU	0.0141	0.0801	0.4212	1.4320	5.0139
	ERR	8.2044e-07	9.2339e-07	8.8030e-07	9.3161e-07	9.8732e-07
GSOR-HSS ($\alpha = 1$)	ω	0.55	0.80	1.25	1.55	1.55
	IT	84	81	61	61	100
	CPU	0.1458	0.9717	3.6481	15.0669	73.4512
	ERR	8.1851e-07	7.7101e-07	8.2311e-07	9.3980e-07	8.1270e-07

the proposed iteration methods. Numerical experiments demonstrate that the proposed methods are computationally feasible and effective.

However, the numerical results reveal that the iteration methods may still converge to the solution even when the spectral norm of T is greater than or equal to 1. Therefore, further investigation is required to derive a more precise sufficient condition for convergence.

Declarations

Ethical approval. This manuscript does not contain any studies with human participants or animals performed by any of the authors.

Availability of supporting data. Enquiries about data availability should be directed to the authors.

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