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APPLICATIONS OF SOLITARY WAVE SOLUTIONS AND CONSERVED FLOWS OF A GENERALIZED GEOPHYSICAL KDV MODEL WITH NONLINEAR DUAL POWER LAW IN GEOPHYSICS AND MARINE SCIENCE

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Abstract This paper analytically examines a generalized geophysical Korteweg-de Vries model with a nonlinear power-law in ocean science. To begin with, we apply Lie symmetry analysis in generating the point symmetries of the equation. This further leads to the model being simplified to a nonlinear ordinary differential equation. Thus, for the very first time as far as we know, we attain diverse solitary wave solutions for the model. Initially, a direct integration technique is adopted to obtain solutions to the equation. Additionally, we obtain more broadly defined exact solutions for the generalized geophysical Korteweg-de Vries model. This is achieved via an extended Jacobi expansion approach with elliptic functions. This is a widely recognized technique for achieving closed-form solutions to evolution equations. As a result, one obtains different cnoidal, snoidal, and dnoidal wave solutions to the less-explored model. The tabulated copolar trio illustrates that these solutions can revert to different hyperbolic and trigonometric functions given specific conditions. Furthermore, various graphical representations of the dynamic characteristics of the obtained results are shown. This is done to achieve a clear comprehension of the physical phenomena of the fundamental model. In the latter section, conserved vectors related to the aforementioned model are obtained using the standard multiplier method as well as Noether's theorem. We adopt Lie symmetry analysis in studying the dual power version of the KdV-type equation for the first time. Besides, the inclusion of Noether theorem as well as multiplier technique makes it novel compared to some other forms of work earlier done.

Keywords Generalized geophysical KdV with dual power law, symmetry analysis, analytical solutions, extended Jacobi elliptic function expansion approach, conserved vectors.

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1. Introduction

It is noted that we encounter numerous complex physical phenomena exhibiting nonlinearity in our environment. Nonlinear partial differential equations (NLNPDEEQs) effectively represent these occurrences, encompassing areas such as epidemiology, population ecology, fluid mechanics, plasma physics, nonlinear circuitry as well as biology. To completely understand these phenomena, it is essential to find answers to the differential equations (DEs) that govern them.

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Consequently, this requires the investigation of solitary wave solutions of these NLNPDEEQs in precise structure. Ongoing extensive research is being carried out on these equations, given their vital importance in representing relationships among different physical quantities observed in nature and human-made constructs. Recent advances in computer technology have greatly improved our ability to develop algorithms for solving NLNPDEEQs. Many of the solution appear in the form of solitary wave solutions [1, 8, 15, 16, 41, 50, 72]. Despite this progress, it is crucial to acknowledge the brilliant minds that have laid the theoretical groundwork for these technologies to succeed.

Lately, many researchers passionate about nonlinear physical effects have investigated accurate solutions of NLNPDEEQs because of their importance in evaluating model results. Examination of analytical results of NLNPDEEQs is essential for comprehending particular physical situations. The spectrum of answers to NLNPDEEQs occupies an important role across various scientific disciplines. These encompass optical fibers, chemical physics, electromagnetic theory, plasma physics, meteorology, heat transfer, hydrodynamics, geochemistry as well as chemical kinetics.

Furthermore, recognizing that many prominent scientists view nonlinear science as an essential frontier for understanding nature more deeply, we introduce relevant models, including the 3D generalized nonlinear potential model YTSF equation in Physics and Engineering, which the authors recently investigated in [10]. Writers explored a different type of generalized NLNPDEEQs referred to as a nonlinear advection-diffusion equation relevant to fluids with power law in [11]. Depictions of motions of floatability-driven plumes in a curved medium is modelled by the equation. Moreover, a research examining generalized version of a KdV-ZK equation as outlined by the authors in [51] was entrenched. This model was employed to examine the blending of warm isentropic fluid alongside cold static components as well as hot isothermal materials within fluid dynamics. Additionally, a study in a different resource concentrated on the adjusted and generalized ZK model, emphasizing waves of ion-acoustic solitary results discovered in a magnetized plasma environment containing electron positron ion particles found in an inherent universe [68].

Furthermore, studies investigated vector solitons (bright) as well as their interactions within the coupled Fokas-Lenells framework in [54]. Inquiries into additionally encompassed femtosecond pulses (optical) integrated within bi-refractive fibers (optical), characterized through NLNPDEEQs modeling. Furthermore, focus was directed towards Boussinesq-Burgers-type model in [52], characterizing waves in shallow waters close to lakes as well as coastlines. Additionally the texts, cite other relevant results for additional investigation. Publications mentioned in [6, 7, 9, 10, 12, 22, 24, 33, 50–52, 54] are available for exploring additional applications of NLNPDEEQs in different manners.

Moreover, it is observed that after thorough investigation, it has been concluded that no singular method exists for attaining exact solutions to NLNPDEEQs. Nevertheless, to tackle this ongoing problem, mathematicians and physicists have created various efficient methods. For example, Sophus Lie (1842–1899) played a crucial role in the discipline through his research on Lie Algebras [43, 44], offering a cohesive method for addressing various differential equations. Recent progress in addressing DEs encompasses modern group analysis [43, 44], Cole-Hopf transformation approach [48], (G'/G)-expansion technique [56], Kudryashov's approach [34, 35], Hirota's bilinear approach [37] as well as power series solution technique [23]. Besides, others involve tanh-coth technique [59], bifurcation approach [13, 66], extended homoclinic test technique [20], simplest equation approach [55], Adomian decomposition technique [57], Bäcklund transformation method [26], rational expansion approach [65], homotopy perturbation [19], sine-Gordon equation expansion technique [18], Darboux transformation approach [67], Painlevé expansion technique [63], F-expansion approach [71], and numerous others have existed.

Since Petviashvili and Kadomtsev developed their hierarchy equation models over five decades ago, many articles have existed and released, each exploring various facets of this intricate area of equations. For example, refer to [21, 32, 36, 39, 40, 62, 69] for additional insights on the issue mentioned.

Among these fascinating framework is well-known Korteweg-de Vries model [61]

$$\Theta_t + \nu\Theta\Theta_x + \sigma\Theta_{xxx} = 0, \;\; \nu, \sigma \neq 0, \tag{1.1}$$

frequently referred to as "KdV". Model (1.1) has garnered attention for ages because of its relevance to different physical happenings. Existence of other versions of (1.1), there are, featuring generalized (power law) as well as modified forms, as indicated in [58, 61, 64]

$$\Theta_t + p\Theta^2\Theta_x + q\Theta_{xxx} = 0 \tag{1.2}$$

and

$$\Theta_t + p\Theta^n\Theta_x + \alpha\Theta^{2n}\Theta_x + q\Theta_{xxx} = 0 \tag{1.3}$$

with constants p, q, α alongside n being real numbers that are not zero. For numerous years, KdV and KdV-associated frameworks along with their solitary waves have been a primary focus of thorough investigation because of their importance in representing various physical scenarios. Currently, numerous articles focus on KdV and KdV-related models along with their solitary waves, particularly the mathematical principles that regulate these kinds of model equations are a common subject of current research. For example, in [61], the writer investigated equations (1.1)–(1.3), where he presented novel methods mixing duo hyperbolic functions to analyze them. Eventually, the study showed that this type of equations yields classical solitons and periodic solutions.

The proposed schemes were also shown to create entirely new groups of solitary wave solutions in addition to the traditional ones. Wazwaz in [61] subsequently suggested that the analysis could be relevant to a wide variety of evolution models (nonlinear). Additionally, in [64], Yan explored (1.2) with both (+) and (−) (focusing and defocusing respectively) branches, where numerous new varieties of binary traveling-wave periodic solutions were derived for the equation using Jacobi elliptic functions, including sine, delta, and cosine amplitude solutions along with their extensions. Moreover, the asymptotic characteristics of several identified solutions were examined. Moreover, utilizing the Miura transformation, Yan also provided the related binary travelling-wave solutions (periodic) of (1.2). Additionally, in reference [58], Wazwaz explored the generalized format of KdV equation featuring power nonlinearities (dual) exhibited as (1.3) in his study. Tanh approach and two sets of ansatze employing hyperbolic functions were introduced for the analytical investigation of the equation. New varieties of solitary wave results were formally derived.

In the same vein, various other generalized forms of KdV equations with nonlinear power-laws have been studied. In [60], Wazwaz examined a fifth-order KdV model with dual power describing motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice which reads as

$$\Theta_t + p\Theta^n\Theta_x + q\Theta^n\Theta_{xxx} + \Theta_{xxxxx} = 0, \tag{1.4}$$

in which $n \geq 1$. The author made an analytic study on the equation in which the tanh method and a sinh-cosh functions ansatz were used. A set of entirely new solitons and periodic solutions was established. The study introduced new ansatz to handle NLNPDEEQs in the solitary wave theory. The nonlinear generalized KdV equation is an important mathematical model with wide applications in quantum mechanics and nonlinear optics. Typical examples are widely used in various fields such as solid state physics, plasma physics, fluid physics and quantum field theory.

Subsequently, Geyer and Quirchmayr introduced another variant of KdV equation in [27,30], referring to it as the geophysical KdV equation

$$\Theta_t - w_0\Theta_x + \frac{3}{2}\Theta\Theta_x + \frac{1}{6}\Theta_{xxx} = 0 \tag{1.5}$$

with the parameter w_0 representing Coriolis effect. Already obtained equation (1.5) is invoked to investigate propagation of waves in ocean. This has garnered significant attention from academic researchers. As an illustration, in [30], Karunakar et al. explored influence of the Coriolis (which is constant) on results of geophysical KdV (1.5). From that analysis, it was found that constant associated to Coriolis has a direct correlation with wave height and an inverse relationship with wavelength. The addition of term of Coriolis in the model results in a notable change in the structure of the outcome. Furthermore, in [47], the writers achieved a lump soliton result for (1.5) by employing the Hirota bilinear procedure. They also achieved lump-kink solitons (resulting from the interplay of a lump with a kink soliton outcome), lump solutions (periodic) formed by the interaction between periodic waves and a lump, and lump-kink solutions (periodic) which emerge from the periodic waves collisions with both a lump and a another soliton (kink). Behaviour of these solutions was analyzed pictorially by choosing key parameters. Additionally, in [14], researchers attained multiple new solitary results to (1.5). The solution acquired from applying the shooting method served effectively as an initial value for the adaptive procedure utilized to create the numerical results of the problem. The derived exact solutions matched the acquired numerical solutions. Moreover, they utilized the Fourier concept to investigate the accuracy and stability of the numerical schemes, which they indicated to be unconditionally stable. Recently, an extended form of the model was investigated in [4], where the author derived multiple exact solutions and conservation laws that are significant aspects in physical sciences. In [27], the authors examined a broader form of (1.5) as

$$\Theta_t - w_0\Theta_x + \frac{3}{2}\Theta\Theta_x + \frac{1}{6}\Theta_{xxx} = -\left(\beta_0 + \beta_1\Theta + \beta_2\Theta^2 + \cdots + \beta_m\Theta^m\right), \tag{1.6}$$

where they officially presented a source, which is a polynomial (with degree m) in the involved function (unknown), within the model. Painlevé analysis indicates that (1.6) with the basis being non-integrable. Under certain essential integrability criteria, multiple kink-type waves (solitary) for specific instances of the controlling model for which $m = 2$ and $m = 4$ are obtained utilizing well-proportioned Kudryashov's technique.

Recently, a general version of (1.5) was examined in [5] in the structure of

$$\Theta_t + a\Theta_x + b\Theta^n\Theta_x + c\Theta_{xxx} = 0, \tag{1.7}$$

in which $a, b, c \neq 0$ are real constants. The author in the referenced paper utilized ansatz. Thereby, transformation of the equation was enhanced into an ordinary differential equation. Subsequently, some standard approaches were invoked to secure various analytic solutions of interest. Besides, conservation laws of (1.7) was also constructed through the use of multiplier technique.

After reviewing the literature, our study investigates generalized version of (1.5), depicted here as gGKdVe, reading

$$\Delta \equiv v_t + av_x + bv^n v_x + cv^{2n} v_x + dv_{xxx} = 0, \tag{1.8}$$

with the representation $\Theta = v$ (v depending on t and x) is made with real numbers $a, b, c, d \neq 0$. It is straightforward to observe that one derives (1.8) from (1.5) if $n = 1$, $a = -w_0$, $b = 3/2$, $c = 0$ and $d = 1/6$. Consequently, latter serves as a broader version of former, suggesting latter includes former.

We declare here that specifically, this is the very first time in which model (1.8) is being investigated via the Lie symmetry analysis standpoint. Although, various other versions of KdV-type model with power laws have been studied as earlier pointed out. This work stands out with the inclusion of Noether approach viz-a-viz multiplier method in getting the conserved vectors associated with the equation. Thus, this work is novel and original compared to other studies earlier conducted regarding the model.

In contrast to the earlier studies discussed in this article, it is unequivocally stated that the research presented here is original. Indeed, the research conducted in the references represents a specific instance of the work presented here. Moreover, the laws of conservation are both universal and distinctive. This suggests that this study is broader and more thorough.

This study provides the implicit outcomes of g-GKdVe (1.8). The structure of the article is as follows; In section 2, an analysis of Lie symmetry for the NLNPDEQ model (1.8) is conducted, leading to a reduction of the equation to a nonlinear ordinary differential equation (NLNODE). The NLNODE is directly integrated. Furthermore, a standard method (extended Jacobi elliptic expansion technique) is employed to obtain general solutions of (1.8) expressed as elliptic cosine, sine, and delta functions. In section 3, the calculated structures of the model's conservation laws are further presented using the multiplier technique. Final comments are detailed in section 4.

2. Symmery analysis and solutions of (1.8)

We start by obtaining the Lie point symmetries of g-GKdVe (1.8), and then we use these symmetries to find exact solutions.

2.1. Lie point symmetries of equation (1.8)

The vector field that will constitute the symmetry group of g-GKdVe (1.8) is given as

$$Q = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \varphi \frac{\partial}{\partial v}$$

with functions (ξ^1, ξ^2, φ) depending on (t, x, v), is a Lie point symmetry of g-GKdVe (1.8) if

$$Q^{[3]} \left[v_t + av_x + bv^n v_x + cv^{2n} v_x + dv_{xxx} = 0 \right] \Big|_{\Delta} = 0 \tag{2.1}$$

where $Q^{[3]}$ connotes Q in the third extension form defined as

$$Q^{[3]} = Q + \zeta_t \frac{\partial}{\partial v_t} + \zeta_x \frac{\partial}{\partial v_x} + \zeta_{xx} \frac{\partial}{\partial v_{xx}} + \zeta_{xxx} \frac{\partial}{\partial v_{xxx}}, \tag{2.2}$$

with the ζ's given as

$$\zeta_t = D_t(\varphi) - v_t D_t(\xi^1) - v_x D_t(\xi^2), \tag{2.3}$$

$$\zeta_x = D_x(\varphi) - v_t D_x(\xi^1) - v_x D_x(\xi^2), \tag{2.4}$$

$$\zeta_{xx} = D_x(\zeta_x) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2), \tag{2.5}$$

$$\zeta_{xxx} = D_x(\zeta_{xx}) - v_{txx} D_x(\xi^1) - v_{xxx} D_x(\xi^2), \tag{2.6}$$

in which involved total derivatives are presented as

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x} + v_{txx} \frac{\partial}{\partial v_{xx}} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v} + v_{xx} \frac{\partial}{\partial v_x} + v_{tx} \frac{\partial}{\partial v_t} + v_{xxxx} \frac{\partial}{\partial v_{xxx}} + \cdots. \end{aligned} \tag{2.7}$$

Through the expansion of equation (2.1) and the separation of its components according to the derivatives of function v, we can attain the subsequent system of linear partial differential equations (overdetermined):

$$\xi^1_t = 0, \quad \xi^1_v = 0, \quad \xi^1_x = 0, \quad \xi^2_t = 0, \quad \xi^2_v = 0, \quad \xi^2_x = 0, \quad \varphi = 0,$$

which can be easily be solved by hand and so the solution to the system yields

$$\xi^1(t, x, v) = B_1 \text{ and } \xi^2(t, x, v) = B_2$$

with B_1 and B_2 as constants (arbitrary). Thus, consequential symmetries are translational, given as

$$Q_1 = \frac{\partial}{\partial t} \text{ and } Q_2 = \frac{\partial}{\partial x}. \tag{2.8}$$

Next, we combine the two symmetries linearly with the purpose of gnerating the travelling wave and obtaining exact solutions.

2.2. Traveling wave solutions of g-GKdVe (1.8)

On considering a combination (linear) of the obtained symmetries (translation), i.e, Q_1 and Q_2 for g-GKdVe (1.8) as; $Q = Q_1 + \nu Q_2$. The symmetry produces the two invariants entrenched as

$$\zeta = x - \nu t \quad \text{and} \quad U = v,$$

yielding group-invariant result $U = U(\zeta)$ with new independent variable ζ. One makes use of the above and successfully transforms model (1.8) to nonlinear ordinary differential equation (NLNODE) of third-order, viz

$$aU'(\zeta) + bU'(\zeta)U(\zeta)^n + cU'(\zeta)U(\zeta)^{2n} + dU'''(\zeta) - \nu U'(\zeta) = 0. \tag{2.9}$$

Remark 2.1. It is to be noted very importantly that, one could have utilized a wave transformation directly in form of an ansatz (that is in a pre-known solution format) instead of finding and associated symmetries and then combining the symmetries but this investigation poised to generate the wave transformation systematically and then apply it eventually.

2.3. Solution of (1.8) through direct integration

This part of the study furnishes general soliton solutions of (1.8) in two parts where bright soliton as well as periodic wave solutions are secured for the general case n. In order that one might secure a result of (1.8), direct integration of NLNODE (2.9) is adopted. Therefore, integrating (2.9) furnishes

$$aU(\zeta) + \frac{b}{n+1}U(\zeta)^{n+1} + \frac{c}{2n+1}U(\zeta)^{2n+1} + dU''(\zeta) - \nu U(\zeta) + A_0 = 0, \tag{2.10}$$

where $n \neq -1, -1/2$ and arbitrary A_0 represents integration constant. Moreover, we adopt some certain measures and repeat the integrating process, thus leading to

$$\begin{aligned}&\frac{1}{2}aU(\zeta)^2 + \frac{b}{(n+1)(n+2)}U(\zeta)^{n+2} + \frac{c}{(2n+1)(2n+2)}U(\zeta)^{2n+2} + \frac{d}{2}U'^2(\zeta)\\ &-\frac{1}{2}\nu U(\zeta)^2 + A_0 U(\zeta) + A_1 = 0\end{aligned} \tag{2.11}$$

with arbitrary integration constant A_1, where $n \neq -1/2, -1, -2$. On solving (2.11), we achieve the result

$$\begin{aligned}&\int_1^{u^n} \left\{ \Big[\Omega^2 \left(-\left(\Omega^n \left(c(n+2)\Omega^n + b(4n+2)\right) + a\left(2n^3+7n^2+7n+2\right)\right.\right.\right.\\ &\left.\left. -\nu\left(2n^3+7n^2+7n+2\right)\right)\right) - 2A_0\left(2n^3+7n^2+7n+2\right)\Omega\\ &\left. -2A_1\left(2n^3+7n^2+7n+2\right)\Big]^{-\frac{1}{2}}\right\} d\Omega\\ &= \pm\frac{x-\nu t}{\sqrt{d\left(2n^3+7n^2+7n+2\right)}} + \mathfrak{B}_0,\end{aligned} \tag{2.12}$$

where arbitrary $\mathfrak{B}_0$ stands for an integration constant.

2.3.1. Soliton solutions of (2.9)

In this part, we further the course, by contemplating the outcome of the NLNODE that produced the implicit solution (2.12). Thus, we achieve

$$\begin{aligned}&\frac{1}{2}aU(\zeta)^2 + \frac{b}{n^2+3n+2}U(\zeta)^{n+2} + \frac{c}{4n^2+6n+2}U(\zeta)^{2n+2} + \frac{d}{2}U'^2(\zeta)\\ &-\frac{1}{2}\nu U(\zeta)^2 = 0.\end{aligned} \tag{2.13}$$

Suppose we allow $U(\zeta) = \Theta^{1/n}(\zeta)$, then (2.13) transforms to

$$\frac{2b}{n^2+3n+2}\Theta(\zeta)^3 + \frac{2c}{4n^2+6n+2}\Theta(\zeta)^4 + \frac{d}{n^2}\Theta'^2(\zeta) - (\nu - a)\,\Theta(\zeta)^2 = 0. \tag{2.14}$$

The reorganization of the newly obtained result (2.14) produces

$$\Theta'^2(\zeta) = L\Theta^2(\zeta) - M\Theta^3(\zeta) - N\Theta^4(\zeta), \tag{2.15}$$

where one has

$$L = \frac{1}{d}\left\{(\nu - a)n^2\right\}, \ M = \frac{2bn^2}{d(n+1)(n+2)}, \ N = \frac{2cn^2}{d(2n+1)(2n+2)}. \tag{2.16}$$

Moreover, if one explicates the result of NLNODE (2.15) in the format

$$P(\zeta) = \frac{1}{\lambda + S(\zeta)}. \tag{2.17}$$

Replacement of P value, found from (2.17) into NLNODE (2.15), one secures

$$\begin{aligned} &S'^2(\zeta) - (L\lambda^2 - M\lambda^3 - N\lambda^4)S^4(\zeta) - (2L\lambda - 3M\lambda^2 - 4N\lambda^3)S^3(\zeta) \\ &- (L - 3M\lambda - 6N\lambda^2)S^2(\zeta) + (M + 4N\lambda)S(\zeta) + N = 0. \end{aligned} \tag{2.18}$$

Equation of diverse coefficients concerning variables S^4 and S^2 to zero procures the system

$$\begin{aligned} &L\lambda^2 - M\lambda^3 - N\lambda^4 = 0, \\ &L - 3M\lambda - 6N\lambda^2 = 0, \end{aligned} \tag{2.19}$$

whose outcome engenders the resulting parameters (constant) N as well as λ as

$$\lambda = \frac{5L}{3M} \quad \text{and} \quad N = -\frac{6M^2}{25L}. \tag{2.20}$$

On engaging the obtained outcomes, occasions (2.18) to transform to

$$S'^2 = -\frac{5L^2}{9M}S^3 + \frac{3M}{5}S + \frac{6M^2}{25L}. \tag{2.21}$$

One now takes the product of (2.21) alongside $-36M/5L^2$, and arrives at

$$G'^2 = 4G^3 - S_2 G - S_3, \tag{2.22}$$

where, we have the portrayal explicated as

$$G(w) = S(\zeta), \quad w = \frac{L}{6}\sqrt{\frac{-5}{M}}\zeta \quad \text{with} \quad S_2 = \frac{108M^2}{25L^2} \quad \text{and} \quad S_3 = \frac{216M^3}{125L^3} \tag{2.23}$$

with S_2 and S_3 connoting elliptic invariants, one observes in this instance that a general solution of (2.22) could be delineated regarding Weierstrass elliptic function, see [31]

$$G(w) = \wp(w; S_2, S_3). \tag{2.24}$$

Therefore, a general solution to (2.15) expresses here as

$$P(\zeta) = \left\{\wp\left(\frac{L}{6}\sqrt{\frac{-5}{M}}\zeta; S_2, S_3\right)\right\}^{-1} + \frac{5L}{3M}. \tag{2.25}$$

Sequel to that, one has the soliton solutions formatted as

$$P(\zeta) = 5L\left\{M\left[3 - \cosh^2\left(\frac{\sqrt{L}}{2}\zeta\right)\right]\right\}^{-1}, \ P(\zeta) = 5L\left\{M\left[3 + \sinh^2\left(\frac{\sqrt{L}}{2}\zeta\right)\right]\right\}^{-1} \tag{2.26}$$

with parameter $L > 0$. Consequently, retrograding to original variables (t, x), solutions to nonlinear model (1.8) with dual-power law, formats as general bright soliton solutions

$$v(t,x) = \left\{ \frac{5(n+1)(n+2)(\nu - a)}{2b\left(3 - \cosh^2\left[\frac{1}{2}\sqrt{\frac{1}{d}\left\{(\nu-a)n^2\right\}}\,(x - \nu t)\right]\right)} \right\}^{1/n}, \tag{2.27}$$

$$v(t,x) = \left\{ \frac{5(n+1)(n+2)(\nu - a)}{2b\left(\sinh^2\left[\frac{1}{2}\sqrt{\frac{1}{d}\left\{(\nu-a)n^2\right\}}\,(x - \nu t)\right] + 3\right)} \right\}^{1/n}, \tag{2.28}$$

where there is constraint criteria $5(n+2)(\nu - a)(n+1) \neq 0$ and $d(\nu - a) > 0$.

After successfully presenting several general solitary wave solutions of g-GKdVe (1.8), the subsequent step is to provide additional periodic solutions of (1.8) by employing a standard technique for certain specific values of n.

Remark 2.2. We note here that adopting the direct integration of the contemplated NLNODE (2.9), hyperbolic solutions can be obtained for some special cases of the involved constants of integration. These solutions are of immense significance in science and engineering fields.

2.4. Periodic soliton solutions of g-GKdVe (1.8) via extended Jacobi elliptic function expansion technique

This section, introduces the analytical solutions of equation (1.8) by utilizing Jacobi elliptic function procedure in an extended structure, described in earlier research. Refer to sources [3,31].

The aim is to find precise periodic solutions for equation (1.8) relying on the fundamental copolar trio of Jacobian elliptic functions. The trio consists of the elliptic sine function $\mathrm{sn}(\zeta|\omega)$, the elliptic cosine function $\mathrm{cn}(\zeta|\omega)$, and the delta amplitude function $\mathrm{dn}(\zeta|\omega)$, all derived from the parameter ω where $0 \leq \omega \leq 1$. These three elliptic functions may also be utilized to establish extra Jacobian functions. The significance of Jacobi elliptic functions resides in their ability to transform into trigonometric, hyperbolic, and eventually exponential functions as demonstrated in Table 1. This alteration makes them significant and relevant in various mathematical contexts. Further sources for extra research can be found in references like Abramowitz and Stegun [2], as well as Gradshteyn and Ryzhik [25].

Table 1. Copolar trio for $\omega = 0$ and $\omega = 1$.

	$\omega = 0$	$\omega = 1$
$\mathrm{sn}(\zeta\|\omega)$	$\sin\zeta$	$\tanh\zeta$
$\mathrm{cn}(\zeta\|\omega)$	$\cos\zeta$	$\mathrm{sech}\,\zeta$
$\mathrm{dn}(\zeta\|\omega)$	1	$\mathrm{sech}\,\zeta$

We observe that $R(\zeta)$ satisfies the first-order NLNODE

$$R'(\zeta) + \sqrt{(1 - R^2(\zeta))(1 - \omega + \omega R^2(\zeta))} = 0, \tag{2.29}$$

$$R'(\zeta) - \sqrt{(1 - R^2(\zeta))(1 - \omega R^2(\zeta))} = 0, \tag{2.30}$$

$$R'(\zeta) + \sqrt{(1 - R^2(\zeta))(\omega - 1 + R^2(\zeta))} = 0, \tag{2.31}$$

whose resolutions can be explained in terms of the Jacobi elliptic sine, delta, and cosine amplitude functions, accordingly, as

$$R(\zeta) = \operatorname{cn}(\zeta|\omega), \; R(\zeta) = \operatorname{sn}(\zeta|\omega), \text{ and } R(\zeta) = \operatorname{dn}(\zeta|\omega). \tag{2.32}$$

In addition, if the third-order NLNODE (2.9) possesses a solution structured as

$$U(\zeta) = \sum_{i=-p}^{p} A_i R(\zeta)^i \tag{2.33}$$

with the aim of determining the value of positive integer p by employing a balancing method, refer to [70], consequently, one might consider the following solitary wave solution routes.

2.4.1. Solutions of g-GKdVe (1.8) when n = 1

This section of the paper focuses on calculating the precise periodic solutions of g-GKdVe (1.8) for a specific instance of the equation when $n = 1$. Consequently, we create the solutions using cosine, sine, and delta amplitude functions.

Cosine amplitude function solutions of (1.8)

In this context, we initially consider the NLNODE (2.9) when $n = 1$. Therefore, the balancing process results in $p = 1$, and as a result, (2.33) takes on the form

$$U(\zeta) = A_{-1}R(\zeta)^{-1} + A_0 + A_1 R(\zeta). \tag{2.34}$$

By replacing the value of $U(\zeta)$ from (2.34) into (2.9) in accordance with (2.29), one obtains an algebraic equation. When this equation is divided into different contributing powers of $R(\zeta)$, a set of seventeen algebraic equations is achieved, presented as:

$$\begin{aligned}
&c\omega\, A_1^3 - 6\, d\omega^2 A_1 = 0,\\
&2\, c\omega\, A_0 A_1^2 + b\omega\, A_1^2 = 0,\\
&2\, cA_{-1}^2 A_0 - 2\, c\omega\, A_{-1}^2 A_0 - b\omega\, A_{-1}^2 + bA_{-1}^2 = 0,\\
&4\, c\omega\, A_{-1}^2 A_0 + 2\, b\omega\, A_{-1}^2 - 2\, cA_{-1}^2 A_0 - bA_{-1}^2 = 0,\\
&2\, cA_0 A_1^2 - 4\, c\omega\, A_0 A_1^2 - 2\, b\omega\, A_1^2 + bA_1^2 = 0,\\
&cA_{-1}^3 - c\omega\, A_{-1}^3 + 6\, d\omega^2 A_{-1} - 12\, d\omega\, A_{-1} + 6\, dA_{-1} = 0,\\
&2\, c\omega\, A_0 A_1^2 - 2\, c\omega\, A_{-1}^2 A_0 - b\omega\, A_{-1}^2 + b\omega\, A_1^2 - 2\, cA_0 A_1^2 - bA_1^2 = 0,\\
&c\omega\, A_{-1} A_1^2 + c\omega\, A_0^2 A_1 - 2\, c\omega\, A_1^3 + b\omega\, A_0 A_1 + cA_1^3 + 14\, d\omega^2 A_1 + a\omega\, A_1\\
&\; - 7\, d\omega\, A_1 - \nu\, \omega\, A_1 = 0,\\
&2\, c\omega\, A_{-1}^3 - c\omega\, A_{-1}^2 A_1 - c\omega\, A_{-1} A_0^2 - b\omega\, A_{-1} A_0 - cA_{-1}^3 + cA_{-1}^2 A_1\\
&\; + cA_{-1} A_0^2 - 14\, d\omega^2 A_{-1} - a\omega\, A_{-1} + bA_{-1} A_0 + 21\, d\omega\, A_{-1} + \nu\, \omega\, A_{-1}\\
&\; + aA_{-1} - 7\, dA_{-1} - \nu\, A_{-1} = 0,\\
&c\omega\, A_1^3 - c\omega\, A_{-1}^2 A_1 - c\omega\, A_{-1} A_0^2 - 2\, c\omega\, A_{-1} A_1^2 - 2\, c\omega\, A_0^2 A_1 - b\omega\, A_{-1} A_0\\
&\; - 2\, b\omega\, A_0 A_1 + cA_{-1} A_1^2 + cA_0^2 A_1 - cA_1^3 - 2\, d\omega^2 A_{-1} - 10\, d\omega^2 A_1 - a\omega\, A_{-1}
\end{aligned}$$

$$
\begin{aligned}
&- 2\,a\omega\,A_1 + bA_0A_1 + d\omega\,A_{-1} + 10\,d\omega\,A_1 + \nu\,\omega\,A_{-1} + 2\,\nu\,\omega\,A_1 + aA_1\\
&- dA_1 - \nu\,A_1 = 0,\\
&2\,c\omega\,A_{-1}^2A_1 - c\omega\,A_{-1}^3 + 2\,c\omega\,A_{-1}A_0^2 + c\omega\,A_{-1}A_1^2 + c\omega\,A_0^2A_1 + 2\,b\omega\,A_{-1}A_0\\
&+ b\omega\,A_0A_1 - cA_{-1}^2A_1 - cA_{-1}A_0^2 - cA_{-1}A_1^2 - cA_0^2A_1 + 10\,d\omega^2A_{-1} + 2\,d\omega^2A_1\\
&+ 2\,a\omega\,A_{-1} + a\omega\,A_1 - bA_{-1}A_0 - bA_0A_1 - 10\,d\omega\,A_{-1} - 3\,d\omega\,A_1 - 2\,\nu\,\omega\,A_{-1}\\
&- \nu\,\omega\,A_1 - aA_{-1} - aA_1 + dA_{-1} + dA_1 + \nu\,A_{-1} + \nu\,A_1 = 0.
\end{aligned}
$$

Invoking a software package, the above system of equations solve to give three sets of solutions; viz

$$
\nu = \frac{1}{4c}\left\{8\,cd\omega + 4\,ac - b^2 - 4\,cd\right\},\ A_{-1} = 0,\ A_0 = -\frac{b}{2c},\ A_1 = \pm\sqrt{\frac{6d\omega}{c}}, \tag{2.35}
$$

$$
\begin{aligned}
&\nu = \pm\frac{1}{4c}\left\{24\,c\sqrt{d^2\omega\,(\omega-1)} - 8\,cd\omega - 4\,ac + b^2 + 4\,cd\right\},\\
&A_{-1} = \pm\sqrt{\frac{6\,d\,(\omega-1)}{c}},\ A_0 = -\frac{b}{2c},\ A_1 = \sqrt{\frac{6d\omega}{c}},
\end{aligned} \tag{2.36}
$$

$$
\begin{aligned}
&\nu = \frac{1}{4c}\left\{8\,cd\omega + 4\,ac - b^2 - 4\,cd\right\},\ A_{-1} = \pm\sqrt{\frac{6\,d\,(\omega-1)}{c}},\\
&A_0 = -\frac{b}{2c},\ A_1 = 0.
\end{aligned} \tag{2.37}
$$

Thus, we attain respectively three solution sets of (1.8) related to (2.35)–(2.37) as

$$
v(t,x) = -\frac{b}{2c} \pm \sqrt{\frac{6d\omega}{c}}\,\mathrm{cn}\left(x - \nu t\middle|\omega\right), \tag{2.38}
$$

$$
v(t,x) = \pm\sqrt{\frac{6\,d\,(\omega-1)}{c}}\,\mathrm{nc}^{-1}\left(x - \nu t\middle|\omega\right) - \frac{b}{2c} + \sqrt{\frac{6d\omega}{c}}\,\mathrm{cn}\left(x - \nu t\middle|\omega\right), \tag{2.39}
$$

$$
v(t,x) = -\frac{b}{2c} \pm \sqrt{\frac{6\,d\,(\omega-1)}{c}}\,\mathrm{nc}^{-1}\left(x - \nu t\middle|\omega\right). \tag{2.40}
$$

One presents the wave motion display of cnoidal wave outcomes in Figures 1, 2, 3 and 4.

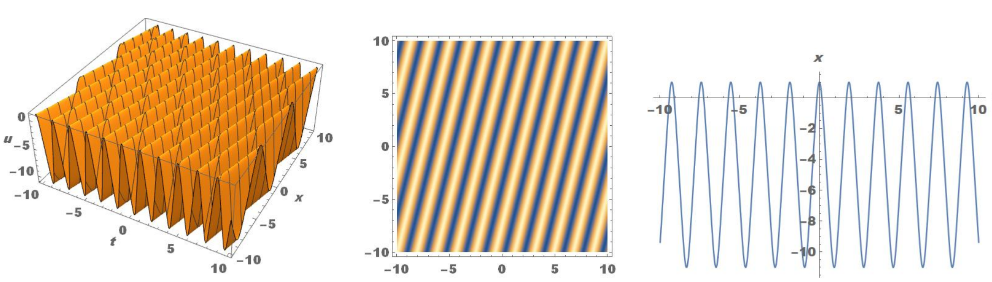

Figure 1. Wave depiction of cnoidal wave solution (2.38) using the assigned values $c = 1$, $b = 10$, $\nu = 4$, $d = 2$, $\omega = 0.5$, where one utilizes the interval $-10 \le t, x, \le 10$. This reveals waves of uniform wavelength and frequency over a wide range of interval.

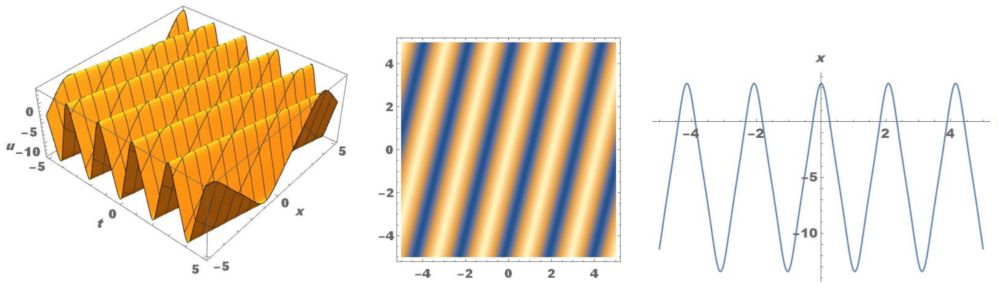

Figure 2. Wave representation of cnoidal wave solution (2.38) using the assigned values $c = 1$, $b = 10$, $\nu = 4$, $d = 2$, $\omega = 0.7$, where one utilizes the interval $-5 \leq t, x, \leq 5$. This reveals waves of uniform wavelength and frequency.

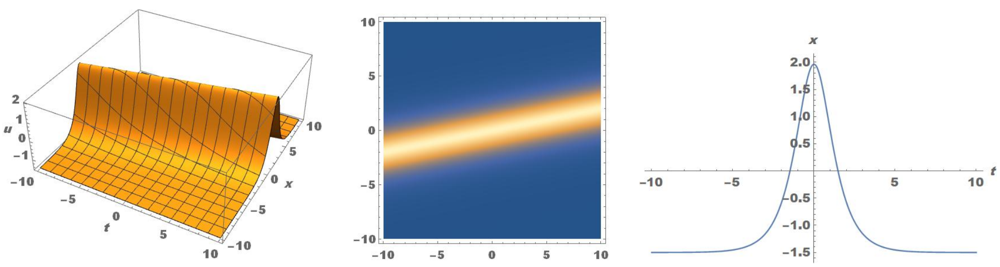

Figure 3. Wave depiction of cnoidal wave solution (2.39) using the assigned values $c = 1$, $b = 3$, $\nu = 0.2$, $d = 2$, $\omega = 1$, where one utilizes the interval $-10 \leq t, x, \leq 10$. This graph is typical of bright soliton: A demonstration that shows a pulse of light that retains its structure as it travels, a result of the equilibrium between optical dispersion and a self-focusing nonlinearity.

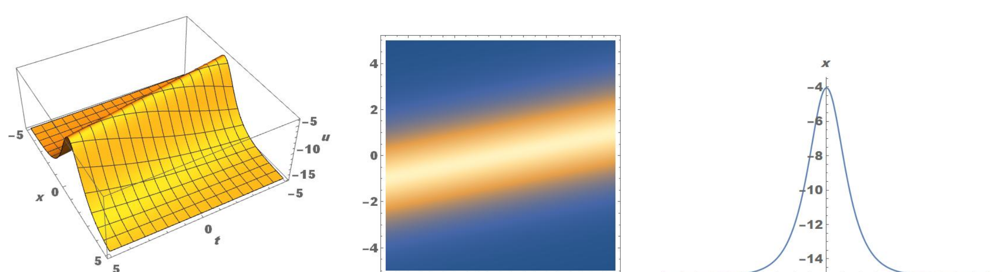

Figure 4. Wave representation of cnoidal wave solution (2.39) using the assigned values $c = 0.1$, $b = 3$, $\nu = 0.2$, $d = 2$, $\omega = 1$, where one utilizes the interval $-5 \leq t, x, \leq 5$. This represents a light pulse that retains its form while traveling, a phenomenon that arises from the equilibrium between optical dispersion and a self-focusing nonlinearity.

Sine amplitude function solutions of (1.8)

This section, aims to attain certain outcomes of NLNODE (2.9) related to the sine amplitude function. Recognizing that the balancing process results in $p = 1$, (2.33) presumes the framework

$$U(\zeta) = A_{-1}R(\zeta)^{-1} + A_0 + A_1R(\zeta). \tag{2.41}$$

By substituting the value of $U(\zeta)$ from (2.34) into (2.9) along with using (2.30), an algebraic equation is obtained. When this equation is divided across different powers of $R(\zeta)$, it results in a system of seventeen equations, yielding three distinct sets of solutions represented as

$$\nu = \frac{1}{4c}\left\{4\,ac - 4\,cd\omega - b^2 - 4\,cd\right\},\ A_{-1} = \pm\sqrt{\frac{-6\,d}{c}},\ A_0 = -\,\frac{b}{2c},\ A_1 = 0, \tag{2.42}$$

$$\nu = \pm\,\frac{1}{4c}\left(24\,c\sqrt{d^2\omega} - 4\,cd\omega + 4\,ac - b^2 - 4\,cd\right),\ A_{-1} = \sqrt{\frac{-6\,d}{c}},$$

$$A_0 = -\,\frac{b}{2c},\ A_1 = \pm\sqrt{\frac{-6\,d\omega}{c}}, \tag{2.43}$$

$$\nu = \frac{1}{4c}\left\{4\,ac - 4\,cd\omega - b^2 - 4\,cd\right\}, A_{-1} = 0,\ A_0 = -\,\frac{b}{2c},\ A_1 = \pm\sqrt{\frac{-6\,d\omega}{c}}. \tag{2.44}$$

Therefore, corresponding to (2.42)–(2.44), we have the set of snoidal solutions

$$v(t,x) = \pm\sqrt{\frac{-6\,d}{c}}\,\mathrm{ns}^{-1}\left(x - \nu t|\omega\right) - \frac{b}{2c}, \tag{2.45}$$

$$v(t,x) = \sqrt{\frac{-6\,d}{c}}\,\mathrm{ns}^{-1}\left(x - \nu t|\omega\right) - \frac{b}{2c} \pm \sqrt{\frac{-6\,d\omega}{c}}\,\mathrm{sn}\left(x - \nu t|\omega\right), \tag{2.46}$$

$$v(t,x) = -\,\frac{b}{2c} \pm \sqrt{\frac{-6\,d\omega}{c}}\,\mathrm{sn}\left(x - \nu t|\omega\right). \tag{2.47}$$

The wave nature depiction of part of the snoidal wave solutions attained here is presented in Figures 5, 6, 7 and 8.

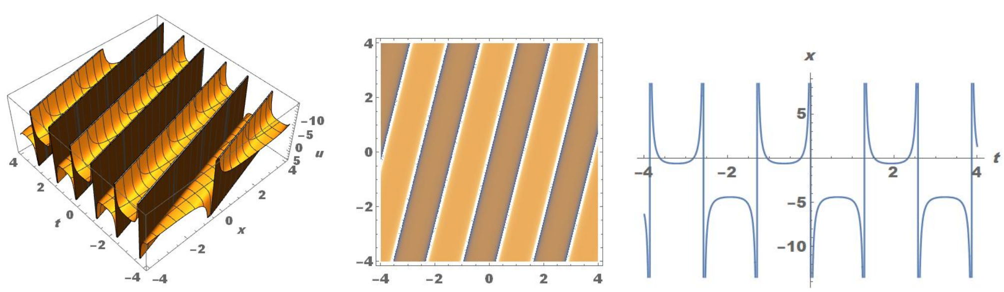

Figure 5. Singular Periodic wave depiction of snoidal wave solution (2.45) using the assigned values$c = 2$, $b = 10$, $\nu = 4$, $d = -1.2$, $\omega = 0.9$, where one utilizes the interval $-5 \le t, x, \le 5$. This displays a singular periodic graph whose points of singularities exist within $-4 \le t, x, \le 4$.

Delta amplitude function solutions of (1.8)

At this point, we proceed to achieve certain outcomes of NLNODE (2.9) regarding the delta amplitude function. We have previously confirmed that the balancing process results in $p = 1$. The result of that is (2.33) assuming the framework

$$U(\zeta) = A_{-1}R(\zeta)^{-1} + A_0 + A_1R(\zeta). \tag{2.48}$$

Substituting the value of $U(\zeta)$ from (2.48) into (2.9) together with (2.31) results in an equation (algebraic) that, when separated according to the different powers of $R(\zeta)$, provides a system of a specific number of equations. We resolve the system and provide three groups of solutions displayed as

$$\nu = \frac{1}{4c}\left\{4\,ac - 4\,cd\omega - b^2 + 8\,cd\right\},\ A_{-1} = 0,\ A_0 = -\,\frac{b}{2c},\ A_1 = \pm\sqrt{\frac{6d}{c}}, \tag{2.49}$$

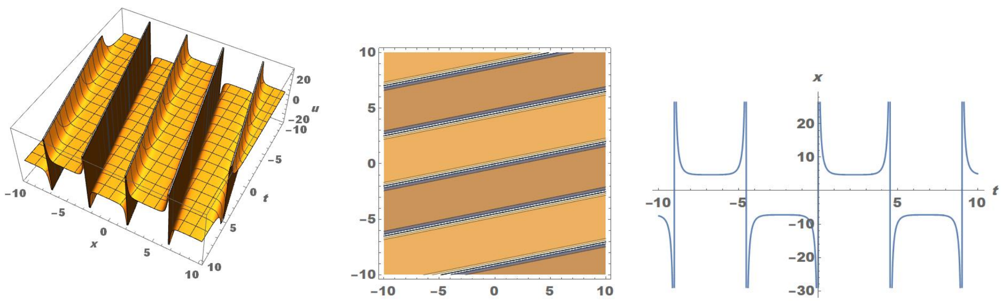

Figure 6. Singular periodic wave depiction of snoidal wave solution (2.46) using the allocated parameter values $c = 1.2$, $b = 3$, $\nu = 0.2$, $d = -2$, $\omega = 0.8$, in the interval $-10 \leq t, x, \leq 10$. This profile reveals a singular periodic plot whose points of singularities exist in the given interval.

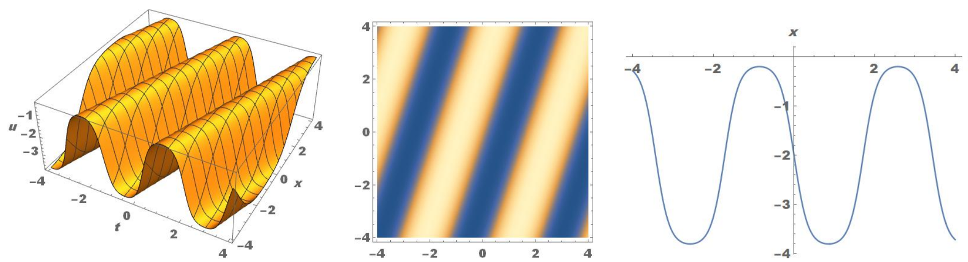

Figure 7. Periodic wave depiction of snoidal wave solution (2.47) using the assigned values $c = 2$, $b = 8$, $\nu = 3$, $d = -1.2$, $\omega = 0.9$, where one utilizes the interval $-4 \leq t, x, \leq 4$. This graph reveals a smooth sinusodial wave movement in which their is uniform frequency and the wavelengths are consistent.

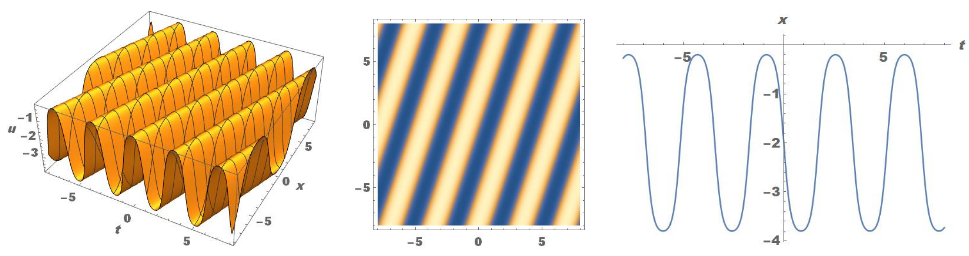

Figure 8. Smooth periodic wave depiction of snoidal wave solution (2.47) using the assigned values $c = 2$, $b = 8$, $\nu = 3$, $d = -1.2$, $\omega = 0.9$, where one utilizes the interval $-8 \leq t, x, \leq 8$. This graph depicts a smooth sinusodial wave motion with uniform frequency and wavelength.

$$\nu = \pm \frac{1}{4c}\left(24\,c\sqrt{d^2\,(1-\omega)} + 4\,cd\omega - 4\,ac + b^2 - 8\,cd\right),$$

$$A_{-1} = \pm\sqrt{\frac{6\,d(1-\omega)}{c}},\;\; A_0 = -\frac{b}{2c},\;\; A_1 = \sqrt{\frac{6d}{c}}, \tag{2.50}$$

$$\nu = \frac{1}{4c}\left\{4\,ac - 4\,cd\omega - b^2 + 8\,cd\right\},\;\; A_{-1} = \pm\sqrt{\frac{6\,d(1-\omega)}{c}},\;\; A_0 = -\frac{b}{2c},\;\; A_1 = 0. \tag{2.51}$$

Therefore, corresponding to (2.49)–(2.51), we have the set of dnoidal solutions

$$v(t,x) = -\frac{b}{2c} \pm \sqrt{\frac{6d}{c}}\,\mathrm{dn}\left(x - \nu t|\omega\right), \tag{2.52}$$

$$v(t,x) = \pm\sqrt{\frac{6\,d(1-\omega)}{c}}\,\mathrm{nd}^{-1}\left(x-\nu t|\omega\right) - \frac{b}{2c} + \sqrt{\frac{6d}{c}}\,\mathrm{dn}\left(x-\nu t|\omega\right), \tag{2.53}$$

$$v(t,x) = \pm\sqrt{\frac{6\,d(1-\omega)}{c}}\,\mathrm{nd}^{-1}\left(x-\nu t|\omega\right) - \frac{b}{2c}. \tag{2.54}$$

The presentation of wave dynamics of part of the dnoidal wave solutions attained here is explicated in Figures 9, 10, and 11.

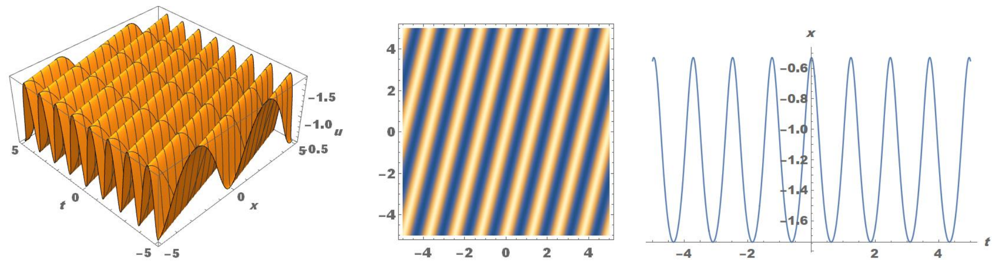

Figure 9. Periodic wave portrayal of dnoidal wave solution (2.52) obtained by invoking the assigned values $c = 2.1$, $b = 10$, $\nu = 4.02$, $d = 1.2$, $\omega = 0.88$, where one utilizes the interval $-5 \leq t, x, \leq 5$. This reveals waves of uniform wavelength and frequency, travelling over a range of interval time.

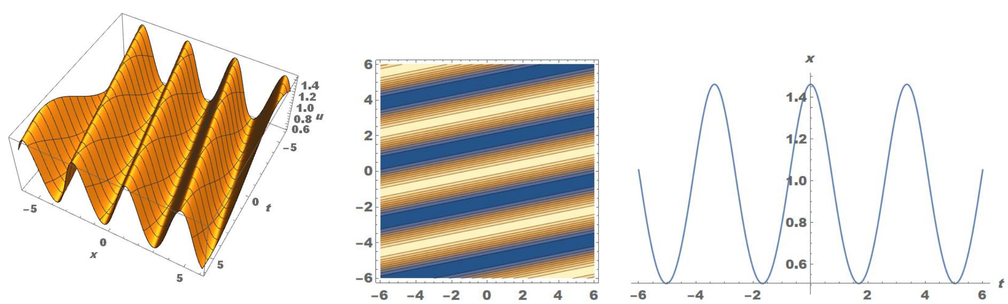

Figure 10. Periodic wave exhibition of dnoidal wave solution (2.53) achieved by engaging the assigned values $c = 1.2$, $b = 3.2$, $\nu = 0.2$, $d = 1.2$, $\omega = 0.98$, with the utilization of interval $-6 \leq t, x, \leq 6$. This graph depicts waves of uniform wavelength and frequency, travelling over a short time interval.

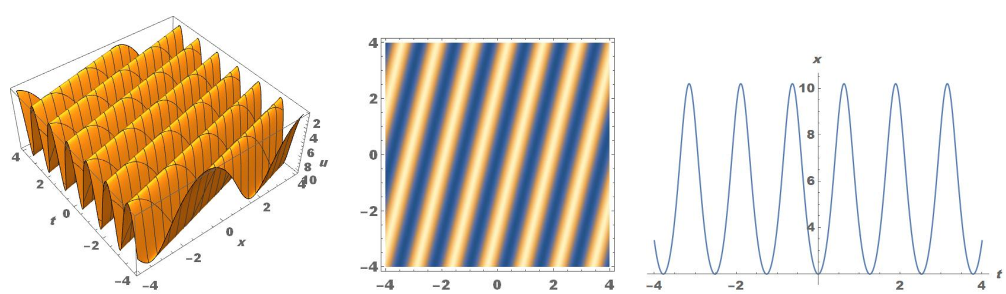

Figure 11. Wave depiction of periodic solution (2.54) using the values of constant parameters as $c = 2.4$, $b = 10.03$, $\nu = 4.02$, $d = 60.4$, $\omega = 0.89$ in the interval $-4 \leq t, x \leq 4$. This plots showcase waves of uniform wavelength and frequency, travelling over a given range of time.

Remark 2.3. We observe here that utilizing the extended Jacobi elliptic function expansion method in solving some particular cases of NLNODE (2.9), diverse periodic solutions of interest can be found. Special limits of these solutions make them disintegrate to basic trigonometry and hyperbolic solutions of note as showcased in some of the graphical depictions.

2.5. Physical significance of cnoidal, snoidal and dnoidal (periodic) solutions in geophysics and marine science

In this study, various periodic solutions have been obtained. They are in the structure of cnoidal, snoidal as well as dnoidal solutions. These solutions are highly significant. Thus, one highlights some of these significance in geophysics and marine science.

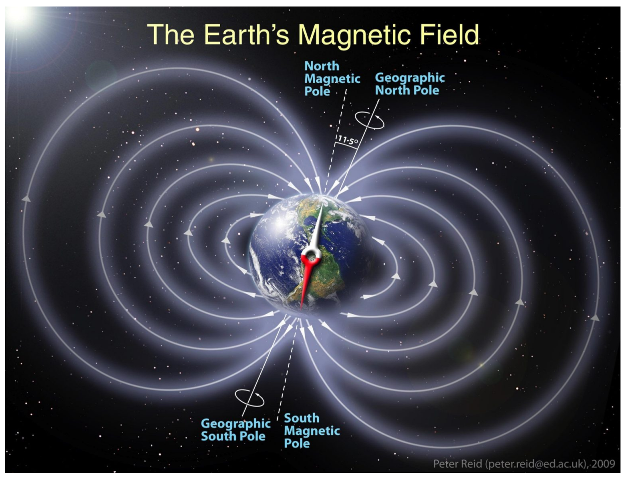

Figure 12. Diagrammatic representation of Earth's magnetic field. https://scite chdaily.com/earths-magnetosphere-protecting-our-planet-from-harmful-space-radiation.

Periodic (cnoidal, snoidal and dnoidal waves) solutions play a crucial role in geophysics for simulating cyclic natural events, including tides, internal oceanic waves, and atmospheric gravity waves. They are crucial for comprehending the stability and behavior of intricate systems such as the Earth's rotation, magnetic field, and fluid dynamics in the oceans and atmosphere (see Figure 12). These methods enable geoscientists to examine wave movement, energy distribution, and events such as seismic precursors, ultimately supporting resource discovery and forecasting natural disasters.

The primary magnetic field of the Earth, generated in the core, is expected to display a clear and consistent gradient with depth in the ocean. Overlaying this consistent gradient could be magnetic signals resulting from diverse sources. These encompass crustal magnetization, the temporary variations generated outside the Earth that lead to secondary induced fields within; and, the emphasis of this paper, magnetic signals produced by the motion induction of seawater flowing in Earth's stable main magnetic field [38].

They assist in understanding the periodic nature of geophysical fields, such as daily and yearly cycles in electromagnetic signals, which can be associated with events like seismic activity or core-crust interactions.

Due to geodynamic processes, electromagnetic fields are linked to the preparatory phase of earthquakes, and current interpretations of the geodynamic activities occurring in the Earth's crust suggest that these processes can be monitored by tracking natural or artificial electromagnetic fields present on the Earth's surface. Geodynamic processes can be monitored electromagnetically through two entirely distinct physical processes. One aspect is the alteration of the electromagnetic characteristics of the geological environment caused by tectonic activities, and another is the conversion of mechano-electric energy into electromagnetic fields during earthquake preparation. They certainly correspond to the first and second type seismic-electric effects [46].

Cnoidal, snoidal and dnoidal solutions which are periodic solutions in marine science are essential for comprehending natural cyclic phenomena such as ocean currents, dynamics of fish populations, and seasonal fluctuations in the carbon cycle, as they facilitate the modeling of recurring processes and forecasting their future conditions. These mathematical instruments assist researchers in examining the stability and long-term dynamics of marine ecosystems and physical processes, offering understanding of how environmental elements affect these cycles and the possible consequences of climate change. We further outline the importance of cnoidal, snoidal and dnoidal solutions in marine science comprehensively under the following:

Physical oceanography

In physical oceanography, periodic solutions illustrate and forecast the essential, recurring motions of ocean water. For example, in tides and tidal currents, the clearest periodic occurrence is the tide, which is influenced by the gravitational pull of the moon and sun. Mathematical models employ a combination of sinusoidal functions or tidal constituents to depict the fluctuations of sea levels and the related tidal currents. Comprehending tidal patterns is essential for secure navigation, coastal construction, and handling coastal flooding.

Moreover, in ocean circulation, periodic solutions are employed to represent large-scale, recurring ocean currents that move heat, nutrients, and carbon throughout the planet. These patterns are determined by Earth's rotation (the Coriolis effect), temperature differences, and wind. Modeling these cycles aids in anticipating climate trends and the allocation of marine organisms.

Additionally, in waves, periodic solutions represent the movement of waves, encompassing shallow-water waves and rogue waves. Examining these solutions enables engineers and physicists to grasp wave energy, its effects on coastal structures, and its possibilities for energy extraction.

Many of the mechanisms leading to unexpectedly large surface waves in deep water (such as dispersive and geometrical focusing, interactions with currents and internal waves, and reflection from caustic regions, etc.) are also present in shallow waters. Only the process of modulational instability is inactive under finite depth conditions. Rather, wave enhancement along specific coastal profiles and the significant reliance of the run-up height on the incoming wave shape may greatly aid in the development of rogue waves in the nearshore. An exclusive origin of enduring rogue waves (which has no counterparts in the deep ocean) is the nonlinear interplay of obliquely traveling solitary shallow-water waves and a corresponding mechanism of Mach reflection of waves off the shore [53].

Marine biology and ecology

Periodic solutions are utilized to represent the recurring biological cycles that marine organisms rely on for survival and reproduction. For instance, it can be noted that numerous marine species have developed to align their behaviors with environmental cycles in biological rhythms. These rhythms include circadian (daily), circatidal (tidal), circalunar (monthly), and seasonal cycles. Mathematical models can assist in forecasting how these rhythms influence feeding patterns, reproduction, and migration.

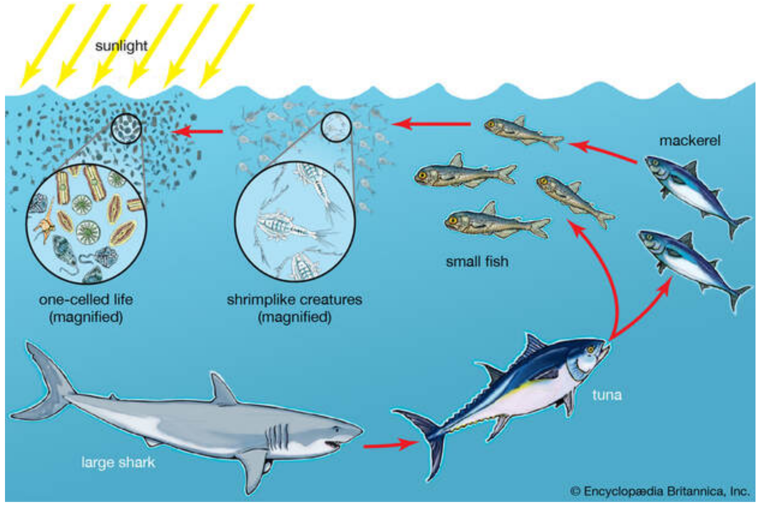

Figure 13. Schematic illustration of phytoplankton diatoms ocean food chains foundations for fishes. https://www.imbrsea.eu/product/74.

In plankton dynamics, population models that include periodic solutions can effectively depict recurring plankton blooms. Incorporating seasonal variations in temperature and light as periodic driving factors allows models to forecast the timing and strength of these blooms, which serve as the foundation for numerous aquatic food webs (see Figure 13 for the diagrammatic representation of marine food chains for fishes).

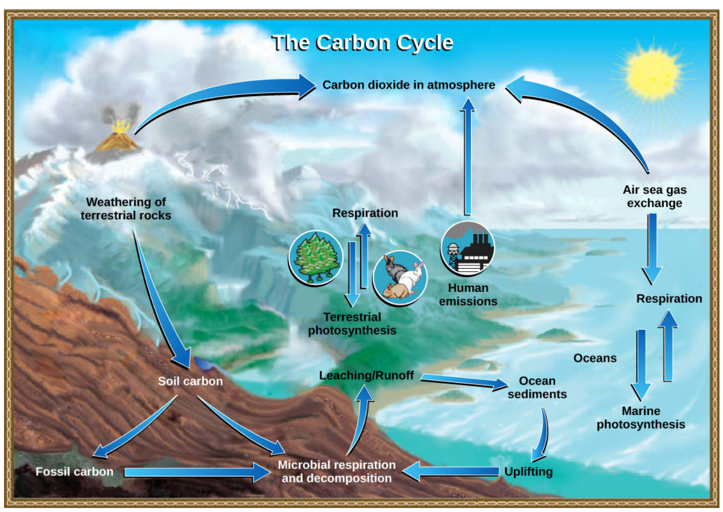

Figure 14. Carbon dioxide is present in the atmosphere and is also dissolved in water. Photosynthesis transforms carbon dioxide gas into organic carbon, while respiration returns the organic carbon to carbon dioxide gas. The enduring storage of organic carbon happens when material from living organisms is buried deep within the earth and transforms into fossils. Volcanic eruptions and, more recently, emissions from humans reintroduce this stored carbon into the carbon cycle. https://openoregon.pressbooks.pub/envirobiology/chapter/3-2-biogeochemical-cycles

In ecosystem stability, researchers can examine the long-term stability and coexistence of species by investigating periodic solutions in population models. For instance, certain models indicate that seasonal variations may enhance the coexistence of various plankton species.

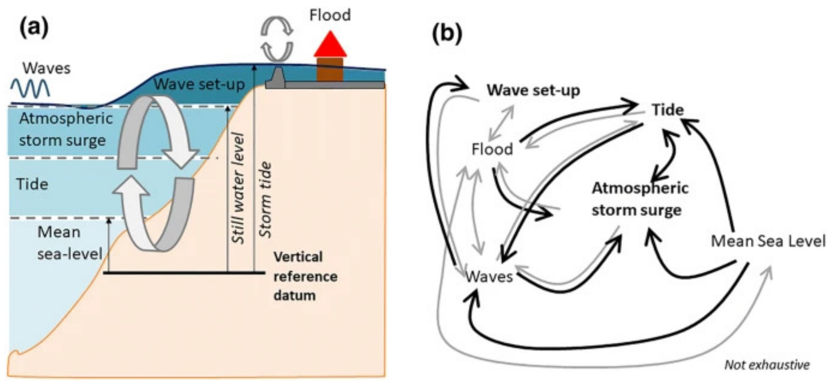

Figure 15. **a** Components of storm surge (tide), terminology and illustration of interactions. **b** Key interactions among mean sea level, waves, storm surges caused by the atmosphere, tides, wave setup, and flooding [29].

Ecological systems typically experience constant disturbances or disruptions. In this context, disturbances typically denote substantial, rare occurrences that can lead to major and enduring alterations in the ecological framework. They have the ability to alter the ecological environment and create a new balance. Perturbations generally indicate minor, brief alterations that assess the system's reaction and robustness. They typically do not lead to lasting alterations in the complete framework of the ecological system. Ecological systems' reactions to disturbances and perturbations are frequently quantified as stability, a core idea based on dynamical systems and control theory [17].

Modeling natural cycles

Periodic solutions are applied in ocean dynamics to represent tidal cycles, ocean currents, and the propagation of waves exhibiting both linear and nonlinear traits. They additionally assist in seasonal and climate variations to comprehend the impact of seasonal fluctuations and climate variability on marine ecosystems, including the seasonal cycling of the marine carbon cycle (see the illustration in Figure 14).

Coastal regions represent the concept of 'vulnerable' land in relation to climate change and sea level rise. Understanding the fluctuations in water levels at the coast due to mean sea level variations, tides, atmospheric surges, and wave setups is essential for evaluating coastal flooding. It has been examined how coastal water levels can be changed by the interplay of sea level rise, tides, storm surges, waves, and flooding (see Figure 15). The primary interaction mechanisms are determined, primarily through the examination of the shallow water equations. According to a review of existing literature, the scales of these interactions are assessed across various environments. The examined interactions demonstrate significant spatiotemporal variability. Depending on the nature of the environments (e.g., morphology, hydrometeorological context), they may attain several tens of centimeters (either positive or negative). As a result, probabilistic forecasts of future coastal water levels and flooding must determine if interaction processes are of primary significance, and when suitable, forecasts should incorporate these interactions via modeling or statistical techniques [29].

Ecological insights

Periodic solutions also play a vital role in examining the population dynamics of marine species. These encompass fish populations and phytoplankton, by illustrating cyclical trends of growth, reproduction, and seasonal movement. Models that include periodic behaviors in food chain interactions can illustrate tri-trophic food chains and examine how seasonal changes affect predator-prey dynamics.

Phytoplankton and zooplankton are essential elements of aquaculture systems, contributing significantly to the health and productivity of aquatic ecosystems. Phytoplankton, being primary producers, constitutes the foundation of the aquatic food web by supplying vital nutrients and oxygen via photosynthesis. Zooplankton, being primary consumers, consume phytoplankton and play a vital role as a food resource for higher trophic levels, such as fish and shellfish. Their presence and equilibrium are crucial for the ideal growth and survival of cultured species [42].

Applications in research and prediction

This involves stability analysis where researchers utilize periodic solutions to ascertain the presence and stability of these cycles, which is essential for comprehending how marine systems react to alterations.

Additionally, in predictive modeling, examining the stability of periodic solutions allows scientists to forecast the long-term dynamics of marine systems and evaluate the impacts of human activities and climate change.

Ultimately, the examination of periodic solutions contributes to the creation of mathematical techniques, such as averaging theory, to estimate intricate systems and recognize periodic orbits, resulting in a deeper comprehension of these phenomena.

The dynamical integrity of a solution in a dynamical system signifies its resilience to external disturbances, with a wider basin of attraction suggesting enhanced robustness. Nevertheless, calculating the complete basin of attraction can be resource-intensive. The local integrity metric provides a effective option by measuring the compact area of the attraction basin surrounding the steady state in question. A swift algorithm was recently presented to assess the dynamical integrity of equilibrium points through the local integrity measure. Additional research has broadened this approach to assess the dynamic integrity of stable periodic orbits in autonomous systems as well as those that are periodically excited. The findings show that the algorithm is both swift and efficient, providing valuable engineering insights and setting the stage for including dynamical integrity as a design factor [45].

3. Conservation laws of g-GKdVe (1.8)

In this part, we develop the Lie point symmetries and conserved vectors for the g-GKdVe (1.8). The significant multiplier method [43] will be examined in safeguarding different conserved vectors for the model.

Take into account a system of ϱ PDEQs of sth order [28]

$$\Xi_{\varepsilon}(z, \varOmega, \varOmega_{(1)}, \dots, \varOmega_{(s)}) = 0, \quad \varepsilon = 1, \dots, \varrho, \tag{3.1}$$

with ς and ϱ (independent and dependent variables accordingly) given as $z = (z^1, z^2, \dots, z^{\varsigma})$ and $\varOmega = (\varOmega^1, \varOmega^2, \dots, \varOmega^{\varrho})$. Operator $\delta/\delta\varOmega^{\varepsilon}$, explicated for each ε, as

$$\frac{\delta}{\delta\varOmega^{\varepsilon}} = \frac{\partial}{\partial\varOmega^{\varepsilon}} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\dots D_{i_s}\frac{\delta}{\delta\varOmega^{\varepsilon}_{i_1,i_2,\dots,i_s}}, \quad i = 1, \dots, \varsigma, \tag{3.2}$$

is Euler-Lagrange operator with

$$D_i = \frac{\partial}{\partial z^i} + \varOmega^{\varepsilon}_i\frac{\partial}{\partial\varOmega^{\varepsilon}} + \varOmega^{\varepsilon}_{ij}\frac{\partial}{\partial\varOmega^{\varepsilon}_j} + \cdots, \;\; i = 1, \dots, \varsigma, \;\; j = 1, \dots, \varsigma \tag{3.3}$$

is total differential operator.

An n-tuple $C = (C^1, C^2, \cdots, C^n)$, $1 = 1, 2, \cdots, n$, such that

$$D_i C^i = 0, \tag{3.4}$$

holds for all solutions of (3.1) is noted to be conserved vectors of equation (3.1). Multiplier Λ for system (3.1) possesses the characteristic that

$$D_i C^i = \Lambda^\varepsilon \Xi_\varepsilon, \quad \varepsilon = 1, \ldots, \varrho. \tag{3.5}$$

The deciding equations for every multiplier involved are derived from

$$\frac{\delta}{\delta \Omega^\varepsilon} (\Lambda^\varepsilon \Xi_\varepsilon) = 0, \quad \varepsilon = 1, \ldots, \varrho. \tag{3.6}$$

As soon as the multipliers are produced through (3.6), the conserved vectors can be attained using (3.5) as the defining equation.

3.1. The multiplier technique

The multiplier approach [43] is highly beneficial since it can be applied regardless of whether a PDEQ has variational principle(s) or not. To employ this method, we consider a zeroth-order multiplier $\Lambda(t, x, v)$ to achieve the conserved vectors of (1.8). One employs the governing equation (3.6), which is

$$\frac{\delta}{\delta u} \{\Lambda(t, x, v)\Delta\} = 0, \tag{3.7}$$

where Euler operator $\delta/\delta v$ in this instance is expressed as

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_x^2 \frac{\partial}{\partial v_{xx}} - D_x^3 \frac{\partial}{\partial v_{xxx}} \tag{3.8}$$

with the definition of total differential operators Expansion of (3.7) and split over various derivatives of u, achieves seven determining equations which solves to produce the result multiplier $\Lambda(t, x, v)$ as

$$\Lambda(t, x, v) = C_1 v + C_2 \tag{3.9}$$

with constants C_1, C_2 arbitrary in the solution. Now, invoking equation (3.5) one achieves conserved vectors associated to the multipliers as

$$
\begin{aligned}
T_1^t &= \frac{1}{2} v^2, \\
T_1^x &= \frac{1}{2(2n^3 + 7n^2 + 7n + 2)} \Big\{ 2an^3 v^2 + 4dn^3 v v_{xx} - 2dn^3 v_x^2 + 7an^2 v^2 + 14dn^2 v v_{xx} \\
&\quad - 7dn^2 v_x^2 + 4bn^2 v^{n+2} + 2cn^2 v^{2n+2} + 7anv^2 + 14dnv v_{xx} - 7dn v_x^2 + 6bnv^{n+2} \\
&\quad + 5cnv^{2n+2} + 2av^2 + 4dv v_{xx} - 2dv_x^2 + 2bv^{n+2} + 2cv^{2n+2} \Big\}, \\
T_2^t &= v, \\
T_2^x &= \frac{1}{2n^3 + 7n^2 + 7n + 2} \Big\{ 7an^2 v + 7anv + 2dn^3 v_{xx} + 7dn^2 v_{xx} + 7dn v_{xx} \\
&\quad + 2bn^2 v^{n+1} + 5bnv^{n+1} + cn^2 v^{2n+1} + 3cnv^{2n+1} + 2an^3 v + 2cv^{2n+1} \\
&\quad + 2bv^{n+1} + 2av + 2dv_{xx} \Big\}.
\end{aligned}
$$

Remark 3.1. We declare here that the multiplier corresponding to $\Lambda(t, x, u) = 1$, produces the model g-GKdVe (1.8) in a conserved structure.

3.2. Conservation laws of (1.8) using the Noether theorem

This part, applies the Noether theorem in deriving the conservation laws applicable to g-GKdVe (1.8). This equation does not have a Lagrangian right now. Applying the theorem, we transform (1.8) into a Lagrangian expression of fourth-order. Consequently, by employing the potential structure $v = p_x$, (1.8) transforms into

$$p_{tx} + ap_{xx} + bp_x^n p_{xx} + cp_x^{2n} p_{xx} + dp_{xxxx} = 0. \tag{3.10}$$

It is straightforward to verify that the second-order Lagrangian for (3.10) is given; viz

$$\mathcal{L} = -\frac{1}{2}p_x p_t - \frac{1}{2}ap_x^2 - \frac{1}{(n+1)(n+2)}bp_x^{n+2} - \frac{1}{(2n+1)(2n+2)}cp_x^{2n+2} + \frac{1}{2}dp_{xx}^2 \tag{3.11}$$

due to the fact that $\delta\mathcal{L}/\delta p = 0$ on (3.10). Here $\delta/\delta p$ exhibits Euler-Lagrange operator

$$\frac{\delta\mathcal{L}}{\delta p} = \frac{\partial}{\partial p} - \mathfrak{D}_t\frac{\partial}{\partial p_t} - \mathfrak{D}_x\frac{\partial}{\partial p_x} + \mathfrak{D}_t^2\frac{\partial}{\partial p_{tt}} + \mathfrak{D}_x^2\frac{\partial}{\partial p_{xx}} + \mathfrak{D}_t\mathfrak{D}_x\frac{\partial}{\partial p_{tx}} - \cdots, \tag{3.12}$$

where operators (differential) $\mathfrak{D}_x$, $\mathfrak{D}_t$ appear in (2.7). One contemplates

$$\mathcal{W} = \xi^1(t,x,p)\frac{\partial}{\partial t} + \xi^2(t,x,p)\frac{\partial}{\partial x} + \eta(t,x,p)\frac{\partial}{\partial p}, \tag{3.13}$$

as vector field, where ξ^1, ξ^2, η all depend on p, t, together with x. To determine Noether symmetries $\mathcal{W}$ of (3.10), one inserts the assertion for $\mathcal{L}$ from (3.11) into

$$\mathcal{W}^{[2]}(\mathcal{L}) + \mathcal{L}[\mathfrak{D}_x(\xi^2) + \mathfrak{D}_t(\xi^1)] = \mathfrak{D}_x(\mathfrak{B}^x) + \mathfrak{D}_t(\mathfrak{B}^t), \tag{3.14}$$

where $\mathfrak{B}^x(t,x,p) = \mathfrak{B}^x$ and $\mathfrak{B}^t(t,x,p) = \mathfrak{B}^t$ are the involved terms (gauge) while $\mathcal{W}^{[2]}$ connotes second extension related to $\mathcal{W}$ which defines as

$$\mathcal{W}^{[2]} = \mathcal{W} + \zeta_x\frac{\partial}{\partial p_x} + \zeta_t\frac{\partial}{\partial p_t} + \zeta_{xx}\frac{\partial}{\partial p_{xx}} + \zeta_{tt}\frac{\partial}{\partial p_{tt}} + \zeta_{tx}\frac{\partial}{\partial p_{tx}} \tag{3.15}$$

with ζ_t and ζ_x defined in this regard as (2.3).

Equation (3.14) becomes

$$\begin{aligned} &-\frac{p_x}{2}\zeta_t - \frac{p_t}{2}\zeta_x - ap_x\zeta_x - \frac{1}{n+1}bp_x^{n+1}\zeta_x - \frac{1}{2n+1}cp_x^{2n+1}\zeta_x + dp_{xx}\zeta_{xx} \\ =&\mathfrak{B}_t^t + \mathfrak{B}_x^x + p_t\mathfrak{B}_u^t + p_x\mathfrak{B}_u^x. \end{aligned} \tag{3.16}$$

The expansion of the equation mentioned above results in a system of fourteen differential equations:

$$\begin{aligned} &\mathfrak{B}_{px}^x = 0,\ \ \mathfrak{B}_{pp}^x = 0,\ \ \xi_t^1 = 0,\ \ \xi_t^2 = 0,\ \ \mathfrak{B}_t^t + \mathfrak{B}_x^x = 0,\ \ \eta_t + 2\mathfrak{B}_p^x = 0, \\ &\xi_x^1 = 0,\ \ \xi_x^2 = 0,\ \ \eta_x = 0,\ \ \xi_p^1 = 0,\ \ \xi_p^2 = 0,\ \ \mathfrak{B}_p^t = 0,\ \ \eta_p = 0. \end{aligned} \tag{3.17}$$

Consequently, resolving the system results in the solution shown as

$$\xi^1(t,x,p) = A_1,\ \ \xi^2(t,x,p) = A_2,\ \ \eta(t,x,p) = f(t),$$

$$\mathfrak{B}^t(t,x,p) = g(t,x), \ \ \mathfrak{B}^x(t,x,p) = Q(t) - \frac{1}{2}f'(t)p + e(t)$$

with $-\int g_t(t,x)dx = Q(t)$, with constants A_1, A_2 being arbitrary. Moreover, $g(t,x)$, $f(t)$, and $e(t)$ can be chosen freely. It is observed that one may select $e(t) = g(t,x) = 0$, since they constitute to the trivial component of the conserved vector, thereby fulfilling (3.17). Therefore, one obtains the subsequent Noether symmetries along with their associated functions (gauge):

$$\begin{aligned}
\mathcal{W}_1 &= \frac{\partial}{\partial t}, \quad \mathfrak{B}^t = 0, \quad \mathfrak{B}^x = 0,\\
\mathcal{W}_2 &= \frac{\partial}{\partial x}, \quad \mathfrak{B}^t = 0, \quad \mathfrak{B}^x = 0,\\
\mathcal{W}_f &= f(t)\frac{\partial}{\partial p}, \quad \mathfrak{B}^t = 0, \quad \mathfrak{B}^x = -\frac{1}{2}f'(t)p.
\end{aligned}$$

Subsequently, we utilize the results mentioned above to determine conserved vectors of (3.10). Utilizing the equations for (T^t, T^x) (conserved vector) outlined in [49], one can derive the three vectors (conserved) linked to the determined Noether symmetries correspondingly as

$$\begin{aligned}
T_1^t &= \frac{1}{2}dp_{xx}^2 - \frac{1}{2}ap_x^2 - \frac{b}{(n+1)(n+2)}p_x^{n+2} - \frac{c}{(2n+1)(2n+2)}p_x^{2n+2},\\
T_1^x &= ap_tp_x + \frac{b}{n+1}p_tp_x^{n+1} + \frac{c}{2n+1}p_tp_x^{2n+1} + dp_tp_{xxx} - dp_{xx}p_{tx} + \frac{1}{2}p_t^2,\\
T_2^t &= \frac{1}{2}p_x^2,\\
T_2^x &= \frac{1}{2}ap_x^2 + \frac{b}{n+1}p_x^{n+2} - \frac{b}{(n+1)(n+2)}p_x^{n+2} + dp_{xxx}p_x - \frac{1}{2}dp_{xx}^2\\
&\quad + \frac{c}{2n+1}p_x^{2n+2} - \frac{c}{(2n+1)(2n+2)}p_x^{2n+2},\\
T_f^t &= -\frac{1}{2}f(t)p_x,\\
T_f^x &= -af(t)p_x - \frac{b}{n+1}f(t)p_x^{n+1} - df(t)p_{xxx} - \frac{1}{2}f(t)p_t + \frac{1}{2}f'(t)p\\
&\quad - \frac{c}{2n+1}f(t)p_x^{2n+1}.
\end{aligned}$$

Returning to the initial variables, we acquire non-local and local conserved vectors of (1.8) represented by

$$\begin{aligned}
C_1^t &= \frac{1}{2}dv_x^2 - \frac{1}{2}av^2 - \frac{b}{(n+1)(n+2)}v^{n+2} - \frac{c}{(2n+1)(2n+2)}v^{2n+2},\\
C_1^x &= av\left(\int v_t\,dx\right) + \frac{b}{n+1}v^{n+1}\left(\int v_t\,dx\right) - dv_tv_x + dv_{xx}\left(\int v_t\,dx\right)\\
&\quad + \frac{c}{2n+1}v^{2n+1}\left(\int v_t\,dx\right) + \frac{1}{2}\left(\int v_t\,dx\right)^2,\\
C_2^t &= \frac{1}{2}v^2,\\
C_2^x &= dv_{xx}v + \frac{1}{2}av^2 + \frac{b}{n+1}v^{n+2} - \frac{b}{(n+1)(n+2)}v^{n+2} - \frac{1}{2}dv_x^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c}{2n+1}v^{2n+2}-\frac{c}{(2n+1)(2n+2)}v^{2n+2},\\
C_f^t =&\ -\frac{1}{2}f(t)v,\\
C_f^x =&\ -af(t)v-\frac{b}{n+1}f(t)v^{n+1}-df(t)v_{xx}-\frac{1}{2}f(t)\int v_t\,dx\\
&-\frac{c}{2n+1}f(t)v^{2n+1}+\frac{1}{2}f'(t)\int vdx.
\end{aligned}
$$

Remark 3.2. It is important to mention that because of the existence of the arbitrary function $f(t)$, there are infinitely many nonlocal conservation laws.

4. Concluding remarks

A thorough analytical examination of model (1.8) featuring dual power-law in marine science is presented, aiming to derive several exact solitary wave solutions as well as conserved vectors of the model. Initially, performance of Lie symmetry analysis of the model was done, resulting in two translational Lie point symmetries. This outcome enables the model to be simplified to a nonlinear ordinary differential equation by means of a combination (linear) of the symmetries. Initially, a direct integration method is utilized to obtain solutions to the model. Additionally, to obtain more comprehensive exact solutions to the generalized geophysical Korteweg-de Vries model, the Jacobi elliptic function procedure in an extended structure is employed, which is a widely accepted technique for deriving analytical solutions to evolution equations. As a result, one obtains different cnoidal, snoidal, and dnoidal wave solutions for the model under consideration. The co-polar trio explained in tabular format shows that these solutions can revert to different hyperbolic and trigonometric functions under specific conditions. Furthermore, various graphical representations of the dynamic characteristics of the obtained results are shown to achieve a clear comprehension of the physical happenings associated with the under study model. In the subsequent section, the conserved vectors of the previously mentioned equation were obtained by invoking conventional multiplier procedure along with Noether's theorem.

Conservation laws play a vital role in marine science and geophysics for forecasting system behavior, including weather patterns and ocean currents, through the application of core principles of mass, momentum, and energy. In marine science, these regulations aid in managing resources, curbing pollution, and assessing ecosystem health, whereas in geophysics, they underpin geophysical fluid dynamics to simulate Earth's oceans and atmosphere, ensuring sustainability for marine ecosystems and living resources.

In marine science, conservation laws facilitate the comprehension and preservation of marine ecosystem health by governing the sustainable utilization of natural resources such as fish populations. Additionally, through the application of mass and energy conservation principles, researchers facilitate Resource Management that allows them to establish sustainable fishing practices and avoid over-fishing, thereby safeguarding marine biodiversity.

In geophysics, the principles of mass, momentum, and energy conservation are essential to geophysical fluid dynamics, which examines fluid movements in the atmosphere and oceans. Moreover, in forecasting ocean currents, the conservation of potential vorticity is an essential concept for comprehending and forecasting ocean currents and extensive fluid motions, as it monitors a particular characteristic of the fluid during its movement.

Declarations

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