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TEN LIMIT CYCLES NEAR A CUBIC HOMOCLINIC LOOP WITH A NILPOTENT CUSP*

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Abstract In this paper, we study the bifurcation of limit cycles near a homoclinic cuspidal loop in a planar cubic near-Hamiltonian system by high-order Melnikov functions. We combine the algebraic structure of Abelian integrals with Picard-Fuchs equations for computing the corresponding asymptotic expansion of Melnikov functions near the cuspidal loop. Using this system as an example, we show that planar cubic systems can have ten limit cycles bifurcating near a homoclinic loop, which is a new lower bound for the number of limit cycles produced by homoclinic bifurcation in cubic systems.

Keywords Homoclinic bifurcation, limit cycle, high-order Melnikov function.

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1. Introduction and main results

One of restricted versions of the second part of Hilbert's 16th problem is to estimate the number of limit cycles bifurcating near a homoclinic or heteroclinic loop in planar polynomial near-integrable systems of the form

$$\begin{aligned} \dot{x} &= \mu^{-1}(x,y)H_y(x,y) + \varepsilon p(x,y,\varepsilon), \\ \dot{y} &= -\mu^{-1}(x,y)H_x(x,y) + \varepsilon q(x,y,\varepsilon), \end{aligned} \tag{1.1}$$

where $|\varepsilon| \ll 1$, $H(x,y)$ is a smooth first integral of the unperturbed system $(1.1)|_{\varepsilon=0}$ with an integrating factor $\mu(x,y)$, perturbation functions $p(x,y,\varepsilon)$ and $q(x,y,\varepsilon)$ are polynomials in (x,y) and analytic in ε. Suppose the integrable system $(1.1)|_{\varepsilon=0}$ has a continuous family of periodic orbits given by

$$\Gamma_h\colon\ H(x,y) = h,\ h \in (\alpha,\beta)$$

with a homoclinic or heteroclinic loop $\Gamma_\beta \subset \{H(x,y) = \beta\}$. Then the number of limit cycles produced near Γ_β can be estimated by the number of isolated zeros of the Melnikov function

$$M(h) = \oint_{\Gamma_h} \mu(x,y)(q(x,y,0)\mathrm{d}x - p(x,y,0)\mathrm{d}y)$$

for $0 < \beta - h \ll 1$.

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To determine the number of isolated zeros of $M(h)$ for $0 < \beta - h \ll 1$, lots of results have been derived for the asymptotic expansion of $M(h)$ near Γ_β, for example see [1,2,9,11,14,19,21, 22,27,31,32]. Formulas were also obtained for the first several coefficients of related asymptotic expansions for Γ_β connecting different types of singular points [10–12,14]. Tian and Han [24] established an algorithm to compute more coefficients of the expansion of Melnikov functions near homoclinic loops with studying bifurcations of small limit cycles near an elementary center simultaneously. This method was extended to some other cases of heteroclinc loops [6], cuspidal loops [20] and loops passing through nilpotent saddles [30,31].

For the quadratic case of system (1.1), there are some known results obtained for limit cycle bifurcations near a homoclinic or heteroclinic loop, for example see [4, 7, 8, 13, 15–18, 28] and references therein. It was proved that the cyclicity of a homoclinic loop is two for quadratic Hamiltonian systems under quadratic perturbations [13, 17, 18]. Gavrilov and Iliev [4] proved that there are at most three limit cycles bifurcating from a heteroclinic loop with two saddles in quadratic near-Hamiltonian systems. Xiong, Han and Xiao [28] showed that three limit cycles can be produced near a triangle loop in quadratic near-integrable systems. For cubic systems, there are few results for finding lower bounds on the number of limit cycles near homoclinic or heterclinic loops. Examples were presented to show the existence of 5 limit cycles near a homoclinic loop in [25], and near a heteroclinic loop in [5].

In this paper, we shall present an example of a planar cubic Hamiltonian system with a cuspidal homoclinic loop from which ten limit cycles bifurcate under suitable cubic perturbations. Consider the following cubic near-Hamiltonian system

$$\dot{x} = H_y(x,y) + \sum_{k=1}^{3} \varepsilon^k P_k(x,y), \quad \dot{y} = -H_x(x,y) + \sum_{k=1}^{3} \varepsilon^k Q_k(x,y), \tag{1.2}$$

where $|\varepsilon| \ll 1$, $H(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4$ and

$$P_k(x,y) = \sum_{i+j=1}^{3} p_{ijk} x^i y^j, \quad Q_k(x,y) = \sum_{i+j=1}^{3} q_{ijk} x^i y^j \tag{1.3}$$

with the coefficients p_{ijk} and q_{ijk} as free parameters.

The unperturbed system of (1.2) has an elementary center at the point $(1,0)$ and a nilpotent cusp at the origin. There is a homoclinic loop Γ_0 defined by $H(x,y) = 0$ passing through the origin. There exist two continuous families of periodic orbits

$$\begin{aligned} \Gamma_h^+ &: \ H(x,y) = h, \quad h \in (0, +\infty), \\ \Gamma_h^- &: \ H(x,y) = h, \quad h \in (-1/12, 0), \end{aligned} \tag{1.4}$$

which are separated by the cuspidal loop Γ_0.

Polynomial perturbations of the Hamiltonian system $(1.2)|_{\varepsilon=0}$ were investigated in [20, 25, 26, 33]. Zhao and Zhang [33] gave an upper bound $[\frac{7n}{2}]$ for the number of zeros of the first order Melnikov functions for perturbations of degree n on the two period annulus. In [25] the existence of 5 limit cycles near Γ_0 was proved for cubic perturbations by using the first order Melnikov function. Here we shall study higher-order Melnikov functions and find 10 limit cycles near Γ_0 for suitable cubic perturbations.

Let $d^\pm(h,\varepsilon)$ be two displacement functions near Γ_0 for $\pm h > 0$, respectively. Then $d^\pm(h,\varepsilon)$ can be expanded as

$$d^\pm(h,\varepsilon) = \varepsilon M_1^\pm(h) + \varepsilon^2 M_2^\pm(h) + \cdots + \varepsilon^k M_k^\pm(h) + \cdots, \tag{1.5}$$

where

$$M_1^{\pm}(h) = \oint_{\Gamma_h^{\pm}} Q_1(x,y)\mathrm{d}x - P_1(x,y)\mathrm{d}y. \tag{1.6}$$

By studying the first nonvanishing Melnikov functions in (1.5) we get the following two theorems.

Theorem 1.1. *Let* (1.5) *hold. Then the following statements hold:*

(I) *If* $M_1^{\pm}(h) \not\equiv 0$*, system* (1.2) *can have 5 limit cycles bifurcating near the homoclinic loop* Γ_0 *with proper perturbations. Furthermore,* $M_1^{\pm}(h) \equiv 0$ *if and only if*

$$p_{121} = -3q_{031}, \quad q_{011} = -p_{101}, \quad q_{111} = -2p_{201}, \quad q_{211} = -3p_{301}. \tag{1.7}$$

(II) *Assume* $M_1^{\pm}(h) \equiv 0$*. If* $M_2^{\pm}(h) \not\equiv 0$*, system* (1.2) *can have 8 limit cycles bifurcating near the homoclinic loop* Γ_0 *with proper perturbations. Furthermore,* $M_2^{\pm}(h) \equiv 0$ *if and only if one of the following conditions holds:*

$$\begin{aligned}
&S_1 : \begin{cases} p_{111} = -2q_{021}, \quad p_{122} = -3q_{032}, \quad q_{012} = -p_{102}, \\ q_{112} = -2p_{202}, \quad q_{121} = -p_{211}, \quad q_{212} = -3p_{302}; \end{cases} \\
&S_2 : \begin{cases} p_{122} = p_{021}(p_{111} + 2q_{021}) - 3q_{032}, \quad q_{012} = -p_{102}, \quad q_{031} = 0, \\ q_{112} = p_{101}(p_{111} + 2q_{021}) - 2p_{202}, \quad q_{121} = -p_{211}, \\ q_{212} = (p_{201} + p_{301})(p_{111} + 2q_{021}) - 3p_{302}; \end{cases} \\
&S_3 : \begin{cases} p_{021} = 0, \quad p_{122} = 2p_{301}(p_{211} + q_{121}) - 3q_{032}, \quad q_{012} = -p_{102}, \\ q_{031} = 0, \quad q_{112} = p_{101}(p_{111} + 2q_{021}) - 2p_{202}, \\ q_{212} = 2p_{101}(p_{211} + q_{121}) + 2(p_{201} + p_{301})(p_{211} + q_{021} + q_{121}) \\ \qquad + p_{111}(p_{201} + p_{301}) - 3p_{302}. \end{cases}
\end{aligned} \tag{1.8}$$

Theorem 1.2. *Let* (1.5) *hold. If* $M_1^{\pm}(h) = M_2^{\pm}(h) \equiv 0$*,* $M_3^{\pm}(h) \not\equiv 0$*, system* (1.2) *can have 10 limit cycles produced near the homoclinic loop* Γ_0 *with proper perturbations.*

To prove Theorems 1.1 and 1.2, we use the algebraic structure of Abelian integrals

$$I_{i,j}^{\pm}(h) = \oint_{\Gamma_h^{\pm}} x^i y^j \mathrm{d}x$$

and Picard-Fuchs equation to study the asymptotic expansion of $M_k^{\pm}(h)$ near $h = 0$, $k = 1, 2, 3$. Recursive formulas are obtained for the coefficients of the expansion of Melnikov functions near Γ_0 without investigating Hopf bifurcation near the center simultaneously.

The paper is organized as follows: In Section 2 we study the algebraic structure of Abelian integrals $I_{i,j}^{\pm}(h)$ in Lemma 2.1 and obtain recursive formulas for the coefficients of the asymptotic expansion of $I_{i,1}^{\pm}(h)$, $i = 0, 1, 2$, in Lemma 2.3 by Picard-Fuchs equation. Then we can get the asymptotic expansion of any linear combination of Abelian integrals $I_{i,j}^{\pm}(h)$ near $h = 0$. In Section 3 we give the proofs of Theorems 1.1 and 1.2, respectively.

2. Preliminaries

We begin by studying Abelian integrals of system (1.2)

$$I_{i,j}^{\pm}(h) = \oint_{\Gamma_h^{\pm}} x^i y^j \mathrm{d}x,$$

where $\Gamma_h^{\pm}$ are given in (1.4). For the algebraic structure of Abelian integrals $I_{i,j}^{\pm}(h)$ we have the following lemma.

Lemma 2.1. *For Abelian integrals $I_{i,j}^{\pm}(h)$ of system* (1.2), *the following identities hold:*

(i) $I_{i,j}^{\pm}(h) \equiv 0$ *for j even;*

(ii) $(i+1)I_{i,j}^{\pm}(h) = j(I_{i+4,j-2}^{\pm}(h) - I_{i+3,j-2}^{\pm}(h))$ *for $i \geq -1$ and $j \geq 1$;*

(iii) $I_{3,1}^{\pm}(h) = I_{2,1}^{\pm}(h)$ *and* $I_{i,1}^{\pm}(h) = \frac{4i+6}{3i+9}I_{i-1,1}^{\pm}(h) + \frac{4i-12}{i+3}hI_{i-4,1}^{\pm}(h)$ *for $i \geq 4$.*

Proof. Because all the periodic orbits $\Gamma_h^{\pm}$ are symmetric with respect to the x-axis, it is straightforward to get the statement (i).

Note that $x^4/4 = H(x,y) - y^2/2 + x^3/3$. For $i \geq -1$ and $j \geq 1$, we have

$$\begin{aligned}
x^{i+4}y^{j-2}\mathrm{d}x &= x^{i+1}y^{j-2}\mathrm{d}(x^4/4)\\
&= x^{i+1}y^{j-2}\mathrm{d}\left(H - y^2/2 + x^3/3\right)\\
&= x^{i+1}y^{j-2}\mathrm{d}H - x^{i+1}y^{j-1}\mathrm{d}y + x^{i+3}y^{j-2}\mathrm{d}x\\
&= x^{i+1}y^{j-2}\mathrm{d}H - \mathrm{d}(x^{i+1}y^j/j) + \frac{i+1}{j}x^iy^j\mathrm{d}x + x^{i+3}y^{j-2}\mathrm{d}x.
\end{aligned}$$

Integrating its both sides along $\Gamma_h^{\pm}$ yields the statement (ii).

Taking $j = 3$ and replacing i by $i-4$ in (ii), for $i \geq 3$ we have

$$\begin{aligned}
I_{i,1}^{\pm}(h) &= I_{i-1,1}^{\pm}(h) + \frac{i-3}{3}I_{i-4,3}^{\pm}(h)\\
&= I_{i-1,1}^{\pm}(h) + \frac{i-3}{3}\oint_{\Gamma_h^{\pm}}\left(2h + \frac{2}{3}x^3 - \frac{1}{2}x^4\right)x^{i-4}y\mathrm{d}x\\
&= I_{i-1,1}^{\pm}(h) + \frac{2(i-3)}{3}hI_{i-4,1}^{\pm}(h) + \frac{2(i-3)}{9}I_{i-1,1}^{\pm}(h) - \frac{i-3}{6}I_{i,1}^{\pm}(h),
\end{aligned}$$

which yields the statement (iii). The proof is completed. □

Then by Lemma 2.1 for any $i \geq 0$ and $j \geq 1$, $I_{i,j}^{\pm}(h)$ can be expressed in the form

$$I_{i,j}^{\pm}(h) = p_1(h)I_{0,1}^{\pm}(h) + p_2(h)I_{1,1}^{\pm}(h) + p_3(h)I_{2,1}^{\pm}(h),$$

where $p_i(h)$, $i = 1, 2, 3$, are polynomials in h. Thus in order to expand $I_{i,j}^{\pm}(h)$ near $h = 0$, we only need to study the asymptotic expansions of $I_{0,1}^{\pm}(h)$, $I_{1,1}^{\pm}(h)$ and $I_{2,1}^{\pm}(h)$. To this end, we derive the Picard-Fuchs equation for $I_{i,1}^{\pm}(h)$, $i = 1, 2, 3$ in the next lemma.

Lemma 2.2. *Let $\boldsymbol{X}^{\pm}(h) = (I_{0,1}^{\pm}(h), I_{1,1}^{\pm}(h), I_{2,1}^{\pm}(h))^T$. Then $\boldsymbol{X}^{\pm}(h)$ satisfies the Picard-Fuchs equation*

$$P(h)\frac{\mathrm{d}}{\mathrm{d}h}\boldsymbol{X}^{\pm}(h) = (\boldsymbol{A}_1h + \boldsymbol{A}_0)\boldsymbol{X}^{\pm}(h), \tag{2.1}$$

where $P(h) = 12h(12h+1)$,

$$\boldsymbol{A}_0 = \begin{pmatrix} 10 & 2 & -15\\ 0 & 14 & -15\\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{A}_1 = \begin{pmatrix} 108 & 0 & 0\\ -12 & 144 & 0\\ -12 & -24 & 180 \end{pmatrix}. \tag{2.2}$$

Proof. Noting $y^2 = 2h + \frac{2}{3}x^3 - \frac{1}{2}x^4$ along $\Gamma_h^\pm$, we have

$$I_{i,1}^\pm(h) = \oint_{\Gamma_h^\pm} \frac{x^i y^2}{y}\,\mathrm{d}x = 2hI_{i,-1}^\pm(h) + \frac{2}{3}I_{i+3,-1}^\pm(h) - \frac{1}{2}I_{i+4,-1}^\pm(h). \tag{2.3}$$

By (ii)$|_{j=1}$ of Lemma 2.1, for $i \geq -1$ we get

$$I_{i+4,-1}^\pm(h) = (i+1)I_{i,1}^\pm(h) + I_{i+3,-1}^\pm(h). \tag{2.4}$$

Then substituting (2.4) into (2.3) yields

$$(3i+9)I_{i,1}^\pm(h) = 12hI_{i,-1}^\pm(h) + I_{i+3,-1}^\pm(h). \tag{2.5}$$

When $i = -1, 0, 1$, by (2.4) we can further get

$$\begin{aligned} I_{3,-1}^\pm(h) &= I_{2,-1}^\pm(h), \\ I_{4,-1}^\pm(h) &= I_{0,1}^\pm(h) + I_{3,-1}^\pm(h) = I_{0,1}^\pm(h) + I_{2,-1}^\pm(h), \\ I_{5,-1}^\pm(h) &= 2I_{1,1}^\pm(h) + I_{4,-1}^\pm(h) = 2I_{1,1}^\pm(h) + I_{0,1}^\pm(h) + I_{2,-1}^\pm(h). \end{aligned} \tag{2.6}$$

Taking $i = 0, 1, 2$ for (2.5) and then removing $I_{3,-1}^\pm(h)$, $I_{4,-1}^\pm(h)$ and $I_{5,-1}^\pm(h)$ by (2.6), we obtain

$$\begin{aligned} 9I_{0,1}^\pm(h) &= 12hI_{0,-1}^\pm(h) + I_{2,-1}^\pm(h), \\ -I_{0,1}^\pm(h) + 12I_{1,1}^\pm(h) &= 12hI_{1,-1}^\pm(h) + I_{2,-1}^\pm(h), \\ -I_{0,1}^\pm(h) - 2I_{1,1}^\pm(h) + 15I_{2,1}(h) &= (12h+1)I_{2,-1}^\pm(h), \end{aligned}$$

which yields (2.1) because $\frac{\mathrm{d}}{\mathrm{d}h}I_{i,1}^\pm(h) = I_{i,-1}^\pm(h)$. The proof is completed. □

Next, we shall use the Picard-Fuchs equation (2.1) to compute the asymptotic expansions of $I_{0,1}^\pm(h)$, $I_{1,1}^\pm(h)$ and $I_{2,1}^\pm(h)$ near $h = 0$. By [14], $\boldsymbol{X}^\pm(h)$ can be expanded in the following form

$$\boldsymbol{X}^\pm(h) = \sum_{j=0}^{\infty} \boldsymbol{a}_j^\pm h^j + \sum_{j=0}^{\infty} (\boldsymbol{b}_j^\pm |h|^{\frac{5}{6}} + \boldsymbol{c}_j^\pm |h|^{\frac{7}{6}})|h|^j, \quad 0 \leq \pm h \ll 1, \tag{2.7}$$

where $\boldsymbol{a}_j^\pm$, $\boldsymbol{b}_j^\pm$ and $\boldsymbol{c}_j^\pm$ are vector coefficients. We have the next lemma.

Lemma 2.3. *Let* $\boldsymbol{X}^\pm(h) = (I_{0,1}^\pm(h), I_{1,1}^\pm(h), I_{2,1}^\pm(h))^T$. *Then for the vector coefficients in* (2.7) *we have*

$$\begin{aligned} &\boldsymbol{A}_0\,\boldsymbol{a}_0^\pm = 0, \qquad (\boldsymbol{A}_0 - 10\,\boldsymbol{E}_3)\boldsymbol{b}_0^\pm = 0, \qquad (\boldsymbol{A}_0 - 14\,\boldsymbol{E}_3)\boldsymbol{c}_0^\pm = 0, \\ &(\boldsymbol{A}_0 - 12j\,\boldsymbol{E}_3)\,\boldsymbol{a}_j^\pm = -(\boldsymbol{A}_1 - 144(j-1)\,\boldsymbol{E}_3)\,\boldsymbol{a}_{j-1}^\pm, \ j \geq 1, \\ &(\boldsymbol{A}_0 - (12j+10)\,\boldsymbol{E}_3)\,\boldsymbol{b}_j^\pm = \mp(\boldsymbol{A}_1 - (144j-24)\,\boldsymbol{E}_3)\,\boldsymbol{b}_{j-1}^\pm, \ j \geq 1, \\ &(\boldsymbol{A}_0 - (12j+14)\,\boldsymbol{E}_3)\,\boldsymbol{c}_j^\pm = \mp(\boldsymbol{A}_1 - (144j+24)\,\boldsymbol{E}_3)\,\boldsymbol{c}_{j-1}^\pm, \ j \geq 1, \end{aligned} \tag{2.8}$$

where $\boldsymbol{A}_0$ *and* $\boldsymbol{A}_1$ *are matrices given in* (2.2), *and* $\boldsymbol{E}_3$ *is the* 3×3 *identity matrix.*

We shall use the Picard-Fuchs equation (2.1) to prove Lemma 2.3. By (2.7) we notice that the derivative $\frac{\mathrm{d}}{\mathrm{d}h}\boldsymbol{X}^{\pm}(h)$ is not defined at $h = 0$ when $\boldsymbol{b}_0^{\pm} \neq 0$. However, the left-hand side of (2.1) can be rewritten as

$$P(h)\frac{\mathrm{d}}{\mathrm{d}h}\boldsymbol{X}^{\pm}(h) = \frac{\mathrm{d}}{\mathrm{d}h}(P(h)\boldsymbol{X}^{\pm}(h)) - P'(h)\boldsymbol{X}^{\pm}(h), \tag{2.9}$$

which can be expanded as a convergent series near $h = 0$ by (2.7). Then $P(h)\frac{\mathrm{d}}{\mathrm{d}h}\boldsymbol{X}^{\pm}(h)$ can be well-defined at $h = 0$, which implies that we can use (2.1) and (2.9) to study the asymptotic expansion (2.7) for $\boldsymbol{X}^{\pm}(h)$.

Proof. Here we only present the proof for $\boldsymbol{X}^{+}(h)$. The case for $\boldsymbol{X}^{-}(h)$ can be similarly proved. By (2.7) we have

$$\begin{aligned} P(h)\boldsymbol{X}^{+}(h) = {} & 12\boldsymbol{a}_0^{+}h + \sum_{j=2}^{\infty}(12\boldsymbol{a}_{j-1}^{+} + 144\boldsymbol{a}_{j-2}^{+})h^{j} + 12\boldsymbol{b}_0^{+}h^{\frac{11}{6}} + 12\boldsymbol{c}_0^{+}h^{\frac{13}{6}} \\ & + \sum_{j=2}^{\infty}\left[(12\boldsymbol{b}_{j-1}^{+} + 144\boldsymbol{b}_{j-2}^{+})h^{\frac{5}{6}} + (12\boldsymbol{c}_{j-1}^{+} + 144\boldsymbol{c}_{j-2}^{+})h^{\frac{7}{6}}\right]h^{j}, \\ P'(h)\boldsymbol{X}^{+}(h) = {} & 12\boldsymbol{a}_0^{+} + \sum_{j=1}^{\infty}(12\boldsymbol{a}_{j}^{+} + 288\boldsymbol{a}_{j-1}^{+})h^{j} + 12\boldsymbol{b}_0^{+}h^{\frac{5}{6}} + 12\boldsymbol{c}_0^{+}h^{\frac{7}{6}} \\ & + \sum_{j=1}^{\infty}\left[(12\boldsymbol{b}_{j}^{+} + 288\boldsymbol{b}_{j-1}^{+})h^{\frac{5}{6}} + (12\boldsymbol{c}_{j}^{+} + 288\boldsymbol{c}_{j-1}^{+})h^{\frac{7}{6}}\right]h^{j}. \end{aligned}$$

Then by (2.9), the left-hand side of (2.1) can be expanded as

$$\begin{aligned} P(h)\frac{\mathrm{d}}{\mathrm{d}h}\boldsymbol{X}^{+}(h) = {} & \sum_{j=1}^{\infty}[12j\boldsymbol{a}_{j}^{+} + 144(j-1)\boldsymbol{a}_{j-1}^{+}]h^{j} + 10\boldsymbol{b}_0^{+}h^{\frac{5}{6}} + 14\boldsymbol{c}_0^{+}h^{\frac{7}{6}} \\ & + \sum_{j=1}^{\infty}[(12j+10)\boldsymbol{b}_{j}^{+} + (144j-24)\boldsymbol{b}_{j-1}^{+}]h^{j+\frac{5}{6}} \\ & + \sum_{j=1}^{\infty}[(12j+14)\boldsymbol{c}_{j}^{+} + (144j+24)\boldsymbol{c}_{j-1}^{+}]h^{j+\frac{7}{6}}. \end{aligned} \tag{2.10}$$

On the other hand, substituting (2.7) into the right-hand side of (2.1) yields

$$\begin{aligned} (\boldsymbol{A}_1 h + \boldsymbol{A}_0)\boldsymbol{X}^{+}(h) = {} & \boldsymbol{A}_0\boldsymbol{a}_0^{+} + \sum_{j=1}^{\infty}(\boldsymbol{A}_0\boldsymbol{a}_{j}^{+} + \boldsymbol{A}_1\boldsymbol{a}_{j-1}^{+})h^{j} + \boldsymbol{A}_0\boldsymbol{b}_0^{+}h^{\frac{5}{6}} \\ & + \boldsymbol{A}_0\boldsymbol{c}_0^{+}h^{\frac{7}{6}} + \sum_{j=1}^{\infty}\left[(\boldsymbol{A}_0\boldsymbol{b}_{j}^{+} + \boldsymbol{A}_1\boldsymbol{b}_{j-1}^{+})h^{j+\frac{5}{6}} + (\boldsymbol{A}_0\boldsymbol{c}_{j}^{+} + \boldsymbol{A}_1\boldsymbol{c}_{j-1}^{+})h^{j+\frac{7}{6}}\right]. \end{aligned} \tag{2.11}$$

Comparing the coefficients in (2.10) and (2.11), we can get (2.8) for $\boldsymbol{a}_j^{+}$, $\boldsymbol{b}_j^{+}$ and $\boldsymbol{c}_j^{+}$, $j \geq 0$. The proof is completed. □

Note that by (2.2) for any $j \geq 1$ we have

$$\det(\boldsymbol{A}_0 - (12j+\lambda)\boldsymbol{E}_3) \neq 0, \quad \lambda \in \{0, 10, 14\}.$$

Then by the last three equations of (2.8), for any $j \geq 1$ the coefficients $\boldsymbol{a}_j^{\pm}$, $\boldsymbol{b}_j^{\pm}$ and $\boldsymbol{c}_j^{\pm}$ are uniquely determined by $\boldsymbol{a}_{j-1}^{\pm}$, $\boldsymbol{b}_{j-1}^{\pm}$ and $\boldsymbol{c}_{j-1}^{\pm}$, respectively. To be precise, for $j \geq 1$ we have

$$\begin{aligned} \boldsymbol{a}_j^{\pm} &= -(\boldsymbol{A}_0 - 12j\,\boldsymbol{E}_3)^{-1}(\boldsymbol{A}_1 - 144(j-1)\,\boldsymbol{E}_3)\,\boldsymbol{a}_{j-1}^{\pm}, \\ \boldsymbol{b}_j^{\pm} &= \mp(\boldsymbol{A}_0 - (12j+10)\,\boldsymbol{E}_3)^{-1}(\boldsymbol{A}_1 - (144j-24)\,\boldsymbol{E}_3)\,\boldsymbol{b}_{j-1}^{\pm}, \\ \boldsymbol{c}_j^{\pm} &= \mp(\boldsymbol{A}_0 - (12j+14)\,\boldsymbol{E}_3)^{-1}(\boldsymbol{A}_1 - (144j+24)\,\boldsymbol{E}_3)\,\boldsymbol{c}_{j-1}^{\pm}. \end{aligned} \tag{2.12}$$

Then when we have the values of $\boldsymbol{a}_0^{\pm}$, $\boldsymbol{b}_0^{\pm}$ and $\boldsymbol{c}_0^{\pm}$, we can get all the remaining coefficients in (2.7) by using (2.12).

Suppose that $\boldsymbol{a}_0^{\pm}$, $\boldsymbol{b}_0^{\pm}$ and $\boldsymbol{c}_0^{\pm}$ are given by

$$\boldsymbol{a}_0^{\pm} = (a_{10}^{\pm}, a_{20}^{\pm}, a_{30}^{\pm})^T, \;\; \boldsymbol{b}_0^{\pm} = (b_{10}^{\pm}, b_{20}^{\pm}, b_{30}^{\pm})^T, \;\; \boldsymbol{c}_0^{\pm} = (c_{10}^{\pm}, c_{20}^{\pm}, c_{30}^{\pm})^T.$$

By the first three equations of (2.8), we can get

$$a_{20}^{\pm} = \frac{5}{6}a_{10}^{\pm}, \quad a_{30}^{\pm} = \frac{7}{9}a_{10}^{\pm}, \quad b_{20}^{\pm} = b_{30}^{\pm} = 0, \quad c_{20}^{\pm} = 2c_{10}^{\pm}, \quad c_{30}^{\pm} = 0. \tag{2.13}$$

Using the method presented in [14], we obtain

$$\begin{aligned} a_{10}^{\pm} &= \oint_{\Gamma_0} y\,\mathrm{d}x = 2\int_0^{\frac{4}{3}} x^{\frac{3}{2}}\sqrt{\frac{2}{3} - \frac{1}{2}x}\,\mathrm{d}x = \frac{4\sqrt{2}}{27}\pi, \\ b_{10}^{\pm} &= \widetilde{r}_{00}B_{00}^{\pm} = 2\sqrt{2} \times (-\frac{1}{3})^{-\frac{1}{3}}B_{00}^{\pm} = -2\sqrt[6]{72}B_{00}^{\pm}, \\ c_{10}^{\pm} &= \widetilde{r}_{10}B_{10}^{\pm} = -\frac{4}{3}\sqrt{2} \times (-\frac{1}{3})^{-\frac{5}{3}} \times \frac{1}{4}B_{10}^{\pm} = \sqrt[6]{648}B_{10}^{\pm}, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} B_{00}^{+} &= -\frac{3}{5}\int_{-\infty}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^3}} < 0, \qquad B_{00}^{-} = \frac{3}{5}\int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x^3)}} > 0, \\ B_{10}^{+} &= \frac{3}{7}\left(\int_1^{-1}\frac{x\mathrm{d}x}{\sqrt{1-x^3}} - \int_0^1 \frac{x^{\frac{3}{2}}\mathrm{d}x}{\sqrt{1+x^3}+1+x^3} - 2\right) < 0, \\ B_{10}^{-} &= -\frac{3}{7}\left(\int_0^1 \frac{x^{\frac{3}{2}}\mathrm{d}x}{\sqrt{1-x^3}+1-x^3} - 2\right) > 0. \end{aligned} \tag{2.15}$$

Then by (2.12), (2.13) and (2.14) we can derive the following lemma.

Lemma 2.4. *Abelian integrals $I_{0,1}^{\pm}(h)$, $I_{1,1}^{\pm}(h)$ and $I_{2,1}^{\pm}(h)$ have the following asymptotic expansions near $h = 0$:*

$$\begin{aligned} I_{0,1}^{\pm}(h) &= \frac{4\sqrt{2}}{27}\pi - 2b_0^{\pm}|h|^{\frac{5}{6}} + b_1^{\pm}|h|^{\frac{7}{6}} \pm \frac{35}{44}b_0^{\pm}|h|^{\frac{11}{6}} \mp \frac{385}{208}b_1^{\pm}|h|^{\frac{13}{6}} + \cdots, \\ I_{1,1}^{\pm}(h) &= \frac{10\sqrt{2}}{81}\pi + \sqrt{8}\pi h + 2b_1^{\pm}|h|^{\frac{7}{6}} \pm \frac{21}{22}b_0^{\pm}|h|^{\frac{11}{6}} \mp \frac{55}{26}b_1^{\pm}|h|^{\frac{13}{6}} + \cdots, \\ I_{2,1}^{\pm}(h) &= \frac{28\sqrt{2}}{243}\pi + \frac{4\sqrt{2}}{3}\pi h \pm \frac{12}{11}b_0^{\pm}|h|^{\frac{11}{6}} \mp \frac{30}{13}b_1^{\pm}|h|^{\frac{13}{6}} + \cdots, \end{aligned} \tag{2.16}$$

where $b_0^{\pm} = \sqrt[6]{72}\,B_{00}^{\pm}$ and $b_1^{\pm} = \sqrt[6]{648}\,B_{10}^{\pm}$ with $B_{00}^{\pm}$ and $B_{10}^{\pm}$ given in (2.15).

Then for any Melnikov functions $M^{\pm}(h,\boldsymbol{\delta})$ of system (1.2) given by

$$M^{\pm}(h,\boldsymbol{\delta})=\sum_{i=0}^{n}\sum_{j=0}^{n-i}a_{ij}(\boldsymbol{\delta})I_{i,j}^{\pm}(h),\quad n\in\mathbb{N}^{+},$$

we can use Lemmas 2.1 and 2.4 to compute the coefficients of the following asymptotic expansions

$$\begin{aligned}M^{+}(h,\boldsymbol{\delta})&=c_0(\boldsymbol{\delta})+\sum_{j\geq 0}\left(c_{3j+1}(\boldsymbol{\delta})h^{j+\frac{5}{6}}+c_{3j+2}(\boldsymbol{\delta})h^{j+1}+c_{3j+3}(\boldsymbol{\delta})h^{j+\frac{7}{6}}\right),\\ M^{-}(h,\boldsymbol{\delta})&=\tilde{c}_0(\boldsymbol{\delta})+\sum_{j\geq 0}\left(\tilde{c}_{3j+1}(\boldsymbol{\delta})|h|^{j+\frac{5}{6}}+\tilde{c}_{3j+2}(\boldsymbol{\delta})h^{j+1}+\tilde{c}_{3j+3}(\boldsymbol{\delta})|h|^{j+\frac{7}{6}}\right),\end{aligned}\tag{2.17}$$

where $\boldsymbol{\delta}\in\mathbb{R}^m$ is a vector consisting of all the free parameters in system (1.2). Note that by (2.12), (2.13) and (2.14) we can get

$$\boldsymbol{a}_j^{+}=\boldsymbol{a}_j^{-},\quad \boldsymbol{b}_j^{+}=(-1)^{j+1}\rho_1\,\boldsymbol{b}_j^{-},\quad \boldsymbol{c}_j^{+}=(-1)^{j+1}\rho_3\,\boldsymbol{c}_j^{-},\quad j\geq 0,$$

where $\rho_1=-\frac{B_{00}^{+}}{B_{00}^{-}}>0$ and $\rho_3=-\frac{B_{10}^{+}}{B_{10}^{-}}>0$. Then for (2.17) using Lemma 2.1 we can further obtain

$$c_0=\tilde{c}_0,\quad c_{3j+2}=\tilde{c}_{3j+2},\quad c_{3j+k}=(-1)^{j+1}\rho_k\,\tilde{c}_{3j+k},\quad k=1,3,\quad j\geq 0.\tag{2.18}$$

The identities in (2.18) show the relation between the coefficients c_k and $\tilde{c}_k$, $k\geq 0$, which can be also obtained by [29]. In [29] Yang and Han presented the relation between the coefficients of the asymptotic expansions of $M^{\pm}(h)$ near a cuspidal loop defined by an analytic Hamiltonian. Based on this result, bifurcations of limit cycles near a cuspidal loop can be investigated by only using the coefficients of the asymptotic expansion of $M^{+}(h)$. The next lemma is derived from [29] for system (1.2).

Lemma 2.5. *Assume that $M^{\pm}(h,\boldsymbol{\delta})$ are the first nonvanishing Melnikov functions in* (1.5) *with* (2.17) *holding. If there exist $\boldsymbol{\delta}_0\in\mathbb{R}^m$ and $3n+1\leq k\leq 3(n+1)$ such that*

$$\begin{aligned}&c_0(\boldsymbol{\delta}_0)=c_1(\boldsymbol{\delta}_0)=\cdots=c_{k-1}(\boldsymbol{\delta}_0)=0,\quad c_k(\boldsymbol{\delta}_0)\neq 0,\\ &\operatorname{rank}\frac{\partial(c_0,c_1,\cdots,c_{k-1})}{\partial\boldsymbol{\delta}}\bigg|_{\boldsymbol{\delta}=\boldsymbol{\delta}_0}=k,\end{aligned}$$

then system (1.2) *can have $2k-(n+1)$ limit cycles near Γ_0 with k (or $k-(n+1)$) limit cycles inside Γ_0 and $k-(n+1)$ (or k) limit cycles outside Γ_0 for some $(\varepsilon,\boldsymbol{\delta})$ near $(0,\boldsymbol{\delta}_0)$.*

3. Proof of Theorems 1.1 and 1.2

In this section, we shall prove Theorems 1.1 and 1.2 by using Lemmas 2.1, 2.4 and 2.5.

Proof of Theorem 1.1. Note that Melnikov functions $M_1^{\pm}(h)$ in (1.6) can be written in the form

$$M_1^{\pm}(h)=\oint_{\Gamma_h^{\pm}}\left(Q_1(x,y)+\int\frac{\partial}{\partial x}P_1(x,y)\mathrm{d}y\right)\mathrm{d}x.\tag{3.1}$$

Then by (1.3) and (3.1) we have

$$M_1^{\pm}(h) = A_1 I_{0,1}^{\pm}(h) + A_2 I_{1,1}^{\pm}(h) + A_3 I_{2,1}^{\pm}(h) + A_4 I_{0,3}^{\pm}(h), \tag{3.2}$$

where

$$\begin{aligned} &A_1 = p_{101} + q_{011},\ A_2 = 2p_{201} + q_{111}, \\ &A_3 = 3p_{301} + q_{211},\ A_4 = \frac{1}{3}p_{121} + q_{031}. \end{aligned} \tag{3.3}$$

By Lemma 2.5 we only need to study the coefficients of the asymptotic expansion of $M_1^+(h)$ for $0 \le h \ll 1$. By the statements (ii) and (iii) of Lemma 2.1, we get

$$\begin{aligned} I_{4,1}^+(h) &= \frac{4}{7}hI_{0,1}^+(h) + \frac{22}{21}I_{3,1}^+(h) = \frac{2}{21}(6hI_{0,1}^+(h) + 11I_{2,1}^+(h)), \\ I_{0,3}^+(h) &= 3I_{4,1}^+(h) - 3I_{3,1}^+(h) = \frac{1}{7}(12hI_{0,1}^+(h) + I_{2,1}^+(h)). \end{aligned} \tag{3.4}$$

Substituting the second identity of (3.4) into (3.2) yields

$$M_1^+(h) = (A_1 + \frac{12}{7}A_4 h)\, I_{0,1}^+(h) + A_2\, I_{1,1}^+(h) + (A_3 + \frac{1}{7}A_4)\, I_{2,1}^+(h).$$

Then using (2.16) we further obtain

$$M_1^+(h) = c_0 + b_0^+ c_1 h^{\frac{5}{6}} + c_2 h + b_1^+ c_3 h^{\frac{7}{6}} + \cdots, \quad 0 < h \ll 1, \tag{3.5}$$

where

$$\begin{aligned} c_0 &= \frac{2\sqrt{2}\pi}{243}(18A_1 + 15A_2 + 14A_3 + 2A_4), \quad c_1 = -2A_1, \\ c_2 &= \frac{2\sqrt{2}\pi}{9}(9A_2 + 6A_3 + 2A_4), \qquad c_3 = A_1 + 2A_2. \end{aligned} \tag{3.6}$$

Then by (3.3) and (3.6) solving $c_0 = c_1 = c_2 = 0$ in q_{011}, q_{111} and q_{211} yields

$$\begin{aligned} &q_{011} = -p_{101}, \quad q_{111} = -2p_{201} - \frac{4}{27}(p_{121} + 3q_{031}), \\ &q_{211} = -3p_{301} + \frac{1}{9}(p_{121} + 3q_{031}), \end{aligned}$$

and $c_3 = -\frac{8}{27}(p_{121} + 3q_{031})$. Because c_0, c_1 and c_2 are linear in q_{011}, q_{111} and q_{211}, by Lemma 2.5 system (1.2) can have 5 limit cycles near Γ_0 with proper perturbations with $p_{121} + 3q_{031} \neq 0$.

By (3.6) it is easy to get

$$c_j = 0,\ j = 0,1,2,3 \iff A_j = 0,\ j = 1,2,3,4.$$

Then by (3.2) and (3.5) we obtain $M_1^{\pm}(h) \equiv 0$ if and only if $A_1 = A_2 = A_3 = A_4 = 0$, which yields (1.7).

Next, we assume $M_1^{\pm}(h) \equiv 0$, and then study the asymptotic expansion of $M_2^+(h)$ near $h = 0$. With (1.7) holding, we get

$$Q_1 \mathrm{d}x - P_1 \mathrm{d}y = r_1 \mathrm{d}H + \mathrm{d}R_1,$$

where

$$r_1(x,y) = -(p_{111}+2q_{021})x - (p_{211}+q_{121})x^2. \tag{3.7}$$

Then using Françoise's algorithm [3], we have

$$M_2^{\pm}(h) = \oint_{\Gamma_h^{\pm}} Q\mathrm{d}x - P\mathrm{d}y,$$

where

$$\begin{aligned} Q(x,y) &= Q_2(x,y) + r_1(x,y)Q_1(x,y),\\ P(x,y) &= P_2(x,y) + r_1(x,y)P_1(x,y). \end{aligned}$$

Then by the statement (i) of Lemma 2.1, $M_2^{\pm}(h)$ can be simplified into the form

$$\begin{aligned} M_2^{\pm}(h) &= \oint_{\Gamma_h^{\pm}} \left(Q + \int \frac{\partial}{\partial x} P\mathrm{d}y\right)\mathrm{d}x\\ &= \sum_{j=1}^{5} B_j I_{j-1,1}^{\pm}(h) + \sum_{j=6}^{8} B_j I_{j-6,3}^{\pm}(h), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} B_1 &= \ p_{102}+q_{012}, \qquad B_2 = 2p_{202}+q_{112}-(p_{111}+2q_{021})p_{101},\\ B_3 &= \ 3p_{302}+q_{212}-(p_{111}+2q_{021})p_{201}-(2p_{211}+2q_{121})p_{101},\\ B_4 &= -(p_{111}+2q_{021})p_{301}-(2p_{211}+2q_{121})p_{201},\\ B_5 &= -2(p_{211}+q_{121})p_{301}, \quad B_6 = \frac{1}{3}p_{122}+q_{032}-\frac{1}{3}(p_{111}+2q_{021})p_{021},\\ B_7 &= \ (p_{111}+2q_{021})q_{031}-\frac{2}{3}(p_{211}+q_{121})p_{021}, \quad B_8 = 2(p_{211}+q_{121})q_{031}. \end{aligned} \tag{3.9}$$

Using (ii) and (iii) of Lemma 2.1, we get

$$\begin{aligned} I_{5,1}^{+} &= \ hI_{1,1}^{+} + \frac{13}{12}I_{4,1}^{+} = \frac{13}{21}hI_{0,1}^{+} + hI_{1,1}^{+} + \frac{143}{126}I_{2,1}^{+},\\ I_{1,3}^{+} &= \ \frac{3}{2}I_{5,1}^{+} - \frac{3}{2}I_{4,1}^{+} = \frac{1}{14}hI_{0,1}^{+} + \frac{3}{2}hI_{1,1}^{+} + \frac{11}{84}I_{2,1}^{+},\\ I_{6,1}^{+} &= \ \frac{4}{3}hI_{2,1}^{+} + \frac{10}{9}I_{5,1}^{+} = \frac{130}{189}hI_{0,1}^{+} + \frac{10}{9}hI_{1,1}^{+} + (\frac{4}{3}h + \frac{715}{567})I_{2,1}^{+},\\ I_{2,3}^{+} &= \ I_{6,1}^{+} - I_{5,1}^{+} = \frac{13}{189}hI_{0,1}^{+} + \frac{1}{9}hI_{1,1}^{+} + (\frac{4}{3}h + \frac{143}{1134})I_{2,1}^{+}. \end{aligned} \tag{3.10}$$

Then substituting (3.4) and (3.10) into (3.8) yields

$$M_2^{+}(h) = (B_1 + C_3h)\,I_{0,1}^{+} + (B_2 + C_4h)\,I_{1,1}^{+} + (C_5 + C_6h)\,I_{2,1}^{+}, \tag{3.11}$$

where

$$\begin{aligned} C_3 &= \ \frac{4}{7}B_5 + \frac{12}{7}B_6 + \frac{1}{14}B_7 + \frac{13}{189}B_8, \qquad C_4 = \frac{3}{2}B_7 + \frac{1}{9}B_8,\\ C_5 &= \ B_3 + B_4 + \frac{22}{21}B_5 + \frac{1}{7}B_6 + \frac{11}{84}B_7 + \frac{143}{1134}B_8, \quad C_6 = \frac{4}{3}B_8. \end{aligned} \tag{3.12}$$

Then by (2.16) we can get

$$M_2^+(h) = \bar{c}_0 + b_0^+ \bar{c}_1 h^{\frac{5}{6}} + \bar{c}_2 h + b_1^+ \bar{c}_3 h^{\frac{7}{6}} + b_0^+ \bar{c}_4 h^{\frac{11}{6}} + \bar{c}_5 h^2 + \cdots,$$

where

$$\begin{aligned}
\bar{c}_0 =& \frac{\sqrt{2}\pi}{19683}(2916B_1 + 2430B_2 + 2268B_3 + 2268B_4 + 2376B_5 \\
& + 324B_6 + 297B_7 + 286B_8), \\
\bar{c}_1 =& -2B_1, \\
\bar{c}_2 =& \frac{2\sqrt{2}\pi}{81}(81B_2 + 54B_3 + 54B_4 + 60B_5 + 18B_6 + 15B_7 + 14B_8), \\
\bar{c}_3 =& B_1 + 2B_2, \\
\bar{c}_4 =& \frac{1}{44}(35B_1 + 42B_2 + 48B_3 + 48B_4 - 144B_6), \\
\bar{c}_5 =& \sqrt{2}\pi(3B_7 + 2B_8).
\end{aligned} \tag{3.13}$$

Note that $\bar{c}_j$, $j = 0, 1, \cdots, 5$, are linear in p_{ij2} and q_{ij2}. By (3.9) and (3.13), from $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = 0$ we can get a unique solution in p_{122}, q_{012}, q_{112} and q_{212}, under which we have

$$\begin{aligned}
\bar{c}_4 =& -\frac{5}{3}p_{021}(q_{121} + p_{211}) \\
& + \frac{1}{54}q_{031}(135p_{111} + 244p_{211} + 270q_{021} + 244q_{121}), \\
\bar{c}_5 =& \sqrt{2}\pi[-2p_{021}(q_{121} + p_{211}) + q_{031}(3p_{111} + 4p_{211} + 6q_{021} + 4q_{121})].
\end{aligned} \tag{3.14}$$

Then if $q_{121} + p_{211} \neq 0$, by (3.14) solving $\bar{c}_4 = 0$ in p_{021} yields

$$p_{021} = \frac{q_{031}(135p_{111} + 244p_{211} + 270q_{021} + 244q_{121})}{90(q_{121} + p_{211})}, \tag{3.15}$$

and $\bar{c}_5 = -\frac{64\sqrt{2}\pi}{45}q_{031}(q_{121} + p_{211})$. Note that

$$\det\left(\frac{\partial(\bar{c}_0, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)}{\partial(p_{122}, q_{012}, q_{112}, q_{212}, p_{021})}\right)\Bigg|_{(3.15)} = \frac{2560\,\pi^2}{19683}(q_{121} + p_{211}).$$

Then by Lemma 2.5 for $q_{031}(q_{121} + p_{211}) \neq 0$ system (1.2) can have 8 limit cycles near Γ_0 with proper perturbations.

By (3.12) and (3.13), it is straightforward to get that $\bar{c}_0 = \bar{c}_1 = \cdots = \bar{c}_5 = 0$ if and only if

$$B_1 = B_2 = B_7 = B_8 = 0, \quad B_3 = 3B_6 - B_4, \quad B_5 = -3B_6, \tag{3.16}$$

which is equivalent to $B_1 = B_2 = 0$, $C_3 = C_4 = C_5 = C_6 = 0$. Then by (3.11), $M_2^+(h) \equiv 0$ if and only if (3.16) holds. By (3.9), we can get all the solutions (1.8) for (3.16). The proof is completed. □

Next, we present the proof for Theorem 1.2.

Proof of Theorem 1.2. To show the existence of ten limit cycles near Γ_0 in system (1.2), for the sake of simplicity we use the following perturbations

$$\begin{aligned}
P_1(x, y) =& y(p_{031}y^2 + p_{211}x^2 + p_{021}y + p_{111}x), \\
P_2(x, y) =& xy[p_{021}(2q_{021} + p_{111})y + p_{212}x], \\
Q_1(x, y) =& -y^2(p_{211}x - q_{021}), \quad Q_2(x, y) = 0,
\end{aligned} \tag{3.17}$$

and $P_3(x,y)$ and $Q_3(x,y)$ are given in (1.3). It is easy to get that P_i and Q_i, $i=1,2$ in (3.17) satisfy (1.7) and the condition S_2 of (1.8). Then by Theorem 1.1 we can get $M_k^{\pm}(h)\equiv 0$, $k=1,2$.

It is straightforward to get

$$\begin{aligned}r_2 = &\frac{2}{5}p_{031}(2q_{021}+p_{111})x^5 - \frac{1}{2}p_{031}(2q_{021}+p_{111})x^4 + \frac{1}{3}p_{211}(2q_{021}+p_{111})x^3\\ &+ (p_{111}^2+3p_{111}q_{021}+2q_{021}^2-p_{212})x^2 + p_{031}(2q_{021}+p_{111})xy^2,\end{aligned}$$

which satisfies

$$(Q_2+r_1Q_1)\mathrm{d}x - (P_2+r_1P_1)\mathrm{d}y = r_2\mathrm{d}H + \mathrm{d}R_2,$$

where $r_1=-(p_{111}+2q_{021})x$ by (3.7), and R_2 is a polynomial of degree 9 in (x,y). Then using Françoise's algorithm [3], we obtain

$$M_3^{\pm}(h) = \oint_{\Gamma_h^{\pm}} \widetilde{Q}_3\mathrm{d}x - \widetilde{P}_3\mathrm{d}y = \oint_{\Gamma_h^{\pm}}\left(\widetilde{Q}_3 + \int \frac{\partial}{\partial x}\widetilde{P}_3\mathrm{d}y\right)\mathrm{d}x, \tag{3.18}$$

where

$$\begin{aligned}\widetilde{Q}_3(x,y) =&\ Q_3(x,y) + r_1(x,y)Q_2(x,y) + r_2(x,y)Q_1(x,y),\\ \widetilde{P}_3(x,y) =&\ P_3(x,y) + r_1(x,y)P_2(x,y) + r_2(x,y)P_1(x,y).\end{aligned}$$

Similarly as in the proof of Theorem 1.1, we shall use Lemmas 2.1 and 2.4 to study the asymptotic expansion of $M_3^+(h)$ in (3.18). By (ii) and (iii) of Lemma 2.1, we derive

$$\begin{aligned}I_{3,3}^+ =&\ \frac{13}{189}hI_{0,1}^+ + \frac{1}{9}hI_{1,1}^+ + \left(\frac{4}{3}h + \frac{143}{1134}\right)I_{2,1}^+,\\ I_{4,3}^+ =&\ \left(\frac{442}{6237}h + \frac{48}{77}h^2\right)I_{0,1}^+ + \frac{34}{297}hI_{1,1}^+ + \left(\frac{221}{1701} + \frac{988}{693}h\right)I_{2,1}^+,\\ I_{0,5}^+ =&\ \left(\frac{65}{6237}h + \frac{240}{77}h^2\right)I_{0,1}^+ + \frac{5}{297}hI_{1,1}^+ + \left(\frac{65}{3402} + \frac{320}{693}h\right)I_{2,1}^+.\end{aligned} \tag{3.19}$$

Then by (3.4), (3.10) and (3.19), $M_3^+(h)$ can be simplified into the form

$$M_3^+(h) = (B_1 + B_2h + B_3h^2)I_{0,1}^+ + (B_4 + B_5h)I_{1,1}^+ + (B_6 + B_7h)I_{2,1}^+,$$

where

$$\begin{aligned}B_1 =&\ p_{103} + q_{013},\\ B_2 =&\ \frac{1}{18711}\big[-1782q_{021}^2 + (130p_{031} - 891p_{111} + 858p_{211})q_{021}\\ &+ (65p_{031} + 429p_{211})p_{111} - 891p_{212}\big]p_{021} + \frac{4}{7}p_{123} + \frac{12}{7}q_{033},\\ B_3 =&\ \frac{80}{77}p_{031}(2q_{021} + p_{111})p_{021}, \qquad B_4 = 2p_{203} + q_{113},\\ B_5 =&\ \frac{1}{891}(5p_{031}p_{111} + 10p_{031}q_{021} + 33p_{111}p_{211} - 891p_{111}q_{021}\\ &+ 66p_{211}q_{021} - 1782q_{021}^2 - 891p_{212})p_{021},\end{aligned} \tag{3.20}$$

$$B_6 = -\frac{1}{10206}\big[1782q_{021}^2 - (130p_{031} - 891p_{111} + 858p_{211})q_{021} - (65p_{031} + 429p_{211})p_{111} + 891p_{212}\big]p_{021} + q_{213} + \frac{1}{21}p_{123} + 3p_{303} + \frac{1}{7}q_{033},$$
$$B_7 = \frac{4}{2079}(2q_{021} + p_{111})(80p_{031} + 231p_{211})p_{021}.$$

Then by (2.16) we get

$$M_3^+(h) = c_0 + b_0^+ c_1 h^{\frac{5}{6}} + c_2 h + b_1^+ c_3 h^{\frac{7}{6}} + b_0^+ c_4 h^{\frac{11}{6}} + c_5 h^2 + b_1^+ c_6 h^{\frac{13}{6}} + \cdots,$$

where

$$\begin{aligned}
c_0 &= \frac{2\sqrt{2}\pi}{243}(18B_1 + 15B_4 + 14B_6), \qquad c_1 = -2B_1,\\
c_2 &= \frac{2\sqrt{2}\pi}{243}(18B_2 + 243B_4 + 15B_5 + 162B_6 + 14B_7),\\
c_3 &= B_1 + 2B_4,\\
c_4 &= \frac{1}{44}(35B_1 - 88B_2 + 42B_4 + 48B_6),\\
c_5 &= \frac{2\sqrt{2}\pi}{3}(3B_5 + 2B_7),\\
c_6 &= -\frac{1}{208}(385B_1 - 208B_2 + 440B_4 - 416B_5 + 480B_6).
\end{aligned} \tag{3.21}$$

By (3.20) and (3.21), solving $c_0 = c_1 = \cdots = c_5 = 0$ in p_{123}, q_{013}, q_{113}, q_{213}, p_{212} and p_{211} yields

$$\begin{aligned}
p_{123} &= \frac{5}{891}p_{021}p_{031}(2q_{021} + p_{111}) - 3q_{033}, \quad p_{211} = \frac{10}{33}p_{031},\\
p_{212} &= \frac{85}{297}p_{031}(2q_{021} + p_{111}) - p_{111}q_{021} - 2q_{021}^2, \quad q_{013} = -p_{103},\\
q_{113} &= -2p_{203}, \quad q_{213} = \frac{5}{891}p_{021}p_{031}(2q_{021} + p_{111}) - 3p_{303},
\end{aligned} \tag{3.22}$$

and then $c_6 = -\frac{160}{297}(2q_{021} + p_{111})p_{031}p_{021}$,

$$\det\left(\frac{\partial(c_0, c_1, c_2, c_3, c_4, c_5)}{\partial(p_{123}, q_{013}, q_{113}, q_{213}, p_{212}, p_{211})}\right)\Bigg|_{(3.22)} = -\frac{16384\sqrt{2}}{531441}\pi^3(2q_{021} + p_{111})p_{021}^2.$$

Then by Lemma 2.5 system (1.2) can have 10 limit cycles produced near Γ_0 with proper perturbations for $|\varepsilon|$ sufficiently small and parameters p_{123}, q_{013}, q_{113}, q_{213}, p_{212}, p_{211} close enough to (3.22) with $(2q_{021} + p_{111})p_{031}p_{021} \neq 0$. The proof is completed. □

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