

## ANALYSIS AND SIMULATION OF FRACTIONAL-ORDER WAVE-LIKE EQUATIONS UNDER CAPUTO FRACTIONAL OPERATOR

Abdul Hamid Ganie<sup>1,†</sup>

**Abstract** This paper examines the two methods for resolving nonlinear Caputo time-fractional wave-like equations with variable coefficients. These two methods have the names Homotopy perturbation transform method and Elzaki transform decomposition method. I first turn the problem into its differential partner with the use of the Elzaki transform, and then used the He's polynomials as well as adomian polynomials to calculate the nonlinear terms. The simplicity and accuracy of given techniques are illustrated by three different numerical examples. The obtained results indicate the reliability and efficacy of the two methods, both of which give estimations with greater accuracy, precession and closed-form solutions. The mentioned techniques can be utilized to generate the solutions to these types of equations as infinite series, and when these series admit closed form, they offer the precise solution. Numerical and graphical simulations are used to confirm the usefulness of the suggested approaches. It is confirmed that the methods yield solutions converging to the exact solution at an adequate rate. Moreover, we include solution profiles that outline the behavior of the acquired findings, helping researchers grasp the impact of the fractional order. It has been shown that the suggested strategies are efficient and successful when it is implemented. The accuracy and efficacy of the technique are examined using three numerical examples.

**Keywords** Elzaki transform, analytical techniques, nonlinear time-fractional wave-like equations, Caputo operator.

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### 1. Introduction

Differentiation and integration are both regarded as discrete mathematical processes because we can differentiate or integrate a function once, twice, or any whole number of times. Fractional order derivatives, however, have the capability to handle even the most complex systems. Fractional order derivatives have a long history that dates back to Leibnitz and L'Hospital. In the area of mathematical analysis, the idea of fractional calculus (FC) is primarily concerned with the study and usage of integrals and derivatives of noninteger order. In fact, this approach offers significant flexibility in while addressing to solving multiple equation types (differential, integral, and integro differential), mathematical physics issues containing special functions, their presumption, and extensions in one or more variables. With the aid of fractional calculus, we can elegantly and symmetrically describe the naturalism of the world in the context of classical calculus. This is because it can explain how memory effects and paradoxical attitudes express themselves in nonlinear processes. The pioneers good work laid the mathematical groundwork

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<sup>†</sup>The corresponding author.

<sup>1</sup>Basic Sciences Department, College of Science and Theoretical Studies, Saudi Electronic University,  
PO Box: 93499, Riyadh 11673, Saudi Arabia  
Email: a.ganie@seu.edu.sa(A. H. Ganie)

for the notion of fractional order derivatives such as Albaidani et al. [2], Ali et al. [4], Caputo [10], Fathima et al. [17], Ganie et al. [18–22], Hasan et al. [24], Higazy et al. [27], Iqbal et al. [28], Jami et al. [31], Liouville [34], Mashaël et al. [35–37], Miller and Ross [38], Murad et al. [39–41], Naik et al. [43], Shah et al. [50], Uli et al. [54], and many others as cited in the text. During the past forty years and beyond, fractional partial differential equations have attracted the attention, understanding, and relevance of many mathematical experts as the most effective means of making scientific and engineering discoveries, notably in the field of chaos theory [23], human diseases [42, 44], Podlubny [46], Reimann [48], biomathematics [49], financial models [53], fluid dynamics [55], and many others. The solutions of fractional differential equations are crucial in describing the potential of nonlinear situations that arise in everyday life.

The identification of memory and heredity properties that classical systems generally neglect is made possible by the use of differential equations with fractional derivatives. The study of dynamical systems can be assisted by fractional-order derivative modelling, which is important for understanding how real-world systems behave. Fractional differential equations are frequently used in applied mathematics and physics to examine and represent a variety of real-world problems. Many techniques include Differential Transform Method [16], Adomian Decomposition Method [3, 29], Homotopy Analysis Method [12]. The new techniques of analytical solution of linear and nonlinear FDEs have been considered by many researchers, via., Sumudu [8], Mahgoub [15], Laplace [32], Elzaki [13, 14], and Natural [52]. While solving nonlinear system FDEs, it was Adomian decomposition approach which was combined to Elzaki transform method [45, 51], to Sumudu transform method [47], to Laplace transform method [5, 57] and with Natural transform method [9, 33]. Elzaki transform is obtained from the standard Fourier integral [13]. Tarig Elzaki developed the Elzaki transform to make it easier to solve ordinary and partial differential equations in the time domain. The Adomian decomposition technique (ADM) [1, 7] is a renowned efficient technique for finding solution of deterministic or stochastic operator equations, which includes ordinary, partial, integral, and integro-differential equations. The ADM is an advanced method that offers quick procedure for numerical simulation and analytical solution in engineering, technology and applied sciences. It was, He [25, 26], who introduced the homotopy perturbation technique in 1998. By this method, the solution can be taken as sum of an infinite series, yielding fast convergence precisely. It has been used to many mathematical issues. The nonlinear Caputo time-fractional wave-like equations with variable coefficients are of the form:

$$\begin{aligned}
 D_{\mu}^{\beta} \mathcal{V} = & \sum_{i,j=1}^n B_{1ij}(\varphi, \mu, \mathcal{V}) \frac{\partial^{k+m}}{\partial \xi_i^k \partial \xi_j^m} B_{2ij}(\mathcal{V}_{\xi_i}, \mathcal{V}_{\xi_j}) \\
 & + \sum_{i=1}^n C_{1i}(\varphi, \mu, \mathcal{V}) \frac{\partial^p}{\partial \xi_i^p} C_{2i}(\mathcal{V}_{\xi_i}) + D(\varphi, \mu, \mathcal{V}) + E(\varphi, \mu),
 \end{aligned}
 \tag{1.1}$$

with initial conditions

$$\mathcal{V}(\varphi, 0) = a_0(\varphi), \mathcal{V}_{\mu}(\varphi, 0) = a_1(\varphi),
 \tag{1.2}$$

where  $D_{\mu}^{\beta}$  is the Caputo fractional derivative operator having order  $\beta$  and  $1 < \beta \leq 2$ ,  $\mathcal{V} = \mathcal{V}(\varphi, \mu)$ ,  $\varphi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ ,  $\mu \geq 0$ ,  $B_{1ij}, C_{1i}, j \in 1, 2, \dots, n$  are nonlinear operators of  $\varphi, \mu$  and  $\mathcal{V}$ ,  $B_{2ij}, C_{2i}$ , are nonlinear operators of derivatives of  $\mathcal{V}$  with respect to  $\xi_i$  and  $\xi_j$   $i, j \in 1, 2, \dots, n$ , respectively. Moreover,  $k, m, p$  are integers while  $D, E$  are nonlinear functions.

It is clear that when  $\beta = 2$ , equation (1) can be reduced to the typical wave-like equations with variable coefficients. These kinds of equations are essential to many applied sciences, like

nonlinear hydrodynamics, plasma physics, astrophysics, engineering biophysics, et al. These equations explain changes in laser light intensity, the creation of microscopic particles moving erratically when submerged in fluids, and the distribution of fluid particle velocities in turbulent flows [56].

Our paper is presented in the manner outlined below. Section 1 includes an introduction. Section 2 presents the basic definitions of FC, the Elzaki transform, and its features. Section 3 presents the notion of HPTM, whereas Section 4 presents the notion of ETDM. Section 5 illustrates its application to the nonlinear time-fractional wave-like equations. In Section 6, we provide a summary of the results.

## 2. Preliminaries

Here, as in [5, 6, 11, 13, 14, 30, 51, 56] and reference therein, we stated various basic definitions associated to our study.

### 2.1. Definition

The fractional derivative operator in Abel-Riemann manner is given by

$$D^\beta \mathcal{V}(\xi) = \begin{cases} \frac{d^\rho}{d\xi^\rho} \mathcal{V}(\xi), & \beta = \rho, \\ \frac{1}{\Gamma(\rho - \beta)} \frac{d}{d\xi^\rho} \int_0^\xi \frac{\mathcal{V}(\xi)}{(\xi - \psi)^{\beta - \rho + 1}} d\psi, & \rho - 1 < \beta < \rho, \end{cases}$$

where  $\rho \in \mathbb{Z}^+, \beta \in \mathbb{R}^+$ .

### 2.2. Definition

The fractional integral operator in Abel-Riemann manner is given by

$$J^\beta \mathcal{V}(\xi) = \frac{1}{\Gamma(\beta)} \int_0^\xi (\xi - \psi)^{\beta - 1} \mathcal{V}(\xi) d\xi, \quad \xi > 0, \quad \beta > 0,$$

with the following properties

$$\begin{aligned} J^\beta \xi^\rho &= \frac{\Gamma(\rho + 1)}{\Gamma(\rho + \beta + 1)} \xi^{\rho + \beta}, \\ D^\beta \xi^\rho &= \frac{\Gamma(\rho + 1)}{\Gamma(\rho - \beta + 1)} \xi^{\rho - \beta}. \end{aligned}$$

### 2.3. Definition

The fractional derivative operator in Caputo manner is as

$$D^\beta \mathcal{V}(\xi) = \begin{cases} \frac{1}{\Gamma(\rho - \beta)} \int_0^\xi \frac{\mathcal{V}^\rho(\psi)}{(\xi - \psi)^{\beta - \rho + 1}} d\psi, & \rho - 1 < \beta < \rho, \\ \frac{d^\rho}{d\xi^\rho} \mathcal{V}(\xi), & \rho = \beta. \end{cases} \tag{2.1}$$

with the following properties

$$\begin{aligned}
 J_\xi^\beta D_\xi^\beta \mathcal{V}(\xi) &= g(\xi) - \sum_{k=0}^m g^k(0^+) \frac{\xi^k}{k!}, \quad \text{for } \xi > 0, \quad \text{and } \rho - 1 < \beta \leq \rho, \quad \rho \in N, \\
 D_\xi^\beta J_\xi^\beta \mathcal{V}(\xi) &= g(\xi).
 \end{aligned}
 \tag{2.2}$$

### 2.4. Definition

The ET of Caputo fractional operator is as

$$\mathbf{E}[D_\xi^\beta \mathcal{V}(\xi)] = s^{-\beta} \mathbf{E}[\mathcal{V}(\xi)] - \sum_{k=0}^{\rho-1} s^{2-\beta+k} \mathcal{V}^{(k)}(0), \quad \text{where } \rho - 1 < \beta < \rho.$$

## 3. General procedure of HPTM

Here, we stated the general procedure of HPTM for solving the FPDE:

$$D_\mu^\beta \mathcal{V}(\xi, \mu) = \mathcal{F}_1[\xi] \mathcal{V}(\xi, \mu) + \mathcal{G}_1[\xi] \mathcal{V}(\xi, \mu), \quad 0 < \beta \leq 1,
 \tag{3.1}$$

having initial condition

$$\mathcal{V}(\xi, 0) = \vartheta(\xi),$$

with  $D_\mu^\beta = \frac{\partial^\beta}{\partial \mu^\beta}$  is the Caputo fractional derivative operator, and  $\mathcal{F}_1[\xi]$ ,  $\mathcal{G}_1[\xi]$  are linear and nonlinear operators.

By using the ET, we have

$$\mathbf{E}[D_\mu^\beta \mathcal{V}(\xi, \mu)] = \mathbf{E}[\mathcal{V}_1[\xi] \mathcal{V}(\xi, \mu) + \mathcal{R}_1[\xi] \mathcal{V}(\xi, \mu)],
 \tag{3.2}$$

$$\frac{1}{u^\beta} \{M(u) - u^2 \mathcal{V}(\xi, 0)\} = \mathbf{E}[\mathcal{F}_1[\xi] \mathcal{V}(\xi, \mu) + \mathcal{G}_1[\xi] \mathcal{V}(\xi, \mu)].
 \tag{3.3}$$

After simplification, we obtain

$$M(u) = u^2 \mathcal{V}(\xi, 0) + u^\beta \mathbf{E}[\mathcal{F}_1[\xi] \mathcal{V}(\xi, \mu) + \mathcal{G}_1[\xi] \mathcal{V}(\xi, \mu)].
 \tag{3.4}$$

By using inverse ET, we have

$$\mathcal{V}(\xi, \mu) = \mathcal{V}(\xi, 0) + \mathbf{E}^{-1}[u^\beta \mathbf{E}[\mathcal{F}_1[\xi] \mathcal{V}(\xi, \mu) + \mathcal{G}_1[\xi] \mathcal{V}(\xi, \mu)]].
 \tag{3.5}$$

Now by the concept of HPM, we get

$$\mathcal{V}(\xi, \mu) = \sum_{j=0}^{\infty} \epsilon^j \mathcal{V}_j(\xi, \mu).
 \tag{3.6}$$

having homotopy parameter  $\epsilon \in [0, 1]$ .

Here nonlinear term is given as

$$\mathcal{G}_1[\xi] \mathcal{V}(\xi, \mu) = \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{V}),
 \tag{3.7}$$

in homotopy polynomial sense and the polynomials are illustrated as [30]:

$$H_k(\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_n) = \frac{1}{\Gamma(n+1)} D_\epsilon^k \left[ \mathcal{G}_1 \left( \sum_{i=0}^{\infty} \epsilon^i \mathcal{V}_i \right) \right]_{\epsilon=0}, \tag{3.8}$$

with  $D_\epsilon^k = \frac{\partial^k}{\partial \epsilon^k}$ .

Using (3.6) and (3.7) in (3.5), we get

$$\sum_{k=0}^{\infty} \epsilon^k \mathcal{V}_k(\xi, \mu) = \mathcal{V}(\xi, 0) + \epsilon \times \left( \mathbf{E}^{-1} \left[ u^\beta \mathbf{E} \{ \mathcal{F}_1 \sum_{k=0}^{\infty} \epsilon^k \mathcal{V}_k(\xi, \mu) + \sum_{k=0}^{\infty} \epsilon^k H_k(\mathcal{V}) \} \right] \right). \tag{3.9}$$

On comparing  $\epsilon$  coefficient both sides, we get

$$\begin{aligned} \epsilon^0 : \mathcal{V}_0(\xi, \mu) &= \mathcal{V}(\xi, 0), \\ \epsilon^1 : \mathcal{V}_1(\xi, \mu) &= \mathbf{E}^{-1} \left[ u^\beta \mathbf{E}(\mathcal{F}_1[\xi] \mathcal{V}_0(\xi, \mu) + H_0(\mathcal{V})) \right], \\ \epsilon^2 : \mathcal{V}_2(\xi, \mu) &= \mathbf{E}^{-1} \left[ u^\beta \mathbf{E}(\mathcal{F}_1[\xi] \mathcal{V}_1(\xi, \mu) + H_1(\mathcal{V})) \right], \\ &\vdots \\ &\vdots \\ \epsilon^k : \mathcal{V}_k(\xi, \mu) &= \mathbf{E}^{-1} \left[ u^\beta \mathbf{E}(\mathcal{F}_1[\xi] \mathcal{V}_{k-1}(\xi, \mu) + H_{k-1}(\mathcal{V})) \right], \quad k > 0, k \in N. \end{aligned} \tag{3.10}$$

At the end, the approximate series form solution is as

$$\mathcal{V}(\xi, \mu) = \lim_{M \rightarrow \infty} \sum_{k=1}^M \mathcal{V}_k(\xi, \mu). \tag{3.11}$$

### 4. General procedure of ETDM

Here, we stated the general procedure of ETDM for solving the FPDE:

$$D_\mu^\beta \mathcal{V}(\xi, \mu) = \mathcal{F}_1(\xi, \mu) + \mathcal{G}_1(\xi, \mu), 0 < \beta \leq 1, \tag{4.1}$$

with initial condition

$$\mathcal{V}(\xi, 0) = \vartheta(\xi),$$

where  $D_\mu^\beta = \frac{\partial^\beta}{\partial \mu^\beta}$  is the Caputo fractional derivative operator and  $\mathcal{F}_1$  and  $\mathcal{G}_1$  are linear and non-linear operators.

By using the ET, we see

$$\begin{aligned} \mathbf{E}[D_\mu^\beta \mathcal{V}(\xi, \mu)] &= \mathbf{E}[\mathcal{F}_1(\xi, \mu) + \mathcal{G}_1(\xi, \mu)], \\ \frac{1}{u^\beta} \{M(u) - u^2 \mathcal{V}(\xi, 0)\} &= \mathbf{E}[\mathcal{F}_1(\xi, \mu) + \mathcal{G}_1(\xi, \mu)]. \end{aligned} \tag{4.2}$$

After simplification, we obtain

$$M(u) = u \mathcal{V}(\xi, 0) + u^\beta \mathbf{E}[\mathcal{F}_1(\xi, \mu) + \mathcal{G}_1(\xi, \mu)]. \tag{4.3}$$

By using inverse ET, we have

$$\mathcal{V}(\xi, \mu) = \mathcal{V}(\xi, 0) + \mathbf{E}^{-1}[u^\beta \mathbf{E}[\mathcal{F}_1(\xi, \mu) + \mathcal{G}_1(\xi, \mu)]]. \tag{4.4}$$

Now by the concept of ADM, we get

$$\mathcal{V}(\xi, \mu) = \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \mu). \tag{4.5}$$

The nonlinear term is given as

$$\mathcal{G}_1(\xi, \mu) = \sum_{m=0}^{\infty} \mathcal{A}_m(\mathcal{V}), \tag{4.6}$$

with

$$\mathcal{A}_m \left( \mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_m \right) = \frac{1}{m!} \left[ \frac{\partial^m}{\partial \ell^m} \left\{ \mathcal{G}_1 \left( \sum_{m=0}^{\infty} \ell^m \mathcal{V}_m \right) \right\} \right]_{\ell=0}, \quad m = 0, 1, 2, \dots \tag{4.7}$$

Using (4.5) and (4.6) in (4.4), we get

$$\sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \mu) = \mathcal{V}(\xi, 0) + \mathbf{E}^{-1} u^\beta \left[ \mathbf{E} \left\{ \mathcal{F}_1 \left( \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \mu) \right) + \sum_{m=0}^{\infty} \mathcal{A}_m(\mathcal{V}) \right\} \right]. \tag{4.8}$$

On comparing both sides, we get

$$\begin{aligned} \mathcal{V}_0(\xi, \mu) &= \mathcal{V}(\xi, 0), \\ \mathcal{V}_1(\xi, \mu) &= \mathbf{E}^{-1} \left[ u^\beta \mathbf{E} \{ \mathcal{F}_1(\mathcal{V}_0) + \mathcal{A}_0 \} \right]. \end{aligned} \tag{4.9}$$

Finally, for  $m \geq 1$  the solution is stated as

$$\mathcal{V}_{m+1}(\xi, \mu) = \mathbf{E}^{-1} \left[ u^\beta \mathbf{E} \{ \mathcal{F}_1(\mathcal{V}_m) + \mathcal{A}_m \} \right].$$

## 5. Applications

### 5.1. problem

Let us assume the nonlinear fractional wave-like equation with variable coefficients:

$$\frac{\partial^\beta \mathcal{V}(\xi, \mu)}{\partial \mu^\beta} = \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \quad \mu > 0, 1 < \beta, \leq 2 \tag{5.1}$$

having initial conditions

$$\mathcal{V}(\xi, 0) = e^\xi, \quad \frac{\partial}{\partial \mu} \mathcal{V}(\xi, 0) = e^\xi.$$

By using the ET, we have

$$\mathbf{E} \left( \frac{\partial^\beta \mathcal{V}}{\partial \mu^\beta} \right) = \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right]. \tag{5.2}$$

Upon simplification, it yields

$$\frac{1}{u^\beta} \{M(u) - u^2 \mathcal{V}(\xi, 0) - u^3 \mathcal{V}_\mu(\xi, 0)\} = \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right], \quad (5.3)$$

$$M(u) = [u^2 \mathcal{V}(\xi, 0) + u^3 \mathcal{V}_\mu(\xi, 0)] + u^\beta \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right]. \quad (5.4)$$

By using inverse ET, we have

$$\begin{aligned} \mathcal{V}(\xi, \mu) &= \mathbf{E}^{-1} [u^2 \mathcal{V}(\xi, 0) + u^3 \mathcal{V}_\mu(\xi, 0)] \\ &\quad + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right] \right\} \right], \\ \mathcal{V}(\xi, \mu) &= (e^\xi + e^\xi \mu) \\ &\quad + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right] \right\} \right]. \end{aligned} \quad (5.5)$$

Now by the concept of HPM, we get

$$\begin{aligned} \sum_{k=0}^{\infty} \epsilon^k \mathcal{V}_k(\xi, \mu) &= \left( (e^\xi + e^\xi \mu) \right) + \left( \mathbf{E}^{-1} \left[ u^\beta \mathbf{E} \left[ \left( \sum_{k=0}^{\infty} \epsilon^k H_k^1(\mathcal{V}) \right) \right. \right. \right. \\ &\quad \left. \left. \left. + \left( \sum_{k=0}^{\infty} \epsilon^k H_k^2(\mathcal{V}) \right) - 18 \left( \sum_{k=0}^{\infty} \epsilon^k H_k^3(\mathcal{V}) \right) + \left( \sum_{k=0}^{\infty} \epsilon^k \mathcal{V}_k(\xi, \mu) \right) \right] \right] \right). \end{aligned} \quad (5.6)$$

The nonlinear terms are given by He’s polynomial  $H_k(\mathcal{V})$  as

$$\begin{aligned} \sum_{k=0}^{\infty} \epsilon^k H_k^1(\mathcal{V}) &= \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}), \\ \sum_{k=0}^{\infty} \epsilon^k H_k^2(\mathcal{V}) &= \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3), \\ \sum_{k=0}^{\infty} \epsilon^k H_k^3(\mathcal{V}) &= \mathcal{V}^5. \end{aligned} \quad (5.7)$$

First few terms are calculated as

$$\begin{aligned} H_0^1(\mathcal{V}) &= \mathcal{V}_0^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi} \mathcal{V}_{0\xi\xi} \mathcal{V}_{0\xi\xi\xi}), \\ H_1^1(\mathcal{V}) &= 2\mathcal{V}_0 \mathcal{V}_1 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi} \mathcal{V}_{0\xi\xi} \mathcal{V}_{0\xi\xi\xi}) + \mathcal{V}_0^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{1\xi} \mathcal{V}_{0\xi\xi} \mathcal{V}_{0\xi\xi\xi} + \mathcal{V}_{0\xi} \mathcal{V}_{1\xi\xi} \mathcal{V}_{0\xi\xi\xi} + \mathcal{V}_{0\xi} \mathcal{V}_{0\xi\xi} \mathcal{V}_{1\xi\xi\xi}), \\ &\vdots \\ H_0^2(\mathcal{V}) &= \mathcal{V}_{0\xi}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi\xi}^3), \\ H_1^2(\mathcal{V}) &= 2\mathcal{V}_{0\xi} \mathcal{V}_{1\xi} \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi\xi}^3) + 3\mathcal{V}_{0\xi}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi\xi}^2 \mathcal{V}_{1\xi\xi}), \end{aligned}$$

$$\begin{aligned} & \vdots \\ H_0^3(\mathcal{V}) &= \mathcal{V}_0^5, \\ H_1^3(\mathcal{V}) &= 5\mathcal{V}_0^4\mathcal{V}_1. \end{aligned}$$

On comparing  $\epsilon$  coefficient both sides, we get

$$\begin{aligned} \epsilon^0 : \mathcal{V}_0(\xi, \mu) &= (e^\xi + e^\xi \mu), \\ \epsilon^1 : \mathcal{V}_1(\xi, \mu) &= \left( \frac{\mu^\beta}{\Gamma(\beta + 1)} + \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} \right) e^\xi, \\ \epsilon^2 : \mathcal{V}_2(\xi, \mu) &= \left( \frac{\mu^{2\beta}}{\Gamma(2\beta + 1)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} \right) e^\xi, \\ & \vdots \end{aligned}$$

At the end, the approximate series form solution is as

$$\begin{aligned} \mathcal{V}(\xi, \mu) &= \mathcal{V}_0(\xi, \mu) + \mathcal{V}_1(\xi, \mu) + \mathcal{V}_2(\xi, \mu) + \dots, \\ \mathcal{V}(\xi, \mu) &= \left( 1 + \mu + \frac{\mu^\beta}{\Gamma(\beta + 1)} + \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} + \frac{\mu^{2\beta}}{\Gamma(2\beta + 1)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} + \dots \right) e^\xi. \end{aligned}$$

**Solution by employing the ETDM**

By using the ET, we get

$$\mathbf{E} \left\{ \frac{\partial^\beta \mathcal{V}}{\partial \mu^\beta} \right\} = \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right]. \tag{5.8}$$

Upon simplification, it yields

$$\frac{1}{u^\beta} \{M(u) - u^2\mathcal{V}(\xi, 0) - u^3\mathcal{V}_\mu(\xi, 0)\} = \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right], \tag{5.9}$$

$$M(u) = [u^2\mathcal{V}(\xi, 0) + u^3\mathcal{V}_\mu(\xi, 0)] + u^\beta \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right]. \tag{5.10}$$

By using inverse ET, we have

$$\begin{aligned} \mathcal{V}(\xi, \mu) &= \mathbf{E}^{-1} [u^2\mathcal{V}(\xi, 0) + u^3\mathcal{V}_\mu(\xi, 0)] \\ &+ \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right] \right\} \right], \\ \mathcal{V}(\xi, \mu) &= (e^\xi + e^\xi \mu) \\ &+ \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) + \mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) - 18\mathcal{V}^5 + \mathcal{V} \right] \right\} \right]. \end{aligned} \tag{5.11}$$

Now by the concept of ADM, we get

$$\mathcal{V}(\xi, \mu) = \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \mu), \tag{5.12}$$

where  $\mathcal{V}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi} \mathcal{V}_{\xi\xi\xi}) = \sum_{m=0}^\infty \mathcal{A}_m$ ,  $\mathcal{V}_\xi^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{\xi\xi}^3) = \sum_{m=0}^\infty \mathcal{B}_m$  and  $\mathcal{V}^5 = \sum_{m=0}^\infty \mathcal{C}_m$  are the Adomian polynomials that show the nonlinear terms, and

$$\begin{aligned} & \sum_{m=0}^\infty \mathcal{V}_m(\xi, \mu) \\ &= \mathcal{V}(\xi, 0) + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \sum_{m=0}^\infty \mathcal{A}_m + \sum_{m=0}^\infty \mathcal{B}_m - 18 \sum_{m=0}^\infty \mathcal{C}_m + \mathcal{V} \right] \right\} \right], \\ & \sum_{m=0}^\infty \mathcal{V}_m(\xi, \mu) \\ &= (e^\xi + e^\xi \mu) + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \sum_{m=0}^\infty \mathcal{A}_m + \sum_{m=0}^\infty \mathcal{B}_m - 18 \sum_{m=0}^\infty \mathcal{C}_m + \mathcal{V} \right] \right\} \right]. \end{aligned} \tag{5.13}$$

First few terms are calculated as

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{V}_0^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi} \mathcal{V}_{0\xi\xi} \mathcal{V}_{0\xi\xi\xi}), \\ \mathcal{A}_1 &= 2\mathcal{V}_0 \mathcal{V}_1 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi} \mathcal{V}_{0\xi\xi} \mathcal{V}_{0\xi\xi\xi}) + \mathcal{V}_0^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{1\xi} \mathcal{V}_{0\xi\xi} \mathcal{V}_{0\xi\xi\xi} + \mathcal{V}_{0\xi} \mathcal{V}_{1\xi\xi} \mathcal{V}_{0\xi\xi\xi} + \mathcal{V}_{0\xi} \mathcal{V}_{0\xi\xi} \mathcal{V}_{1\xi\xi\xi}), \\ & \vdots \\ \mathcal{B}_0 &= \mathcal{V}_{0\xi}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi\xi}^3), \\ \mathcal{B}_1 &= 2\mathcal{V}_{0\xi} \mathcal{V}_{1\xi} \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi\xi}^3) + 3\mathcal{V}_{0\xi}^2 \frac{\partial^2}{\partial \xi^2} (\mathcal{V}_{0\xi\xi}^2 \mathcal{V}_{1\xi\xi}), \\ & \vdots \\ \mathcal{C}_0 &= \mathcal{V}_0^5, \\ \mathcal{C}_1 &= 5\mathcal{V}_0^4 \mathcal{V}_1. \end{aligned}$$

On comparing both sides, we get

$$\mathcal{V}_0(\xi, \mu) = (e^\xi + e^\xi \mu).$$

On  $m = 0$ ,

$$\mathcal{V}_1(\xi, \mu) = \left( \frac{\mu^\beta}{\Gamma(\beta + 1)} + \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} \right) e^\xi.$$

On  $m = 1$

$$\mathcal{V}_2(\xi, \mu) = \left( \frac{\mu^{2\beta}}{\Gamma(2\beta + 1)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} \right) e^\xi.$$

At the end, the approximate series form solution is as

$$\mathcal{V}(\xi, \mu) = \sum_{m=0}^\infty \mathcal{V}_m(\xi, \mu) = \mathcal{V}_0(\xi, \mu) + \mathcal{V}_1(\xi, \mu) + \mathcal{V}_2(\xi, \mu) + \dots,$$

$$\mathcal{V}(\xi, \mu) = \left( 1 + \mu + \frac{\mu^\beta}{\Gamma(\beta + 1)} + \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} + \frac{\mu^{2\beta}}{\Gamma(2\beta + 1)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} + \dots \right) e^\xi.$$

By choosing  $\beta = 2$  we obtain

$$\mathcal{V}(\xi, \mu) = e^{\xi+\mu}. \tag{5.14}$$

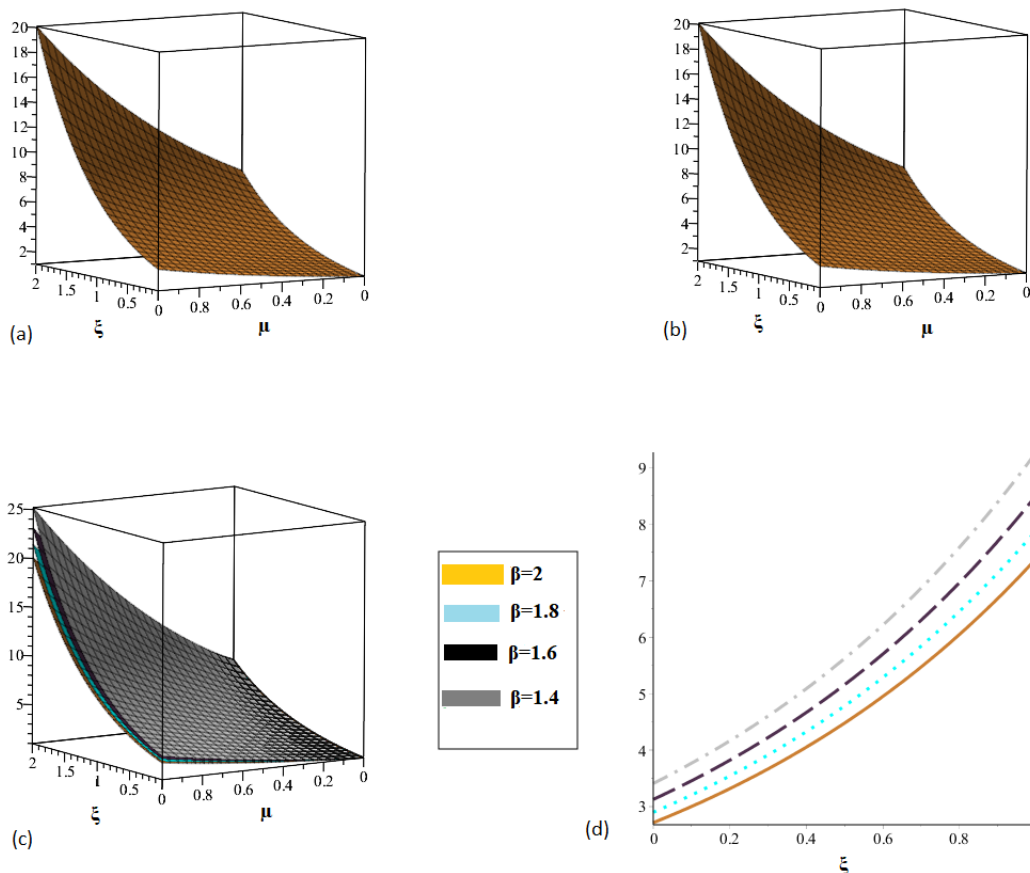


Figure 1. The graphical nature of  $\mathcal{V}(\xi, \mu)$  of example 1.

Figures 1a and 1b plot depicts how the precise and proposed solutions to the problem behaved at  $\beta = 2$ . Figure 1c illustrates our method’s solution for various fractional-orders of  $\beta = 2, 1.9, 1.8, 1.7,$  and  $0 \leq \xi \leq 2$  while figure 1d at  $\mu = 1$  and  $0 \leq \xi \leq 2$  for example 1, respectively. The graphical representation exhibits the nice agreement between the precise solution and the assumed approaches solution.

### 5.2. problem

Let us assume the non-linear fractional wave like equation with variable coefficients:

$$\frac{\partial^\beta \mathcal{V}(\xi, \mu)}{\partial \mu^\beta} = \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \quad \mu > 0, 1 < \beta \leq 2, \tag{5.15}$$

having initial conditions

$$\mathcal{V}(\xi, 0) = 0, \frac{\partial}{\partial \mu} \mathcal{V}(\xi, 0) = \xi^2.$$

By using the ET, we have

$$\mathbf{E} \left( \frac{\partial^\beta \mathcal{V}}{\partial \mu^\beta} \right) = \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right]. \tag{5.16}$$

Upon simplification, it yields

$$\frac{1}{u^\beta} \{M(u) - u^2 \mathcal{V}(\xi, 0) - u^3 \mathcal{V}_\mu(\xi, 0)\} = \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right], \tag{5.17}$$

$$M(u) = [u^2 \mathcal{V}(\xi, 0) + u^3 \mathcal{V}_\mu(\xi, 0)] + u^\beta \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right]. \tag{5.18}$$

Therefore, taking inverse ET, one can see

$$\begin{aligned} \mathcal{V}(\xi, \mu) &= \mathbf{E}^{-1} [u^2 \mathcal{V}(\xi, 0) + u^3 \mathcal{V}_\mu(\xi, 0)] + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right] \right\} \right], \\ \mathcal{V}(\xi, \mu) &= (\xi^2 \mu) + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right] \right\} \right]. \end{aligned} \tag{5.19}$$

Now by the concept of HPM, we get

$$\begin{aligned} &\sum_{k=0}^{\infty} \epsilon^k \mathcal{V}_k(\xi, \mu) \\ &= \left( \xi^2 \mu \right) + \left( \mathbf{E}^{-1} \left[ u^\beta \mathbf{E} \left[ \left( \sum_{k=0}^{\infty} \epsilon^k H_k^1(\mathcal{V}) \right) - \left( \sum_{k=0}^{\infty} \epsilon^k H_k^2(\mathcal{V}) \right) - \left( \sum_{k=0}^{\infty} \epsilon^k \mathcal{V}_k(\xi, \mu) \right) \right] \right] \right). \end{aligned} \tag{5.20}$$

The nonlinear terms are given by He’s polynomial  $H_k(\mathcal{V})$  as

$$\begin{aligned} \sum_{k=0}^{\infty} \epsilon^k H_k^1(\mathcal{V}) &= \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}), \\ \sum_{k=0}^{\infty} \epsilon^k H_k^2(\mathcal{V}) &= \xi^2 (\mathcal{V}_{\xi\xi})^2. \end{aligned} \tag{5.21}$$

First few terms are calculated as

$$\begin{aligned} H_0^1(\mathcal{V}) &= \mathcal{V}_{0\xi} \mathcal{V}_{0\xi\xi}, \\ H_1^1(\mathcal{V}) &= \mathcal{V}_{0\xi} \mathcal{V}_{1\xi\xi} + \mathcal{V}_{1\xi} \mathcal{V}_{0\xi\xi}, \\ H_2^1(\mathcal{V}) &= \mathcal{V}_{0\xi} \mathcal{V}_{2\xi\xi} + \mathcal{V}_{1\xi} \mathcal{V}_{1\xi\xi} + \mathcal{V}_{2\xi} \mathcal{V}_{0\xi\xi}, \\ &\vdots \\ H_0^2(\mathcal{V}) &= \mathcal{V}_{0\xi\xi}^2, \\ H_1^2(\mathcal{V}) &= 2\mathcal{V}_{0\xi\xi} \mathcal{V}_{1\xi\xi}, \end{aligned}$$

$$H_2^2(\mathcal{V}) = 2\mathcal{V}_{0\xi\xi}\mathcal{V}_{2\xi\xi} + \mathcal{V}_{1\xi\xi}^2.$$

On comparing  $\epsilon$  coefficient both sides, we get

$$\begin{aligned} \epsilon^0 : \mathcal{V}_0(\xi, \mu) &= \xi^2 \mu, \\ \epsilon^1 : \mathcal{V}_1(\xi, \mu) &= -\frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} \xi^2, \\ \epsilon^2 : \mathcal{V}_2(\xi, \mu) &= \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} \xi^2, \\ \epsilon^3 : \mathcal{V}_3(\xi, \mu) &= -\frac{\mu^{3\beta+1}}{\Gamma(3\beta + 2)} \xi^2, \\ &\vdots \end{aligned}$$

At the end, the approximate series form solution is as

$$\begin{aligned} \mathcal{V}(\xi, \mu) &= \mathcal{V}_0(\xi, \mu) + \mathcal{V}_1(\xi, \mu) + \mathcal{V}_2(\xi, \mu) + \mathcal{V}_3(\xi, \mu) + \dots, \\ \mathcal{V}(\xi, \mu) &= \xi^2 \left( \mu - \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} - \frac{\mu^{3\beta+1}}{\Gamma(3\beta + 2)} + \dots \right). \end{aligned}$$

**Solution by employing the ETDM**

By using the ET, we get

$$\mathbf{E} \left\{ \frac{\partial^\beta \mathcal{V}}{\partial \mu^\beta} \right\} = \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right]. \tag{5.22}$$

Upon simplification, it yields

$$\frac{1}{u^\beta} \{ M(u) - u^2 \mathcal{V}(\xi, 0) - u^3 \mathcal{V}_\mu(\xi, 0) \} = \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right], \tag{5.23}$$

$$M(u) = [u^2 \mathcal{V}(\xi, 0) + u^3 \mathcal{V}_\mu(\xi, 0)] + u^\beta \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right]. \tag{5.24}$$

Therefore, taking inverse ET, one can see

$$\begin{aligned} \mathcal{V}(\xi, \mu) &= \mathbf{E}^{-1} [u^2 \mathcal{V}(\xi, 0) + u^3 \mathcal{V}_\mu(\xi, 0)] + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right] \right\} \right], \\ \mathcal{V}(\xi, \mu) &= (\xi^2 \mu) + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \xi^2 \frac{\partial}{\partial \xi} (\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) - \xi^2 (\mathcal{V}_{\xi\xi})^2 - \mathcal{V} \right] \right\} \right]. \end{aligned} \tag{5.25}$$

Now by the concept of ADM, we get

$$\mathcal{V}(\xi, \mu) = \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \mu), \tag{5.26}$$

where  $(\mathcal{V}_\xi \mathcal{V}_{\xi\xi}) = \sum_{m=0}^{\infty} \mathcal{A}_m$  and  $(\mathcal{V}_{\xi\xi})^2 = \sum_{m=0}^{\infty} \mathcal{B}_m$  are the Adomian polynomials that show the nonlinear terms, and

$$\begin{aligned} \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \mu) &= \mathcal{V}(\xi, 0) + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \sum_{m=0}^{\infty} \mathcal{A}_m - \sum_{m=0}^{\infty} \mathcal{B}_m - \mathcal{V} \right] \right\} \right], \\ \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \mu) &= (\xi^2 \mu) + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \sum_{m=0}^{\infty} \mathcal{A}_m - \sum_{m=0}^{\infty} \mathcal{B}_m - \mathcal{V} \right] \right\} \right]. \end{aligned} \tag{5.27}$$

First few terms are calculated as

$$\begin{aligned}
 \mathcal{A}_0 &= \mathcal{V}_{0\xi} \mathcal{V}_{0\xi\xi}, \\
 \mathcal{A}_1 &= \mathcal{V}_{0\xi} \mathcal{V}_{1\xi\xi} + \mathcal{V}_{1\xi} \mathcal{V}_{0\xi\xi}, \\
 \mathcal{A}_2 &= \mathcal{V}_{0\xi} \mathcal{V}_{2\xi\xi} + \mathcal{V}_{1\xi} \mathcal{V}_{1\xi\xi} + \mathcal{V}_{2\xi} \mathcal{V}_{0\xi\xi}, \\
 &\vdots \\
 \mathcal{B}_0 &= \mathcal{V}_{0\xi\xi}^2, \\
 \mathcal{B}_1 &= 2\mathcal{V}_{0\xi\xi} \mathcal{V}_{1\xi\xi}, \\
 \mathcal{B}_2 &= 2\mathcal{V}_{0\xi\xi} \mathcal{V}_{2\xi\xi} + \mathcal{V}_{1\xi\xi}^2.
 \end{aligned}$$

On comparing both sides, we get

$$\mathcal{V}_0(\xi, \mu) = \xi^2 \mu.$$

On  $m = 0$ ,

$$\mathcal{V}_1(\xi, \mu) = -\frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} \xi^2.$$

On  $m = 1$

$$\mathcal{V}_2(\xi, \mu) = \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} \xi^2.$$

On  $m = 2$

$$\mathcal{V}_3(\xi, \mu) = -\frac{\mu^{3\beta+1}}{\Gamma(3\beta + 2)} \xi^2.$$

At the end, the approximate series form solution is as

$$\begin{aligned}
 \mathcal{V}(\xi, \mu) &= \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \mu) = \mathcal{V}_0(\xi, \mu) + \mathcal{V}_1(\xi, \mu) + \mathcal{V}_2(\xi, \mu) + \mathcal{V}_3(\xi, \mu) + \dots, \\
 \mathcal{V}(\xi, \mu) &= \xi^2 \left( \mu - \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} - \frac{\mu^{3\beta+1}}{\Gamma(3\beta + 2)} + \dots \right).
 \end{aligned}$$

By choosing  $\beta = 2$  we obtain

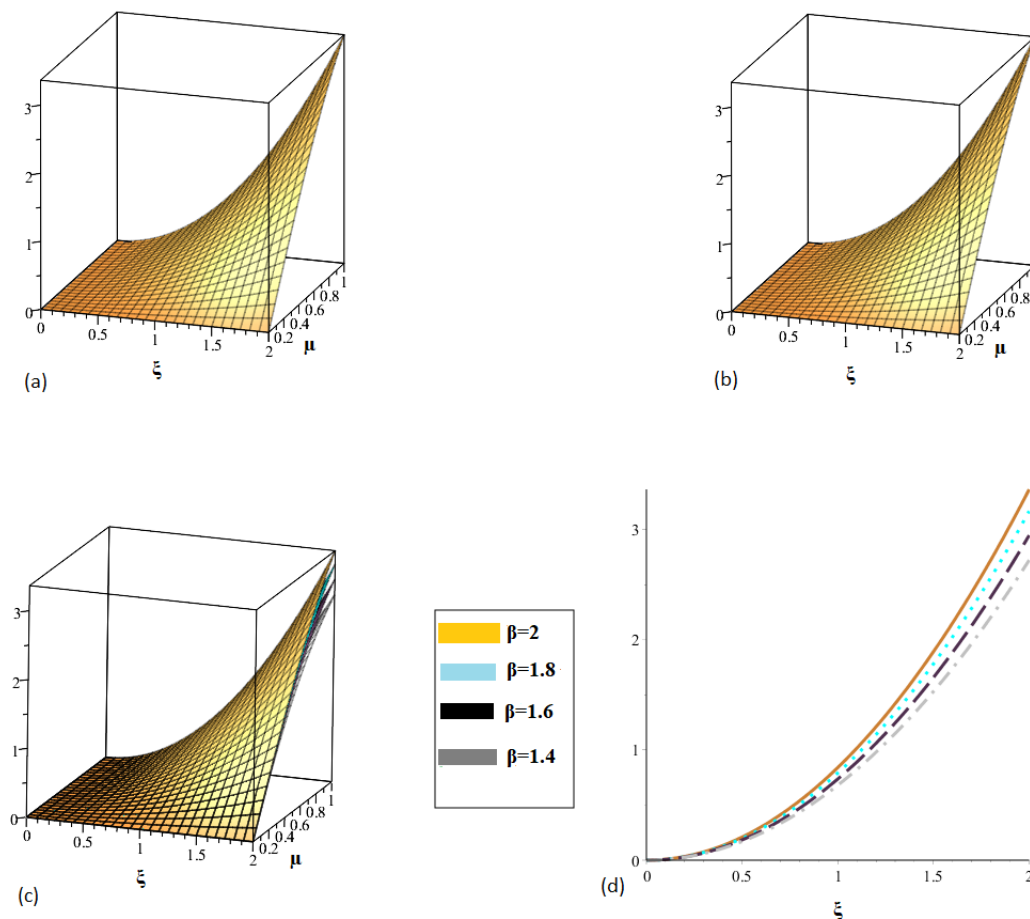
$$\mathcal{V}(\xi, \mu) = \xi^2 \sin \mu. \tag{5.28}$$

The plots in Figures 2a and 2b illustrate how the precise and proposed solutions to the problem behaved at  $\beta = 2$ . Figure 2c illustrates our method’s solution for various fractional-orders of  $\beta = 2, 1.9, 1.8, 1.7$ , and  $0 \leq \xi \leq 2$  while figure 2d at  $\mu = 1$  and  $0 \leq \xi \leq 2$  for example 2, respectively. The graphs demonstrate tight agreement between the exact solution and the proposed method’s solution.

### 5.3. problem

Let us assume the 2-dimensional nonlinear fractional wave-like equation with variable coefficients:

$$\frac{\partial^\beta \mathcal{V}(\xi, \zeta, \mu)}{\partial \mu^\beta} = \frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial \xi \partial \zeta} (\xi \zeta \mathcal{V}_\xi \mathcal{V}_\zeta) - \mathcal{V} \quad \mu > 0, 1 < \beta \leq 2, \tag{5.29}$$



**Figure 2.** The graphical nature of  $\mathcal{V}(\xi, \mu)$  of example 2.

having initial conditions

$$\mathcal{V}(\xi, \zeta, 0) = e^{\xi\zeta}, \frac{\partial}{\partial \mu} \mathcal{V}(\xi, \zeta, 0) = e^{\xi\zeta}, (\xi, \zeta) \in \mathbb{R}^2.$$

By using the ET, we have

$$\mathbf{E} \left( \frac{\partial^\beta \mathcal{V}}{\partial \mu^\beta} \right) = \mathbf{E} \left[ \frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial \xi \partial \zeta} (\xi\zeta \mathcal{V}_\xi \mathcal{V}_\zeta) - \mathcal{V} \right]. \tag{5.30}$$

Upon simplification, it yields

$$\frac{1}{u^\beta} \{M(u) - u^2 \mathcal{V}(\xi, \zeta, 0) - u^3 \mathcal{V}_\mu(\xi, \zeta, 0)\} = \mathbf{E} \left[ \frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial \xi \partial \zeta} (\xi\zeta \mathcal{V}_\xi \mathcal{V}_\zeta) - \mathcal{V} \right], \tag{5.31}$$

$$M(u) = [u^2 \mathcal{V}(\xi, \zeta, 0) + u^3 \mathcal{V}_\mu(\xi, \zeta, 0)] + u^\beta \mathbf{E} \left[ \frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial \xi \partial \zeta} (\xi\zeta \mathcal{V}_\xi \mathcal{V}_\zeta) - \mathcal{V} \right]. \tag{5.32}$$

Therefore, taking inverse ET, one can see

$$\begin{aligned} \mathcal{V}(\xi, \zeta, \mu) &= \mathbf{E}^{-1}[u^2\mathcal{V}(\xi, \zeta, 0) + u^3\mathcal{V}_\mu(\xi, \zeta, 0)] \\ &\quad + \mathbf{E}^{-1}\left[u^\beta\left\{\mathbf{E}\left[\frac{\partial^2}{\partial\xi\partial\zeta}(\mathcal{V}_{\xi\xi}\mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial\xi\partial\zeta}(\xi\zeta\mathcal{V}_\xi\mathcal{V}_\zeta) - \mathcal{V}\right]\right\}\right], \end{aligned} \tag{5.33}$$

$$\mathcal{V}(\xi, \zeta, \mu) = ((e^{\xi\zeta} + e^{\xi\zeta}\mu)) + \mathbf{E}^{-1}\left[u^\beta\left\{\mathbf{E}\left[\frac{\partial^2}{\partial\xi\partial\zeta}(\mathcal{V}_{\xi\xi}\mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial\xi\partial\zeta}(\xi\zeta\mathcal{V}_\xi\mathcal{V}_\zeta) - \mathcal{V}\right]\right\}\right].$$

Now by the concept of HPM, we get

$$\begin{aligned} &\sum_{k=0}^{\infty} \epsilon^k \mathcal{V}_k(\xi, \zeta, \mu) \\ &= \left( (e^{\xi\zeta} + e^{\xi\zeta}\mu) + \left( \mathbf{E}^{-1}\left[ u^\beta \mathbf{E}\left[ \left( \sum_{k=0}^{\infty} \epsilon^k H_k^1(\mathcal{V}) \right) - \left( \sum_{k=0}^{\infty} \epsilon^k H_k^2(\mathcal{V}) \right) + \left( \sum_{k=0}^{\infty} \epsilon^k \mathcal{V}_k(\xi, \mu) \right) \right] \right] \right) \right). \end{aligned} \tag{5.34}$$

The nonlinear terms are given by He’s polynomial  $H_k(\mathcal{V})$  as

$$\begin{aligned} \sum_{k=0}^{\infty} \epsilon^k H_k^1(\mathcal{V}) &= \frac{\partial^2}{\partial\xi\partial\zeta}(\mathcal{V}_{\xi\xi}\mathcal{V}_{\zeta\zeta}), \\ \sum_{k=0}^{\infty} \epsilon^k H_k^2(\mathcal{V}) &= \frac{\partial^2}{\partial\xi\partial\zeta}(\xi\zeta\mathcal{V}_\xi\mathcal{V}_\zeta). \end{aligned} \tag{5.35}$$

First few terms are calculated as

$$\begin{aligned} H_0^1(\mathcal{V}) &= \mathcal{V}_{0\xi\xi}\mathcal{V}_{0\zeta\zeta}, \\ H_1^1(\mathcal{V}) &= \mathcal{V}_{0\xi\xi}\mathcal{V}_{1\zeta\zeta} + \mathcal{V}_{1\xi\xi}\mathcal{V}_{0\zeta\zeta}, \\ &\vdots \\ H_0^2(\mathcal{V}) &= \xi\zeta\mathcal{V}_{0\xi}\mathcal{V}_{0\zeta}, \\ H_1^2(\mathcal{V}) &= \xi\zeta\mathcal{V}_{0\xi}\mathcal{V}_{1\zeta} + \xi\zeta\mathcal{V}_{1\xi}\mathcal{V}_{0\zeta}. \end{aligned}$$

On comparing  $\epsilon$  coefficient both sides, we get

$$\begin{aligned} \epsilon^0 : \mathcal{V}_0(\xi, \zeta, \mu) &= (e^{\xi\zeta} + e^{\xi\zeta}\mu), \\ \epsilon^1 : \mathcal{V}_1(\xi, \zeta, \mu) &= -\left(\frac{\mu^\beta}{\Gamma(\beta + 1)} + \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)}\right)e^{\xi\zeta}, \\ \epsilon^2 : \mathcal{V}_2(\xi, \zeta, \mu) &= \left(\frac{\mu^{2\beta}}{\Gamma(2\beta + 1)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)}\right)e^{\xi\zeta}, \\ &\vdots \end{aligned}$$

At the end, the approximate series form solution is as

$$\begin{aligned} \mathcal{V}(\xi, \zeta, \mu) &= \mathcal{V}_0(\xi, \zeta, \mu) + \mathcal{V}_1(\xi, \zeta, \mu) + \mathcal{V}_2(\xi, \zeta, \mu) + \dots, \\ \mathcal{V}(\xi, \zeta, \mu) &= \left(1 + \mu - \frac{\mu^\beta}{\Gamma(\beta + 1)} - \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} + \frac{\mu^{2\beta}}{\Gamma(2\beta + 1)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} + \dots\right)e^{\xi\zeta}. \end{aligned}$$

**Solution by employing the ETDM**

By using the ET, we get

$$\mathbf{E} \left\{ \frac{\partial^\beta \mathcal{V}}{\partial \mu^\beta} \right\} = \mathbf{E} \left[ \frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial \xi \partial \zeta} (\xi \zeta \mathcal{V}_\xi \mathcal{V}_\zeta) - \mathcal{V} \right]. \tag{5.36}$$

Upon simplification, it yields

$$\frac{1}{u^\beta} \{M(u) - u^2 \mathcal{V}(\xi, \zeta, 0) - u^3 \mathcal{V}_\mu(\xi, \zeta, 0)\} = \mathbf{E} \left[ \frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial \xi \partial \zeta} (\xi \zeta \mathcal{V}_\xi \mathcal{V}_\zeta) - \mathcal{V} \right], \tag{5.37}$$

$$M(u) = [u^2 \mathcal{V}(\xi, \zeta, 0) + u^3 \mathcal{V}_\mu(\xi, \zeta, 0)] + u^\beta \mathbf{E} \left[ \frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial \xi \partial \zeta} (\xi \zeta \mathcal{V}_\xi \mathcal{V}_\zeta) - \mathcal{V} \right]. \tag{5.38}$$

Therefore, taking inverse ET, one can see

$$\begin{aligned} \mathcal{V}(\xi, \zeta, \mu) &= \mathbf{E}^{-1} [u^2 \mathcal{V}(\xi, \zeta, 0) + u^3 \mathcal{V}_\mu(\xi, \zeta, 0)] \\ &\quad + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial \xi \partial \zeta} (\xi \zeta \mathcal{V}_\xi \mathcal{V}_\zeta) - \mathcal{V} \right] \right\} \right], \\ \mathcal{V}(\xi, \mu) &= (e^{\xi\zeta} + e^{\xi\zeta} \mu) \\ &\quad + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) - \frac{\partial^2}{\partial \xi \partial \zeta} (\xi \zeta \mathcal{V}_\xi \mathcal{V}_\zeta) - \mathcal{V} \right] \right\} \right]. \end{aligned} \tag{5.39}$$

Now by the concept of ADM, we get

$$\mathcal{V}(\xi, \zeta, \mu) = \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \zeta, \mu), \tag{5.40}$$

where  $\frac{\partial^2}{\partial \xi \partial \zeta} (\mathcal{V}_{\xi\xi} \mathcal{V}_{\zeta\zeta}) = \sum_{m=0}^{\infty} \mathcal{A}_m$  and  $\frac{\partial^2}{\partial \xi \partial \zeta} (\xi \zeta \mathcal{V}_\xi \mathcal{V}_\zeta) = \sum_{m=0}^{\infty} \mathcal{B}_m$  are the Adomian polynomials that show the nonlinear terms, and

$$\begin{aligned} \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \zeta, \mu) &= \mathcal{V}(\xi, \zeta, 0) + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \sum_{m=0}^{\infty} \mathcal{A}_m - \sum_{m=0}^{\infty} \mathcal{B}_m - \mathcal{V} \right] \right\} \right], \\ \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \zeta, \mu) &= (e^{\xi\zeta} + e^{\xi\zeta} \mu) + \mathbf{E}^{-1} \left[ u^\beta \left\{ \mathbf{E} \left[ \sum_{m=0}^{\infty} \mathcal{A}_m - \sum_{m=0}^{\infty} \mathcal{B}_m - \mathcal{V} \right] \right\} \right]. \end{aligned} \tag{5.41}$$

The first several terms are computed as

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{V}_{0\xi\xi} \mathcal{V}_{0\zeta\zeta}, \\ \mathcal{A}_1 &= \xi \zeta \mathcal{V}_{0\xi} \mathcal{V}_{0\zeta}, \\ &\vdots \\ \mathcal{B}_0 &= \mathcal{V}_{0\xi\xi} \mathcal{V}_{1\zeta\zeta} + \mathcal{V}_{1\xi\xi} \mathcal{V}_{0\zeta\zeta}, \\ \mathcal{B}_1 &= \xi \zeta \mathcal{V}_{0\xi} \mathcal{V}_{1\zeta} - \xi \zeta \mathcal{V}_{1\xi} \mathcal{V}_{0\zeta}. \end{aligned}$$

On comparing both sides, we get

$$\mathcal{V}_0(\xi, \zeta, \mu) = (e^{\xi\zeta} + e^{\xi\zeta} \mu).$$

On  $m = 0$ ,

$$\mathcal{V}_1(\xi, \zeta, \mu) = -\left(\frac{\mu^\beta}{\Gamma(\beta + 1)} + \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)}\right) e^{\xi\zeta}.$$

On  $m = 1$

$$\mathcal{V}_2(\xi, \zeta, \mu) = \left(\frac{\mu^{2\beta}}{\Gamma(2\beta + 1)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)}\right) e^{\xi\zeta}.$$

At the end, the approximate series form solution is as

$$\begin{aligned} \mathcal{V}(\xi, \zeta, \mu) &= \sum_{m=0}^{\infty} \mathcal{V}_m(\xi, \mu) = \mathcal{V}_0(\xi, \zeta, \mu) + \mathcal{V}_1(\xi, \zeta, \mu) + \mathcal{V}_2(\xi, \zeta, \mu) + \dots, \\ \mathcal{V}(\xi, \zeta, \mu) &= \left(1 + \mu - \frac{\mu^\beta}{\Gamma(\beta + 1)} - \frac{\mu^{\beta+1}}{\Gamma(\beta + 2)} + \frac{\mu^{2\beta}}{\Gamma(2\beta + 1)} + \frac{\mu^{2\beta+1}}{\Gamma(2\beta + 2)} + \dots\right) e^{\xi\zeta}. \end{aligned}$$

By choosing  $\beta = 2$  we obtain

$$\mathcal{V}(\xi, \zeta, \mu) = (\cos \mu + \sin \mu) e^{\xi\zeta}. \tag{5.42}$$

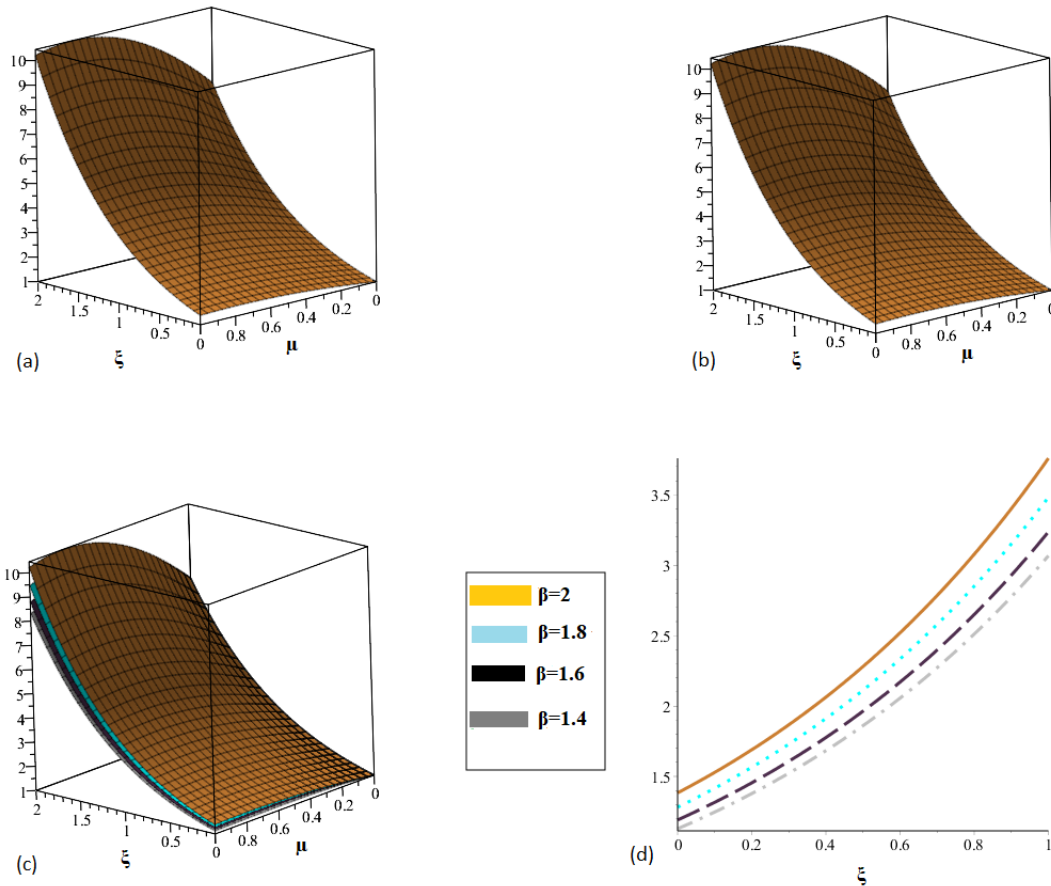


Figure 3. The graphical nature of  $\mathcal{V}(\xi, \zeta, \mu)$  of example 3.

The plots in Figures 3a and 3b illustrate how the precise and proposed solutions to the problem behaved at  $\beta = 2$ . Figure 3c illustrates our method's solution for various fractional-orders of  $\beta = 2, 1.9, 1.8, 1.7$ , and  $0 \leq \xi \leq 2$  while figure 3d at  $\mu = \zeta = 1$  and  $0 \leq \xi \leq 2$  for example 3, respectively. The plots demonstrate strong agreement between the precise solution and the proposed method's solution

## 6. Conclusion

In this article, two methods for solving problems resembling nonlinear Caputo time-fractional waves are examined. The exactness and usefulness of the offered methods are evaluated using three numerical examples. In this study, the exact solutions for nonlinear time-fractional wave-like equations with variable coefficients were represented in a new way using the Elzaki transform decomposition method and the Homotopy perturbation transform method. The methods were applied to three numerical instances. Our approaches offered the results as infinite series in the numerical frameworks, and when this series admits a closed form, it yields the precise solution of the associated equations. The outcome of the figures and tables demonstrate that there is a fair amount of agreement between the precise and suggested solutions. The current approaches have shown to be a quick and easy process. Also, the suggested approaches required less calculations and can be applied to resolve additional fractional-order issues. Researchers can address nonlinear real-world problems by using Caputo fractional order derivatives. These studies can be extended to include other fractional derivatives, such as conformable, Atangana-Baleanu, and Caputo-Fabrizio derivatives. Based on advantages of the existing operator, extending this framework to new operators merits future investigation. The present approach is beneficial and gives new opportunities for engineering and the sciences.

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**Data availability.** The numerical data used to support the findings of this study are included within the article.

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