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EXISTENCE OF SOLUTIONS TO BOUNDARY VALUE PROBLEM FOR NONLINEAR TEMPERED FRACTIONAL DIFFERENTIAL EQUATION WITH DELAY AND P-LAPLACIAN OPERATOR

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Abstract In this article, we study the existence of solutions to boundary value problem for nonlinear tempered fractional differential equation with p-Laplacian operator and delay. By introducing an appropriate operator, we establish uniqueness of the solution for this problem via the Banach contraction mapping principle and obtain several existence results by employing the Schaefer's fixed-point theorem, the Leray–Schauder principle, and the Leray–Schauder nonlinear alternative theorem. Finally, a concrete problem is provided to demonstrate the applicability of the obtained theoretical results.

Keywords Tempered fractional derivative, delay, p-Laplacian operator, fixed point theorem, existence of solutions.

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1. Introduction

In this paper, we study the boundary value problem (BVP) involving the p-Laplacian operator with delay and tempered fractional derivatives as follows:

$$\begin{cases} {}_{0}^{R}\mathbb{D}_t^{\theta,\lambda}(\varphi_p({}_{0}^{R}\mathbb{D}_t^{\sigma,\lambda}y(t))) = \gamma f(t, y(t-\tau)), \quad t \in (0,1] \setminus \{\tau\}, \\ y(t) = \xi(t), \quad t \in [-\tau, 0], \\ y(0) = y'(0) = y''(0) = \cdots = y^{(n-2)}(0) = 0, \\ y(1) = \varsigma \displaystyle\int_0^1 e^{-\lambda(1-t)} y(t)dt, \\ {}_{0}^{R}\mathbb{D}_t^{\sigma,\lambda}y(0) = 0, \end{cases} \tag{1.1}$$

where $0 < \theta \le 1$; $n-1 < \sigma \le n$ $(n \ge 3)$; $\varsigma < \sigma$; γ is a positive constant; $\varphi_p(s) = |s|^{p-2}s$ is a p-Laplacian operator with $\frac{1}{p} + \frac{1}{q} = 1, q \ge 2$; $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function; $\xi \in C[-\tau, 0]$, and $\xi(t) > 0$ for t $\in [-\tau, 0)$, $\xi(0) = 0$; ${}_{0}^{R}\mathbb{D}_t^{\theta,\lambda}$ and ${}_{0}^{R}\mathbb{D}_t^{\sigma,\lambda}$ are tempered fractional derivatives and can be defined by

$${}_{0}^{R}\mathbb{D}_t^{\theta,\lambda}y(t) = e^{-\lambda t}D_t^{\theta}(e^{\lambda t}y(t)), \quad {}_{0}^{R}\mathbb{D}_t^{\sigma,\lambda}y(t) = e^{-\lambda t}D_t^{\sigma}(e^{\lambda t}y(t)),$$

where D_t^{θ} and D_t^{σ} are standard Riemann-Liouville derivatives.

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The notion of fractional calculus was initially conceived in the late 1600s by Gottfried Wilhelm Leibniz and Isaac Newton, who deliberated on the potential existence of derivatives of fractional orders. Over time, prominent mathematicians such as Joseph Liouville, Leonhard Euler, and Joseph Fourier extended differentiation and integration to non-integer orders.

Fractional differential equations have attracted widespread attention from mathematicians because fractional-order models overcome the limitations of the locality inherent in integer-order models and can accurately describe numerous longterm and large-range natural phenomena, such as finance, biology, analytical chemistry, biological systems, and time-frequency analysis [2, 5, 11, 15, 32, 36, 37]. In fact, owing to its broad applications, this field has experienced rapid development in recent years [1, 3, 4, 17, 29, 31, 35].

In order to use different fractional differential equations to describe different physical phenomena, people have introduced various generalizations of fractional calculus. The classical fractional derivatives, the Riemann-Liouville [39], Caputo [26], Hilfer [34], and Hadamard types [19] among others, primarily rely on power law convolutions. However, these derivatives are sometimes inadequate for accurately modeling the limits of random walk processes. Thus, it is necessary to multiply an exponential factor into the power law kernel to obtain the tempered derivatives with both mathematical and practical advantages [27]. As a generalization of the standard fractional derivative, this derivative integrates a parameter λ. Especially, for $\lambda = 0$, this tempered fractional derivative reduces to classical fractional derivative. In recent years, the tempered fractional derivative has received considerable attention, as we can see from the literature [6, 20, 25, 28, 40].

In [42], Zhou et al. studied the following p-Laplacian BVP with tempered fractional derivatives:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\sigma,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\theta,\lambda}y(t))) = f(t, y(t), y(t)) + g(t, y(t)), \quad t \in [0,1], \\ y(0) = y'(0) = \cdots = y^{(n-2)}(0) = 0, \\ \varphi_p({}_0^R\mathbb{D}_t^{\theta,\lambda}y)(0) = 0, \\ y(1) = \sigma \displaystyle\int_0^1 e^{-\lambda(1-t)} y(t)\, dt, \\ {}_0^R\mathbb{D}_t^{\gamma_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\theta,\lambda}y))(1) = \int_0^\eta m(t) {}_0^R\mathbb{D}_t^{\gamma_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\theta,\lambda}y(t))] dM(t), \end{cases} \tag{1.2}$$

where $n-1 < \theta \le n, 1 < \sigma \le 2, 0 < \gamma_2 < \gamma_1 < \sigma{-}1, \sigma < \theta,\ \lambda > 0,\ \varphi_p(s) = |s|^{p-2}s$ denotes a p-Laplacian operator. ${}_0^R\mathbb{D}_t^{\theta,\lambda}$ and ${}_0^R\mathbb{D}_t^{\sigma,\lambda}$ denote tempered fractional derivatives. $M(t)$ is a bounded variation function and $\int_0^\eta a(t) {}_0^R\mathbb{D}_t^{\gamma_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\theta,\lambda}y(t))] dM(t)$ is a Riemann–Stieltjes integral regarding $M(t)$. By means of the fixed point theorem, the uniqueness of a positive solution of the problem (1.2) was given. Zhou et al. also employed an iterative scheme to approximate the unique positive solution.

In [40], Zhang et al. investigated the following singular tempered fractional equation involving the p-Laplacian operator:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\theta,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\sigma,\lambda}u(t))) = f(t, u(t)), \\ u(0) = 0, \\ {}_0^R\mathbb{D}_t^{\sigma,\lambda}u(0) = 0, \\ u(1) = \displaystyle\int_0^1 e^{-\lambda(1-t)} u(t) dt, \end{cases} \tag{1.3}$$

where $0 < \theta \leq 1, 1 < \sigma \leq 2$, $\lambda > 0$, $\varphi_p(s) = |s|^{p-2}s$ denotes a p-Laplacian operator, ${}_0^R\mathbb{D}_t^{\theta,\lambda}$ and ${}_0^R\mathbb{D}_t^{\sigma,\lambda}$ are tempered fractional derivatives, and $f(t,u)$ is decreasing in u. Applying the upper and lower solutions approach, it has been demonstrated that the problem (1.3) admits positive solutions, including both singular and nonsingular cases. Furthermore, Zhang et al. conducted an analysis of the solution's asymptotic properties.

On the other hand, delay constitutes a crucial and nonnegligible factor in practical problems, as it effectively expresses the influence of historical states on the current situation. Indeed, the behavior of a system is determined by its current status and its previous states. Therefore, the investigation of fractional differential equations with delay is essential for solving many practical problems, and these equations are widely applied in control theory [10], signal processing [24], biology [7] and finance [16]. Consequently, delay differential equations have attracted significant interest [12, 13, 18, 30, 33].

In [21], Mu et al. studied singular Reimann-Liouville nonlinear fractional BVP with delay as follows:

$$\begin{cases} D^{\theta}y(t) + \lambda f(t, y(t-\tau)) = 0, \quad t \in (0,1) \setminus \{\tau\}, \\ y(t) = \xi(t), \quad t \in [-\tau, 0], \\ y'(1) = y'(0) = 0, \end{cases}$$

where $\lambda > 0, 2 < \theta \leq 3, \xi \in C([-\tau, 0]), \xi(t) > 0$ for $t \in [-\tau, 0)$ and $\xi(0) = 0$. The function f is continuous and may change sign and be singular at $t = 0$, $t = 1$, and $y = 0$. Based on the Guo-Krasnoselskii fixed point theorem, it has been proved that there exist eigenvalue intervals. Moreover, corresponding positive solutions are derived.

In [8], Bai et al. investigated the following BVP with sign-changing nonlinearity and delay:

$$\begin{cases} -D_{0^+}^{\theta}y(t) + ay(t) = \lambda f(t, y(t-\tau)), \quad t \in (0,1) \setminus \{\tau\}, \\ y(t) = \xi(t), \quad t \in [-\tau, 0], \\ y(0) = y'(0) = y''(0) = \cdots = y^{(n-2)}(0) = 0, \\ y(1) = 0, \end{cases} \tag{1.4}$$

where $n–1 < \theta < n, n = [\theta] + 1 (n \geq 3), a > 0, \lambda > 0, \xi \in C[-\tau, 0], \xi(0) = 0$ and $\xi(t) > 0$ for $t \in [-\tau, 0)$, the continuous function f may be singular at $t = 0$, $t = 1$ and $y = 0$ and change sign. Via fixed point theorems, the existence and multiplicity of solutions were given.

Motivated by the above work, we study BVP for nonlinear tempered fractional differential equation with delay and p-Laplacian operator (1.1). Compared to the problems (1.2) and (1.4), we establish a significant relationship between tempered fractional calculus and delay. To our knowledge, this connection has been scarcely studied in current literature. Furthermore, we integrate these elements with the p-Laplacian operator. Therefore, we can say that the problem (1.1) is an interesting problem. This problem may provide a theoretical support to solve issues that reflect realistic phenomena. In this work, we explore the problem (1.1) by employing several fixed-point theorems, thereby demonstrate the existence of solutions under corresponding conditions.

This article takes the following structure. In Section 2, we provide the essential preliminaries, including key lemmas and preliminary results. In Section 3, we obtain uniqueness of solution by applying the Banach contraction mapping principle and several existence results for problem (1.1) by employing multiple fixed-point theorems. Finally, in Section 4, we give a concrete problem as an application of the theorems presented in this paper.

2. Preliminaries

In this part, we list essential preliminaries to support the proofs of our main conclusions.

Lemma 2.1 ([23]). *For a function f and an order $\theta > 0$, the Riemann–Liouville fractional integral is expressed as*

$$I_0^{\theta} f(y) = \frac{1}{\Gamma(\theta)} \int_0^y (y-s)^{\theta-1} f(s)\, ds.$$

Lemma 2.2 ([41]). *For $p(t) \in C[0,1] \cap L^1[0,1]$, $\theta > 0$, we have*

$$I_t^{\theta} D_t^{\theta} p(t) = p(t) + \sum_{i=1}^{n} a_i t^{\theta-i},$$

where $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ with $n = [\theta] + 1$.

Lemma 2.3 ([42]). *If $q(t) \in C[0,1]$, then the tempered fractional equation*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\sigma,\lambda} y(t) + q(t) = 0, \quad n-1 < \sigma \leq n, \\ y(0) = y'(0) = \cdots = y^{(n-2)}(0) = 0, \\ y(1) = \varsigma \displaystyle\int_0^1 e^{-\lambda(1-t)} y(t)\, dt, \end{cases}$$

has the unique solution

$$y(t) = \int_0^1 G(t,s) q(s)\, ds,$$

where $\varsigma < \sigma$ and we have the Green's function

$$G(t,s) = \begin{cases} \dfrac{\sigma(1-s)^{\sigma-1}(\sigma-\varsigma+\varsigma s)e^{\lambda s} t^{\sigma-1} - \sigma(\sigma-\varsigma)e^{\lambda s}(t-s)^{\sigma-1}}{(\sigma-\varsigma)\Gamma(\sigma+1)} e^{-\lambda t}, & 0 \leq s \leq t \leq 1, \\ \dfrac{\sigma(1-s)^{\sigma-1}(\sigma-\varsigma+\varsigma s)e^{\lambda s}}{(\sigma-\varsigma)\Gamma(\sigma+1)} e^{-\lambda t} t^{\sigma-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.1}$$

Lemma 2.4. *Let $w(t) \in C[0,1]$, the BVP with delay and tempered fractional derivatives*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\theta,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\sigma,\lambda} y(t))) = w(t), \quad t \in (0,1] \setminus \{\tau\}, \\ y(0) = y'(0) = y''(0) = \cdots = y^{(n-2)}(0) = 0, \\ y(1) = \varsigma \displaystyle\int_0^1 e^{-\lambda(1-t)} y(t) dt, \\ {}_0^R\mathbb{D}_t^{\sigma,\lambda} y(0) = 0, \end{cases} \tag{2.2}$$

is expressed as the fractional integral equation

$$y(t) = \int_0^1 G(t,s)\Big[\int_0^s \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda \eta} w(\eta)\, d\eta\Big]^{q-1} ds.$$

Proof. Let $k(t) = {}_0^R\mathbb{D}_t^{\sigma,\lambda} y(t)$, $l(t) = \varphi_p(k(t))$, we obtain

$$\begin{cases} {}_0^R\mathbb{D}_t^{\theta,\lambda} l(t) = w(t), \quad t \in (0,1] \setminus \{\tau\}, \\ l(0) = 0. \end{cases}$$

From the definition of ${}^{R}_{0}\mathbb{D}^{\theta,\lambda}_{t}$, we get

$${}^{R}_{0}\mathbb{D}^{\theta,\lambda}_{t} l(t) = e^{-\lambda t} D^{\theta}_{t}(e^{\lambda t} l(t)).$$

Thus, we have

$$D^{\theta}_{t}(e^{\lambda t} l(t)) = e^{\lambda t} w(t).$$

Applying Lemma 2.2, we obtain

$$e^{\lambda t} l(t) = \int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} e^{\lambda s} w(s)\, ds - l_1 t^{\theta-1}, \quad t \in (0,1] \setminus \{\tau\}.$$

Since $l(0) = 0$, we find that $l_1 = 0$, that is

$$l(t) = \int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} e^{\lambda s} e^{-\lambda t} w(s)\, ds, \quad t \in (0,1] \setminus \{\tau\}.$$

Furthermore, by $k(t) = {}^{R}_{0}\mathbb{D}^{\sigma,\lambda}_{t} y(t)$, $l(t) = \varphi_p(k(t))$, we obtain

$$l(t) = \varphi_p({}^{R}_{0}\mathbb{D}^{\sigma,\lambda}_{t}) = \int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} e^{\lambda s} e^{-\lambda t} w(s)\, ds.$$

Thus, the BVP for fractional differential equation (2.2) can be reformulated as:

$$\begin{cases} {}^{R}_{0}\mathbb{D}^{\sigma,\lambda}_{t} y(t) = \varphi_p^{-1}(\int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} e^{\lambda s} e^{-\lambda t} w(s)\, ds) \quad t \in (0,1] \setminus \{\tau\}, \\ y(0) = y'(0) = y''(0) = \cdots = y^{(n-2)}(0) = 0, \\ y(1) = \varsigma \int_0^1 e^{-\lambda(1-t)} y(t) dt. \end{cases}$$

By Lemma 2.3, we conclude that

$$y(t) = \int_0^1 G(t,s) \varphi_p^{-1} [\int_0^s \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{\lambda \eta} e^{-\lambda s} w(\eta)\, d\eta]\, ds.$$

On the other hand, since $w(s) \geq 0,\ s \in (0,1)$, we get that the unique solution to problem (2.2) is

$$y(t) = \int_0^1 G(t,s) [\int_0^s \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda \eta} w(\eta)\, d\eta]^{q-1}\, ds,$$

where $G(t,s)$ is given as (2.1). □

Lemma 2.5. *The Green's function $G(t,s)$ is continuous on $(0,1) \times (0,1)$ and its characteristics are as follows:*
(1) $G(t,s) \geq 0$,
(2) $G(t,s) \geq M_1 s(1-s)^{\sigma-1} e^{-\lambda t} t^{\sigma-1}$,
(3) $G(t,s) \leq M_2 (1-s)^{\sigma-1} e^{-\lambda t} t^{\sigma-1}$,
where

$$M_1 = \frac{\sigma\varsigma}{(\sigma-\varsigma)\Gamma(\sigma+1)},\ M_2 = \frac{\sigma^2 e^{\lambda}}{(\sigma-\varsigma)\Gamma(\sigma+1)}.$$

Proof. For any $(t,s)\in(0,1)\times(0,1)$, we have

$$\begin{aligned} G(t,s) &\leq \frac{\sigma(1-s)^{\sigma-1}(\sigma-\varsigma+\varsigma s)t^{\sigma-1}}{(\sigma-\varsigma)\Gamma(\sigma+1)}e^{-\lambda t}e^{\lambda s} \\ &\leq \frac{\sigma^2 e^{\lambda}}{(\sigma-\varsigma)\Gamma(\sigma+1)}(1-s)^{\sigma-1}e^{-\lambda t}t^{\sigma-1}. \end{aligned}$$

Furthermore, when $0\leq s\leq t\leq 1$ we obtain $t-s\leq t-ts$. Consequently, $(t-s)^{\sigma-1}\leq t^{\sigma-1}(1-s)^{\sigma-1}$. Based on this inequality we have

$$\begin{aligned} G(t,s) &= \frac{[\sigma^2(1-s)^{\sigma-1}-\sigma\varsigma(1-s)^{\sigma}]t^{\sigma-1}-\sigma(\sigma-\varsigma)(t-s)^{\sigma-1}}{(\sigma-\varsigma)\Gamma(\sigma+1)}e^{-\lambda t}e^{\lambda s} \\ &\geq \frac{[\sigma^2(1-s)^{\sigma-1}-\sigma\varsigma(1-s)^{\sigma}]t^{\sigma-1}-\sigma(\sigma-\varsigma)t^{\sigma-1}(1-s)^{\sigma-1}}{(\sigma-\varsigma)\Gamma(\sigma+1)}e^{-\lambda t}e^{\lambda s} \\ &= \frac{\sigma\varsigma s(1-s)^{\sigma-1}t^{\sigma-1}}{(\sigma-\varsigma)\Gamma(\sigma+1)}e^{-\lambda t}e^{\lambda s} \\ &\geq \frac{\sigma\varsigma}{(\sigma-\varsigma)\Gamma(\sigma+1)}s(1-s)^{\sigma-1}e^{-\lambda t}t^{\sigma-1}. \end{aligned}$$

When $0\leq t\leq s\leq 1$ we get

$$\begin{aligned} G(t,s) &= \frac{[\sigma^2(1-s)^{\sigma-1}-\sigma\varsigma(1-s)^{\sigma}]t^{\sigma-1}}{(\sigma-\varsigma)\Gamma(\sigma+1)}e^{-\lambda t}e^{\lambda s} \\ &\geq \frac{[\sigma^2(1-s)^{\sigma-1}-\sigma\varsigma(1-s)^{\sigma}]t^{\sigma-1}-\sigma(\sigma-\varsigma)t^{\sigma-1}(1-s)^{\sigma-1}}{(\sigma-\varsigma)\Gamma(\sigma+1)}e^{-\lambda t}e^{\lambda s} \\ &= \frac{\sigma\varsigma s(1-s)^{\sigma-1}t^{\sigma-1}}{(\sigma-\varsigma)\Gamma(\sigma+1)}e^{-\lambda t}e^{\lambda s} \\ &\geq \frac{\sigma\varsigma}{(\sigma-\varsigma)\Gamma(\sigma+1)}s(1-s)^{\sigma-1}e^{-\lambda t}t^{\sigma-1}. \end{aligned}$$

Hence, the proof is complete. □

Now, define a Banach space $\mathscr{D}=C([-\tau,1],\mathbb{R})$ endowed with the usual norm

$$\|y\|=\max_{-\tau\leq t\leq 1}|y(t)|,\quad y\in\mathscr{D},$$

and let

$$\overline{\xi}(t)=\begin{cases}\xi(t), & t\in[-\tau,0],\\ 0, & t\in(0,1].\end{cases} \tag{2.3}$$

The preceding lemmas show that a solution to problem (1.1) satisfies:

$$y(t)=\begin{cases}\displaystyle\int_0^1 G(t,s)[\int_0^s \gamma\frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}f(\eta,y(\eta-\tau))\,d\eta]^{q-1}\,ds, & t\in(0,1],\\ \xi(t), & t\in[-\tau,0].\end{cases}$$

Under the condition of homogeneous initial history, the operator $\mathcal{T}$ is constructed as follows:

$$(\mathcal{T}y)(t)=\begin{cases}\displaystyle\int_0^1 G(t,s)[\int_0^s \gamma\frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}f(\eta,y(\eta-\tau))\,d\eta]^{q-1}\,ds, & t\in(0,1],\\ 0, & t\in[-\tau,0].\end{cases} \tag{2.4}$$

Lemma 2.4 implies that solving the following problem can be reduced to the fixed point problem $\overline{y} = \mathcal{T}\overline{y}$:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\theta,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\sigma,\lambda}\overline{y}(t))) = \gamma f(t, \overline{y}(t-\tau)), & t \in (0,1] \setminus \{\tau\}, \\ \overline{y}(t) = 0, & t \in [-\tau, 0], \\ \overline{y}(0) = \overline{y}'(0) = \overline{y}''(0) = \cdots = \overline{y}^{(n-2)}(0) = 0, \\ \overline{y}(1) = \varsigma \displaystyle\int_0^1 e^{-\lambda(1-t)}\overline{y}(t)dt, \\ {}_0^R\mathbb{D}_t^{\sigma,\lambda}\overline{y}(0) = 0. \end{cases} \tag{2.5}$$

Let

$$y(t) = \overline{y}(t) + \overline{\xi}(t). \tag{2.6}$$

Lemma 2.6. *Suppose $\overline{y}(t) = y(t) - \overline{\xi}(t)$ is a solution of problem* (2.5)*, thus $y(t)$ is a solution of the BVP with delay and p-Laplacian operator* (1.1)*.*

Proof. Firstly, when $t \in [-\tau, 0]$, from (2.3) and (2.6) we obtain

$$y(t) = 0 + \xi(t) = \xi(t).$$

Then, when $t \in (0,1] \setminus \{\tau\}$, we get

$$\begin{aligned} {}_0^R\mathbb{D}_t^{\theta,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\sigma,\lambda}y(t))) &= {}_0^R\mathbb{D}_t^{\theta,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\sigma,\lambda}(\overline{y}(t) + \overline{\xi}(t)))) \\ &= {}_0^R\mathbb{D}_t^{\theta,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\sigma,\lambda}(\overline{y}(t) + 0))) \\ &= \gamma f(t, \overline{y}(t-\tau)) \\ &= \gamma f(t, y(t-\tau)). \end{aligned}$$

Therefore $y(t)$ is a solution of boundary value problem (1.1). □

3. Main results

This section focuses on the existence of solutions to problem (1.1). Depending on different assumptions, we give detailed proofs by employing corresponding theorems.

Theorem 3.1. *Suppose that*

(H_1) there exists a $\phi \in C([0,1], \mathbb{R}^+)$ satisfying

$$|f(t,y)| \le \phi(t), \forall (t,y) \in [0,1] \times \mathbb{R};$$

(H_2) there exists $\ell \in \mathbb{R}^+$ satisfying

$$|f(t,y) - f(t,z)| \le \ell|y - z|, \forall t \in [0,1], y, z \in \mathbb{R},$$

where ℓ satisfies $\ell\frac{(q-1)M_2\|\phi\|^{q-2}\gamma^{q-1}}{\sigma\Gamma^{q-1}(\theta+1)} < 1$, and the function ϕ is defined by (H_1).

Then problem (1.1) *admits a unique solution.*

Proof. Firstly, choose $r \geq \frac{M_2}{\sigma}[\frac{\gamma\|\phi\|}{\Gamma(\theta+1)}]^{q-1}$, and let $K_r = \{y \in \mathscr{D} : \|y\| \leq r\}$. Next we show that $\mathcal{T} K_r \subset K_r$.

For $y \in K_r$, it holds that $\mathcal{T} y(t) = 0$ on $[-\tau, 0]$. Then considering $t \in [0,1]$, we get

$$\begin{aligned}\|\mathcal{T} y(t)\| &\leq \int_0^1 M_2(1-s)^{\sigma-1} e^{-\lambda t} t^{\sigma-1} \left[\int_0^s |\gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta} f(\eta, y(\eta-\tau))|\, d\eta\right]^{q-1} ds \\ &= M_2 \int_0^1 (1-s)^{\sigma-1} e^{-\lambda t} t^{\sigma-1} \left[\int_0^s |\gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta} f(\eta, y(\eta-\tau))|\, d\eta\right]^{q-1} ds.\end{aligned}$$

Let $m = s - \eta$, then it follows from (H_1) that

$$\begin{aligned}&\int_0^s |\gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta} f(\eta, y(\eta-\tau))|\, d\eta \\ &\leq \max_{\eta\in[0,1]} |f(\eta, y(\eta-\tau))| \cdot \int_0^s \gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta}\, d\eta \\ &\leq \|\phi\| \frac{\gamma}{\Gamma(\theta)} \int_0^s (s-\eta)^{\theta-1} e^{-\lambda(s-\eta)}\, d\eta \\ &\leq \|\phi\| \frac{\gamma}{\Gamma(\theta)} \int_0^s m^{\theta-1} e^{-\lambda m}\, dm \\ &\leq \|\phi\| \frac{\gamma}{\Gamma(\theta)} \int_0^s m^{\theta-1}\, dm \\ &\leq \|\phi\| \frac{\gamma}{\Gamma(\theta+1)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|\mathcal{T} y(t)\| &\leq M_2 \int_0^1 (1-s)^{\sigma-1} e^{-\lambda t} t^{\sigma-1} [\frac{\gamma\|\phi\|}{\Gamma(\theta+1)}]^{q-1} ds \\ &\leq M_2 [\frac{\gamma\|\phi\|}{\Gamma(\theta+1)}]^{q-1} \int_0^1 (1-s)^{\sigma-1} ds \\ &= \frac{M_2}{\sigma} [\frac{\gamma\|\phi\|}{\Gamma(\theta+1)}]^{q-1} \\ &\leq r,\end{aligned}$$

which implies that $\mathcal{T} K_r \subset K_r$.

Next, for any $y, z \in \mathscr{D}$ and $t \in [0,1]$, the mean value theorem gives

$$\begin{aligned}&\|\mathcal{T} y - \mathcal{T} z\| \\ &= \max_{t\in[0,1]} |\mathcal{T} y(t) - \mathcal{T} z(t)| \\ &\leq \int_0^1 G(t,s) \left\{ [\int_0^s \gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta} f(\eta, y(\eta-\tau))\, d\eta]^{q-1} \right. \\ &\quad \left. - \int_0^s \gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta} f(\eta, z(\eta-\tau))\, d\eta]^{q-1} \right\} ds \\ &\leq (q-1)\delta^{q-2} \int_0^1 G(t,s) \int_0^s \gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta} \left[f(\eta, y(\eta-\tau)) - f(\eta, z(\eta-\tau))\right] d\eta\, ds\end{aligned}$$

$$\leq (q-1)M_2[\frac{\gamma\|\phi\|}{\Gamma(\theta+1)}]^{q-2}\int_0^1 (1-s)^{\sigma-1}e^{-\lambda t}t^{\sigma-1}\int_0^s \gamma\frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}$$
$$\times [f(\eta, y(\eta-\tau)) - f(\eta, z(\eta-\tau))]\, d\eta\, ds$$
$$\leq \ell(q-1)\frac{M_2\gamma^{q-1}\|\phi\|^{q-2}}{\Gamma^{q-2}(\theta+1)}\|y-z\|\int_0^1 (1-s)^{\sigma-1}e^{-\lambda t}t^{\sigma-1}\int_0^s \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}\, d\eta\, ds$$
$$\leq \ell(q-1)\frac{M_2\gamma^{q-1}\|\phi\|^{q-2}}{\Gamma^{q-1}(\theta+1)}\|y-z\|\int_0^1 (1-s)^{\sigma-1}e^{-\lambda t}t^{\sigma-1}\, ds$$
$$\leq \ell\frac{(q-1)M_2\|\phi\|^{q-2}\gamma^{q-1}}{\sigma\Gamma^{q-1}(\theta+1)}\|y-z\|,$$

where δ between $\int_0^s \gamma\frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}f(\eta, y(\eta-\tau))\, d\eta$ and $\int_0^s \gamma\frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}f(\eta, z(\eta-\tau))\, d\eta$, and from the boundedness of δ and the assumption (H_1), the estimate

$$0 \leq \delta \leq \frac{\gamma\|\phi\|}{\Gamma(\theta+1)}$$

holds uniformly for all $y, z \in \mathscr{D}$.

By assumption, the condition $\ell\frac{(q-1)M_2\|\phi\|^{q-2}\gamma^{q-1}}{\sigma\Gamma^{q-1}(\theta+1)} < 1$ holds. Thus the operator $\mathcal{T}$ is a contraction on $\mathscr{D}$. Consequently, Banach's fixed-point theorem yields the uniqueness of a solution for problem (1.1). This concludes the proof. □

Lemma 3.1 ([9]). *Suppose the operator $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ is completely continuous on Banach space $\mathcal{E}$. If the set*

$$\mathcal{K} = \{y \in \mathscr{D} \mid y = k\mathcal{T}y, 0 < k < 1\}$$

is bounded, then $\mathcal{T}$ has a fixed point in $\mathcal{E}$.

Theorem 3.2. *Assume the continuous function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies $|f(t,y)| \leq M$ for a constant $M > 0$ and all $t \in [0,1]$, $y \in \mathbb{R}$. Then the problem (1.1) has at least one solution.*

Proof. Firstly, since f is continuous, the operator $\mathcal{T}$ is continuous. Furthermore, consider a bounded set $\mathscr{H} \subset \mathscr{D}$. The boundedness of f implies

$$|\mathcal{T}y(t)| \leq M_2\int_0^1 (1-s)^{\sigma-1}e^{-\lambda t}t^{\sigma-1}\left[\int_0^s |\gamma\frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}f(\eta, y(\eta-\tau))|\, d\eta\right]^{q-1} ds$$
$$\leq M_2\int_0^1 (1-s)^{\sigma-1}e^{-\lambda t}t^{\sigma-1}\left[M\int_0^s \gamma\frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}\ d\eta\right]^{q-1} ds$$
$$\leq M_2\int_0^1 (1-s)^{\sigma-1}e^{-\lambda t}t^{\sigma-1}\left[M\frac{\gamma}{\Gamma(\theta+1)}\right]^{q-1} ds$$
$$\leq M_2\left[M\frac{\gamma}{\Gamma(\theta+1)}\right]^{q-1}\int_0^1 (1-s)^{\sigma-1}\, ds$$
$$\leq \frac{M_2}{\sigma}[M\frac{\gamma}{\Gamma(\theta+1)}]^{q-1}.$$

Thus, we get

$$\|\mathcal{T}y\| = \max_{t\in[0,1]}|\mathcal{T}y(t)| \leq \frac{M_2}{\sigma}[\frac{\gamma M}{\Gamma(\theta+1)}]^{q-1}.$$

Consequently, $\mathcal{T}(\mathscr{H})$ is uniformly bounded.

Then, the uniform continuity of the Green's function $G(t,s)$ yields: Given every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|G(t,s) - G(l,s)| < \varepsilon[\frac{\gamma M}{\Gamma(\theta+1)}]^{1-q}$$

for all $t, l, s \in [0,1]$ satisfying $|t-l| < \delta$.

Let $t, l \in [0,1]$, then

$$\begin{aligned}
|\mathcal{T}y(t) - \mathcal{T}y(l)| &\leq \int_0^1 |G(t,s) - G(l,s)| \left[\int_0^s \gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda \eta} f(\eta, y(\eta-\tau))\, d\eta\right]^{q-1} ds \\
&\leq \int_0^1 |G(t,s) - G(l,s)| \left[M \int_0^s \gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda \eta}\, d\eta\right]^{q-1} ds \\
&< \varepsilon[\frac{\gamma M}{\Gamma(\theta+1)}]^{1-q} \left[\frac{\gamma M}{\Gamma(\theta+1)}\right]^{q-1} \\
&= \varepsilon.
\end{aligned}$$

Therefore $\mathcal{T}(\mathscr{H})$ is equicontinuous. Since it satisfies all the conditions of the Arzelà-Ascoli theorem, $\mathcal{T}$ is completely continuous.

Let $\mathcal{K} = \{y \in \mathscr{D} \mid y = k\mathcal{T}y, 0 < k < 1\}$. If $y \in \mathcal{K}$, then for some $k \in (0,1)$, the equality $y = k\mathcal{T}y$ holds.

For any $t \in [0,1]$, by the boundedness of $\mathcal{T}$ established previously, we get

$$|y(t)| = k|\mathcal{T}y(t)| \leq |\mathcal{T}y(t)| \leq \frac{M_2}{\sigma}[\frac{\gamma M}{\Gamma(\theta+1)}]^{q-1}.$$

Hence, the set $\mathcal{K}$ is bounded.

As Lemma 3.1 ensures the operator $\mathcal{T}$ has a fixed point, the problem (1.1) admits at least one solution. □

Theorem 3.3. *Let the continuous function* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *satisfy* $|f(t,y)| \leq [\zeta|y| + L]^{1-q}$ *for some constants* $0 \leq \zeta < \frac{1}{\delta}$ *and* $L > 0$, *where* $t \in [0,1]$, $y \in \mathscr{D}$ *and* $\delta = \frac{M_2}{\sigma}[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1}$. *Then the problem* (1.1) *has at least one solution.*

Proof. Set $\mathscr{H} = \{y \in \mathscr{D} \mid \|y\| < r\}$ where $r = \frac{\delta L}{1-\zeta\delta} + 1$. Consider the operator $\mathcal{T} : \overline{\mathscr{H}} \to \mathscr{D}$ given by (2.4). Now, assume that $y = \mu\mathcal{T}y$ for some $\mu \in [0,1]$ and all $t \in [0,1]$. From this assumption, we get

$$\begin{aligned}
|y(t)| &= |\mu\mathcal{T}y(t)| \\
&\leq M_2 \int_0^1 (1-s)^{\sigma-1} e^{-\lambda t} t^{\sigma-1} \left[\int_0^s |\gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda \eta} f(\eta, y(\eta-\tau))|\, d\eta\right]^{q-1} ds \\
&\leq \frac{M_2}{\sigma}[\max_{\eta\in[0,1]} |f(\eta, y(\eta-\tau))|]^{q-1}[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1} \\
&\leq \frac{M_2}{\sigma}(\zeta|y| + L)[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1}.
\end{aligned}$$

Therefore, taking the norm, we have

$$\|y\| = \max_{t\in[0,1]} |y(t)| \leq \frac{M_2}{\sigma}(\zeta\,\|y\| + L)[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1} = (\zeta\,\|y\| + L)\delta,$$

where $\delta = \frac{M_2}{\sigma}[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1}$.

The definition of r and the above inequality give

$$\|y\| \le \frac{\delta L}{1-\zeta\delta} < \frac{\delta L}{1-\zeta\delta} + 1 = r.$$

Thus, we can conclude that

$$y \ne \mu\mathcal{T}y, \quad \forall y \in \partial\mathscr{H} \text{ and } \forall \mu \in [0,1].$$

Next, we show the operator $\mu\mathcal{T}$ is completely continuous.

The continuity of f implies that $\mu\mathcal{T}$ is continuous. Moreover, for any $y \in \overline{\mathscr{H}}$, we have

$$\|\mu\mathcal{T}y\| \le \|\mathcal{T}y\| \le \frac{M_2}{\sigma}(\zeta r + L)[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1} = (\zeta r + L)\delta,$$

hence, $\mu\mathcal{T}(\overline{\mathscr{H}})$ is uniformly bounded.

Since $G(t,s)$ is uniformly continuous in t on $[0,1]$, we know for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|t-l| < \delta$ for $t, l, s \in [0,1]$, the inequality

$$|G(t,s) - G(l,s)| < \varepsilon[\frac{\gamma}{\Gamma(\theta+1)}]^{1-q}(\zeta r + L)^{-1}$$

holds.

Let $t, l \in [0,1]$, then

$$\begin{aligned}
&|\mu\mathcal{T}y(t) - \mu\mathcal{T}y(l)| \\
&\le \int_0^1 |G(t,s) - G(l,s)| \left[\int_0^s \gamma\frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}f(\eta, y(\eta-\tau))\,d\eta\right]^{q-1} ds \\
&\le \int_0^1 |G(t,s) - G(l,s)| \left[\max_{\eta\in[0,1]} |f(\eta, y(\eta-\tau))| \int_0^s \gamma\frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)}e^{-\lambda s}e^{\lambda\eta}\,d\eta\right]^{q-1} ds \\
&\le \int_0^1 |G(t,s) - G(l,s)| \left[\frac{\gamma(\zeta r + L)}{\Gamma(\theta+1)}\right]^{q-1} ds \\
&< \varepsilon[\frac{\gamma}{\Gamma(\theta+1)}]^{1-q}(\zeta r + L)^{-1}\left[\frac{\gamma(\zeta r + L)}{\Gamma(\theta+1)}\right]^{q-1} \\
&= \varepsilon.
\end{aligned}$$

Therefore $\mathcal{T}(\overline{\mathscr{H}})$ is equicontinuous. Hence, the Arzelà-Ascoli theorem yields that $\mu\mathcal{T}$ is completely continuous.

Let $\nu_\mu(y) = y - \mu\mathcal{T}y$ and denote the unit operator by I. Based on the preceding proof, we obtain well-defined Leray-Schauder degrees as follows. Using the homotopy invariance we obtain

$$\begin{aligned}
\deg(\nu_\mu, \mathscr{H}, 0) &= \deg(I - \mu\mathcal{T}, \mathscr{H}, 0) \\
&= \deg(\nu_0, \mathscr{H}, 0) \\
&= \deg(I, \mathscr{H}, 0) \\
&= 1 \\
&\ne 0,\ 0 \in \mathscr{H}.
\end{aligned}$$

It follows from the solvability of the Leray-Schauder degree that $\nu_1(y) = 0$ possesses at least one solution on $\mathscr{H}$. Thus, problem (1.1) has at least one solution. The proof is finished. □

Remark 3.1. Theorem 3.2 relies on the strong uniform bound $|f(t,y)| \leq M$ for Krasnoselskii's theorem, whereas Theorem 3.3 admits the more general growth $|f(t,y)| \leq [\zeta|y| + L]^{1-q}$. Exploring weaker alternatives to the condition in Theorem 3.2 is an interesting problem.

Lemma 3.2 ([14]). *If L_1 is a closed, convex subset in the Banach space L and M is an open set in L_1 satisfying $0 \in M$. Suppose that $\mathcal{V} : \overline{M} \to L_1$ is a continuous compact map. Then either*

(i) $\mathcal{V}$ has a fixed point in $\overline{M}$, or

(ii) there are an $y \in \partial M$ and $k \in (0,1)$ with $y = k\mathcal{V}(y)$.

Theorem 3.4. *Suppose that:*

(H_3) there exist $h : [0,1] \to \mathbb{R}^+$ is continuous and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, which satisfy:

$$|f(t,y)| \leq h^{1-q}(t)\psi^{1-q}(|y|), \quad \forall (t,y) \in [0,1] \times \mathbb{R};$$

(H_4) there exists $A \in \mathbb{R}^+$ satisfying

$$\frac{A}{\psi(A)\|h\|\delta} > 1,$$

where $\delta = \frac{M_2}{\sigma}[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1}$.

Then the problem (1.1) has at least one solution.

Proof. Let $K_r = \{y \in \mathscr{D} \mid \|y\| \leq r\}$, where r is a positive number. Then, for $y \in K_r$ and $t \in [0,1]$, assumptions (H_3) and (H_4) imply that

$$\begin{aligned}
|\mathcal{T}y(t)| &\leq M_2 \int_0^1 (1-s)^{\sigma-1} e^{-\lambda t} t^{\sigma-1} \left[\int_0^s \gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta} f(\eta, y(\eta-\tau))\, d\eta\right]^{q-1} ds \\
&\leq \frac{M_2}{\sigma}[\max_{\eta\in[0,1]} |f(\eta, y(\eta-\tau))|]^{q-1}[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1} \\
&\leq \frac{M_2}{\sigma}[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1} \|h\|\, \psi(\|y\|) \\
&\leq \delta\, \|h\|\, \psi(r),
\end{aligned}$$

where $\delta = \frac{M_2}{\sigma}[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1}$, hence $\mathcal{T} : K_r \to \mathscr{D}$ is well-defined and $\mathcal{T}(K_r)$ is uniformly bounded.

The uniformly continuous of $G(t,s)$ shows that given every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|G(t,s) - G(l,s)| < \varepsilon[\frac{\gamma}{\Gamma(\theta+1)}]^{1-q} \|h\|^{-1} \psi(r)^{-1}$$

for all $t, l, s \in [0,1]$ satisfying $|t-l| < \delta$.

Further, for $t, l \in [0,1]$, it follows that

$$\begin{aligned}
|\mathcal{T}y(t) - \mathcal{T}y(l)| &\leq \int_0^1 |G(t,s) - G(l,s)| \left[\int_0^s \gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta} f(\eta, y(\eta-\tau))\, d\eta\right]^{q-1} ds \\
&\leq \int_0^1 |G(t,s) - G(l,s)| \left[\max_{\eta\in[0,1]} |f(\eta, y(\eta-\tau))| \int_0^s \gamma \frac{(s-\eta)^{\theta-1}}{\Gamma(\theta)} e^{-\lambda s} e^{\lambda\eta}\, d\eta\right]^{q-1} ds \\
&\leq \int_0^1 |G(t,s) - G(l,s)|\, \|h\|\, \psi(r) \left[\frac{\gamma}{\Gamma(\theta+1)}\right]^{q-1} ds
\end{aligned}$$

$$< \varepsilon[\frac{\gamma}{\Gamma(\theta+1)}]^{1-q}\left\|h\right\|^{-1}\psi(r)^{-1}\left\|h\right\|\psi(r)\left[\frac{\gamma}{\Gamma(\theta+1)}\right]^{q-1}$$
$$= \varepsilon.$$

Therefore $\mathcal{T}(K_r)$ is equicontinuous. Consequently, $\mathcal{T}$ is completely continuous.

For contradiction, suppose that $y = \mu\mathcal{T}y$ holds for some $\mu \in (0,1)$ and y. It follows that

$$\|y\| = \|\mu(\mathcal{T}y)\| \leq \psi(\|y\|)\|h\|\delta.$$

Choose $K = \{y \in \mathscr{D} \mid \|y\| < A\}$. Then $\mathcal{T} : \overline{K} \to \mathscr{D}$ is completely continuous and for any $y \in \partial K$, we know $\|y\| = A$. But from the condition (H_4), we obtain a contradiction:

$$\|y\| = \sup_{t\in[0,1]} |\mu(\mathcal{T}y)(t)| \leq \psi(A)\|h\|\delta < A.$$

Thus, $y = \mu\mathcal{T}(y)$ has no solution $y \in \partial K$ for $\mu \in (0,1)$.

According to Lemma 3.2, $\mathcal{T}$ has a fixed point $y \in \overline{K}$, providing a solution to problem (1.1). The proof is complete. □

4. Example

We examine a BVP for p-Laplacian fractional differential equation with delay and tempered fractional derivatives below:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\frac{1}{2},1}(\varphi_2({}_0^R\mathbb{D}_t^{\frac{9}{2},1}y(t))) = t - \sin y, \quad t \in (0,1) \setminus \left\{\frac{1}{4}\right\}, \\ y(t) = t^8, \quad t \in [-\frac{1}{4}, 0], \\ y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0, \\ y(1) = \int_0^1 e^{t-1}y(t)dt, \\ {}_0^R\mathbb{D}_t^{\frac{9}{2},1}y(0) = 0, \end{cases} \tag{4.1}$$

where $\theta = \frac{1}{2}, \sigma = \frac{9}{2}, p = 2, \lambda = \gamma = \varsigma = 1, \tau = \frac{1}{4}, \xi(t) = t^8$.

(1) Let us verify that the problem (4.1) satisfies the condition (H_1) and (H_2).

Choose $\phi(t) = t + 1$, we observe that $|t - \sin y| \leq t + 1$, thus condition (H_1) holds. Then, take $l = 1$, by a simple computation, we get

$$|f(t,y) - f(t,z)| = |\sin y - \sin z| \leq |y - z|, \ y, z \in \mathbb{R},$$
$$\ell\frac{(q-1)M_2\left\|\phi\right\|^{q-2}\gamma^{q-1}}{\sigma\Gamma^{q-1}(\theta+1)} = \frac{M_2\left\|\phi\right\|}{\sigma\Gamma(\theta+1)} = 0.1505 < 1,$$

which implies that (H_2) holds.

In conclusion, problem (4.1) satisfies Theorem 3.1 and hence has a unique solution.

(2) In the following, take $M = 2$, for the continuous function $f(t,y) = t - \sin y$, we get

$$|f(t,y)| \leq 2, \ \forall t \in [0,1], y \in \mathbb{R}.$$

Hence, the conditions of Theorem 3.2 are satisfied. By applying this theorem, problem (4.1) has at least one solution.

(3) Choose $\zeta = 0, L = \frac{1}{4}$. By means of computation, we have $\delta = \frac{M_2}{\sigma}[\frac{\gamma}{\Gamma(\theta+1)}]^{q-1} = \frac{2}{15\sqrt{\pi}}$. Thus, the values of ζ and L satisfy

$$L > 0,$$
$$0 \leq \zeta < \frac{1}{\delta} = \frac{15\sqrt{\pi}}{2} \approx 13.29.$$

Furthermore, we get $[\zeta|y| + L]^{-1} = 4$, and

$$|f(t, y)| \leq 2 < [\zeta|y| + L]^{-1}.$$

So, the problem (4.1) satisfies all the conditions of Theorem 3.3. We obtain that problem (4.1) has at least one solution.

(4) We investigate that the problem (4.1) satisfies the condition (H_3) and (H_4).

Take continuous function $h(t) = 1+t$ and nondecreasing function $\psi(y) = \frac{1}{4}(1-e^{-y})$, for any $t \in [0, 1], y \in \mathbb{R}$, we find $1 \leq h(t) \leq 2, 0 < \psi(y) < \frac{1}{4}$. Therefore, we get

$$2 < [h(t)\psi(|y|)]^{-1} < \infty.$$

Thus, we get

$$|f(t, y)| \leq 2 < [h(t)\psi(|y|)]^{-1}, \ \forall(t, y) \in [0, 1] \times \mathbb{R},$$

which implies that (H_3) holds.

Again choose $A = 1$, and by using the computed result $\delta = \frac{2}{15\sqrt{\pi}}$, we can see that A satisfies

$$\psi(1)\|h\|\delta = \frac{1 - e^{-1}}{15\sqrt{\pi}} < 1.$$

Hence, (H_4) is valid. By Theorem 3.4, we obtain that problem (4.1) has at least one solution.

5. Conclusions

This study examined the existence and uniqueness of solutions for a boundary value problem. The problem involved the p-Laplacian operator, included a delay, and used a tempered fractional derivative. We created a suitable operator using Green's function and a few fixed point methods to get complete solvability results. The uniqueness of the solution was demonstrated through the Banach contraction mapping principle, contingent upon a Lipschitz condition on the nonlinear term. Existence results were derived utilizing Schaefer's fixed point theorem and the Leray–Schauder degree, along with alternative principles within appropriately defined bounded sets. These findings add to and expand upon existing research on fractional differential equations that include delay and tempered fractional derivatives. Nevertheless, this particular study is limited to an examination of the existence of solutions for the defined boundary value problem. Future research endeavors could investigate the following directions:

(1) The existence and multiplicity of positive solutions under more general boundary conditions and nonlinearities.

(2) The feasibility of developing numerical or approximation methods derived from iterative fixed-point algorithms to devise solutions for the p-Laplacian fractional boundary value problem and examining their stability and convergence characteristics, as inspired by [38].

(3) The possibility of investigating the solvability of the examined boundary value problem within generalized metric frameworks, such as interpolative metric spaces recently introduced by Patel et al. [22], and ascertaining whether fixed point theorems in these contexts can provide more precise existence or multiplicity criteria.

(4) Obtaining solutions of the problem via iterative methods remains an interesting issue.

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