

EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR DISCRETE (P, Q)-LAPLACIAN PROBLEM WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract In the present paper, we investigate the existence of positive solutions for a class of singular semi-positive discrete (p, q) -Laplacian problem of second order. Specially, the nonlinear term is singular at $u = 0$ and may approach $-\infty$ as $u \rightarrow 0^+$. To overcome the lack of maximum principle, we construct a new comparison theorem. Then, by using Krasnosel'skii type fixed point theorem, we obtain the existence, multiplicity and nonexistence of positive solutions for this kind of problem.

Keywords Discrete (p, q) -Laplacian, positive solutions, nonlinear boundary conditions, Krasnosel'skii-type fixed point theorem, existence.

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1. Introduction

The study of the existence of solutions to boundary value problems of ODE has last a long history. Specially, since Erbe and Wang [9] employed the fixed point theorem in cones to study the existence of positive solutions to the following Sturm-Liouville problem

$$\begin{cases} u'' + a(t)f(u(t)) = 0, & t \in (0, 1), \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases} \quad (1.1)$$

a great deal of results have been established concerning the existence of positive solutions to the boundary value problems of ODE, see, for instance, [1, 2, 4, 5, 9, 11–14, 17, 18, 23, 24, 26–28]. Meanwhile, as its discrete analogous, the existence of positive solutions of discrete boundary value problems has also been discussed by several authors [3, 6–8, 10, 15, 16, 19–22, 25, 29]. In particular, because its important application in several applied subject, the problems with nonlinear boundary conditions of ODE and its discrete analogous also has been discussed by several authors. The earliest result of the existence of positive solutions to the problems with nonlinear boundary conditions of ODE could be dated in 1997, Wang [26] considered the existence of positive solutions of the following problem

$$(\phi(u'))' + a(t)f(u) = 0, \quad t \in (0, 1)$$

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subject to one of the following three pairs of nonlinear boundary conditions:

$$\begin{aligned} u(0) - H_1(u'(0)) &= 0, & u(1) - H_2(u'(1)) &= 0; \\ u(0) - H_1(u'(0)) &= 0, & u'(1) &= 0; \\ u'(0) &= 0, & u(1) - H_2(u'(1)) &= 0, \end{aligned}$$

where f is nonsingular and nonnegative, $\phi(x) := |x|^{p-2}x$, $p > 1$. If the nonlinearity term f is singular and may satisfies some semi-positive conditions, which means f may approach $-\infty$ as u approaches 0 from the right, Hai and Wang [14] established the existence results of positive solution to the following problem

$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t)f(u), & t \in (0, 1), \\ a_1 u(0) - H_1(u'(0)) = 0, \\ a_2 u(1) + H_2(u'(1)) = 0. \end{cases}$$

Later, Hai [11, 12], Sim and Son [24] considered this kind of problems under different cases. For other related work on singular, nonsingular semipositone problems on bounded domains, we refer to the references [1, 2, 4, 9, 11, 12, 17, 18, 27, 28].

For the discrete case, since 1995, Merdivenci [22] considered the existence of two positive solutions to a kind of discrete nonlinear second order Sturm-Liouville problems, there are still several results on the existence of positive solutions to the discrete boundary value problems, see, for instance, [3, 5–8, 10, 15, 16, 19–21, 25, 29]. Specially, in [3], the authors considered a kind of problems with nonlinear boundary conditions as follows,

$$\begin{cases} -\Delta[p(t-1)\Delta u(t-1)] = \lambda g(t)f(u), & t \in [1, T]_{\mathbb{Z}}, \\ a_1 u(0) - H_1(\Delta u(0)) = 0, \\ a_2 u(T+1) + H_2(\Delta u(T)) = 0, \end{cases}$$

where $T > 4$ is an integer, $[1, T]_{\mathbb{Z}} = \{1, 2, \dots, T\}$, a_1 and a_2 are nonnegative constants with $a_1^2 + a_2^2 > 0$ and $\lambda > 0$ is a parameter, $g : [1, T]_{\mathbb{Z}} \rightarrow (0, \infty)$, $H_i : \mathbb{R} \rightarrow \mathbb{R}$ is odd, nondecreasing functions with $a_i + |H_i| \neq 0$ for $i = 1, 2$, $f : (0, \infty) \rightarrow \mathbb{R}$ is continuous and there exists a constant $\delta \in [0, 1)$ such that

$$\limsup_{z \rightarrow 0^+} z^\delta |f(z)| < \infty.$$

It is noticed that the nonlinear term in [3] is non-positive and may approach $-\infty$ as z approaches 0^+ . Under some different growth restriction at ∞ and 0, the authors obtained the existence and asymptote behavior of positive solutions to the above problem.

However, there are no existence results of positive solutions on the discrete problems with (p, q) -Laplace operators and nonlinear boundary conditions. Thus, in this paper, we try to consider the existence, multiplicity and nonexistence of positive solutions to discrete (p, q) -Laplacian problem

$$\begin{cases} -\Delta(\varphi_p(\Delta u(t-1))) - \Delta(\varphi_q(\Delta u(t-1))) = \lambda h(t)f(u(t)), & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = 0 = a\Delta u(T) + g(\lambda, u(T+1))u(T+1). \end{cases} \quad (1.2)$$

Notice that $\varphi(s) = |s|^{p-2}s + |s|^{q-2}s$ is an odd and increasing homeomorphism on \mathbb{R} . Then the problem (1.2) can be equivalently transformed into the following problem

$$\begin{cases} -\Delta(\varphi(\Delta u(t-1))) = \lambda h(t)f(u(t)), & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = 0 = a\Delta u(T) + g(\lambda, u(T+1))u(T+1). \end{cases} \quad (1.3)$$

We assume that

(H1) $h : [1, T]_{\mathbb{Z}} \rightarrow (0, \infty)$;

(H2) $f : (0, \infty) \rightarrow \mathbb{R}$ with $\lim_{s \rightarrow 0^+} f(s) \in (-\infty, 0) \cup \{-\infty\}$ and there exists $\alpha \in (0, 1), \sigma > 0, c_1 > 0$ such that $|f(s)| < \frac{c_1}{s^\alpha}$ for $s < \sigma$ and $f(s)$ is positive and nondecreasing for $s \geq \sigma$.

(H3) $g : (0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ such that $g(r, s)s$ is nondecreasing with respect to $s \in [0, \infty)$.

Meanwhile, we also shall adopt the following assumptions:

(H_a) $\lim_{s \rightarrow \infty} \frac{f(s)}{s^{r-1}} = A \in (0, \infty), 1 < r < q$;

(H_b) $\lim_{s \rightarrow \infty} \frac{f(s)}{s^{q-1}} = \infty$;

(H_c) $\lim_{s \rightarrow \infty} \frac{f(s)}{s^{q-1}} = B \in (0, \infty)$.

2. Preliminary

In the rest of the paper, we let $[a, b]_{\mathbb{Z}} = \{a, a+1, \dots, b\}$ for integers $a < b$, and we also let

$$E = \{u|u : [1, T+1]_{\mathbb{Z}} \rightarrow \mathbb{R}, u(0) = 0 = a\Delta u(T) + g(\lambda, u(T+1))u(T+1)\}$$

with the norm $\|u\|_{\infty} = \max_{t \in [0, T+1]_{\mathbb{Z}}} |u(t)|$. Then $(E, \|\cdot\|_E)$ is a Banach space. Now, we present a Krasnosel'skii type fixed point theorem.

Lemma 2.1. *Let X be a Banach space and $K : X \rightarrow X$ be a completely continuous operator. Suppose that there exist a nonzero element $z \in X$ and positive constants w and r with $w \neq r$ such that*

(a) *if $y \in X$ satisfies $y = Ky + \xi z$ for $\xi \geq 0$, then $\|y\|_X \neq w$;*

(b) *if $y \in X$ satisfies $y = \theta Ky$ for $\theta \in (0, 1]$, then $\|y\|_X \neq r$.*

Then K has a fixed point $y \in X$ with $\min\{r, w\} < \|y\|_X < \max\{r, w\}$.

In the following, we consider a modified problem associated with problem (1.2), namely,

$$\begin{cases} -\Delta(\varphi(\Delta x(t-1))) = \lambda h(t)f^*(y(t)), & t \in [1, T]_{\mathbb{Z}}, \\ x(0) = 0 = a\Delta x(T) + g(\lambda, |x(T+1)|)x(T+1), \end{cases} \quad (2.1)$$

where $y(t) \in E, f^*(y(t)) := f(\max\{y(t), \sigma d(t)\}), d(t) := \min\{\frac{t}{T+1}, \frac{T+1-t}{T+1}\}, t \in [1, T]_{\mathbb{Z}}$. Notice that, for $t \in [1, T]_{\mathbb{Z}}$,

$$\begin{aligned} h(t)|f^*(y(t))| &\leq h(t)\frac{c_1}{\sigma^\alpha d(t)^\alpha} + h(t)f(\max\{y(t), \sigma\}) \\ &\leq h^*(t) + h(t)f(\max\{\sigma, \|y\|_{\infty}\}), \end{aligned} \quad (2.2)$$

where $h^*(t) := h(t) \frac{c_1}{\sigma^\alpha d(t)^\alpha}$. Then $h(t)f^*(y(t))$ is bounded.

Define $g_a^*(s) : \mathbb{R} \rightarrow (0, \infty)$ by $g_a^*(s) = 1$ when $a = 0$ and $g_a^*(s) = \frac{g(\lambda, |s|)}{a}$ when $a > 0$. We also define $T_\lambda : [0, T + 1]_{\mathbb{Z}} \rightarrow [0, T + 1]_{\mathbb{Z}}$ by

$$T_\lambda y(t) = \sum_{s=1}^t \varphi^{-1} \left(-\varphi(g_a^*(m_y)m_y) + \lambda \sum_{\tau=s}^T h(\tau)f^*(y(\tau)) \right),$$

where

$$m_y = \begin{cases} 0, & a = 0, \\ \sum_{s=1}^{T+1} \varphi^{-1} \left(-\varphi(g_a^*(m_y)m_y) + \lambda \sum_{\tau=s}^T h(\tau)f^*(y(\tau)) \right), & a > 0. \end{cases} \quad (2.3)$$

By (H3), we know that m_y is uniquely determined and there exists a constant $M > 0$ such that $|m_y| \leq M$.

Lemma 2.2. *Let (H1)-(H3) hold. Then $T_\lambda : [0, T + 1]_{\mathbb{Z}} \rightarrow [0, T + 1]_{\mathbb{Z}}$ is a completely continuous operator.*

Proof. First, we can prove that $T_\lambda y(t)$ is bounded. By (H3), we know that m_y is uniquely determined and there exists constant $M > 0$ such that $|m_y| \leq M$. Furthermore, by (2.2), we can deduce

$$\begin{aligned} |T_\lambda y(t)| &= \left| \sum_{s=1}^t \varphi^{-1} \left(-\varphi(g_a^*(m_y)m_y) + \lambda \sum_{\tau=s}^T h(\tau)f^*(y(\tau)) \right) \right| \\ &\leq \sum_{s=1}^t \varphi^{-1} \left(\left| \varphi(g_a^*(m_y)m_y) \right| + \left| \lambda \sum_{\tau=1}^T h(\tau)f^*(y(\tau)) \right| \right) \\ &\leq \sum_{s=1}^t \varphi^{-1} \left(\varphi(g_a^*(M)M) + \lambda \left| \sum_{\tau=1}^T h(\tau)f^*(y(\tau)) \right| \right) \\ &\leq T\varphi^{-1} \left(\varphi(g_a^*(M)M) + \lambda \sum_{\tau=1}^T h(\tau) \left| f^*(y(\tau)) \right| \right) \\ &\leq T\varphi^{-1} \left(\varphi(g_a^*(M)M) + \lambda \sum_{\tau=1}^T (h^*(\tau) + h(\tau)f(\max\{\sigma, \|y\|_\infty\})) \right) \\ &< \infty. \end{aligned} \quad (2.4)$$

Thus, $T_\lambda y(t)$ is bounded. On the other hand, for $\varepsilon > 0$, there exists

$$\delta = \varepsilon/\varphi^{-1} \left(\varphi(g_a^*(M)M) + \lambda \sum_{\tau=1}^T (h^*(\tau) + h(\tau)f(\max\{\sigma, \|y\|_\infty\})) \right) > 0$$

such that for any $1 \leq t_1 < t_2 \leq T$ satisfying $|t_1 - t_2| < \delta$ and for any $y \in E$,

$$\begin{aligned} &|T_\lambda y(t_2) - T_\lambda y(t_1)| \\ &= \left| \sum_{s=1}^{t_2} \varphi^{-1} \left(-\varphi(g_a^*(m_y)m_y) + \lambda \sum_{\tau=s}^T h(\tau)f^*(y(\tau)) \right) \right| \end{aligned}$$

$$\begin{aligned} & \left| -\sum_{s=1}^{t_1} \varphi^{-1} \left(-\varphi(g_a^*(m_y)m_y) + \lambda \sum_{\tau=s}^T h(\tau)f^*(y(\tau)) \right) \right| \\ = & \left| \sum_{s=t_1+1}^{t_2} \varphi^{-1} \left(-\varphi(g_a^*(m_y)m_y) + \lambda \sum_{\tau=s}^T h(\tau)f^*(y(\tau)) \right) \right| \\ \leq & \sum_{s=t_1+1}^{t_2} \varphi^{-1} \left(\varphi(g_a^*(M)M) + \lambda \sum_{\tau=1}^T (h^*(\tau) + h(\tau)f(\max\{\sigma, \|y\|_\infty\})) \right) \\ \leq & \varepsilon. \end{aligned}$$

Then, by Arzelà-Ascoli Theorem, $T_\lambda : [0, T + 1]_{\mathbb{Z}} \rightarrow [0, T + 1]_{\mathbb{Z}}$ is a compact operator.

Now, we prove that T_λ is continuous. Let $y_n \rightarrow y$ as $n \rightarrow \infty$ in E . Then by (2.3), we know that m_{y_n} and m_y can be uniquely determined by y_n and y . Therefore, we have $m_{y_n} \rightarrow m_y$ as $n \rightarrow \infty$. Moreover, $T_\lambda y_n \rightarrow T_\lambda y$ as $n \rightarrow \infty$, which completes the proof. \square

Remark 2.1. In view of boundary condition in problem (1.3), there exists $t_m \in [1, T]_{\mathbb{Z}}$ such that $\|u\|_\infty = |u(t_m)| = \max_{t \in [0, T+1]_{\mathbb{Z}}} |u(t)|$. Then $T_\lambda y$ can be expressed as

$$T_\lambda y(t) = \begin{cases} T_\lambda y(t_*) + \sum_{s=t_*+1}^t \varphi^{-1} \left(\varphi(\Delta u(t_m)) + \lambda \sum_{\tau=s}^{t_m} h(\tau)f^*(y(\tau)) \right), & t \in [t_* + 1, t_m]_{\mathbb{Z}}, \\ T_\lambda y(t^*) - \sum_{s=t+1}^{t^*} \varphi^{-1} \left(\varphi(\Delta u(t_m)) + \lambda \sum_{\tau=s}^{t_m} h(\tau)f^*(y(\tau)) \right), & t \in [t_m + 1, t^*]_{\mathbb{Z}}. \end{cases}$$

Lemma 2.3. If $u, v : [0, T + 1]_{\mathbb{Z}} \rightarrow \mathbb{R}$ satisfy

$$\begin{cases} -\Delta(\varphi(\Delta u(t - 1))) \geq -\Delta(\varphi(\Delta v(t - 1))), & t \in [1, T]_{\mathbb{Z}}, \\ u(0) \geq v(0), u(T + 1) \geq v(T + 1), \end{cases} \tag{2.5}$$

then $u(t) \geq v(t), t \in [1, T]_{\mathbb{Z}}$.

Proof. Denote $z(t) = u(t) - v(t)$. If the Lemma is invalid, then there exists $t_0 \in [1, T]_{\mathbb{Z}}$ such that $z(t_0) = \min_{t \in [1, T]_{\mathbb{Z}}} z(t) < 0$. Then $\Delta z(t_0 - 1) \leq 0$. In (2.5), summing from $s = t_0$ to $s = t, t \in [t_0, T]_{\mathbb{Z}}$, we can obtain $\varphi(\Delta u(t)) - \varphi(\Delta u(t_0 - 1)) \leq \varphi(\Delta v(t)) - \varphi(\Delta v(t_0 - 1))$. To be precise, $\varphi(\Delta u(t)) - \varphi(\Delta v(t)) \leq \varphi(\Delta u(t_0 - 1)) - \varphi(\Delta v(t_0 - 1)) \leq 0$. Therefore, $\Delta z(t) = \Delta u(t) - \Delta v(t) \leq 0, t \in [t_0, T]_{\mathbb{Z}}$ and then we have $z(t_0) \geq z(T + 1) \geq 0$, which contradicts $z(t_0) < 0$. Thus, $u(t) \geq v(t)$ for $t \in [1, T]_{\mathbb{Z}}$. \square

Lemma 2.4. Let $g_0 > 0$ and $k(t) \in E$ be a nonnegative function. Suppose that $v(t)$ satisfy

$$\begin{cases} -\Delta(\varphi(\Delta v(t - 1))) \geq -k(t), & t \in [1, T]_{\mathbb{Z}}, \\ v(0) \geq 0, a\Delta v(T) + g_0v(T + 1) \geq 0. \end{cases} \tag{2.6}$$

If $\|v\|_\infty > (T + 1)\varphi^{-1} \left(2 \sum_{s=1}^T k(s) \right)$, then, for $t \in [0, T + 1]$, there holds $v(t) \geq \frac{\|v\|_\infty}{\varphi_p^{-1}(2)} d(t)$, where $d(t) := \min \left\{ \frac{t}{T+1}, \frac{T+1-t}{T+1} \right\}$.

Proof. Let $t_m \in [0, T + 1]_{\mathbb{Z}}$ be such that $\|v\|_{\infty} = |v(t_m)|$, we claim that $v(t_m) > 0$. In fact, $\|v\|_{\infty} > (T+1)\varphi^{-1}\left(2\sum_{s=1}^T k(s)\right)$, and then we have $v(t_m) \neq 0$. Moreover, suppose that $v(t_m) < 0$. We distinguish the following three cases.

Case 1. If $t_m = 0$, then $v(t_m) = v(0) \geq 0$, which contradicts $v(t_m) < 0$.

Case 2. If $t_m = T + 1$, then $\Delta v(T) = v(T + 1) - v(T) \leq 0$, and $a\Delta v(T) + g_0v(T + 1) < 0$, which contradicts $a\Delta v(T) + g_0v(T + 1) \geq 0$.

Case 3. If $t_m \in [1, T]_{\mathbb{Z}}$, then, summing from $s = t$ to $s = t_m$ in (2.6), we have $-\sum_{s=t}^{t_m} \Delta(\varphi(\Delta v(s - 1))) \geq -\sum_{s=t}^{t_m} k(s)$, and further $\varphi(\Delta v(t - 1)) - \varphi(\Delta v(t_m)) \geq -\sum_{s=1}^T k(s)$. Since $v(t_m) < 0$, $\Delta v(t_m) = v(t_m + 1) - v(t_m) \geq 0$, $\varphi(\Delta v(t_m)) \geq 0$, we have

$$\varphi(\Delta v(t - 1)) \geq -\sum_{s=1}^T k(s), \quad \Delta v(t - 1) \geq -\varphi^{-1}\left(\sum_{s=1}^T k(s)\right).$$

Summing from $s = 1$ to $s = t$ at both sides of the above equation, we obtain

$$\sum_{s=1}^t \Delta v(s - 1) \geq -\sum_{s=1}^t \varphi^{-1}\left(\sum_{\tau=1}^T k(\tau)\right) \geq -\sum_{s=1}^{T+1} \varphi^{-1}\left(\sum_{\tau=1}^T k(\tau)\right),$$

which implies

$$v(t) - v(0) \geq -(T + 1)\varphi^{-1}\left(\sum_{s=1}^T k(s)\right).$$

From $v(0) \geq 0$, one has $v(t) \geq -(T + 1)\varphi^{-1}\left(\sum_{s=1}^T k(s)\right)$. Then

$$\|v\|_{\infty} = -v(t_m) \leq (T + 1)\varphi^{-1}\left(\sum_{s=1}^T k(s)\right) \leq (T + 1)\varphi^{-1}\left(2\sum_{s=1}^T k(s)\right),$$

which is a contradiction. Hence, $v(t_m) > 0$.

In the following, we will show that for $t \in [0, T + 1]_{\mathbb{Z}}$, $v(t) \geq \frac{\|v\|_{\infty}}{\varphi_p^{-1}(2)}d(t)$. For $t = 0$ or $t = T + 1$, it is obvious. We will divide into the case $t \in [1, t_m]_{\mathbb{Z}}$ and $t \in [t_m + 1, T]_{\mathbb{Z}}$.

For the case $t \in [1, t_m]_{\mathbb{Z}}$, let $\omega(t) \in E$ be the solution of the following problem

$$\begin{cases} -\Delta(\varphi(\Delta\omega(t - 1))) = -k(t), t \in [1, t_m]_{\mathbb{Z}}, \\ \omega(0) = 0, \omega(t_m) = \|v\|_{\infty}. \end{cases} \tag{2.7}$$

Then there exists $t_* \in [2, t_m - 1]_{\mathbb{Z}}$ such that $\Delta\omega(t_*) \geq \frac{\omega(t_m) - \omega(0)}{t_m} \geq \frac{\omega(t_m)}{T + 1} = \frac{\|v\|_{\infty}}{T + 1}$. Summing (2.7) from $s = 1$ to $s = t_*$, we obtain $-\sum_{s=1}^{t_*} \Delta(\varphi(\Delta\omega(s - 1))) = -\sum_{s=1}^{t_*} k(s)$. Moreover,

$$\varphi(\Delta\omega(0)) = \varphi(\Delta\omega(t_*)) - \sum_{s=1}^{t_*} k(s)$$

$$\begin{aligned}
&\geq \varphi(\Delta\omega(t_*)) - \sum_{s=1}^T k(s) \\
&\geq \varphi\left(\frac{\|v\|_\infty}{T+1}\right) - \sum_{s=1}^T k(s) \\
&\geq \varphi\left(\frac{\|v\|_\infty}{T+1}\right) - \frac{1}{2}\varphi\left(\frac{\|v\|_\infty}{T+1}\right) \\
&= \frac{1}{2}\varphi\left(\frac{\|v\|_\infty}{T+1}\right).
\end{aligned}$$

Once again summing from $s = 1$ to $s = t - 1$, we have $-\sum_{s=1}^{t-1} \Delta(\varphi(\Delta\omega(s-1))) = -\sum_{s=1}^{t-1} k(s)$, and further $\Delta\omega(t-1) \geq \varphi^{-1}\left(\frac{1}{2}\varphi\left(\frac{\|v\|_\infty}{T+1}\right)\right) \geq \frac{\|v\|_\infty}{(T+1)\varphi_p^{-1}(2)}$. Summing from $s = 1$ to $s = t$ at both sides of the above equation, we have

$$\omega(t) \geq \frac{t\|v\|_\infty}{(T+1)\varphi_p^{-1}(2)} \geq \frac{\|v\|_\infty}{\varphi_p^{-1}(2)}d(t).$$

By Lemma 2.3, for $t \in [1, t_m]_{\mathbb{Z}}$, there holds $v(t) \geq \omega(t)$. Therefore, for $t \in [1, t_m]_{\mathbb{Z}}$, we have $v(t) \geq \frac{\|v\|_\infty}{\varphi_p^{-1}(2)}d(t)$. Similarly, for $t \in [t_m + 1, T]_{\mathbb{Z}}$ and $v(T+1) \geq 0$, we can prove that $v(t) \geq \omega(t)$. At present case, we only show that $v(T+1) \geq 0$. We distinguish the following two cases.

Case 1. For $a = 0$, it is obvious.

Case 2. For $a > 0$, without loss of generality, we can assume $v(T+1) < 0$. Then $\Delta v(T) = v(T+1) - v(T) > 0$, which means that $t_m \in [0, T]_{\mathbb{Z}}$. For $t \in [t_m, T+1]_{\mathbb{Z}}$, let $z(t) \in E$, $t \in [t_m, T+1]_{\mathbb{Z}}$ be the solution of the following problem

$$\begin{cases} -\Delta(\varphi(\Delta z(t-1))) = -k(t), t \in [t_m + 1, T]_{\mathbb{Z}}, \\ z(t_m) = \|v\|_\infty, z(T+1) = v(T+1). \end{cases} \quad (2.8)$$

In (2.8), summing from $s = t$ to $s = T$, we have $-\sum_{s=t}^T \Delta(\varphi(\Delta z(s-1))) = -\sum_{s=t}^T k(s)$, and further $\Delta z(t-1) = \varphi^{-1}\left(\varphi(\Delta z(T)) - \sum_{s=t}^T k(s)\right)$. Reproducing the above argument, we obtain

$$z(t_m) = z(T+1) + \sum_{s=t_m+1}^{T+1} \varphi^{-1}\left(-\varphi(\Delta z(T)) + \sum_{\tau=s}^T k(\tau)\right).$$

By the boundary condition $z(t_m) = \|v\|_\infty$, $z(T+1) = v(T+1)$, we have

$$\begin{aligned}
\|v\|_\infty &= v(T+1) + \sum_{s=t_m+1}^{T+1} \varphi^{-1}\left(-\varphi(\Delta z(T)) + \sum_{\tau=s}^T k(\tau)\right) \\
&\leq v(T+1) + \sum_{s=1}^{T+1} \varphi^{-1}\left(-\varphi(\Delta z(T)) + \sum_{\tau=1}^T k(\tau)\right)
\end{aligned}$$

$$\begin{aligned} &\leq \sum_{s=1}^{T+1} \varphi^{-1} \left(-\varphi(\Delta z(T)) + \sum_{\tau=1}^T k(\tau) \right) \\ &= (T+1)\varphi^{-1} \left(-\varphi(\Delta z(T)) + \sum_{\tau=1}^T k(\tau) \right). \end{aligned}$$

This means that $\Delta z(T) < 0$ since $\|v\|_\infty > (T+1)\varphi^{-1} \left(2 \sum_{s=1}^T k(s) \right)$. By Lemma 2.3, we deduce that, for $t \in [t_m + 1, T]_{\mathbb{Z}}$, $z(t) \leq v(t)$. This implies $\Delta v(T) \leq \Delta z(T) < 0$, which contradicts $\Delta v(T) > 0$. Thus, $v(T+1) \geq 0$. This completes the proof. \square

3. Main results

Theorem 3.1. *Assume that (H1)-(H3) and (H_a) hold. Then we have the following results.*

(i) *For $\lambda \approx 0$, problem (1.3) has no positive solution and for $\lambda \gg 1$, problem (1.3) has positive solution $u(t)$ such that $\|u\|_\infty \rightarrow \infty$ as $\lambda \rightarrow \infty$.*

(ii) *If there exists $\epsilon \geq \sigma$ and $\eta > 0$ such that*

$$\sigma \varphi_p^{-1}(2) \leq \eta < 4\epsilon \varphi_p^{-1}(2), \quad f(\epsilon) \geq \frac{2 \sum_{s=1}^T h^*(s)}{h_*}, \quad f(\eta) \geq \max \left\{ 1, \frac{2 \sum_{s=1}^T h^*(s)}{\sum_{s=1}^T h^*(s) + \sum_{s=1}^T h(s)} \right\},$$

$$\frac{f(\epsilon)}{\varphi\left(\frac{\epsilon}{T+1}\right)} / \frac{f(\eta)}{\varphi\left(\frac{\eta}{T+1}\right)} > \frac{\left(\sum_{s=1}^T h^*(s) + \sum_{s=1}^T h(s) \right) 2\varphi_q(16\varphi_p^{-1}(2))}{h_*},$$

then for $\lambda \in (\lambda_*, \lambda^*)$, problem (1.3) at least two positive solution, where

$$h^*(s) := \frac{c_1}{\sigma^\alpha d(s)^\alpha}, \quad h_* := \min \left\{ \sum_{t=a}^{c-1} h(t), \sum_{t=c+1}^b h(t) \right\},$$

$$\lambda_* := \frac{2\varphi_q(16\varphi_p^{-1}(2)) \varphi\left(\frac{\epsilon}{T+1}\right)}{f(\epsilon)h_*}, \quad \lambda^* := \frac{\varphi\left(\frac{\eta}{T+1}\right)}{f(\eta) \left(\sum_{s=1}^T h(s) + \sum_{s=1}^T h^*(s) \right)},$$

$[a, b]_{\mathbb{Z}} \subset [0, T+1]_{\mathbb{Z}}$ is the largest interval satisfying $\frac{1}{4} \leq d(t) \leq \frac{1}{2}$, $c = \lceil \frac{T+1}{2} \rceil$, ($\lceil x \rceil$ denotes the largest integer not exceeding x .)

Proof. We first show that the nonexistence result for $\lambda \approx 0$. If not, then (1.3) has a positive solution u . In view of (H2) and (H_a), there exists $M_1 > 0$ such that $f(s) \leq M_1 s^{q-1}$. Besides, by the boundary condition, there exists $t_m \in [1, T]_{\mathbb{Z}}$ such that $\|u\|_\infty = |u(t_m)|$. We divide the proof into the case $\|u\|_\infty = u(t_m) > 0$ and $\|u\|_\infty = -u(t_m) > 0$.

Case 1. If $\|u\|_\infty = u(t_m) > 0$, then $\Delta u(t_m) = u(t_m + 1) - u(t_m) \leq 0$. Summing (1.3) from $s = t$ to $s = t_m$, we obtain $-\sum_{s=t}^{t_m} \Delta(\varphi(\Delta u(s-1))) = \lambda \sum_{s=t}^{t_m} h(s)f(u(s))$, and further, $\varphi(\Delta u(t-1)) = \varphi(\Delta u(t_m)) + \lambda \sum_{s=t}^{t_m} h(s)f(u(s)) \leq \lambda \sum_{s=t}^{t_m} h(s)f(u(s))$. Then $\Delta u(t-1) \leq \varphi^{-1} \left(\lambda \sum_{s=t}^{t_m} h(s)f(u(s)) \right)$.

Denote $H = \sum_{\tau=1}^T h(\tau)$, summing from $s = 1$ to $s = t$ at both sides of the above equation, we can obtain

$$\begin{aligned} u(t) &\leq \sum_{s=1}^t \varphi^{-1} \left(\lambda \sum_{\tau=s}^{t_m} h(\tau) f(u(\tau)) \right) \\ &\leq \sum_{s=1}^T \varphi^{-1} \left(\lambda \sum_{\tau=1}^T h(\tau) M_1 u^{q-1} \right) \\ &\leq \sum_{s=1}^T \varphi^{-1} (\lambda T H M_1 \|u\|_{\infty}^{q-1}) \\ &= T \varphi^{-1} (\lambda T H M_1 \|u\|_{\infty}^{q-1}). \end{aligned}$$

From this, we have $\lambda M_1 T^q H \geq \|u\|_{\infty}^{p-q} + 1 > 1$. Obviously, the right side of the inequality is always greater than 1, which is a contradiction for $\lambda \approx 0$.

Case 2. For $\|u\|_{\infty} = -u(t_m) > 0$, i.e., $u(t_m) < 0$, it can be similarly proved.

With a summary, there exists no positive solution for $\lambda \approx 0$.

Next, we prove that the existence result for $\lambda \gg 1$. To this end, we apply Lemma 2.1 with $X = E, z = 1, K = T_{\lambda}$ and find two constants which satisfy (a) and (b). We first need to find a constant ω satisfying (a). Let $\tau \geq 0$ and $u \in E$ be a solution of $u = T_{\lambda} u + \tau$. By (H2), we know

that there exists positive constant $M_2 \geq \sigma$ such that $f(s) > \max \left\{ \frac{2 \sum_{s=1}^T h^*(s)}{h_*}, \frac{A s^{r-1}}{2} \right\}$, $s > M_2$.

Let $\lambda \gg 1$ be such that $\|u\|_{\infty} > (T+1)\varphi^{-1} \left(2\lambda \sum_{t=1}^T h^*(t) \right) \geq \max \{T+1, 4M_2\varphi_p^{-1}(2)\}$ and

$$\frac{1}{2} \left(\frac{\lambda A h_*(T+1)^{q-1}}{2^{1+2q}(4\varphi_p^{-1}(2))^{r-1}} \right)^{\frac{1}{q-r}} > (T+1)\varphi^{-1} \left(2\lambda \sum_{t=1}^T h^*(t) \right).$$

Let $[a, b]_{\mathbb{Z}} \subset [0, T+1]_{\mathbb{Z}}$ be the maximum interval satisfying $\frac{1}{4} \leq d(t) \leq \frac{1}{2}$. Obviously, $a \geq \frac{T+1}{4}, b \leq \frac{3}{4}(T+1)$. Denote $c = \lceil \frac{T+1}{2} \rceil$, Then $c \leq \frac{T+1}{2}$. Notice that $u(t)$ satisfies

$$\begin{cases} -\Delta(\varphi(\Delta u(t-1))) = \lambda h(t) f^*(u) \geq -\lambda h^*(t), & t \in [1, T]_{\mathbb{Z}}, \\ u(0) \geq 0, a\Delta u(T) + g(\lambda, |T_{\lambda} u(T+1)|)u(T+1) \geq 0. \end{cases} \quad (3.1)$$

Since

$$\|u\|_{\infty} > (T+1)\varphi^{-1} \left(2\lambda \sum_{t=1}^T h^*(t) \right) > 4\sigma\varphi_p^{-1}(2),$$

in view of Lemma 2.3, $u(t) > \frac{\|u\|_{\infty}}{\varphi_p^{-1}(2)} d(t)$. Then $\|u\|_{\infty} = u(t_m) > 0$, $\Delta u(t_m - 1) = u(t_m) - u(t_m - 1) \geq 0$. If $t \in [a, b]_{\mathbb{Z}}$, then $f^*(u) = f(u) \geq f \left(\frac{\|u\|_{\infty}}{\varphi_p^{-1}(2)} d(t) \right) \geq f \left(\frac{\|u\|_{\infty}}{4\varphi_p^{-1}(2)} \right)$. If $t_m \geq \frac{T+1}{2}$, then

$$\begin{aligned} \|u\|_{\infty} &\geq \|T_{\lambda} u\|_{\infty} \\ &= \max_{t \in [0, T+1]_{\mathbb{Z}}} \left| \sum_{s=1}^t \varphi^{-1} \left(\varphi(\Delta u(t_m - 1)) + \lambda \sum_{\tau=s}^{t_m-1} h(\tau) f^*(u(\tau)) \right) \right| \end{aligned}$$

$$\begin{aligned}
 &\geq \left| \sum_{s=1}^a \varphi^{-1} \left(\varphi(\Delta u(t_m - 1)) + \lambda \sum_{\tau=a}^{t_m-1} h(\tau) f^*(u(\tau)) \right) \right| \\
 &\geq a \left| \varphi^{-1} \left(\lambda \sum_{\tau=a}^{t_m-1} h(\tau) f^*(u(\tau)) \right) \right| \\
 &\geq a \varphi^{-1} \left(\lambda \sum_{\tau=a}^{c-1} h(\tau) f(u(\tau)) \right) \\
 &\geq \frac{T+1}{4} \varphi^{-1} \left(\frac{\lambda h_*}{2} f \left(\frac{\|u\|_\infty}{4\varphi_p^{-1}(2)} \right) \right).
 \end{aligned}$$

By a similar argument, if $t_m < \frac{T+1}{2}$, then $\|u\|_\infty \geq \frac{T+1}{4} \varphi^{-1} \left(\frac{\lambda h_*}{2} f \left(\frac{\|u\|_\infty}{4\varphi_p^{-1}(2)} \right) \right)$, which means that

$$\varphi \left(\frac{4\|u\|_\infty}{T+1} \right) \geq \frac{\lambda h_*}{2} f \left(\frac{\|u\|_\infty}{4\varphi_p^{-1}(2)} \right). \tag{3.2}$$

Since $\|u\|_\infty \geq \max\{T+1, 4M_2\varphi^{-1}(2)\}$, we have

$$\varphi \left(\frac{4\|u\|_\infty}{T+1} \right) = \left(\frac{4\|u\|_\infty}{T+1} \right)^{q-1} + \left(\frac{4\|u\|_\infty}{T+1} \right)^{p-1} \leq 2 \left(\frac{4\|u\|_\infty}{T+1} \right)^{q-1}.$$

Therefore,

$$2 \left(\frac{4\|u\|_\infty}{T+1} \right)^{q-1} \geq \varphi \left(\frac{4\|u\|_\infty}{T+1} \right) \geq \frac{\lambda h_*}{2} f \left(\frac{\|u\|_\infty}{4\varphi_p^{-1}(2)} \right) \geq \frac{\lambda A h_*}{4} \left(\frac{\|u\|_\infty}{4\varphi_p^{-1}(2)} \right)^{r-1}.$$

By above relation, we have $\|u\|_\infty \geq \left(\frac{\lambda A h_* (T+1)^{q-1}}{2^{1+2q} (4\varphi_p^{-1}(2))^{r-1}} \right)^{\frac{1}{q-r}}$. Noticing that for $\lambda \gg 1$,

$$\frac{1}{2} \left(\frac{\lambda A h_* (T+1)^{q-1}}{2^{1+2q} (4\varphi_p^{-1}(2))^{r-1}} \right)^{\frac{1}{q-r}} > (T+1) \varphi^{-1} \left(2\lambda \sum_{t=1}^T h^*(t) \right),$$

we obtain $\|u\|_\infty \neq \frac{1}{2} \left(\frac{\lambda A h_* (T+1)^{q-1}}{2^{1+2q} (4\varphi_p^{-1}(2))^{r-1}} \right)^{\frac{1}{q-r}}$ for $\lambda \gg 1$.

Next, to find a constant r satisfying (b), let $\theta \in (0, 1]$ and $u \in E$ be a solution of $u = \theta T_\lambda u$. Then we have $\Delta u(t_m) = u(t_m + 1) - u(t_m) \leq 0$. Since

$$u(t) = \theta \sum_{s=1}^t \varphi^{-1} \left(\varphi(\Delta u(t_m)) + \lambda \sum_{\tau=s}^{t_m} h(\tau) f^*(u(\tau)) \right),$$

we have

$$\begin{aligned}
 \|u\|_\infty &= u(t_m) \\
 &= \theta \sum_{s=1}^{t_m} \varphi^{-1} \left(\varphi(\Delta u(t_m)) + \lambda \sum_{\tau=s}^{t_m} h(\tau) f^*(u(\tau)) \right)
 \end{aligned}$$

$$\begin{aligned}
&\leq \theta \sum_{s=1}^{t_m} \varphi^{-1} \left(\lambda \sum_{\tau=s}^{t_m} h(\tau) f^*(u(\tau)) \right) \\
&\leq \sum_{s=1}^{t_m} \varphi^{-1} \left(\lambda \sum_{\tau=s}^{t_m} h(\tau) \left| f^*(u(\tau)) \right| \right) \\
&\leq \sum_{s=1}^{T+1} \varphi^{-1} \left(\lambda \sum_{\tau=1}^T h(\tau) \left| f^*(u(\tau)) \right| \right) \\
&\leq \sum_{s=1}^{T+1} \varphi^{-1} \left(\lambda \sum_{\tau=1}^T h^*(\tau) + \lambda \sum_{\tau=1}^T h(\tau) f(\max\{\sigma, \|u\|_\infty\}) \right) \\
&\leq (T+1) \varphi^{-1} \left(\lambda \sum_{\tau=1}^T h^*(\tau) + \lambda f(\max\{\sigma, \|u\|_\infty\}) \sum_{\tau=1}^T h(\tau) \right). \tag{3.3}
\end{aligned}$$

By (H2) and (H_a), there exists $r_\lambda \gg T+1$ such that $r_\lambda > \|u\|_\infty$, $r_\lambda > \frac{1}{2} \left(\frac{\lambda A h_*(T+1)^{q-1}}{2^{1+2q} (4\varphi_p^{-1}(2))^{r-1}} \right)^{\frac{1}{q-r}}$ for each λ . By Lemma 2.1, T_λ has a fixed point $u \in E$ for $\lambda \gg 1$ such that

$$(T+1) \varphi^{-1} \left(2\lambda \sum_{t=1}^T h^*(t) \right) < \frac{1}{2} \left(\frac{\lambda A h_*(T+1)^{q-1}}{2^{1+2q} (4\varphi_p^{-1}(2))^{r-1}} \right)^{\frac{1}{q-r}} < \|u\|_\infty < r_\lambda.$$

Since $(T+1) \varphi^{-1} \left(2\lambda \sum_{t=1}^T h^*(t) \right) \geq 4M_2 \varphi_p^{-1}(2) > 0$, we can prove that $u(t)$ is a positive solution of problem (1.3) for $\lambda \gg 1$ by Lemma 2.1. Moreover, since $\|u\|_\infty \geq (T+1) \varphi^{-1} \left(2\lambda \sum_{t=1}^T h^*(t) \right)$, $\|u\|_\infty \rightarrow \infty$ as $\lambda \rightarrow \infty$.

Finally, we prove the multiplicity result for $\lambda \in (\lambda_*, \lambda^*)$, where

$$\lambda_* := \frac{2\varphi_q(16\varphi_p^{-1}(2)) \varphi(\frac{\epsilon}{T+1})}{f(\epsilon) h_*}, \quad \lambda^* := \frac{\varphi\left(\frac{\eta}{T+1}\right)}{f(\eta) \left(\sum_{s=1}^T h(s) + \sum_{s=1}^T h^*(s) \right)}.$$

Once again, we will apply Lemma 2.1. To find a constant ω satisfying (a), let $\tau \geq 0$ and $u \in E$ be a solution of $u = T_\lambda u + \tau$. Suppose that $\|u\|_\infty = 4\epsilon \varphi_p^{-1}(2)$. By (H2), there exists positive constant $\epsilon \geq \sigma$ such that

$$f(s) > \max \left\{ \frac{A s^{r-1}}{2}, \frac{2 \sum_{s=1}^T h^*(s)}{h_*} \right\}$$

for $s > \epsilon$. Noticing that for $\lambda < \lambda^*$, we have $4\epsilon \varphi_p^{-1}(2) > (T+1) \varphi^{-1} \left(2\lambda \sum_{t=1}^T h^*(t) \right)$. Then by following the above arguments, u satisfies (3.2) and we have $\frac{\lambda h_* f(\epsilon)}{2} \leq \varphi \left(\frac{16\epsilon \varphi_p^{-1}(2)}{T+1} \right) \leq \varphi_q(16\varphi_p^{-1}(2)) \varphi(\frac{\epsilon}{T+1})$. More, precisely, $\lambda \leq \frac{2\varphi_q(16\varphi_p^{-1}(2)) \varphi(\frac{\epsilon}{T+1})}{f(\epsilon) h_*} = \lambda_*$. This is a contradiction. Therefore, $\|u\|_\infty \neq 4\epsilon \varphi_p^{-1}(2)$ for $\lambda \in (\lambda_*, \lambda^*)$.

Next, we show that there exists a constant greater than $4\epsilon\varphi_p^{-1}(2)$ and a constant less than $4\epsilon\varphi_p^{-1}(2)$ which satisfies (b). To find it, let $\theta \in (0, 1]$ and $u \in E$ be a solution of $u = \theta T_\lambda u$. By (3.3), (H2) and (H_a), there exists $r_\lambda \gg T + 1$ such that $r_\lambda > \max\{\|u\|_\infty, 4\epsilon\varphi_p^{-1}(2)\}$ for each $\lambda > 0$. Assume $\|u\|_\infty = \eta \geq \sigma\varphi_p^{-1}(2)$, $f(\eta) \geq 1$. Then $\varphi\left(\frac{\eta}{T+1}\right) \leq \lambda f(\eta) \left(\sum_{s=1}^T h(s) + \sum_{s=1}^T h^*(s)\right)$ and further we have $\lambda \geq \lambda^*$, which is a contradiction. Thus, $\|u\|_\infty \neq \eta$ for $\lambda \in (\lambda_*, \lambda^*)$. In view of Lemma 2.1, T_λ has fixed points $u_1, u_2 \in E$ for $\lambda \in (\lambda_*, \lambda^*)$ such that $\eta < \|u_1\|_\infty < 4\epsilon\varphi_p^{-1}(2) < \|u_2\| < r_\lambda$. Furthermore, since $f(\eta) \geq \frac{2 \sum_{s=1}^T h^*(s)}{\sum_{s=1}^T h^*(s) + \sum_{s=1}^T h(s)}$, we have $\eta > (T + 1)\varphi^{-1}\left(2\lambda \sum_{s=1}^T h^*(s)\right)$ for $\lambda \in (\lambda_*, \lambda^*)$. This means that problem (1.3) has at least two positive solution. \square

Theorem 3.2. *Assume (H1)-(H3) and (H_b) hold. Then we have the following results.*

- (i) *Problem (1.3) has a positive solution $u(t)$ for $\lambda \approx 0$ such that $\|u\|_\infty \rightarrow \infty$ as $\lambda \rightarrow 0$.*
- (ii) *If there exists $\gamma \geq \sigma$ and $\delta \geq \sigma$ such that*

$$\delta > 4\gamma\varphi_p^{-1}(2), f(\gamma) \geq \max \left\{ 1, \frac{2\varphi_q(16\varphi_p^{-1}(2)) \sum_{s=1}^T h^*(s)}{\varphi_p(4)h_*} \right\},$$

$$\frac{f(\gamma)}{\varphi\left(\frac{\gamma}{T+1}\right)} / \frac{f(\delta)}{\varphi\left(\frac{\delta}{T+1}\right)} > \frac{\left(\sum_{s=1}^T h^*(s) + \sum_{s=1}^T h(s)\right) 2\varphi_q(16\varphi_p^{-1}(2))}{h_*},$$

then problem (1.3) has at least two positive solutions u_1, u_2 for $\lambda \in (\lambda_\Delta, \lambda^\Delta)$ such that $4\gamma\varphi_p^{-1}(2) < \|u_1\|_\infty < \delta < \|u_2\|_\infty$, where $\lambda_\Delta := \frac{2\varphi_q(16\varphi_p^{-1}(2))\varphi\left(\frac{\gamma}{T+1}\right)}{f(\gamma)h_*}$,

$$\lambda^\Delta := \min \left\{ \frac{\varphi\left(\frac{\delta}{T+1}\right)}{f(\delta)\left(\sum_{s=1}^T h(s) + \sum_{s=1}^T h^*(s)\right)}, \frac{\varphi_p(4)\varphi\left(\frac{\gamma}{T+1}\right)}{\sum_{s=1}^T h^*(s)} \right\}.$$

Proof. We first prove that the existence result for $\lambda \approx 0$. To apply Lemma 2.1, we need to find a constant satisfying (b). Let $\theta \in (0, 1]$ and $u \in E$ be a solution of $u = \theta T_\lambda u$. Let $\lambda < \tilde{\lambda} := \frac{\varphi\left(\frac{\sigma}{T+1}\right)}{\sum_{s=1}^T h^*(s) + \sum_{s=1}^T h(s)f(\sigma)}$. In view of (3.3), if $\|u\|_\infty = \sigma$, then we have

$$\varphi\left(\frac{\sigma}{T+1}\right) \leq \lambda \left(\sum_{s=1}^T h^*(s) + \sum_{s=1}^T h(s)f(\sigma)\right),$$

which is a contradiction for $\lambda < \tilde{\lambda}$. Thus $\|u\|_\infty \neq \sigma$ for $\lambda < \tilde{\lambda}$.

Next, we prove that there exists a constant satisfying (a) when $z = 1$. Let $\tau \geq 0$ and $u \in E$ be a solution of $u = T_\lambda u + \tau$. By (H2), there exists positive constant $M_3 > \sigma$ such that

$M_3 > 4\sigma\varphi_p^{-1}(2)$, $f\left(\frac{M_3}{4\varphi_p^{-1}(2)}\right) > \frac{2\sum_{s=1}^T h^*(s)}{h_*}$. Suppose that

$$\|u\|_\infty > \max\left\{T + 1, M_3, (T + 1)\varphi^{-1}\left(2\lambda\sum_{t=1}^T h^*(t)\right)\right\}.$$

Since $u(t)$ satisfies (3.1) and $\|u\|_\infty > (T + 1)\varphi^{-1}\left(2\lambda\sum_{t=1}^T h^*(t)\right)$, we have $u(t) \geq \frac{\|u\|_\infty}{\varphi_p^{-1}(2)}d(t)$ by Lemma 2.4. By following the argument in the proof of Theorem 3.1, we can prove that $u(t)$ satisfies (3.2) and $2\left(\frac{4\|u\|_\infty}{T+1}\right)^{q-1} \geq \frac{\lambda h_*}{2}f\left(\frac{\|u\|_\infty}{4\varphi_p^{-1}(2)}\right)$. By (H_b) , there exists $r_\lambda \gg \sigma$ such that $\|u\|_\infty < r_\lambda$ for each λ . In view of Lemma 2.1, T_λ has a fixed point u for $\lambda < \tilde{\lambda}$ such that $\sigma < \|u\|_\infty < r_\lambda$. By (3.3), we know that $u(t)$ satisfies

$$1 \leq \frac{\lambda\left(\sum_{\tau=1}^T h^*(\tau) + \lambda f(\|u\|_\infty)\sum_{\tau=1}^T h(\tau)\right)}{\varphi\left(\frac{\|u\|_\infty}{T+1}\right)} \leq \frac{\lambda\left(\sum_{\tau=1}^T h^*(\tau) + \lambda f(\|u\|_\infty)\sum_{\tau=1}^T h(\tau)\right)}{\left(\frac{\|u\|_\infty}{T+1}\right)^{q-1}},$$

which means that $\|u\|_\infty \rightarrow \infty$ as $\lambda \rightarrow 0$. Then by Lemma 2.4, $u(t)$ is a positive solution of problem (1.3) for $\lambda \approx 0$.

Now, we show the multiplicity result for $\lambda \in (\lambda_\Delta, \lambda^\Delta)$. Similar to the proof of Theorem 3.1, we first find a constant satisfying (b) in Lemma 2.1. Let $\theta \in (0, 1]$, and $u \in E$ be a solution of $u = \theta T_\lambda u$. Suppose that $\|u\|_\infty = \delta \geq \sigma$, $f(\delta) > 1$. By (3.3), we have

$$\varphi\left(\frac{\delta}{T+1}\right) \leq \lambda f(\delta)\left(\sum_{s=1}^T h^*(s) + \sum_{s=1}^T h(s)\right).$$

Then $\lambda \geq \frac{\varphi\left(\frac{\delta}{T+1}\right)}{f(\delta)\left(\sum_{s=1}^T h^*(s) + \sum_{s=1}^T h(s)\right)} = \lambda^\Delta$, which is a contradiction. Thus, $\|u\|_\infty \neq \delta$ for $\lambda \in (\lambda_\Delta, \lambda^\Delta)$.

Next, we show that there exists a constant greater than δ and a constant less than δ satisfying (a). Let $z = 1, \tau \geq 0$ and $u \in E$ be a solution of $u = T_\lambda u + \tau$. By following the arguments in the proof of Theorem 3.1 and (H_b) , we can prove that there exists $r_\lambda \gg T + 1$ such that $r_\lambda > \|u\|_\infty$ for each $\lambda > 0$. Assume $\|u\|_\infty = 4\gamma\varphi_p^{-1}(2)$. Then $u(t)$ satisfies (3.2) since

$$\gamma \geq \sigma, f(\gamma) > \frac{2\varphi_q(16\varphi_p^{-1}(2))\sum_{s=1}^T h^*(s)}{\varphi_p(4)h_*} > \frac{2\sum_{s=1}^T h^*(s)}{h_*}$$

and $4\gamma\varphi_p^{-1}(2) > (T + 1)\varphi^{-1}\left(2\lambda\sum_{t=1}^T h^*(t)\right)$ for $\lambda < \lambda^\Delta$. Thus, we have

$$\frac{\lambda h_* f(\gamma)}{2} \leq \varphi\left(\frac{16\gamma\varphi_p^{-1}(2)}{T+1}\right) \leq \varphi_q(16\varphi_p^{-1}(2))\varphi\left(\frac{\gamma}{T+1}\right).$$

And further, $\lambda \leq \frac{2\varphi_q(16\varphi_p^{-1}(2))\varphi\left(\frac{\gamma}{T+1}\right)}{f(\gamma)h_*} = \lambda_\Delta$, which is a contradiction. Hence, $\|u\|_\infty \neq 4\gamma\varphi_p^{-1}(2)$ for $\lambda \in (\lambda_\Delta, \lambda^\Delta)$. By Lemma 2.1, T_λ has fixed points $v_1, v_2 \in E$ for $\lambda \in (\lambda_\Delta, \lambda^\Delta)$ such that $4\gamma\varphi_p^{-1}(2) < \|v_1\| < \delta < \|v_2\|_\infty < r_\lambda$. □

Theorem 3.3. *Assume that (H1)-(H3) and (Hc) hold. Then we have the following results.*

- (i) *Problem (1.3) has no positive solution for $\lambda \approx 0$;*
- (ii) *If there exists positive constant $\mu \geq \sigma$ such that*

$$f(\mu) > \frac{2\varphi_q(16\varphi_p^{-1}(2)) \sum_{s=1}^T h^*(s)}{\varphi_p(4)h_*}, \quad \frac{f(\mu)}{\varphi(\frac{\mu}{T+1})} > \frac{4B\varphi_q(16\varphi_p^{-1}(2)) \sum_{s=1}^T h(s)}{h_*},$$

then problem (1.3) has a positive solution $u(t)$ for $\lambda \in (\underline{\lambda}, \bar{\lambda})$, where

$$\underline{\lambda} := \frac{2\varphi_q(16\varphi_p^{-1}(2)) \varphi(\frac{\mu}{T+1})}{h_* f(\mu)}, \quad \bar{\lambda} := \min \left\{ \frac{1}{2B \sum_{s=1}^T h(s)}, \frac{\varphi_p(4)\varphi(\frac{\mu}{T+1})}{\sum_{s=1}^T h^*(s)} \right\}.$$

Proof. The proof of the nonexistence result for $\lambda \approx 0$ follows the argument in the proof of Theorem 3.1. Next, we prove that the existence result for $\lambda \in (\underline{\lambda}, \bar{\lambda})$. Similar to argument in the proof of Theorem 3.1 and Theorem 3.2, we need to find two constants satisfying (a) and (b) in Lemma 2.1. Let $u \in E$ be a solution of $u = T_\lambda u + \tau$. Notice that $4\mu\varphi_p^{-1}(2) >$

$(T + 1)\varphi^{-1} \left(2\lambda \sum_{t=1}^T h^*(t) \right)$ for $\lambda < \bar{\lambda}$ and $f(\mu) > \frac{2\varphi_q(16\varphi_p^{-1}(2)) \sum_{s=1}^T h^*(s)}{\varphi_p(4)h_*} > \frac{2 \sum_{s=1}^T h^*(s)}{h_*}$. Then we can prove that $\|u\| \neq 4\mu\varphi_p^{-1}(2)$ for $\lambda \in (\underline{\lambda}, \bar{\lambda})$. Let $\tau \geq 0$ and $u \in E$ be a solution of $u = \theta T_\lambda u$.

From (2.3), we know that $1 \leq \frac{\lambda \left(\sum_{\tau=1}^T h^*(\tau) + \lambda f(\max(\sigma, \|u\|_\infty)) \sum_{\tau=1}^T h(\tau) \right)}{\varphi \left(\frac{\|u\|_\infty}{T+1} \right)}$. Then by (H2) and (Hc), there

exists $r_\lambda \gg T + 1$ such that $r_\lambda > \max\{\|u\|_\infty, 4\mu\varphi_p^{-1}(2)\}$ for $\lambda > \underline{\lambda}$. Consequently, T_λ has a fixed point $v(t) \in E$ for $\lambda \in (\underline{\lambda}, \bar{\lambda})$ such that $4\mu\varphi_p^{-1}(2) < \|u\|_\infty < r_\lambda$. □

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