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THE BLOW-UP PROBLEM FOR A CLASS OF QUASILINEAR SCHRÖDINGER EQUATIONS

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Abstract We consider the blow-up results of the solution in $W^{2,2}(\mathbb{R}^N)$ for the following quasilinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + uf(|u|^2) = 0, & x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where h and f are real functions which related to various physical models. We prove that the $W^{2,2}(\mathbb{R}^N)$ solutions must blow up if $|x|u_0 \in L^2(\mathbb{R}^N)$(finite variance), and we give the upper bound of the blow-up time. We also show that without the finite variance assumption, the radial symmetric solutions in $W^{2,2}(\mathbb{R}^N)$ must blow up in finite time for the whole class of initial data with strictly negative energy.

Keywords Quasilinear Schrödinger equation, blow up, radial symmetric solution.

MSC(2010) 35Q55, 35B44, 35B06.

1. Introduction

In this paper, we consider the following Cauchy problem of quasilinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + uf(|u|^2) = 0, & x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

where $u = u(x,t) : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{C}$ is a complex valued function, $h, f : \mathbb{R}_+ \to \mathbb{R}$ are given real smooth functions, $i^2 = -1$. $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial^2 x_j}$ is the standard Laplacian operator. Equations of the form (1.1) often appear in many phenomena of physics like plasma physics and fluid mechanics [19, 20, 26, 31], dissipative quantum mechanics [14], condensed matter theory [23] and in the theory of Heisenberg ferromagnets and magnons [1, 18, 21, 32, 36]. For example, the case of $h(s) = \sqrt{1+s}$ models the self-channeling of a high power, ultra-short laser pulse in matter (see [3, 34]) whereas if $h(s) = \sqrt{s}$, equation (1.1) is applied to physical phenomena in dissipative quantum mechanics [4, 14]. In the case $h(s) = s$, equation (1.1) is used for the superfluid flim equation in plasma physics [19].

The well-posedness of the problem can refer to [6, 17, 30, 35] and the references therein. By the known results, we have the following result.

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Theorem 1.1. (Local well-posedness) [35] *Let $N \geq 1$. Assuming that $u_0 \in H^{L+2}(\mathbb{R}^N)$ and $h, f \in C^{L+2}(\mathbb{R}_+)$ for $L \geq N+2$, then there exists a $T_L > 0$ and a unique solution to* (1.1) *satisfying*

$$u \in L^\infty([0,T_L]; H^{L+2}(\mathbb{R}^N)) \cup C([0,T_L]; H^L(\mathbb{R}^N)).$$

Regarding the blow-up phenomena of the nonlinear Schrödinger equations, many scholars have investigated and obtained a series of valuable results [8,10,29,35,37,38]. In 1977, Glassey considered the following Cauchy problem

$$\begin{cases} iu_t = \Delta u + F(|u|^2)u, & x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.2}$$

based primarily on the finite variance assumption ($\int_{\mathbb{R}^N} |x|^2|u_0|^2 \mathrm{d}x < \infty$), and obtained the valuable result that the corresponding solution blows up in finite time for the first time. In his paper [11], the solution to (1.2) satisfies the mass and energy conservation laws. The most critical condition for the blow up of the solution to (1.2) is that there exists a constant $c_N > 1+\frac{2}{N}$ such that $sF(s) \geq c_N G(s)$ for all $s \geq 0$, where $G(s) = \int_0^s F(\eta)\mathrm{d}\eta$.

When $F(u)$ is a typical typical nonlinear term $\lambda|u|^p$, (1.2) becomes the following equation

$$\begin{cases} iu_t + \Delta u = \lambda|u|^p u, & x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.3}$$

where $\lambda \in \mathbb{R}, p > 0$. The case $\lambda > 0(\lambda < 0)$ corresponds the defocusing (focusing) case. Using the similar method as in [11] and without the assumption $\int_{\mathbb{R}^N} |x|^2|u_0|^2 \mathrm{d}x < \infty$, T.Ogawa and Y.Tsutsumi [27,28] obtained the blow up results for radially symmetric H^1 solutions with negative energy for (1.3) in finite time. Moreover, F. Merle and P. Raphäel obtained that the solutions blow up in finite time for the entire class of initial data in H^1 with strictly negative energy in [25], and established the existence of a universal blow-up profile which attracts blow up solutions in the vicinity of blow up time in [24]. In 2009, the similar result was obtained by P. Raphäel and J. Szeftel [33] for $p = 4$ in any dimension. Later, the blow up criterion for the focusing case of (1.3) with negative energy had been studied in [9].

If $h(s) = s, f(s) = \beta s^{\frac{p-2}{2}}$, we obtain a special case of (1.1) as follows

$$\begin{cases} iu_t + \Delta u + \beta|u|^{p-2}u + \theta(\Delta|u|^2)u = 0, & x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.4}$$

In [13], Guo, Chen and Su obtained the solution of (1.4) must blow up in finite time if $4 + \frac{4}{N} \leq p < 2 \cdot 2^*$ under some assumptions, here $2^* = \frac{2N}{N-2}$. Later, in 2009, Chen and Guo [5] used the variational method to prove the one-dimensional blow up results of (1.4) in finite time, and obtained the $H^1(\mathbb{R})$ strong instability of standing waves. The problem also considered in [7].

Our purpose is to extend the blow-up results of the standard nonlinear Schrödinger equation ($h \equiv$ *constant*) to the general quasilinear case. Consequently, the second-order quasilinear term $uh'(|u|^2)\Delta h(|u|^2)$ gives rise to relatively complex terms in the calculation. Meanwhile, determining appropriate assumptions for f and g is also a difficulty which needs to be overcome to obtain the blow-up results. Early blow-up results generally relied on the weighted condition $|x|u_0 \in L^2(\mathbb{R}^N)$, but some numerical computations have indicated that this condition is not

necessary. In this paper, we use the classical method from Glassey in [11] to study the "blowing up" phenomena for the Cauchy problem (1.1). For applications of the same method to obtain the blow-up solutions of nonlinear equations in finite time can refer to [2,12,15,16,37]. Moreover, the solution of equation (1.1) obeys the mass and energy conversation laws, which read as
(i) Mass conversation

$$M(t)=\int_{\mathbb{R}^N}|u(\cdot,t)|^2\mathrm{d}x=M(0).$$

(ii) Energy conversation

$$E(t)=\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+\int_{\mathbb{R}^N}[\nabla h(|u|^2)]^2\mathrm{d}x-\int_{\mathbb{R}^N}G(|u|^2)\mathrm{d}x=E(0).$$

We will prove the conservations of mass and energy in Section 2.

Now, we state our main results as follows.

Theorem 1.2. *Suppose that $u(t)\in W^{2,2}(\mathbb{R}^N)(N\geq 1)$ is a solution of* (1.1) *and*
(1) $u_0\in W^{2,2}(\mathbb{R}^N), Im\int\overline{u_0}x\nabla u_0<0$, and $|x|u_0\in L^2(\mathbb{R}^N)$;
(2) $E(0)=\int(|\nabla u_0|^2+|\nabla h(|u_0|^2)|^2-G(|u_0|^2))\leq 0$, where $G(s)=\int_0^s f(\delta)d\delta$;
(3) there is a constant $C_N\geq 2+\frac{2}{N}$ such that $sf(s)\geq C_N G(s)$ for all $s\geq 0, x\in\mathbb{R}^N$;
(4) for all $s\geq 0$, there holds $h'(s)\geq 0, h''(s)\leq 0$ or $h'(s)\leq 0, h''(s)\geq 0$, that is $h'(s)$ and $h''(s)$ have opposite signs;
here and after, $\overline{u}$ is the complex conjugate of u. Then, there exists a $T_0>0$ such that

$$\lim_{t\to T_0^-}\|\nabla u\|^2_{L^2(\mathbb{R}^N)}=+\infty.$$

By removing the finite variance assumption in Theorem 1.2 and considering the radially symmetric solutions in $W^{2,2}(\mathbb{R}^N)$, we obtain the following resluts.

Theorem 1.3. *Suppose that $u(t)\in W_r^{2,2}(\mathbb{R}^N)=\{u\in W^{2,2}(\mathbb{R}^N); u(x)=u(|x|)\}(N>2)$ is a radially symmetric solution of* (1.1) *and*
(1) $u_0\in W_r^{2,2}(\mathbb{R}^N), \nabla|u_0|^2\in L^2(\mathbb{R}^N)$;
(2) $E(0)=\int(|\nabla u_0|^2+|\nabla h(|u_0|^2)|^2-G(|u_0|^2))<0$, where $G(s)=\int_0^s f(\delta)d\delta$;
(3) there is a constant $C_N\geq 2+\frac{2}{N}$ such that $sf(s)\geq C_N G(s)$ for all $s\geq 0, x\in\mathbb{R}^N$;
(4) for all $s\geq 0$, there holds $h'(s)\geq 0, h''(s)\leq 0$ or $h'(s)\leq 0, h''(s)\geq 0$, that is $h'(s)$ and $h''(s)$ have opposite signs and $\|h'\|_\infty, \|h''\|_\infty$ is limited;
(5) $f(0)=0$ and $\|f'\|_\infty$ is limited;
then, there is $T>0$ such that

$$\lim_{t\to T^-}\|\nabla u\|^2_{L^2(\mathbb{R}^N)}=+\infty.$$

Notations. Throughout this paper, all integrals are taken over $\mathbb{R}^N$ unless stated otherwise. Re(Im) denotes the real(imaginary) part for the complex value. C stands for a generic positive constant, which may be different from line to line.

This paper is organized as follows. In Section 2, we construct the mass and energy conversation laws for solutions to (1.1), then we prove that the solutions will blow up in finite time based on the finite variance assumption and give an upper bound estimate for the blowing up time. In Section 3, we present several lemmas to be used in the proof of Theorem 1.3. In Section 4, we prove Theorem 1.3 which establishes sufficient conditions for the finite-time blow-up of radially symmetric solutions without the finite variance assumption.

2. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Based on the finite variance and negative energy assumptions, we establish the existence of a blow-up time and its upper bound estimate. First, we demonstrate that the solutions to (1.1) satisfy the conservation laws of mass and energy.

Lemma 2.1. *Assume that $u(\cdot,t)$ is a solution to* (1.1). *Then, in the time interval* $[0,t]$ *when it exists,* $u(\cdot,t)$ *satisfies*
(i) Mass conversation

$$M(t)=\int |u(\cdot,t)|^2\,dx = M(0).$$

(ii) Energy conversation

$$E(t)=\int |\nabla u|^2\,dx+\int [\nabla h(|u|^2)]^2\,dx-\int G(|u|^2)\,dx=E(0),$$

where $G(s)=\int_0^s f(\delta)\,d\delta$.

Proof. (i) Multiplying $(1.1)_1$ by $\bar u$ and integrating on $\mathbb{R}^N$, we obtain that

$$i\int \bar u u_t \mathrm{d}x+\int \bar u\Delta u\mathrm{d}x+2\int |u|^2 h'(|u|^2)\Delta h(|u|^2)\mathrm{d}x+\int |u|^2 f(|u|^2)\mathrm{d}x=0, \tag{2.1}$$

integrating by parts and considering the imaginary part yields

$$\mathrm{Re}\int \bar u u_t\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\mathrm{d}x=0,$$

that is,

$$M(t)=\int |u|^2\mathrm{d}x=\int |u_0|^2\mathrm{d}x=M(0). \tag{2.2}$$

(ii) Multiplying $(1.1)_1$ by $\bar u_t$, we have

$$i\int |u_t|^2\mathrm{d}x+\int \bar u_t\Delta u\mathrm{d}x+2\int u h'(|u|^2)\Delta h(|u|^2)\bar u_t\mathrm{d}x+\int u f(|u|^2)\bar u_t\mathrm{d}x=0, \tag{2.3}$$

where

$$\begin{aligned}
\mathrm{Re}\int \bar u_t\Delta u\mathrm{d}x&=-\mathrm{Re}\int \nabla u\cdot\nabla\bar u_t\mathrm{d}x=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla u|^2\mathrm{d}x,\\
2\mathrm{Re}\int \bar u_t u h'(|u|^2)\Delta h(|u|^2)\mathrm{d}x&=\int |u|_t^2 h'(|u|^2)\Delta h(|u|^2)\mathrm{d}x\\
&=\int [h(|u|^2)]_t\Delta h(|u|^2)\mathrm{d}x\\
&=-\int \nabla[h(|u|^2)]_t\nabla h(|u|^2)\mathrm{d}x\\
&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int [\nabla h(|u|^2)]^2\mathrm{d}x,
\end{aligned}$$

$$\mathrm{Re}\int \bar{u}_t u f(|u|^2)\mathrm{d}x = \frac{1}{2}\int |u|_t^2 f(|u|^2)\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int G(|u|^2)\mathrm{d}x,$$

and $G(s) = \int_0^s f(\delta)\mathrm{d}\delta$. Taking the real part of (2.3) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla u|^2\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int [\nabla h(|u|^2)]^2\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int G(|u|^2)\mathrm{d}x = 0,$$

thus we can obtain the conserved quantity as follows

$$E(t) = \int |\nabla u|^2\mathrm{d}x + \int [\nabla h(|u|^2)]^2\mathrm{d}x - \int G(|u|^2)\mathrm{d}x = E(0). \tag{2.4}$$

□

Based on the conservation laws in Lemma 2.1, we will prove Theorem 1.2.

Proof. Denote $D(t) = \int r^2|u|^2\mathrm{d}x$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}D(t) &= 2\mathrm{Re}\int r^2\bar{u}u_t\mathrm{d}x\\ &= -2\mathrm{Im}\int r^2\bar{u}\Delta u\mathrm{d}x\\ &= 2\mathrm{Im}\int (r^2|\nabla u|^2 + 2x\bar{u}\nabla u)\mathrm{d}x\\ &= 4\mathrm{Im}\int x\bar{u}\nabla u\mathrm{d}x\\ &=: -4D_1(t),\end{aligned} \tag{2.5}$$

in here $D_1(t) = -\mathrm{Im}\int x\bar{u}\nabla u\mathrm{d}x$. A direct calculation yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}D_1(t) =& -\mathrm{Im}\int (x\bar{u}_t\nabla u + x\bar{u}\nabla u_t)\mathrm{d}x\\ =& -\mathrm{Im}\int x\bar{u}_t\nabla u\mathrm{d}x + \mathrm{Im}\int (N\bar{u}u_t + x\nabla\bar{u}u_t)\mathrm{d}x\\ =& 2\mathrm{Im}\int x\nabla\bar{u}u_t\mathrm{d}x + \mathrm{Im}\int N\bar{u}u_t\mathrm{d}x\\ =& \mathrm{Re}\int (2x\nabla\bar{u} + N\bar{u})[\Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + uf(|u|^2)]\mathrm{d}x.\end{aligned} \tag{2.6}$$

Note that

$$\begin{aligned}\mathrm{Re}\int [2x\nabla\bar{u} + N\bar{u}]\Delta u\mathrm{d}x &= -\mathrm{Re}\int (\nabla(2x\nabla\bar{u})\nabla u + N\nabla\bar{u}\nabla u)\mathrm{d}x\\ &= -N\int |\nabla u|^2\mathrm{d}x - \mathrm{Re}\int (2|\nabla u|^2 + 2x\nabla^2\bar{u}\nabla u)\mathrm{d}x\\ &= -(N+2)\int |\nabla u|^2\mathrm{d}x - \int x\nabla|\nabla u|^2\mathrm{d}x\\ &= -2\int |\nabla u|^2\mathrm{d}x,\end{aligned} \tag{2.7}$$

$$2\mathrm{Re}\int (2x\nabla\bar{u} + N\bar{u})uh'(|u|^2)\Delta h(|u|^2)\mathrm{d}x$$

$$
\begin{aligned}
&= 2\int x\nabla|u|^2h'(|u|^2)\Delta h(|u|^2)\mathrm{d}x - 2N\int \nabla(|u|^2h'(|u|^2))\nabla h(|u|^2)\mathrm{d}x\\
&= 2\int x\nabla h(|u|^2)\Delta h(|u|^2)\mathrm{d}x - 2N\int |\nabla h(|u|^2)|^2\mathrm{d}x - 2N\int |u|^2\nabla h'(|u|^2)\nabla h(|u|^2)\mathrm{d}x\\
&= -(N+2)\int |\nabla h(|u|^2)|^2\mathrm{d}x - 2N\int |u|^2|\nabla|u|^2|^2h'(|u|^2)h''(|u|^2)\mathrm{d}x, \qquad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Re}\int(2x\nabla\bar{u}+N\bar{u})uf(|u|^2)\mathrm{d}x &= \int x\cdot\nabla|u|^2f(|u|^2)\mathrm{d}x + N\int|u|^2f(|u|^2)\mathrm{d}x\\
&= -N\int G(|u|^2)\mathrm{d}x + N\int |u|^2f(|u|^2)\mathrm{d}x, \qquad (2.9)
\end{aligned}
$$

thus, (2.6) can be written as

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}D_1(t)\\
=&-2\int|\nabla u|^2\mathrm{d}x-(N+2)\int|\nabla h(|u|^2)|^2\mathrm{d}x-2N\int|u|^2|\nabla|u|^2|^2h'(|u|^2)h''(|u|^2)\mathrm{d}x\\
&+N\int(|u|^2f(|u|^2)-G(|u|^2))\mathrm{d}x. \qquad (2.10)
\end{aligned}
$$

Combining this with the hypothesis (2)-(4), we derive that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}D_1(t)\geq&-N(C_N-1)E(0)+(NC_N-N-2)\int|\nabla u|^2\mathrm{d}x\\
&-2N\int|u|^2|\nabla|u|^2|^2h'(|u|^2)h''(|u|^2)\mathrm{d}x\\
>&0, \qquad (2.11)
\end{aligned}
$$

therefore,

$$
D_1(t)\geq D_1(0)=-\mathrm{Im}\int\bar{u}_0x\nabla\bar{u}_0\mathrm{d}x>0,
$$

it now follows from (2.5) that $\mathrm{d}D(t)/\mathrm{d}t<0$ and

$$
D(t)\leq D(0)=\int r^2|u_0|^2\mathrm{d}x. \qquad (2.12)
$$

Because of

$$
\begin{aligned}
|D_1(t)|&=D_1(t)\\
&=-\mathrm{Im}\int\bar{u}x\nabla u\mathrm{d}x\\
&\leq(\int r^2|u|^2\mathrm{d}x)^{\frac{1}{2}}(\int|\nabla u|^2\mathrm{d}x)^{\frac{1}{2}}\\
&\leq D(0)^{\frac{1}{2}}\|\nabla u\|_{L^2(\mathbb{R}^N)}, \qquad (2.13)
\end{aligned}
$$

together with (2.11), it follows that

$$
D_1(t)^2\leq D(0)\|\nabla u\|^2_{L^2(\mathbb{R}^N)}\leq\frac{D(0)}{NC_N-N-2}\frac{\mathrm{d}D_1(t)}{\mathrm{d}t},
$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}D_1(t) \geq \frac{NC_N - N - 2}{D(0)} D_1(t)^2. \tag{2.14}$$

Combining with the expression of $D_1(0) = -\mathrm{Im}\int \bar{u_0} x \nabla \bar{u_0} \mathrm{d}x > 0$, we obtain that

$$D_1(t) \geq \frac{1}{\frac{1}{D_1(0)} - \frac{(NC_N - N-2)t}{D(0)}}. \tag{2.15}$$

Let $T^* = \frac{D(0)}{(NC_N - N-2)D_1(0)} < \infty$, then there exists $T_0 \leq T^*$ such that

$$\lim_{t\to T_0^-} \|\nabla u\|^2_{L^2(\mathbb{R}^N)} \geq \lim_{t\to T_0^-} \frac{D_1(t)^2}{D(0)} = +\infty, \tag{2.16}$$

so that $\|u\|_{W^{2,2}(\mathbb{R}^N)}$ blows up in finite time. □

3. Some lemmas

In this section, we state two lemmas which are useful for the proof of Theorem 1.3. The first is concerned with a radially symmetric function in $W^{1,2}(\mathbb{R}^N)$ which is due to [22].

Lemma 3.1. *Let u be radially symmetric in $W^{1,2}(\mathbb{R}^N)$ and $N \geq 2$. Then for any $R > 0$, u satisfies*

$$\|u\|_{L^\infty(R<r)} \leq CR^{-\frac{N-1}{2}} \|u\|^{\frac{1}{2}}_{L^2(R<r)} \|\nabla u\|^{\frac{1}{2}}_{L^2(R<r)},$$

where $r = |x|$ and C is a constant independent of u and R.

The second lemma is the key identity to obtain our result. The following is the further generalization of the estimates obtained in [13, 22, 27].

Lemma 3.2. *Suppose that $\Psi = (\Psi_1, \Psi_2, \cdots, \Psi_N)$ is a vector valued function in $(W^{3,\infty}(\mathbb{R}^{\mathbb{N}}))^N$, then the $W^{2,2}(\mathbb{R}^{\mathbb{N}})$ solution $u(t)$ of (1.1) satisfies*

$$\begin{aligned}
&Im\int \Psi u_0 \nabla \bar{u}_0 \, dx - Im\int \Psi u(t) \nabla \bar{u}(t) \, dx \\
&= \int_0^t \{2Re\int \sum_{k,j} u_k \Psi_{kj} \bar{u}_j \, dx + 2\int \sum_{k,j} h(|u|^2)_k \Psi_{kj} h(|u|^2)_j \, dx \\
&\quad + \int (\nabla\cdot\Psi) G(|u|^2)\, dx - \frac{1}{2}\int \Delta(\nabla\cdot\Psi)|u|^2 \, dx + \int \nabla(\nabla\cdot\Psi)\nabla|u|^4 |h'(|u|^2)|^2 \, dx \\
&\quad + \int (\nabla\cdot\Psi)|\nabla h(|u|^2)|^2 \, dx + 2\int (\nabla\cdot\Psi)|u|^2 |\nabla |u|^2|^2 h'(|u|^2) h''(|u|^2)\, dx \\
&\quad - \int (\nabla\cdot\Psi)|u|^2 f(|u|^2)\, dx\}\, d\tau,
\end{aligned} \tag{3.1}$$

where $u_k = (\partial/\partial x_k)u$, $h(|u|^2)_k = (\partial/\partial x_k)h(|u|^2)$, and $\Psi_{kj} = (\partial/\partial x_j)\Psi_k$, all summations are taken from 1 to N.

Proof. It is observed that

$$iu_t = -\Delta u - 2uh'(|u|^2)\Delta h(|u|^2) - uf(|u|^2), \tag{3.2}$$
$$-i\bar{u}_t = -\Delta \bar{u} - 2\bar{u}h'(|u|^2)\Delta h(|u|^2) - \bar{u}f(|u|^2). \tag{3.3}$$

Multiplying the terms on the left hand side of (3.2) by $\Psi \cdot \nabla \bar{u}$ and integrating by parts, we obtain

$$i\int u_t(\Psi \cdot \nabla \bar{u})\mathrm{d}x = i\frac{\mathrm{d}}{\mathrm{d}t}\int \Psi u \nabla \bar{u}\mathrm{d}x - i\int \Psi u \nabla \bar{u}_t \mathrm{d}x.$$

Then it follows that

$$\begin{aligned}
&i\frac{\mathrm{d}}{\mathrm{d}t}\int \Psi u \nabla \bar{u}\mathrm{d}x + i\int (\nabla \cdot \Psi) u \bar{u}_t \mathrm{d}x \\
=&i\int u_t(\Psi \nabla \bar{u})\mathrm{d}x + i\int \Psi u \nabla \bar{u}_t \mathrm{d}x + i\int (\nabla \cdot \Psi) u \bar{u}_t \mathrm{d}x \\
=&i\int u_t(\Psi \nabla \bar{u})\mathrm{d}x - i\int \Psi \bar{u}_t \nabla u \mathrm{d}x.
\end{aligned}$$

Combining this with (3.2)(3.3), we can derive that

$$\begin{aligned}
&i\frac{\mathrm{d}}{\mathrm{d}t}\int \Psi u \nabla \bar{u}\mathrm{d}x + i\int (\nabla \cdot \Psi) u \bar{u}_t \mathrm{d}x \\
=&\int [-\Delta u - 2uh'(|u|^2)\Delta h(|u|^2) - uf(|u|^2)](\Psi \nabla \bar{u})\mathrm{d}x \\
&+\int [-\Delta \bar{u} - 2\bar{u}h'(|u|^2)\Delta h(|u|^2) - \bar{u}f(|u|^2)](\Psi \nabla u)\mathrm{d}x.
\end{aligned} \tag{3.4}$$

A direct calculation yields

$$-\int \Delta u(\Psi \nabla \bar{u})\mathrm{d}x = \int \nabla u \nabla(\Psi \nabla \bar{u})\mathrm{d}x = \int (\sum_{k,j} u_k \Psi_{kj} \bar{u}_j + \sum_j \Psi_j \sum_k u_k \bar{u}_{kj})\mathrm{d}x,$$
$$-\int \Delta \bar{u}(\Psi \nabla u)\mathrm{d}x = \int \nabla \bar{u} \nabla(\Psi \nabla u)\mathrm{d}x = \int (\sum_{k,j} \bar{u}_k \Psi_{kj} u_j + \sum_j \Psi_j \sum_k \bar{u}_k u_{kj})\mathrm{d}x,$$

hence

$$\begin{aligned}
&\int [-\Delta u(\Psi \nabla \bar{u}) - \Delta \bar{u}(\Psi \nabla u)]\mathrm{d}x \\
=&2\mathrm{Re}\int \sum_{k,j} u_k \Psi_{kj} \bar{u}_j \mathrm{d}x + \int \sum_j \Psi_j \sum_k (u_k \bar{u}_{kj} + \bar{u}_k u_{kj})\mathrm{d}x \\
=&2\mathrm{Re}\int \sum_{k,j} u_k \Psi_{kj} \bar{u}_j \mathrm{d}x - \int (\nabla \cdot \Psi)|\nabla u|^2 \mathrm{d}x.
\end{aligned} \tag{3.5}$$

Similarly, we obtain that

$$\int [2uh'(|u|^2)\Delta h(|u|^2)(\Psi \nabla \bar{u}) + 2\bar{u}h'(|u|^2)\Delta h(|u|^2)(\Psi \nabla u)]\mathrm{d}x$$

$$
\begin{aligned}
&= \int 2h'(|u|^2)\Delta h(|u|^2)\nabla|u|^2\Psi \mathrm{d}x\\
&= - \int 2\nabla h(|u|^2)\nabla[\Psi\nabla h(|u|^2)]\mathrm{d}x\\
&= - 2\int \sum_{k,j} h(|u|^2)_k\Psi_{kj}h(|u|^2)_j\mathrm{d}x - 2\int \sum_{j}\Psi_j\sum_{k} h(|u|^2)_k h(|u|^2)_{kj}\mathrm{d}x\\
&= - 2\int \sum_{k,j} h(|u|^2)_k\Psi_{kj}h(|u|^2)_j\mathrm{d}x + \int (\nabla\cdot\Psi)|\nabla h(|u|^2)|^2\mathrm{d}x, \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
\int [-uf(|u|^2)(\Psi\cdot\nabla\bar{u}) - \bar{u}f(|u|^2)(\Psi\cdot\nabla u)]\mathrm{d}x &= - \int f(|u|^2)\nabla|u|^2\cdot\Psi\mathrm{d}x\\
&= \int (\nabla\cdot\Psi)G(|u|^2)\mathrm{d}x. \qquad (3.7)
\end{aligned}
$$

Therefore, (3.4) can be written as

$$
\begin{aligned}
&i\frac{\mathrm{d}}{\mathrm{d}t}\int \Psi u\nabla\bar{u}\mathrm{d}x + i\int(\nabla\cdot\Psi)u\bar{u}_t\mathrm{d}x\\
=&2\mathrm{Re}\int\sum_{k,j}u_k\Psi_{kj}\bar{u}_j\mathrm{d}x - \int(\nabla\cdot\Psi)(|\nabla u|^2 + |\nabla h(|u|^2)|^2)\mathrm{d}x + \int(\nabla\cdot\Psi)G(|u|^2)\mathrm{d}x\\
&+ 2\int\sum_{k,j}h(|u|^2)_k\Psi_{kj}h(|u|^2)_j\mathrm{d}x. \qquad (3.8)
\end{aligned}
$$

Then, multiplying (3.3) by $(\nabla\cdot\Psi)u$ shows that

$$
i\int(\nabla\cdot\Psi)u\bar{u}_t\mathrm{d}x = \int(\nabla\cdot\Psi)[u\Delta\bar{u} + 2|u|^2h'(|u|^2)\Delta h(|u|^2) + |u|^2f(|u|^2)]\mathrm{d}x, \qquad (3.9)
$$

similar to the calculation of (3.5) – (3.7), it follows that

$$
\int(\nabla\cdot\Psi)u\Delta\bar{u}\mathrm{d}x = - \int\nabla(\nabla\cdot\Psi)u\nabla\bar{u}\mathrm{d}x - \int(\nabla\cdot\Psi)|\nabla u|^2\mathrm{d}x, \qquad (3.10)
$$

and

$$
\begin{aligned}
&\int 2(\nabla\cdot\Psi)|u|^2h'(|u|^2)\Delta h(|u|^2)\mathrm{d}x\\
=& - \int\nabla(\nabla\cdot\Psi)\nabla|u|^4|h'(|u|^2)|^2\mathrm{d}x - 2\int(\nabla\cdot\Psi)|\nabla h(|u|^2)|^2\mathrm{d}x\\
&- 2\int(\nabla\cdot\Psi)|u|^2|\nabla|u|^2|^2h'(|u|^2)h''(|u|^2)\mathrm{d}x. \qquad (3.11)
\end{aligned}
$$

Combining (3.8) – (3.11) and taking real part, we finally obtain that

$$
\begin{aligned}
&i\frac{\mathrm{d}}{\mathrm{d}t}\int\Psi u\cdot\nabla\bar{u}\mathrm{d}x\\
=&2\mathrm{Re}\int\sum_{k,j}u_k\Psi_{kj}\bar{u}_j\mathrm{d}x + 2\int\sum_{k,j}h(|u|^2)_k\Psi_{kj}h(|u|^2)_j\mathrm{d}x + \int(\nabla\cdot\Psi)G(|u|^2)\mathrm{d}x
\end{aligned}
$$

$$-\frac{1}{2}\int\Delta(\nabla\cdot\Psi)|u|^2\mathrm{d}x+\int\nabla(\nabla\cdot\Psi)\nabla|u|^4|h'(|u|^2)|^2\mathrm{d}x+\int(\nabla\cdot\Psi)|\nabla h(|u|^2)|^2\mathrm{d}x$$
$$-\int(\nabla\cdot\Psi)|u|^2f(|u|^2)\mathrm{d}x+2\int(\nabla\cdot\Psi)|u|^2|\nabla|u|^2|^2h'(|u|^2)h''(|u|^2)\mathrm{d}x. \tag{3.12}$$

Integrating (3.12) with respect to t can get the desired result. □

4. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 by choosing an appropriate weight function Ψ and using the method of contradiction.

Let $\phi:[0,+\infty)\to\mathbb{R}_+$ be a continuous function with bounded third order derivatives and such that

$$\phi(s)=\begin{cases} s, & 0\le s<\frac{1}{2},\\ s-(s-\frac{1}{2})^3, & \frac{1}{2}\le s<\frac{1}{2}+\frac{\sqrt{3}}{3},\\ \text{smooth},\ \phi'<0, & \frac{1}{2}+\frac{\sqrt{3}}{3}\le s<2,\\ 0, & 2\le s\end{cases}$$

and $|\phi'(s)|\le 1, \phi''(s)\le 0$. Let m be a large positive constant to be determined later, we set

$$\phi_m(r)=m\phi(\frac{r}{m}),$$

clearly

$$\left|\frac{\partial^\alpha}{\partial r^\alpha}\phi_m(r)\right|\le\frac{C_\alpha}{m^{\alpha-1}},\qquad \text{for}\quad \alpha=0,1,2,3. \tag{4.1}$$

Denote $r=|x|$ and define Ψ as follows

$$\Psi(x)=\frac{x}{r}\phi_m(r),\quad \Psi_k(x)=\frac{x_k}{r}\phi_m(r).$$

Direct computations show that

$$\Psi_{kj}=\begin{cases} \delta_{kj}, \quad 0\le r<\frac{1}{2}m,\\ (\frac{1}{r}\delta_{kj}-\frac{1}{r^3}x_kx_j)\phi_m(r)+\frac{1}{r^2}x_kx_j\phi_m'(r), \quad \frac{1}{2}m\le r<(\frac{1}{2}+\frac{\sqrt{3}}{3})m,\\ (\frac{1}{r}\delta_{kj}-\frac{1}{r^3}x_kx_j)\phi_m(r)+\frac{1}{r^2}x_kx_j\phi_m'(r), \quad \phi'<0\ (\frac{1}{2}+\frac{\sqrt{3}}{3})m\le r<2m,\\ 0, \quad 2m\le r\end{cases} \tag{4.2}$$

where $k, j = 1, 2, \cdots, N$ and

$$
\begin{aligned}
&\Delta(\nabla\cdot\Psi)\\
=&\sigma(r)\\
=&\begin{cases}
0, \quad 0\le r<\frac{1}{2}m,\\
\phi_m^{(3)}+(N-1)\left\{\frac{2}{r}\phi_m''(r)+\frac{N-3}{r^2}\phi_m'(r)-\frac{N-3}{r^3}\phi_m(r)\right\}, \quad \frac{1}{2}m\le r<(\frac{1}{2}+\frac{\sqrt{3}}{3})m,\\
\phi_m^{(3)}+(N-1)\left\{\frac{2}{r}\phi_m''(r)+\frac{N-3}{r^2}\phi_m'(r)-\frac{N-3}{r^3}\phi_m(r)\right\}, \quad \phi'<0\ (\frac{1}{2}+\frac{\sqrt{3}}{3})m\le r<2m,\\
0. \quad 2m\le r.
\end{cases}
\end{aligned}
\tag{4.3}
$$

Proof. We prove Theorem 1.3 by contradiction. Assuming $u(t)\in W_r^{2,2}(\mathbb{R}^N)$ and $\|\nabla u\|_{L^2(\mathbb{R}^N)}$ exists globally, that is $\|\nabla u\|_{L^2(\mathbb{R}^N)}\neq+\infty$. Substituting (4.2) (4.3) into (3.1) in Lemma 3.2 and using (2.4), we obtain

$$
\begin{aligned}
&\mathrm{Im}\int\Psi u_0\nabla\bar{u_0}\mathrm{d}x-\mathrm{Im}\int\Psi u(t)\nabla\bar{u}(t)\mathrm{d}x\\
=&\int_0^t\Big\{2\int_{r<\frac{m}{2}}|\nabla u|^2\mathrm{d}x+2\int_{r\ge\frac{m}{2}}|\nabla u|^2\phi_m'(r)\mathrm{d}x+(N+2)E(\tau)-(N+2)\int|\nabla u|^2\mathrm{d}x\\
&+(N+2)\int G(|u|^2)\mathrm{d}x-\frac{1}{2}\int_{r\ge\frac{m}{2}}\sigma(r)|u|^2\mathrm{d}x+N\int_{r<\frac{m}{2}}\left[G(|u|^2)-|u|^2f(|u|^2)\right]\mathrm{d}x\\
&+\int_{r\ge\frac{m}{2}}\left(\frac{N-1}{r}\phi_m(r)+\phi_m'(r)\right)\left[G(|u|^2)-|u|^2f(|u|^2)\right]\mathrm{d}x\\
&+\int_{r\ge\frac{m}{2}}\left(\frac{1}{r}\phi_m''(r)+\frac{N-1}{r^2}\phi_m'(r)-\frac{N-1}{r^3}\phi_m(r)\right)(x\cdot\nabla|u|^4)|h'(|u|^2)|^2\mathrm{d}x\\
&+\int_{r\ge\frac{m}{2}}\left(\frac{N-1}{r}\phi_m(r)+3\phi_m'(r)-N-2\right)|\nabla h(|u|^2)|^2\mathrm{d}x\\
&+2N\int_{r<\frac{m}{2}}|u|^2|\nabla|u|^2|^2h'(|u|^2)h''(|u|^2)\mathrm{d}x\\
&+2\int_{r\ge\frac{m}{2}}\left(\frac{N-1}{r}\phi_m(r)+\phi_m'(r)\right)|u|^2|\nabla|u|^2|^2h'(|u|^2)h''(|u|^2)\mathrm{d}x\Big\}\mathrm{d}\tau.
\end{aligned}
\tag{4.4}
$$

Now from the expression of ϕ, we know that

$$0<\phi_m(r)<1,\quad \frac{N-1}{r}\phi_m(r)+3\phi_m'(r)-N-2\le 0,\quad \text{for}\quad r\ge\frac{m}{2}$$

and

$$2\int_{r<\frac{m}{2}}|\nabla u|^2\mathrm{d}x+2\int_{r\ge\frac{m}{2}}|\nabla u|^2\phi_m'(r)\mathrm{d}x-(N+2)\int|\nabla u|^2\mathrm{d}x\le -N\int|\nabla u|^2\mathrm{d}x.$$

Equations (4.2) and (4.3) imply that, for some constant C,

$$|\sigma(r)|\le Cm^{-2},$$

then, combining these with $h'h'' \le 0$, we obtain that

$$\begin{aligned}
&\mathrm{Im}\int \Psi u_0 \nabla \bar{u}_0 \mathrm{d}x - \mathrm{Im}\int \Psi u(t)\nabla \bar{u}(t)\mathrm{d}x \\
\le &\int_0^t \Big\{(N+2)E(\tau) - N\int |\nabla u|^2 \mathrm{d}x + (N+2)\int G(|u|^2)\mathrm{d}x + Cm^{-2}\|u\|_{L^2}^2 \\
&+ \int_{r\ge \frac{m}{2}} \phi_m'(r)\left[G(|u|^2) - |u|^2 f(|u|^2)\right]\mathrm{d}x \\
&+ 2\int_{r\ge \frac{m}{2}} \phi_m'(r)|u|^2|\nabla |u|^2|^2 h'(|u|^2)h''(|u|^2)\mathrm{d}x \\
&+ \int_{r\ge \frac{m}{2}} \left(\frac{1}{r}\phi_m''(r) + \frac{N-1}{r^2}\phi_m'(r) - \frac{N-1}{r^3}\phi_m(r)\right)(x\cdot\nabla |u|^4)|h'(|u|^2)|^2\mathrm{d}x\Big\}\mathrm{d}\tau \\
\le &\int_0^t \Big\{(N+2)E(0) - N\int |\nabla u|^2\mathrm{d}x + Cm^{-2}\|u\|_{L^2}^2 + (N+4)\int_{r\ge\frac{m}{2}} |u|^2 f(|u|^2)\mathrm{d}x \\
&+ 2\|h'\|_\infty \|h''\|_\infty \int_{r\ge \frac{m}{2}} |u|^4|\nabla u|^2\mathrm{d}x + \|h'\|_\infty^2 \int_{r\ge\frac{m}{2}} \sigma(r)|u|^4\mathrm{d}x\Big\}\mathrm{d}\tau. \qquad (4.5)
\end{aligned}$$

From Lemma 3.1, and assuming

$$\|f'\|_\infty < m, \quad \|h'\|_\infty < m, \quad \|h''\|_\infty < m,$$

we arrive at

$$\begin{aligned}
&\int_{r\ge\frac{m}{2}} |u|^2 f(|u|^2)\mathrm{d}x \le \|u\|_{L^\infty[r>(m/2)]}^2 \|f'\|_\infty \int_{r\ge\frac{m}{2}} |u|^2\mathrm{d}x \le Cm^{2-N}\|u\|_{L^2}\|\nabla u\|_{L^2}, \\
&\int_{r\ge\frac{m}{2}} |u|^4|\nabla u|^2\mathrm{d}x \le \|u\|_{L^\infty[r>(m/2)]}^4 \int_{r\ge\frac{m}{2}} |\nabla u|^2\mathrm{d}x \le Cm^{2-2N}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4, \\
&\int_{r\ge\frac{m}{2}} \sigma(r)|u|^4\mathrm{d}x \le Cm^{-2}\|u\|_{L^\infty[r>(m/2)]}^2 \int_{r\ge\frac{m}{2}} |u|^2\mathrm{d}x \le Cm^{-N-1}\|u\|_{L^2}^3\|\nabla u\|_{L^2},
\end{aligned}$$

thus, we can conclude that for sufficiently large m, there exists a positive constant η such that

$$\mathrm{Im}\int \Psi u_0\nabla\bar{u}_0\mathrm{d}x - \mathrm{Im}\int \Psi u(t)\nabla \bar{u}(t)\mathrm{d}x \le -\eta t. \qquad (4.6)$$

Set $\Phi(r) = \int_0^r \phi_m(s)\mathrm{d}s$. Since $\Phi \in L^\infty(\mathbb{R}^N)$ and

$$\nabla\Phi(r) = \frac{x}{r}\phi_m(r) = \Psi(x),$$

using (3.4) we derive that

$$\begin{aligned}
i\int \Phi u\bar{u}_t\mathrm{d}x &= \int \Phi u[\Delta \bar{u} + 2\bar{u}h'(|u|^2)\Delta h(|u|^2) + \bar{u}f(|u|^2)]\mathrm{d}x \\
&= -\int \Psi u\nabla\bar{u}\mathrm{d}x - \int \Phi(|\nabla u|^2 - |u|^2 f(|u|^2) - 2|u|^2h'(|u|^2)\Delta h(|u|^2))\mathrm{d}x. \qquad (4.7)
\end{aligned}$$

Taking the imaginary part of (4.7), we arrive at

$$\mathrm{Re}\int \Phi u\bar{u}_t\mathrm{d}x = -\mathrm{Im}\int \Psi u\nabla\bar{u}\mathrm{d}x.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \Phi|u|^2\mathrm{d}x = \int \Phi(u_t\bar{u} + \bar{u}_t u)\mathrm{d}x = 2\mathrm{Re}\int \Phi u\bar{u}_t\mathrm{d}x, \quad t \geq 0$$

it follows that

$$\begin{aligned}\int \Phi|u|^2\mathrm{d}x =& 2\mathrm{Re}\int_0^t\int \Phi u\bar{u}_t\mathrm{d}x\mathrm{d}\tau + \int \Phi|u_0|^2\mathrm{d}x\\ =& -2\mathrm{Im}\int_0^t\int \Psi u\nabla\bar{u}\mathrm{d}x\mathrm{d}\tau + \int \Phi|u_0|^2\mathrm{d}x\\ \leq& 2\int_0^t(-\eta\tau - \mathrm{Im}\int \Psi u_0\nabla\bar{u}_0\mathrm{d}x)\mathrm{d}\tau + \int \Phi|u_0|^2\mathrm{d}x\\ \leq& -\eta t^2 - 2t\mathrm{Im}\int \Psi u_0\nabla\bar{u}_0\mathrm{d}x\mathrm{d}\tau + \int \Phi|u_0|^2\mathrm{d}x. \end{aligned} \tag{4.8}$$

Therefore the left-hand side of (4.8) becomes negative in finite time, which implies a contradiction since $\Phi(r) > 0$ except when $r = 0$. Hence $\|\nabla u\|_{L^2(\mathbb{R}^N)}$ must blow up in finite time. And we complete the proof. □

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