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ENHANCED HERMITE–HADAMARD INEQUALITIES FOR INTERVAL–VALUED FUNCTIONS VIA GENERALIZED FRACTIONAL OPERATORS AND ARTIFICIAL NEURAL NETWORK PREDICTIONS

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Abstract In this study we explore the development of a branch of the theory of fractional integration by presenting a new methodological framework based on generalized fractional integrals, focusing on interval functions. This framework enables the more systematic and clear derivation of advanced Hermit-Hadamard inequalities suitable for convex functions. This work integrates Riemann-Liouville fractional inequalities with the classical form of Hermit-Hadamard inequalities for LR convex functions within a unified structure, eliminating the need for separate case studies or independent proofs for each inequality. This research also extends to the development of new Hermite-Hadamard inequalities targeting LR convex functions with interval values, further increasing the comprehensiveness of this proposed framework. The theoretical results obtained in this research are validated through discussions on examples and illustrations that show their application in the field of fractional integration inequalities. Artificial neural networks were also used in this research to correctly forecast the boundaries associated with inequalities in this field, increasing the reliability of this research's results.

Keywords Hermite-Hadamard inequalities, fractional operators, ANNs predictions.

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1. Introduction

Fractional calculus has got aremarkable development,and it becomes a powerful mathematical tool for describing phenomena characterized by memory effects, nonlocal interactions, and

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genetic properties. Its applications have expanded to include the research of nonlinear waves [14, 21], the analysis of advanced fractional fluid models such as the multidimensional Navier-Stokes system [47], the modeling of epidemic dynamics using generalized fractional operators [3, 41],and discrete interaction and diffusion systems exhibiting patterns of synchronization and complex diffusion behavior [1, 20]. Recent advances have also highlighted the importance of generalized nuclei and relative fractional operators, which offer greater model flexibility and more accurate representation of long-range dependencies [3,24]. These efforts collectively underscore the growing role of fractional tools in providing more realistic representations of physical, biological, human health, and engineering systems, particularly in cases where usual derivatives fall short of providing accurate descriptions. On the other hand, the Hermit-Hadamard inequality (H-HI), whose beginning were guided by Hermit and Hadamard [12, 44], is a cornerstone in the analysis of convex functions, as it merges geometric significance and applied value at the same time. . It is obvious that for a convex function $f : \mathcal{D} \to \mathbb{R}$ defined over the interval $\mathcal{D}$, and for each different points μ and π within $\mathcal{D}$ with $\mu < \pi$, the following inequalities are valid:

$$f\left(\frac{\mu+\pi}{2}\right) \leq \frac{1}{\pi-\mu}\int_{\mu}^{\pi} f(\mathfrak{l})d\mathfrak{l} \leq \frac{f(\mu)+f(\pi)}{2}. \tag{1.1}$$

The convex case is related to the direction of the inequalities of the concave function (f). In the course of time, many authors have contributed significantly to the development, generalization, and improvement of Hermit-Hadamard inequalities (H-HTIs). The Hermit-Hadamard inequality can be noted as a generalization of the concept of convexity, and it is also related to the Jensen inequality, which can be derived and explained through a unified theoretical framework. Recently, the renewed interest in Hermit-Hadamard inequalities for convex applications led to a plethora of new results, studies, and generalizations. For further information, please refer to the relevant literature of this field, [2, 8, 13, 43, 48, 53, 54].

Interval analysis, which is a technique used in mathematics and computer models to deal with uncertain information in intervals, is an important field of research. The history of interval analysis may be traced back to Archimedes in his measurement of the circle, but in the 1950s, detailed research and exploration of this field started. Moore, who is credited with developing calculus of intervals, wrote the first book on interval analysis in 1966 [38]. Since then, many researchers have worked on this field of interval analysis. Unlike real-valued inequalities that rely on single-point quantities, interval-valued inequalities represent data by ranges, allowing uncertainty and variation to be incorporated directly into the analysis. This leads to more reliable bounds and a more realistic mathematical description when exact values are not available.

In this article, we define $\mathbb{R}_{\mathcal{I}}^{+}$ to denote the family of positive intervals within $\mathbb{R}$. We use $\mathcal{IR}_{([\mu,\pi])}$ to denote the collection of I-VFs that are Riemann integrable over the interval $[\mu, \pi]$, while $\mathcal{R}_{([\mu,\pi])}$ refers to the set of real-valued functions over the same interval. The theorem presented below highlights the relationship between functions that are integrable in the (IR) sense and those that are Riemann integrable ($\mathbb{R}$-integrable). Additionally, for intervals $\left[\underline{\Pi}, \overline{\Pi}\right]$ and $\left[\underline{\Sigma}, \overline{\Sigma}\right]$ within $\mathbb{R}_{\mathcal{I}}^{+}$, we use the notation "$\subseteq$" to show that $\left[\underline{\Pi}, \overline{\Pi}\right]$ is contained within $\left[\underline{\Sigma}, \overline{\Sigma}\right]$, which is true if $\underline{\Sigma} \leq \underline{\Pi}$ and $\overline{\Pi} \leq \overline{\Sigma}$. Recent research has increasingly focused on integral inequalities related to I-VFs. Sadowska [46] has extended the H-HI to set–valued functions, providing a broader framework for interval-valued mappings.

Theorem 1.1. [46] *Consider a convex I-VF* $F : [\mu, \pi] \to \mathbb{R}_{\mathcal{I}}^{+}$ *where* $F(\mathfrak{z}) = \left[\underline{F}(\mathfrak{z}), \overline{F}(\mathfrak{z})\right]$. *The next inequalities hold:*

$$F\left(\frac{\mu+\pi}{2}\right) \supseteq \frac{1}{\pi-\mu}(IR)\int_{\mu}^{\pi} F(\mathfrak{l})d\mathfrak{l} \supseteq \frac{F(\mu)+F(\pi)}{2}. \tag{1.2}$$

Additionally, several classical inequalities, including those by Ostrowski, Minkowski, and Beckenbach, have been modified for application to I-VFs (I-VFs), as explored in [6,7,17] and [15]. In [5], Budak et al. developed inequalities that focus on interval-valued R-L fractional integrals. Liu and his team in [34] presented the idea of interval-valued harmonically convex functions and derived Hermite-Hadamard-type inequalities (H-HTIs) that incorporate interval fractional integrals. The studies in [9] and [10] proposed a modification of Jensen's inequality tailored for I-VFs using fuzzy integrals, along with proving several integral inequalities. In [37], Mitroi et al. explored H-HTIs specifically for set-valued functions. Generalized forms of convex I-VFs were utilized to demonstrate H-HTIs in [40] and [18]. Román Flores et al. established Gronwall-type inequalities related to I-VFs in [16], while Zhao et al. introduced a variety of integral inequalities applicable to I-VFs in [58] and [59]. Recent works [25–28, 49] have broadened the concept of interval-valued convexity, where various forms of LR-convexity for I-VFs were introduced. These studies also provided a number of H-HTIs applicable to LR-convex and I-VFs.

Artificial Neural Networks are computer systems that are modeled after the structure and function of biological neural networks. Artificial Neural Networks are composed of different interconnected components, called neurons, which are arranged in layers and process information in a sequential manner. One of the advantages of using ANN is that it learns from data and adapts to new and unseen situations, and this feature of ANN allows it to be applied to various fields, including science and engineering.

The structure of ANN consists of different layers: Input layer, hidden layer, and output layer. The input layer of ANN handles the input data, the hidden layer of ANN handles the processing of data, and the output layer of ANN handles the display of results. During the learning process, ANN automatically changes its internal parameters using an optimization algorithm, and this process requires the minimization of the difference between the expected and actual values. The optimization algorithms have been developed in such a manner that the Adam optimization algorithm allows ANN to converge faster [31].

In recent years, ANNs have been employed in various fields of engineering and scientific research to solve highly nonlinear and computationally expensive problems. For instance, Periasamy et al. [45] applied artificial intelligence techniques to enhance bioconvection by using nano-encapsulation with oxytactic microorganisms, significantly improving the accuracy and efficiency of the process. Another notable study by Lee et al. [32] integrated Incompressible Smoothed Particle Hydrodynamics (ISPH) simulations with ANNs to model free surface flow over porous media on slopes, showcasing the capability of ANNs in handling complex fluid dynamics problems. Both studies highlight the transformative potential of artificial intelligence, particularly ANNs, in advancing the modeling and simulation of intricate physical phenomena.

This paper makes significant contributions to the research on fractional integrals and inequalities. Firstly, it introduces the concept of fractional generalized integrals for I-VFs, which results in the development of new H-HTIs that can be applied to convex functions. Secondly, it unifies both the R-L fractional H-HTIs and classical H-HTIs for LR-convex functions, which streamlines the derivation process by removing the need to prove both results individually. Thirdly, it extends the derivation process to establish novel H-HTIs. Lastly, it employs ANNs to predict the boundary values of the proposed inequalities, which is an efficient method to verify the results. The results are further reinforced by providing examples, which, along with their graphical representations, offer an extensive discussion on their importance and applications.

The layout of this work is as follows: The introduction presents the research background and objectives. Section 2 covers the necessary theoretical foundations, including key definitions and relevant theorems. In Section 3, the main results are introduced, focusing on the development of new H-HTIs for I-VFs using generalized fractional operators, along with numerical examples. Section 4 describes the implementation of an ANN model for predicting the behavior of inequalities across varying conditions. Finally, Section 5 concludes the main findings.

2. Fundamental concepts

This section reviews the fundamental definitions, results, and properties that will be employed throughout the paper.

2.1. Various fractional integrals for real valued functions

Definition 2.1. [30] Consider $f \in L_1[\mu, \pi]$. The R-L integrals $\mathbb{J}^{\varrho}_{\mu+}f$ and $\mathbb{J}^{\varrho}_{\pi-}f$ of order $\varrho > 0$ are characterized by

$$\mathbb{J}^{\varrho}_{\mu+}f(\mathfrak{l}) = \frac{1}{\Gamma(\varrho)} \int_{\mu}^{\mathfrak{l}} (\mathfrak{l} - \mathfrak{z})^{\varrho-1} f(\mathfrak{z}) d\mathfrak{z}, \quad \mathfrak{l} > \mu \tag{2.1}$$

and

$$\mathbb{J}^{\varrho}_{\pi-}f(\mathfrak{l}) = \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{l}}^{\pi} (\mathfrak{z} - \mathfrak{l})^{\varrho-1} f(\mathfrak{z}) d\mathfrak{z}, \quad \mathfrak{l} < \pi, \tag{2.2}$$

respectively. Here, $\Gamma(\varrho)$ signifies the Gamma function and it is described as follows:

$$\Gamma(\varrho) = \int_0^{\infty} u^{\varrho-1} \exp(-u) du.$$

Let us also note that $\mathbb{J}^{0}_{\mu+}f(\mathfrak{l}) = \mathbb{J}^{0}_{\pi-}f(\mathfrak{l}) = f(\mathfrak{l})$.

Different versions of the H-HI obtained using R-L fractional operators are as follows.

Theorem 2.1. [50] *If $f : [\mu, \pi] \to \mathbb{R}$ is a convex function, then, for $\varrho > 0$, we have the next H-HIs for R-L fractional integrals*

$$f\left(\frac{\mu+\pi}{2}\right) \leq \frac{\Gamma(\varrho+1)}{2(\pi-\mu)^{\varrho}} \left[\mathbb{J}^{\varrho}_{\mu+}f(\pi) + \mathbb{J}^{\varrho}_{\pi-}f(\mu)\right] \leq \frac{f(\mu)+f(\pi)}{2}.$$

Theorem 2.2. [51] *Let the conditions of Theorem 2.1 be assumed. The subsequent inequality is derived:*

$$f\left(\frac{\mu+\pi}{2}\right) \leq \frac{2^{\varrho-1}\Gamma(\varrho+1)}{(\pi-\mu)^{\varrho}} \left[\mathbb{J}^{\varrho}_{\frac{\mu+\pi}{2}+}f(\pi) + \mathbb{J}^{\varrho}_{\frac{\mu+\pi}{2}-}f(\mu)\right] \leq \frac{f(\mu)+f(\pi)}{2}.$$

Theorem 2.3. [11] *Considering the conditions of Theorem 2.1, then the subsequent inequality is obtained:*

$$f\left(\frac{\mu+\pi}{2}\right) \leq \frac{2^{\varrho-1}\Gamma(\varrho+1)}{(\pi-\mu)^{\varrho}} \left[\mathbb{J}^{\varrho}_{\mu+}f\left(\frac{\mu+\pi}{2}\right) + \mathbb{J}^{\varrho}_{\pi-}f\left(\frac{\mu+\pi}{2}\right)\right] \leq \frac{f(\mu)+f(\pi)}{2}. \tag{2.3}$$

Jarad et al. [23] proposed a class of fractional generalized integral operators in 2017. They analyzed their properties and established links between these operators and other fractional operators discussed in the literature. The definitions of these fractional generalized integral operators are given below.

Definition 2.2. [23] For $f \in L^1[\mu,\pi]$, the fractional generalized integral operators with order $\zeta \in \mathbb{C}$, $Re(\zeta) > 0$ and $\varrho \in (0,1]$ are respectively recognized by

$$
\begin{aligned}
{}^{\zeta}_{\mu+}\Psi^{\varrho} f(\mathfrak{l}) &= \frac{1}{\Gamma(\zeta)} \int_{\mu}^{\mathfrak{l}} \left(\frac{(\mathfrak{l}-\mu)^{\varrho} - (\mathfrak{z}-\mu)^{\varrho}}{\varrho} \right)^{\zeta-1} \frac{f(\mathfrak{z})}{(\mathfrak{z}-\mu)^{1-\varrho}} d\mathfrak{z}, \\
{}^{\zeta}\Psi^{\varrho}_{\pi-} f(\mathfrak{l}) &= \frac{1}{\Gamma(\zeta)} \int_{\mathfrak{l}}^{\pi} \left(\frac{(\pi-\mathfrak{l})^{\varrho} - (\pi-\mathfrak{z})^{\varrho}}{\varrho} \right)^{\zeta-1} \frac{f(\mathfrak{z})}{(\pi-\mathfrak{z})^{1-\varrho}} d\mathfrak{z}.
\end{aligned}
$$

Theorem 2.4. [52] *If $f : [\mu,\pi] \to \mathbb{R}$ is a convex function on $[\mu,\pi]$, then the next double inequalities are satisfied*

$$
f\left(\frac{\mu+\pi}{2}\right) \le \frac{\Gamma(\zeta+1)\varrho^{\zeta}}{2(\pi-\mu)^{\varrho\zeta}} \left[{}^{\zeta}_{\mu+}\Psi^{\varrho} f(\pi) + {}^{\zeta}\Psi^{\varrho}_{\pi-} f(\mu) \right] \le \frac{f(\mu)+f(\pi)}{2}.
$$

Here, $\zeta > 0$, $\varrho \in (0,1]$ and Γ is Euler Gamma fuction.

Theorem 2.5. [19] *Employing the hypotheses of Theorem 2.4 leads to the next inequality:*

$$
f\left(\frac{\mu+\pi}{2}\right) \le \frac{2^{\varrho\zeta-1}\Gamma(\zeta+1)\varrho^{\zeta}}{(\pi-\mu)^{\varrho\zeta}} \left[{}^{\zeta}_{\frac{\mu+\pi}{2}+}\Psi^{\varrho} f(\pi) + {}^{\zeta}\Psi^{\varrho}_{\frac{\mu+\pi}{2}-} f(\mu) \right] \le \frac{f(\mu)+f(\pi)}{2}.
$$

Theorem 2.6. [22] *Under the hypotheses of Theorem 2.4, the ensuing inequality is derived:*

$$
f\left(\frac{\mu+\pi}{2}\right) \le \frac{2^{\varrho\zeta-1}\Gamma(\zeta+1)\varrho^{\zeta}}{(\pi-\mu)^{\varrho\zeta}} \left[{}^{\zeta}_{\mu+}\Psi^{\varrho} f\left(\frac{\mu+\pi}{2}\right) + {}^{\zeta}\Psi^{\varrho}_{\pi-} f\left(\frac{\mu+\pi}{2}\right) \right] \le \frac{f(\mu)+f(\pi)}{2}. \quad (2.4)
$$

2.2. Various fractional integrals for I-VFs

A positive interval is an interval that tells you that the boundaries of the interval are also positive. Let $\mathbb{R}^+_{\mathcal{I}}$ represent the collection of all positive intervals found within $\mathbb{R}$. The Hausdorff distance, which quantifies the disparity between the intervals $[\underline{\mathcal{O}},\overline{\mathcal{O}}]$ and $[\underline{\mathcal{P}},\overline{\mathcal{P}}]$, is defined as follows:

$$
d([\underline{\mathcal{O}},\overline{\mathcal{O}}],[\underline{\mathcal{P}},\overline{\mathcal{P}}]) = \max\left\{ |\underline{\mathcal{O}}-\underline{\mathcal{P}}|, |\overline{\mathcal{O}}-\overline{\mathcal{P}}| \right\}.
$$

The pair $(\mathbb{R}_{\mathcal{I}}, d)$ represents a complete metric space. For additional information and fundamental notations regarding I-VFs, refer to [39,57].

We will now present the characteristics of the essential operations in interval analysis for the intervals Π and Σ as outlined below:

$$
\begin{aligned}
&\Pi + \Sigma = \left[\underline{\Pi}+\underline{\Sigma}, \overline{\Pi}+\overline{\Sigma}\right], \\
&\Pi - \Sigma = \left[\underline{\Pi}-\overline{\Sigma}, \overline{\Pi}-\underline{\Sigma}\right], \\
&\Pi.\Sigma = [\min\Lambda, \max\Lambda] \text{ where } \Lambda = \left\{\underline{\Pi}\,\underline{\Sigma}, \underline{\Pi}\,\overline{\Sigma}, \overline{\Pi}\,\underline{\Sigma}, \overline{\Pi}\,\overline{\Sigma}\right\}, \\
&\Pi/\Sigma = [\min\pi, \max\pi] \text{ where } \pi = \left\{\underline{\Pi}/\underline{\Sigma}, \underline{\Pi}/\overline{\Sigma}, \overline{\Pi}/\underline{\Sigma}, \overline{\Pi}/\overline{\Sigma}\right\} \text{ and } 0 \notin \Sigma.
\end{aligned}
$$

For $\theta \in \mathbb{R}$, the scalar product for the interval Π is expressed as:

$$
\theta\Pi = \theta\left[\underline{\Pi},\overline{\Pi}\right] = \begin{cases} \left[\theta\underline{\Pi}, \theta\overline{\Pi}\right], & \theta > 0, \\ \{0\}, & \theta = 0, \\ \left[\theta\overline{\Pi}, \theta\underline{\Pi}\right], & \theta < 0. \end{cases}
$$

For $\left[\underline{\Pi},\overline{\Pi}\right],\left[\underline{\Sigma},\overline{\Sigma}\right]\in\mathbb{R}_{\mathcal{I}}^{+}$, the inclusion "$\subseteq$" is defined by $\left[\underline{\Pi},\overline{\Pi}\right]\subseteq\left[\underline{\Sigma},\overline{\Sigma}\right]$, and only if, $\underline{\Sigma}\leq\underline{\Pi}$, $\overline{\Pi}\leq\overline{\Sigma}$.

1. The relation "$\leq_p$" defined on $\mathbb{R}_{\mathcal{I}}$ by $\left[\underline{\Pi},\overline{\Pi}\right]\leq_p\left[\underline{\Sigma},\overline{\Sigma}\right]$ if and only if $\underline{\Pi}\leq\underline{\Sigma}$, and $\overline{\Pi}\leq\overline{\Sigma}$, for all $\left[\underline{\Pi},\overline{\Pi}\right],\left[\underline{\Sigma},\overline{\Sigma}\right]\in\mathbb{R}_{\mathcal{I}}^{+}$, it is an pseudo order relation. For given $\left[\underline{\Pi},\overline{\Pi}\right],\left[\underline{\Sigma},\overline{\Sigma}\right]\in\mathbb{R}_{\mathcal{I}}$, we say that $\left[\underline{\Pi},\overline{\Pi}\right]\leq_p\left[\underline{\Sigma},\overline{\Sigma}\right]$ if and only if $\underline{\Pi}\leq\underline{\Sigma}$, and $\overline{\Pi}\leq\overline{\Sigma}$.
2. It can be easily seen that "$\leq_p$" looks like "left and right" on real line $\mathbb{R}$, so we call "$\leq_p$" is "left and right "(or "LR" order, in short).

It is noteworthy that Moore [38] was instrumental in defining the Riemann integral for I-VFs (I-VFs). The collection of all I-VFs that are Riemann integrable, along with real-valued functions on the interval $[\mu,\pi]$, is denoted by $\mathcal{IR}_{([\mu,\pi])}$ and $\mathcal{R}_{([\mu,\pi])}$, respectively. The subsequent theorem demonstrates the relationship between (IR)-integrable functions and those that are Riemann integrable (R-integrable) [39].

Theorem 2.7. *Assume $F:[\mu,\pi]\to\mathbb{R}_{\mathcal{I}}$ be an interval–valued function (I-VF) with $F(\mathfrak{z})=\left[\underline{F}(\mathfrak{z}),\overline{F}(\mathfrak{z})\right]$. $F\in\mathcal{IR}_{([\mu,\pi])}$ if and only if $\underline{F}(\mathfrak{z})$, $\overline{F}(\mathfrak{z})\in\mathcal{R}_{([\mu,\pi])}$ and*

$$(IR)\int_{\mu}^{\pi}F(\mathfrak{z})d\mathfrak{z}=\left[(R)\int_{\mu}^{\pi}\underline{F}(\mathfrak{z})d\mathfrak{z},(R)\int_{\mu}^{\pi}\overline{F}(\mathfrak{z})d\mathfrak{z}\right].$$

In [57,58], Zhao et al. proposed a sort of convex I-VF as follows.

Definition 2.3. Consider the non–negative function h from $[c,d]$ into $\mathbb{R}$ with $[c,d]\supseteq(0,1)$ and $h\neq 0$. The function $F:[\mu,\pi]\to\mathbb{R}_{\mathcal{I}}^{+}$ is h–convex I-VF, if for all $\mathfrak{l},y\in[\mu,\pi]$ and $\mathfrak{z}\in(0,1)$, we have

$$h(\mathfrak{z})F(\mathfrak{l})+h(1-\mathfrak{z})F(y)\subseteq F(\mathfrak{z}\mathfrak{l}+(1-\mathfrak{z})y). \tag{2.5}$$

By $SX(h,[\mu,\pi],\mathbb{R}_{\mathcal{I}}^{+})$ we denote the collection of h–convex I-VFs.

The standard concept of convex I-VFs is related to equation (2.5) where $h(\mathfrak{z})=\mathfrak{z}$, as noted in [46]. Additionally, if we set $h(\mathfrak{z})=\mathfrak{z}^s$ in equation (2.5), Definition 2.3 yields another type of convex I-VF introduced by Breckner, as referenced in [4].

Furthermore, Zhao et al. derived the subsequent H-HIs for I-VFs utilizing h-convexity:

Theorem 2.8. [57] *Let $F:[\mu,\pi]\to\mathbb{R}_{\mathcal{I}}^{+}$ be an I-VF with $F(\mathfrak{z})=[\underline{F}(\mathfrak{z}),\overline{F}(\mathfrak{z})]$ and $F\in\mathcal{IR}_{([\mu,\pi])}$, $h:[0,1]\to\mathbb{R}$ be a function with $h\geq 0$ and $h\left(\frac{1}{2}\right)\neq 0$. If $F\in SX(h,[\mu,\pi],\mathbb{R}_{\mathcal{I}}^{+})$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}F\left(\frac{\mu+\pi}{2}\right)\supseteq\frac{1}{\pi-\mu}(IR)\int_{\mu}^{\pi}F(\mathfrak{l})d\mathfrak{l}\supseteq[F(\mu)+F(\pi)]\int_{0}^{1}h(\mathfrak{z})d\mathfrak{z}. \tag{2.6}$$

Remark 2.1.
1. If $h(\mathfrak{z})=\mathfrak{z}$, then (2.6) reduces to the next result:
$$F\left(\frac{\mu+\pi}{2}\right)\supseteq\frac{1}{\pi-\mu}(IR)\int_{\mu}^{\pi}F(\mathfrak{l})d\mathfrak{l}\supseteq\frac{F(\mu)+F(\pi)}{2},$$
which is provided in [46].
2. If $h(\mathfrak{z})=\mathfrak{z}^s$, then (2.6) reduces to the next outcome:
$$2^{s-1}F\left(\frac{\mu+\pi}{2}\right)\supseteq\frac{1}{\pi-\mu}(IR)\int_{\mu}^{\pi}F(\mathfrak{l})d\mathfrak{l}\supseteq\frac{F(\mu)+F(\pi)}{s+1},$$
which is established in [18].

In [35], Lupulescu introduced the subsequent interval-valued left R-L fractional integral.

Definition 2.4. Let $F:[\mu,\pi]\to\mathbb{R}_{\mathcal{I}}$ be an I-VF such that $F(\mathfrak{z})=\left[\underline{F}(\mathfrak{z}),\overline{F}(\mathfrak{z})\right]$ and let $\varrho>0$. The interval-valued left R-L fractional integral of function f is defined by

$$\mathcal{J}^{\varrho}_{\mu+}F(\mathfrak{l})=\frac{1}{\Gamma(\varrho)}\int_{\mu}^{\mathfrak{l}}(\mathfrak{l}-s)^{\varrho-1}F(\mathfrak{z})d\mathfrak{z},\quad \mathfrak{l}>\mu.$$

Using the definition provided by Lupulescu, Budak et al. [5] proposed the formula of the interval-valued fractional integral of right-sided R-Ltype as follows:

$$\mathcal{J}^{\varrho}_{\pi-}F(\mathfrak{l})=\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{l}}^{\pi}(s-\mathfrak{l})^{\varrho-1}F(\mathfrak{z})d\mathfrak{z},\quad \mathfrak{l}<\pi.$$

Definition 2.5. [55] The I-VF $F:I\to\mathbb{R}^{+}_{\mathcal{I}}$ is called LR-convex I-VF on convex set I if for all $\mu,\pi\in I$ and $\mathfrak{z}\in[0,1]$ we have

$$F(\mathfrak{z}\mu+(1-\mathfrak{z})\pi)\leq_p \mathfrak{z}F(\mu)+(1-\mathfrak{z})F(\pi). \tag{2.7}$$

Theorem 2.9. [29] *Let $F:[\mu,\pi]\to\mathbb{R}^{+}_{\mathcal{I}}$ be a LR-convex I-VF on $[\mu,\pi]$ and given by $F(\mathfrak{l})=\left[\underline{F}(\mathfrak{l}),\overline{F}(\mathfrak{l})\right]$ for all $\mathfrak{l}\in[\mu,\pi]$. If $F\in L\left([\mu,\pi],\ \mathbb{R}^{+}_{\mathcal{I}}\right)$, then*

$$F\left(\frac{\mu+\pi}{2}\right)\leq_p\frac{1}{\pi-\mu}\int_{\mu}^{\pi}F(\mathfrak{l})d\mathfrak{l}\leq_p\frac{F(\mu)+F(\pi)}{2}. \tag{2.8}$$

Theorem 2.10. [55] *Let $\mathcal{D}$ be an convex set and $F:\mathcal{D}\to\mathbb{R}^{+}_{\mathcal{I}}$ be an I-VF such that*

$$F(\mathfrak{z})=\left[\underline{F}(\mathfrak{z}),\overline{F}(\mathfrak{z})\right],\quad \forall\mathfrak{z}\in\mathcal{D}.$$

Then F is LR-convex I-VF on I, if and only if, $\underline{F}(\mathfrak{z})$ and $\overline{F}(\mathfrak{z})$ both are convex functions.

If the function $F(\mathfrak{z})=\left[\underline{F}(\mathfrak{z}),\ \overline{F}(\mathfrak{z})\right]$ and $G(\mathfrak{l})=\left[\underline{G}(\mathfrak{z}),\ \overline{G}(\mathfrak{z})\right]$ are LR-convex, $F(\mathfrak{z})\leq_p G(\mathfrak{z})$, $c\in\mathbb{R}$, the upcoming property is valid:

$$cF(\mathfrak{z})\leq_p cG(\mathfrak{z}).$$

Definition 2.6. [56] Let $F:[\mu,\pi]\to\mathbb{R}^{+}_{\mathcal{I}}$ be an I-VF on $[\mu,\pi]$, which is given by $F(\mathfrak{l})=\left[\underline{F}(\mathfrak{l}),\ \overline{F}(\mathfrak{l})\right]$. For $F\in L\left([\mu,\pi],\ \mathbb{R}^{+}_{\mathcal{I}}\right)$, the generalized fractional integral operators for interval valued functions of order $\zeta>0$ and $\varrho\in(0,1]$ are respectively given by

$$\begin{aligned}{}^{\zeta}_{\mu+}\Psi^{\varrho}F(\mathfrak{l})&=\frac{1}{\Gamma(\zeta)}\int_{\mu}^{\mathfrak{l}}\left(\frac{(\mathfrak{l}-\mu)^{\varrho}-(\mathfrak{z}-\mu)^{\varrho}}{\varrho}\right)^{\zeta-1}\frac{F(\mathfrak{z})}{(\mathfrak{z}-\mu)^{1-\varrho}}d\mathfrak{z},\\ {}^{\zeta}\Psi^{\varrho}_{\pi-}F(\mathfrak{l})&=\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{l}}^{\pi}\left(\frac{(\pi-\mathfrak{l})^{\varrho}-(\pi-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}\frac{F(\mathfrak{z})}{(\pi-\mathfrak{z})^{1-\varrho}}d\mathfrak{z}.\end{aligned}$$

3. Hermite-Hadamard type inequalities for LR-convex interval-valued function

In this section, we derive several new H-HTIs for interval-valued LR-convex functions.

Theorem 3.1. *Let $F:[\mu,\pi]\to\mathbb{R}_{\mathcal{I}}^{+}$ be a LR-convex I-VF on $[\mu,\pi]$ given by $F(\mathfrak{l})=\left[\underline{F}(\mathfrak{l}),\ \overline{F}(\mathfrak{l})\right]$ for all $\mathfrak{l}\in[\mu,\pi]$. Then, we have the next inequality for generalized fractional integral operators of interval valued functions*

$$\begin{aligned}&F\left(\frac{\mu+\pi}{2}\right)\\&\leq_p \frac{\varrho^{\zeta}2^{\varrho\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\varrho\zeta}}\left[\,{}_{\mu+\mathfrak{z}}^{\zeta}{}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)+\,{}_{\pi-\mathfrak{z}}^{\zeta}{}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)\right]\\&\leq_p \frac{F(\mu)+F(\pi)}{2}\end{aligned}\tag{3.1}$$

for $\zeta>0$ and $\varrho\in(0,1]$.

Proof. Since F is a LR-convex I-VF, then we have

$$F\left(\frac{u+v}{2}\right)\leq_p \frac{F(u)+F(v)}{2},\tag{3.2}$$

for $u,v\in[\mu,\pi]$. By choosing

$$u=\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi \text{ and } v=\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi \text{ for } \mathfrak{z}\in[0,1],$$

we get

$$F\left(\frac{\mu+\pi}{2}\right)\leq_p \frac{1}{2}\left[F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)+F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\right].$$

Since F is a LR-convex I-VF, we can write

$$F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\leq_p \frac{1+\mathfrak{z}}{2}F(\mu)+\frac{1-\mathfrak{z}}{2}F(\pi)$$

and

$$F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\leq_p \frac{1-\mathfrak{z}}{2}F(\mu)+\frac{1+\mathfrak{z}}{2}F(\pi).$$

Thus we have

$$\begin{aligned}F\left(\frac{\mu+\pi}{2}\right)&\leq_p \frac{1}{2}\left[F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)+F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\right]\\&\leq_p \frac{1}{2}\left[\frac{1+\mathfrak{z}}{2}F(\mu)+\frac{1-\mathfrak{z}}{2}F(\pi)+\frac{1-\mathfrak{z}}{2}F(\mu)+\frac{1+\mathfrak{z}}{2}F(\pi)\right]\\&=\frac{F(\mu)+F(\pi)}{2}.\end{aligned}$$

That is

$$\begin{aligned}F\left(\frac{\mu+\pi}{2}\right)&\leq_p \frac{1}{2}\left[F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)+F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\right]\\&\leq_p \frac{F(\mu)+F(\pi)}{2}.\end{aligned}\tag{3.3}$$

Multiplying both sides of the inequalities (3.3) with $\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}$, and then integrating the resulting inequality over $[0,1]$ with respect to $\mathfrak{z}$, we get

$$\begin{aligned} & F\left(\frac{\mu+\pi}{2}\right)\int_0^1\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z} \\ \leq_p\ & \frac{1}{2}\left[\int_0^1 F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\right. \\ & \left.+\int_0^1 F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\right] \\ \leq_p\ & \frac{F(\mu)+F(\pi)}{2}\int_0^1\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}. \end{aligned} \tag{3.4}$$

By changing variable and Definition 2.6, we have

$$\begin{aligned} & \int_0^1 F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z} \\ =\ & \int_{\mu}^{\frac{\mu+\pi}{2}} F(\mathfrak{l})\left(\frac{1-\left(\frac{2}{\pi-\mu}(\mathfrak{l}-\mu)\right)^{\varrho}}{\varrho}\right)^{\zeta-1}\left(\frac{2}{\pi-\mu}(\mathfrak{l}-\mu)\right)^{\varrho-1}\frac{2}{\pi-\mu}d\mathfrak{l} \\ =\ & \left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta}\int_{\mu}^{\frac{\mu+\pi}{2}} F(\mathfrak{l})\left(\frac{\left(\frac{\pi-\mu}{2}\right)^{\varrho}-(\mathfrak{l}-\mu)^{\varrho}}{\varrho}\right)^{\zeta-1}(\mathfrak{l}-\mu)^{\varrho-1}\,d\mathfrak{l} \\ =\ & \left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta}\Gamma(\zeta)\ {}^{\zeta}_{\mu+}\mathfrak{J}^{\varrho}F\left(\frac{\mu+\pi}{2}\right) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & \int_0^1 F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z} \\ =\ & \int_{\frac{\mu+\pi}{2}}^{\pi} F(\mathfrak{l})\left(\frac{1-\left(\frac{2}{\pi-\mu}(\pi-\mathfrak{l})\right)^{\varrho}}{\varrho}\right)^{\zeta-1}\left(\frac{2}{\pi-\mu}(\pi-\mathfrak{l})\right)^{\varrho-1}\frac{2}{\pi-\mu}d\mathfrak{l} \\ =\ & \left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta}\int_{\frac{\mu+\pi}{2}}^{\pi} F(\mathfrak{l})\left(\frac{\left(\frac{\pi-\mu}{2}\right)^{\varrho}-(\pi-\mathfrak{l})^{\varrho}}{\varrho}\right)^{\zeta-1}(\pi-\mathfrak{l})^{\varrho-1}\,d\mathfrak{l} \\ =\ & \left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta}\Gamma(\zeta)\ {}^{\zeta}_{\pi-}\mathfrak{J}^{\varrho}F\left(\frac{\mu+\pi}{2}\right). \end{aligned} \tag{3.6}$$

On the other hand, we also get

$$\int_0^1\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}=\frac{1}{\zeta\cdot\varrho^{\varrho}}. \tag{3.7}$$

If we substitute the equalities (3.5)-(3.7) in (3.4), then we have

$$\frac{1}{\zeta\cdot\varrho^{\varrho}}F\left(\frac{\mu+\pi}{2}\right)$$

$$\leq_p \frac{1}{2}\left[\left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta}\Gamma(\zeta)\left[{}_{\mu+}^{\;\zeta}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)+{}_{\pi-}^{\;\zeta}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)\right]\right]$$
$$\leq_p \frac{1}{\zeta\cdot\varrho^{\zeta}}\frac{F(\mu)+F(\pi)}{2}.$$

This completes the proof. □

Remark 3.1. If we take $\varrho=1$ in Theorem 3.1, then we have the upcoming inequality for R-L fractional integral operators of interval valued functions

$$F\left(\frac{\mu+\pi}{2}\right)\leq_p \frac{2^{\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\zeta}}\left[\mathcal{J}_{\mu+}^{\zeta}F\left(\frac{\mu+\pi}{2}\right)+\mathcal{J}_{\pi-}^{\zeta}F\left(\frac{\mu+\pi}{2}\right)\right]\leq_p \frac{F(\mu)+F(\pi)}{2}. \tag{3.8}$$

Remark 3.2. If we pick $\varrho=1$ and $\zeta=1$ in Theorem 3.1, then the inequality (3.1) reduces to inequality (2.8).

Remark 3.3. If we pick $\underline{F}(\mathfrak{l})=\overline{F}(\mathfrak{l})=f(\mathfrak{l})$ for all $\mathfrak{l}\in[\mu,\pi]$ in Theorem 3.1, then inequality (3.1) reduces to inequality (2.4).

Example 3.1. Let us consider the function $F:[0,1]\to\mathbb{R}_{\mathcal{I}}^{+}$ defined by $F(\mathfrak{l})=\left[\mathfrak{l}^3,\mathfrak{l}^2\right]$. Then the function F is LR-convex function.Under these assuption, the left term and right term of the inequality (3.1) reduce to

$$F\left(\frac{\mu+\pi}{2}\right)=F\left(\frac{1}{2}\right)=\left[\frac{1}{8},\frac{1}{4}\right]$$

and

$$\frac{F(\mu)+F(\pi)}{2}=\frac{F(0)+F(1)}{2}=\left[\frac{1}{2},\frac{1}{2}\right],$$

respectively. By Definition 2.6, we have

$$\begin{aligned}{}_{\mu+}^{\;\zeta}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)&={}_{0+}^{\;\zeta}\mathfrak{z}^{\varrho}F\left(\frac{1}{2}\right)\\&=\frac{1}{\Gamma(\zeta)}\int_0^{\frac{1}{2}}\left(\frac{\left(\frac{1}{2}\right)^{\varrho}-\mathfrak{z}^{\varrho}}{\varrho}\right)^{\zeta-1}\mathfrak{z}^{\varrho-1}\left[\mathfrak{z}^3,\mathfrak{z}^2\right]d\mathfrak{z}\\&=\frac{1}{\Gamma(\zeta)}\frac{1}{\varrho^{\zeta}}\frac{1}{2^{\varrho\zeta+3}}\left[\mathcal{B}\left(\zeta,\frac{3}{\varrho}+1\right),2\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)\right]\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}{}_{\pi-}^{\;\zeta}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)&={}_{1-}^{\;\zeta}\mathfrak{z}^{\varrho}F\left(\frac{1}{2}\right)\\&=\frac{1}{\Gamma(\zeta)}\int_{\frac{1}{2}}^{1}\left(\frac{\left(\frac{1}{2}\right)^{\varrho}-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\left[\mathfrak{z}^3,\mathfrak{z}^2\right]d\mathfrak{z}\\&=\frac{1}{\Gamma(\zeta)}\frac{1}{\varrho^{\zeta}}\frac{1}{2^{\varrho\zeta}}\left[-\frac{1}{8}\mathcal{B}\left(\zeta,\frac{3}{\varrho}+1\right)+\frac{3}{4}\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)-\frac{3}{2}\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\frac{1}{\zeta},\right.\\&\quad\left.\frac{1}{4}\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)-\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\frac{1}{\zeta}\right].\end{aligned}\tag{3.10}$$

By the equalities (3.9) and (3.10), we can write the mid-term of the inequality (3.1) as

$$\begin{aligned}&\frac{\varrho^{\zeta}2^{\varrho\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\varrho\zeta}}\left[{}^{\zeta}_{\mu+}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)+{}^{\zeta}_{\pi-}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)\right]\\&=\left[\Omega_1(\varrho,\zeta),\Omega_2(\varrho,\zeta)\right],\end{aligned}$$

where

$$\Omega_1(\varrho,\zeta)=\frac{3\zeta}{8}\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)-\frac{3\zeta}{4}\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\frac{1}{2}$$

and

$$\Omega_2(\varrho,\zeta)=\frac{\zeta}{4}\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)-\frac{\zeta}{2}\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\frac{1}{2}.$$

Therefore, we have the inequality

$$\left[\frac{1}{8},\frac{1}{4}\right]\leq_p\left[\Omega_1(\varrho,\zeta),\Omega_2(\varrho,\zeta)\right]\leq_p[1,1]. \tag{3.11}$$

One can see the validity of the inequality (3.11) in Figure 1 and Figure 2 for $\varrho\in[0,1]$ and $\zeta\in[0,5]$. On the other hand, one can see $[\Omega_1(\varrho,\zeta),\Omega_2(\varrho,\zeta)]$ is an I-VF Figure 3.

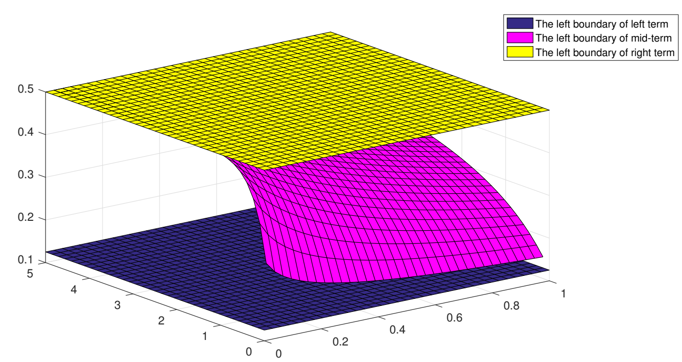

Figure 1. Comparison of left boundaries for Example 3.1.

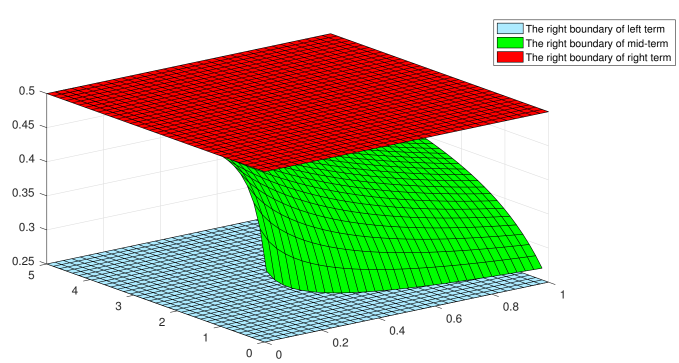

Figure 2. Comparison of right boundaries Example 3.1.

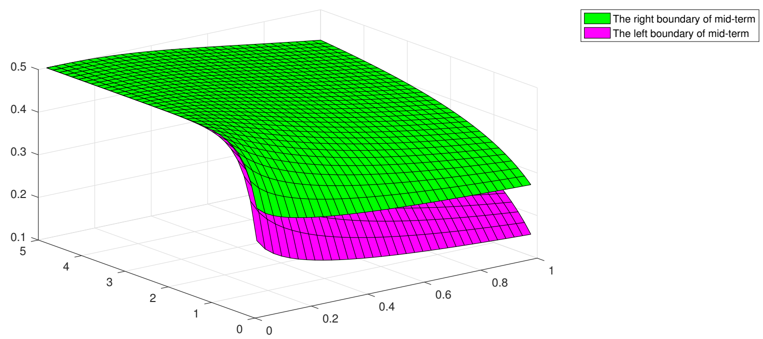

Figure 3. Graph of the mid-term of the inequality (3.11).

Theorem 3.2. *Consider two LR-convex I-VFs F and G mapping from $[\mu,\pi]$ to $\mathbb{R}^+_{\mathcal{I}}$, defined as $F(\mathfrak{l}) = [\underline{F}(\mathfrak{l}), \overline{F}(\mathfrak{l})]$ and $G(\mathfrak{l}) = [\underline{G}(\mathfrak{l}), \overline{G}(\mathfrak{l})]$ for each $\mathfrak{l} \in [\mu,\pi]$. We then establish the upcoming inequality involving generalized fractional integral operators:*

$$\begin{aligned} &\frac{\varrho^{\zeta} 2^{\varrho\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\varrho\zeta}}\left[{}^{\zeta}_{\mu+}\mathfrak{J}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)+{}^{\zeta}_{\pi-}\mathfrak{J}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)\right] \qquad (3.12)\\ &\leq_p \frac{\zeta}{4}\left[M(\mu,\pi)\left(\frac{2}{\zeta}-2\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)\right)\right.\\ &\quad\left.+N(\mu,\pi)\left(2\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)-\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)\right)\right], \end{aligned}$$

$\zeta > 0$ and $\varrho \in (0,1]$, where

$$M(\mu,\pi) = F(\mu)G(\mu) + F(\pi)G(\pi)$$

and

$$N(\mu,\pi) = F(\mu)G(\pi) + F(\pi)G(\mu).$$

Proof. Since the functions F and G are LR-convex I-VFs, then we have

$$F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) \leq_p \frac{1+\mathfrak{z}}{2}F(\mu)+\frac{1-\mathfrak{z}}{2}F(\pi)$$

and

$$G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) \leq_p \frac{1+\mathfrak{z}}{2}G(\mu)+\frac{1-\mathfrak{z}}{2}G(\pi).$$

It is clear that

$$\begin{aligned} &F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) \qquad (3.13)\\ &\leq_p \frac{1}{4}\left[(1+\mathfrak{z})^2F(\mu)G(\mu)+\left(1-\mathfrak{z}^2\right)\left(F(\mu)G(\pi)+F(\pi)G(\mu)\right)+(1-\mathfrak{z})^2F(\pi)G(\pi)\right]. \end{aligned}$$

On the other hand, since F and G are LR-convex interval-valued functions

$$F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) \leq_p \frac{1-\mathfrak{z}}{2}F(\mu)+\frac{1+\mathfrak{z}}{2}F(\pi)$$

and

$$G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\leq_{p}\frac{1-\mathfrak{z}}{2}G\left(\mu\right)+\frac{1+\mathfrak{z}}{2}G\left(\pi\right).$$

Then, we can write

$$\begin{aligned}&F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) \qquad (3.14)\\ &\leq_{p}\frac{1}{4}\left[\left(1-\mathfrak{z}\right)^{2}F\left(\mu\right)G\left(\mu\right)+\left(1-\mathfrak{z}^{2}\right)\left(F\left(\mu\right)G\left(\pi\right)+F\left(\pi\right)G\left(\mu\right)\right)+\left(1+\mathfrak{z}\right)^{2}F\left(\pi\right)G\left(\pi\right)\right].\end{aligned}$$

By the inequalities (3.13) and (3.14), we get

$$\begin{aligned}&F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) \qquad (3.15)\\ &+F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\\ &\leq_{p}\frac{1}{2}\left[\left(1+\mathfrak{z}^{2}\right)M(\mu,\pi)+\left(1-\mathfrak{z}^{2}\right)N(\mu,\pi)\right].\end{aligned}$$

Multiplying both sides of the inequalities (3.15) with $\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}$, and then integrating the resulting inequality over $[0,1]$ with respect to $\mathfrak{z}$, we get

$$\begin{aligned}&\int_{0}^{1}F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z} \quad (3.16)\\ &+\int_{0}^{1}F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\\ &\leq_{p}\frac{1}{2}\left[M(\mu,\pi)\int_{0}^{1}\left(1+\mathfrak{z}^{2}\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\right.\\ &\left.\quad N(\mu,\pi)\int_{0}^{1}\left(1-\mathfrak{z}^{2}\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\right].\end{aligned}$$

By changing variable and Definition 2.6, we have

$$\begin{aligned}&\int_{0}^{1}F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z} \quad (3.17)\\ &=\int_{\mu}^{\frac{\mu+\pi}{2}}F\left(\mathfrak{l}\right)G\left(\mathfrak{l}\right)\left(\frac{1-\left(\frac{2}{\pi-\mu}\left(\mathfrak{l}-\mu\right)\right)^{\varrho}}{\varrho}\right)^{\zeta-1}\left(\frac{2}{\pi-\mu}\left(\mathfrak{l}-\mu\right)\right)^{\varrho-1}\frac{2}{\pi-\mu}d\mathfrak{l}\\ &=\left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta}\int_{\mu}^{\frac{\mu+\pi}{2}}F\left(\mathfrak{l}\right)G\left(\mathfrak{l}\right)\left(\frac{\left(\frac{\pi-\mu}{2}\right)^{\varrho}-\left(\mathfrak{l}-\mu\right)^{\varrho}}{\varrho}\right)^{\zeta-1}\left(\mathfrak{l}-\mu\right)^{\varrho-1}d\mathfrak{l}\\ &=\left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta}\Gamma\left(\zeta\right)\ {}^{\zeta}_{\mu+}\mathfrak{J}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)\end{aligned}$$

and similarly

$$\int_{0}^{1}F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z} \quad (3.18)$$

$$= \left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta} \Gamma(\zeta)\ {}^{\zeta}_{\pi-}\mathfrak{J}^{\varrho} F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right).$$

We also have the facts that

$$\int_0^1 \left(1+\mathfrak{z}^2\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z} = \frac{1}{\varrho^{\zeta}}\left[\frac{2}{\zeta} - 2\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) + \mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right)\right] \tag{3.19}$$

and

$$\int_0^1 \left(1-\mathfrak{z}^2\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z} = \frac{1}{\varrho^{\zeta}}\left[2\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) - \mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right)\right]. \tag{3.20}$$

By substituting the equalities (3.15)-(3.20) in (3.16), then we have

$$\begin{aligned}
&\frac{\varrho^{\zeta} 2^{\varrho\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\varrho\zeta}}\left[{}^{\zeta}_{\mu+}\mathfrak{J}^{\varrho} F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right) + {}^{\zeta}_{\pi-}\mathfrak{J}^{\varrho} F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right)\right] \\
&\leq_p \frac{\zeta}{4}\left[M(\mu,\pi)\left(\frac{2}{\zeta} - 2\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) + \mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right)\right)\right. \\
&\quad \left. + N(\mu,\pi)\left(2\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) - \mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right)\right)\right].
\end{aligned}$$

□

Remark 3.4. If we assign $G(\mathfrak{l}) = [1,1]$ for all $\mathfrak{l} \in [\mu, \pi]$ in Theorem 3.2, then the inequality (3.12) reduces to the right side of the inequality (3.1).

Corollary 3.1. *If we take $\varrho = 1$ in Theorem 3.2, then we have the upcoming inequality for R-L fractional integral operators of interval valued functions*

$$\begin{aligned}
&\frac{2^{\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\zeta}}\left[\mathcal{J}^{\zeta}_{\mu+} F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right) + \mathcal{J}^{\zeta}_{\pi-} F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right)\right] \\
&\leq_p \frac{\zeta+1}{2(\zeta+2)} M(\mu,\pi) + \frac{1}{2(\zeta+2)} N(\mu,\pi).
\end{aligned} \tag{3.21}$$

Remark 3.5. If we assign $G(\mathfrak{l}) = [1,1]$ for all $\mathfrak{l} \in [\mu, \pi]$ in Corollary 3.1, then the inequality (3.21) reduces to the right side of the inequality (3.8).

Remark 3.6. If we pick $\varrho = 1$ and $\zeta = 1$ in Theorem 3.2, then the inequality (3.12) reduces to the inequality

$$\frac{1}{\pi-\mu}\int_{\mu}^{\pi} F(\mathfrak{l})\, G(\mathfrak{l})\, d\mathfrak{l} \leq_p \frac{1}{3} M(\mu,\pi) + \frac{1}{6} N(\mu,\pi) \tag{3.22}$$

which is proved by Khan et al. in [29, Theorem 5].

Corollary 3.2. *If we set* $\underline{F}(\mathfrak{l}) = \overline{F}(\mathfrak{l}) = f(\mathfrak{l})$ *and* $\underline{G}(\mathfrak{l}) = \overline{G}(\mathfrak{l}) = g(\mathfrak{l})$ *for all* $\mathfrak{l} \in [\mu, \pi]$ *in Theorem 3.2, then we have the H-HI for generalized fractional integrals of real-valued functions*

$$\frac{\varrho^{\zeta} 2^{\varrho\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\varrho\zeta}}\left[{}^{\zeta}_{\mu+}\Psi^{\varrho} f\left(\frac{\mu+\pi}{2}\right) g\left(\frac{\mu+\pi}{2}\right) + {}^{\zeta}\Psi^{\varrho}_{\pi-} f\left(\frac{\mu+\pi}{2}\right) g\left(\frac{\mu+\pi}{2}\right)\right] \tag{3.23}$$
$$\leq \frac{\zeta}{4}\left[C(\mu,\pi)\left(\frac{2}{\zeta} - 2\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) + \mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right)\right)\right.$$
$$\left. + D(\mu,\pi)\left(2\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) - \mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right)\right)\right],$$

where

$$C(\mu,\pi) = f(\mu)\, g(\mu) + f(\pi)\, g(\pi)$$

and

$$D(\mu,\pi) = f(\mu)\, g(\pi) + f(\pi)\, g(\mu).$$

Remark 3.7. If we place $g(\mathfrak{l}) = 1$ for all $\mathfrak{l} \in [\mu, \pi]$ in Corollary 3.2, then the inequality (3.23) reduces the right hand side of the inequality (2.4).

Corollary 3.3. *If we take* $\varrho = 1$ *in Corollary 3.2, then we we have the H-HI for R-L fractional integrals of real-valued functions*

$$\frac{2^{\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\zeta}}\left[\mathbb{J}^{\zeta}_{\mu+} f\left(\frac{\mu+\pi}{2}\right) g\left(\frac{\mu+\pi}{2}\right) + \mathbb{J}^{\zeta}_{\pi-} f\left(\frac{\mu+\pi}{2}\right) g\left(\frac{\mu+\pi}{2}\right)\right] \tag{3.24}$$
$$\leq \frac{\zeta+1}{2(\zeta+2)} C(\mu,\pi) + \frac{1}{2(\zeta+2)} D(\mu,\pi).$$

Remark 3.8. If we set $g(\mathfrak{l}) = 1$ for all $\mathfrak{l} \in [\mu, \pi]$ in Corollary 3.3, then the inequality (3.24) reduces to the right hand side of the inequality (2.3).

Remark 3.9. If we select $\varrho = 1$ and $\zeta = 1$ in Corollary 3.2, then the inequality (3.23) reduces to the inequality

$$\frac{1}{\pi-\mu}\int_{\mu}^{\pi} f(\mathfrak{l})\, g(\mathfrak{l})\, d\mathfrak{l} \leq \frac{1}{3} C(\mu,\pi) + \frac{1}{6} D(\mu,\pi) \tag{3.25}$$

which is proved by Pachpatte in [42].

Example 3.2. Let us consider the functions $F, G : [1,3] \to \mathbb{R}^{+}_{\mathcal{I}}$ defined by $F(\mathfrak{l}) = \left[1, \mathfrak{l}^3\right]$ and $G(\mathfrak{l}) = \left[\mathfrak{l}^2, 10\right]$. Then, the functions F and G areLR-convex. By Definition 2.6, we have

$$ {}^{\zeta}_{\mu+}\mathfrak{z}^{\varrho} F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right) \tag{3.26}$$
$$= {}^{\zeta}_{1+}\mathfrak{z}^{\varrho} F(2)\, G(2)$$
$$= \frac{1}{\Gamma(\zeta)}\int_{1}^{2}\left(\frac{1-(\mathfrak{z}-1)^{\varrho}}{\varrho}\right)^{\zeta-1}(\mathfrak{z}-1)^{\varrho-1}\left[1, \mathfrak{z}^3\right]\left[\mathfrak{z}^2, 10\right] d\mathfrak{z}$$
$$= \frac{1}{\varrho^{\zeta-1}\Gamma(\zeta)}\int_{1}^{2}\left(1-(\mathfrak{z}-1)^{\varrho}\right)^{\zeta-1}(\mathfrak{z}-1)^{\varrho-1}\left[\mathfrak{z}^2, 10\mathfrak{z}^3\right] d\mathfrak{z}$$
$$= \frac{1}{\varrho^{\zeta}}\frac{1}{\Gamma(\zeta)}\left[\mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right) + 2\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) + \frac{1}{\zeta},\right.$$

$$10\mathcal{B}\left(\zeta,\frac{3}{\varrho}+1\right)+30\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)+30\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\frac{10}{\zeta}\right]$$

and similarly

$$\begin{aligned}
&{}^{\zeta}_{\pi-}\mathfrak{Z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right) \\
&= {}^{\zeta}_{3-}\mathfrak{Z}^{\varrho}F\left(2\right)G\left(2\right) \\
&= \frac{1}{\varrho^{\zeta-1}}\frac{1}{\Gamma\left(\zeta\right)}\int_2^3\left(1-(3-\mathfrak{z})^{\varrho}\right)^{\zeta-1}(3-\mathfrak{z})^{\varrho-1}\left[\mathfrak{z}^2,10\mathfrak{z}^3\right]d\mathfrak{z} \\
&= \frac{1}{\Gamma\left(\zeta\right)}\frac{1}{\varrho^{\zeta}}\left[\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)-6\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\frac{9}{\zeta},\right. \\
&\left.+10\left(-\mathcal{B}\left(\zeta,\frac{3}{\varrho}+1\right)+9\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)-27\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\frac{27}{\zeta}\right)\right].
\end{aligned}\tag{3.27}$$

By the equalities (3.26) and (3.27), one can calculete the left term of the inequality (3.12) as

$$\begin{aligned}
&\frac{\varrho^{\zeta}2^{\varrho\zeta-1}\Gamma\left(\zeta+1\right)}{(\pi-\mu)^{\varrho\zeta}}\left[{}^{\zeta}_{\mu+}\mathfrak{Z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)+{}^{\zeta}_{\pi-}\mathfrak{Z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)\right] \\
&= \left[\Omega_3(\varrho,\zeta),\Omega_4(\varrho,\zeta)\right],
\end{aligned}$$

where

$$\Omega_3(\varrho,\zeta)=\zeta\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)-2\zeta\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+5$$

and

$$\Omega_4(\varrho,\zeta)=60\zeta\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)-120\zeta\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+140.$$

On the other hand, we have

$$M(\mu,\pi)=N(\mu,\pi)=[10,280].\tag{3.28}$$

Then, the right term of the inequality (3.12) reduce to

$$\begin{aligned}
&\frac{\zeta}{4}\left[M(\mu,\pi)\left(\frac{2}{\zeta}-2\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)\right)\right. \\
&\left.+N(\mu,\pi)\left(2\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)-\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)\right)\right] \\
&= [5,140].
\end{aligned}$$

Therefore, we have the inequality

$$\left[\Omega_3(\varrho,\zeta),\Omega_4(\varrho,\zeta)\right]\leq_p [5,140].\tag{3.29}$$

The accuracy of inequality (3.29) is illustrated in Figure 4 and Figure 5 for $\varrho\in[0,1]$ and $\zeta\in[0,5]$.

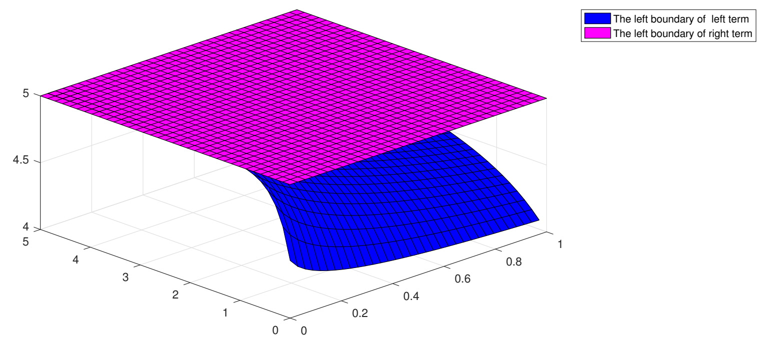

Figure 4. Comparison of left boundaries for Example 3.2.

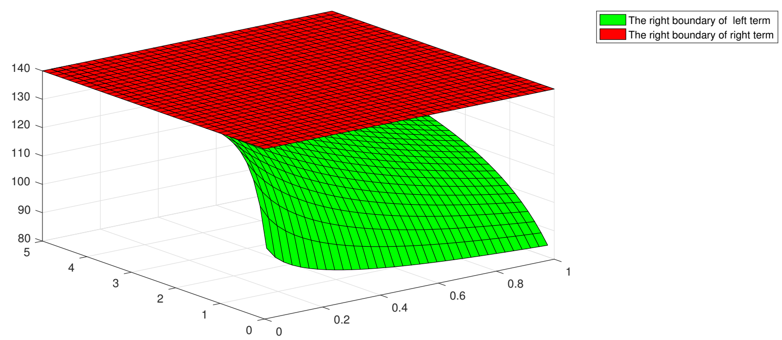

Figure 5. Comparison of left boundaries for Example 3.2.

Theorem 3.3. *Let* $F, G : [\mu, \pi] \to \mathbb{R}_{\mathcal{I}}^{+}$ *be two LR-convex I-VF on* $[\mu, \pi]$ *given by* $F(\mathfrak{l}) = \left[\underline{F}(\mathfrak{l}),\ \overline{F}(\mathfrak{l})\right]$ *and* $G(\mathfrak{l}) = \left[\underline{G}(\mathfrak{l}),\ \overline{G}(\mathfrak{l})\right]$ *for all* $\mathfrak{l} \in [\mu, \pi]$. *We gain the upcoming inequality for generalized fractional integral operators of interval valued functions*

$$\begin{aligned}
&2F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right) \qquad (3.30)\\
\leq_p &\frac{\varrho^{\zeta} 2^{\varrho\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\varrho\zeta}}\left[{}_{\mu+}^{\ \zeta}\mathfrak{J}^{\varrho} F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right) + {}_{\pi-}^{\ \zeta}\mathfrak{J}^{\varrho} F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right)\right]\\
&+\zeta\left[2\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) - \mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right)\right] M(\mu, \pi)\\
&+\zeta\left[\frac{2}{\zeta} - 2\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) + \mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right)\right] N(\mu, \pi)\Bigg]
\end{aligned}$$

for $\zeta > 0$ *and* $\varrho \in (0, 1]$, *where* $M(\mu, \pi)$ *and* $N(\mu, \pi)$ *are defined as in Theorem 3.2.*

Proof. Since the functions F and G are LR-convex I-VFs, then we have

$$\begin{aligned}
F\left(\frac{\mu+\pi}{2}\right) &= F\left(\frac{1}{2}\left(\frac{1+\mathfrak{z}}{2}\mu + \frac{1-\mathfrak{z}}{2}\pi\right) + \frac{1}{2}\left(\frac{1-\mathfrak{z}}{2}\mu + \frac{1+\mathfrak{z}}{2}\pi\right)\right)\\
&\leq_p \frac{1}{2}\left[F\left(\frac{1+\mathfrak{z}}{2}\mu + \frac{1-\mathfrak{z}}{2}\pi\right) + F\left(\frac{1-\mathfrak{z}}{2}\mu + \frac{1+\mathfrak{z}}{2}\pi\right)\right]
\end{aligned}$$

and similarly

$$G\left(\frac{\mu+\pi}{2}\right) \leq_p \frac{1}{2}\left[G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)+G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\right].$$

It is clear that

$$\begin{aligned}
& F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right) \\
\leq_p & \frac{1}{4}\left[F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)+F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\right] \\
& \times\left[G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)+G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\right] \\
= & \frac{1}{4}\left[F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\right. \\
& +F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) \\
& +F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) \\
& \left.+F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\right].
\end{aligned}$$

Since the functions F and G are LR-convex I-VFs, we get

$$\begin{aligned}
& F\left(\frac{\mu+\pi}{2}\right) G\left(\frac{\mu+\pi}{2}\right) \qquad (3.31)\\
\leq_p & \frac{1}{4}\left[F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\right. \\
& +F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) \\
& +F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) \\
& \left.+F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\right] \\
\leq_p & \frac{1}{4}\left[F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\right. \\
& +F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) \\
& +\left(\frac{1+\mathfrak{z}}{2}F(\mu)+\frac{1-\mathfrak{z}}{2}F(\pi)\right)\left(\frac{1-\mathfrak{z}}{2}G(\mu)+\frac{1+\mathfrak{z}}{2}G(\pi)\right) \\
& \left.+\left(\frac{1-\mathfrak{z}}{2}F(\mu)+\frac{1+\mathfrak{z}}{2}F(\pi)\right)\left(\frac{1+\mathfrak{z}}{2}G(\mu)+\frac{1-\mathfrak{z}}{2}G(\pi)\right)\right] \\
= & \frac{1}{4}\left[F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right) G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\right. \\
& +F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right) G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)
\end{aligned}$$

$$+\frac{1-\mathfrak{z}^2}{2}M(\mu,\pi)+\frac{1+\mathfrak{z}^2}{2}N(\mu,\pi)\Bigg].$$

Multiplying both sides of the inequalities (3.31) with $\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}$, and then integrating the resulting inequality over $[0,1]$ with respect to $\mathfrak{z}$, we get

$$\begin{aligned}
&F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)\int_0^1\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\\
&\leq_p\frac{1}{4}\Bigg[\int_0^1 F\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)G\left(\frac{1+\mathfrak{z}}{2}\mu+\frac{1-\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\\
&+\int_0^1 F\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)G\left(\frac{1-\mathfrak{z}}{2}\mu+\frac{1+\mathfrak{z}}{2}\pi\right)\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\\
&+M(\mu,\pi)\int_0^1\frac{1-\mathfrak{z}^2}{2}\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\\
&+N(\mu,\pi)\int_0^1\frac{1+\mathfrak{z}^2}{2}\left(\frac{1-(1-\mathfrak{z})^{\varrho}}{\varrho}\right)^{\zeta-1}(1-\mathfrak{z})^{\varrho-1}\,d\mathfrak{z}\Bigg].
\end{aligned}$$

By the equalities (3.7), (3.18)-(3.20), we have

$$\begin{aligned}
&F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)\frac{1}{\zeta\cdot\varrho^{\varrho}}\\
&\leq_p\frac{1}{4}\Bigg[\left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta}\Gamma(\zeta)\ {}^{\zeta}_{\mu+}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)\\
&+\left(\frac{2}{\pi-\mu}\right)^{\varrho\zeta}\Gamma(\zeta)\ {}^{\zeta}_{\pi-}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)\\
&+M(\mu,\pi)\frac{1}{2\cdot\varrho^{\zeta}}\left[2\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)-\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)\right]\\
&+N(\mu,\pi)\frac{1}{2\cdot\varrho^{\zeta}}\left[\frac{2}{\zeta}-2\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)\right]\Bigg].
\end{aligned}$$

This completes the proof. □

Corollary 3.4. *If we assign $G(\mathfrak{l})=[1,1]$ for all $\mathfrak{l}\in[\mu,\pi]$ in Theorem 3.3, then we have the upcoming inequality*

$$2F\left(\frac{\mu+\pi}{2}\right)\leq_p\frac{\varrho^{\zeta}2^{\varrho\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\varrho\zeta}}\left[{}^{\zeta}_{\mu+}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)+{}^{\zeta}_{\pi-}\mathfrak{z}^{\varrho}F\left(\frac{\mu+\pi}{2}\right)\right]+\frac{F(\mu)+F(\pi)}{2}.$$

Corollary 3.5. *If we take $\varrho=1$ in Theorem 3.3, then we have the next inequality for R-L fractional integral operators of interval valued functions*

$$\begin{aligned}
&2F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)\\
&\leq_p\frac{2^{\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\zeta}}\left[\mathcal{J}^{\zeta}_{\mu+}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)+\mathcal{J}^{\zeta}_{\pi-}F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right)\right]\\
&+\frac{1}{2(\zeta+2)}M(\mu,\pi)+\frac{\zeta+1}{2(\zeta+2)}N(\mu,\pi).
\end{aligned}$$

Corollary 3.6. *If we assign $G(\mathfrak{l}) = [1,1]$ for all $\mathfrak{l} \in [\mu,\pi]$ in Corollary 3.5, then wehave the next inequality*

$$2F\left(\frac{\mu+\pi}{2}\right) \leq_p \frac{2^{\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\zeta}}\left[\mathcal{J}^{\zeta}_{\mu+}F\left(\frac{\mu+\pi}{2}\right)+\mathcal{J}^{\zeta}_{\pi-}F\left(\frac{\mu+\pi}{2}\right)\right]+\frac{F(\mu)+F(\pi)}{2}.$$

Remark 3.10. If we pick $\varrho = 1$ and $\zeta = 1$ in Theorem 3.2, then the inequality (3.12) reduces to the inequality

$$2F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right) \leq_p \frac{1}{\pi-\mu}\int_{\mu}^{\pi} F(\mathfrak{l})\,G(\mathfrak{l})\,d\mathfrak{l}+\frac{1}{6}M(\mu,\pi)+\frac{1}{3}N(\mu,\pi)$$

which is proved by Khan et al. in [29, Theorem 6].

Corollary 3.7. *If we set $\underline{F}(\mathfrak{l}) = \overline{F}(\mathfrak{l}) = f(\mathfrak{l})$ and $\underline{G}(\mathfrak{l}) = \overline{G}(\mathfrak{l}) = g(\mathfrak{l})$ for all $\mathfrak{l} \in [\mu,\pi]$ in Theorem 3.3, then we have the H-HI for generalized fractional integrals of real-valued functions*

$$\begin{aligned}
&2f\left(\frac{\mu+\pi}{2}\right)g\left(\frac{\mu+\pi}{2}\right)\\
&\leq \frac{\varrho^{\zeta}2^{\varrho\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\varrho\zeta}}\left[\,{}^{\zeta}_{\mu+}\Psi^{\varrho} f\left(\frac{\mu+\pi}{2}\right)g\left(\frac{\mu+\pi}{2}\right)+\,{}^{\zeta}\Psi^{\varrho}_{\pi-}f\left(\frac{\mu+\pi}{2}\right)g\left(\frac{\mu+\pi}{2}\right)\right]\\
&+\zeta\left[2\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)-\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)\right]C(\mu,\pi)\\
&+\zeta\left[\frac{2}{\zeta}-2\mathcal{B}\left(\zeta,\frac{1}{\varrho}+1\right)+\mathcal{B}\left(\zeta,\frac{2}{\varrho}+1\right)\right]D(\mu,\pi)\Bigg]
\end{aligned}$$

where $C(\mu,\pi)$ and $D(\mu,\pi)$ are defined by as in Corollary 3.2.

Remark 3.11. If we place $g(\mathfrak{l}) = 1$ for all $\mathfrak{l} \in [\mu,\pi]$ in Corollary 3.7, then we get the next inequality

$$2f\left(\frac{\mu+\pi}{2}\right) \leq \frac{\varrho^{\zeta}2^{\varrho\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\varrho\zeta}}\left[\,{}^{\zeta}_{\mu+}\Psi^{\varrho} f\left(\frac{\mu+\pi}{2}\right)+\,{}^{\zeta}\Psi^{\varrho}_{\pi-}f\left(\frac{\mu+\pi}{2}\right)\right]+\frac{f(\mu)+f(\pi)}{2}.$$

Corollary 3.8. *If we take $\varrho = 1$ in Corollary 3.7, then we we have the H-HI for R-L fractional integrals of real-valued functions*

$$\begin{aligned}
&2f\left(\frac{\mu+\pi}{2}\right)g\left(\frac{\mu+\pi}{2}\right)\\
&\leq \frac{2^{\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\zeta}}\left[\mathbb{J}^{\zeta}_{\mu+}f\left(\frac{\mu+\pi}{2}\right)g\left(\frac{\mu+\pi}{2}\right)+\mathbb{J}^{\zeta}_{\pi-}f\left(\frac{\mu+\pi}{2}\right)g\left(\frac{\mu+\pi}{2}\right)\right]\\
&+\frac{1}{2(\zeta+2)}C(\mu,\pi)+\frac{\zeta+1}{2(\zeta+2)}D(\mu,\pi).
\end{aligned}$$

Remark 3.12. If we take $g(\mathfrak{l}) = 1$ for all $\mathfrak{l} \in [\mu,\pi]$ in Corollary 3.8, we gain the inequality

$$2f\left(\frac{\mu+\pi}{2}\right) \leq \frac{2^{\zeta-1}\Gamma(\zeta+1)}{(\pi-\mu)^{\zeta}}\left[\mathbb{J}^{\zeta}_{\mu+}f\left(\frac{\mu+\pi}{2}\right)+\mathbb{J}^{\zeta}_{\pi-}f\left(\frac{\mu+\pi}{2}\right)\right]+\frac{f(\mu)+f(\pi)}{2}.$$

Remark 3.13. If we set $\varrho = 1$ and $\zeta = 1$ in Corollary 3.7, the inequality (3.23) reduces to the inequality

$$2f\left(\frac{\mu+\pi}{2}\right)g\left(\frac{\mu+\pi}{2}\right) \leq \frac{1}{\pi-\mu}\int_{\mu}^{\pi} f(\mathfrak{l})\, g(\mathfrak{l})\, d\mathfrak{l} + \frac{1}{6}C(\mu,\pi) + \frac{1}{3}D(\mu,\pi)$$

which is proved by Pachpatte in [42].

Example 3.3. Let consider the functions F and G as in Example 3.2. Then we have

$$2F\left(\frac{\mu+\pi}{2}\right)G\left(\frac{\mu+\pi}{2}\right) = 2F(2)\,G(2) = 2\,[1,8]\,[4,10] = [8,160]\,. \tag{3.32}$$

By using the equalities (3.26), (3.27), (3.28) and (3.32) in (3.30), then we have the inequality

$$[8,160] \leq_p [\Omega_3(\varrho,\zeta), \Omega_4(\varrho,\zeta)] \tag{3.33}$$

where

$$\Omega_5(\varrho,\zeta) = \zeta\mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right) - 2\zeta\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) + 10$$

and

$$\Omega_6(\varrho,\zeta) = 60\zeta\mathcal{B}\left(\zeta, \frac{2}{\varrho}+1\right) - 120\zeta\mathcal{B}\left(\zeta, \frac{1}{\varrho}+1\right) + 280.$$

The correctness of inequality (3.33) is demonstrated in Figure 6 and Figure 7 for $\varrho \in [0,1]$ and $\zeta \in [0,5]$.

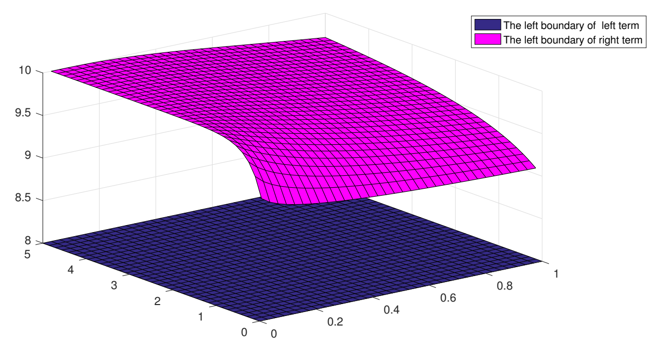

Figure 6. Comparison of left boundaries for Example 3.2.

4. ANN model

In this section, an ANN is utilized to predict the left boundary of the midterm in inequality (3.11) and the left boundary of the left term in inequality (3.29) for fractional Hermite-Hadamard type inequalities with interval-valued LR-convex functions. Computing these boundary terms directly for each (ϱ, ζ) requires repeated numerical evaluation of the generalized fractional integrals and associated special-function terms across many parameter pairs. This is necessary when performing dense parameter sweeps to examine the influence of the fractional orders, verify the inequalities, and generate the numerical figures. Such repeated calculations increase the computational cost and may accumulate numerical error when the grid becomes fine. Therefore,

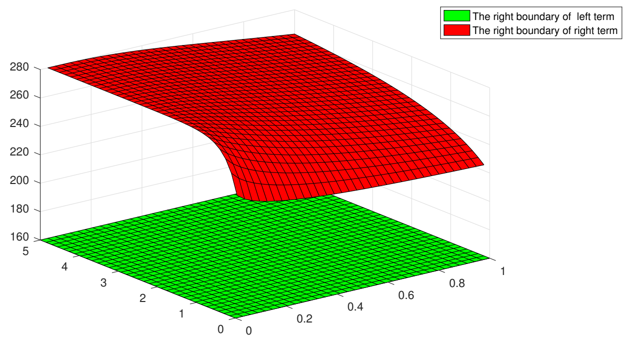

Figure 7. Comparison of right boundaries for Example 3.2.

we employ an ANN as a surrogate model; once trained using accurately computed samples, it provides fast and stable predictions for new (ϱ, ζ) values without repeating the full numerical evaluation. The ANN is designed with a two-layer architecture, consisting of an input layer with two inputs (fractional order parameters ϱ and ζ), a hidden layer with 14 neurons, and an output layer. The hidden layer used the *logsig* (log-sigmoid) activation function, defined as:

$$f(x) = \frac{1}{1+e^{-x}} \tag{4.1}$$

and the output layer employed the *purelin* (linear) activation function, defined as:

$$f(x) = x. \tag{4.2}$$

This architecture was chosen to capture the non-linear relationships between the fractional order parameters and the boundaries of the inequalities. Preprocessing of the input data was done using the *removeconstantrows* and *mapminmax* functions to normalize the inputs, mapping the data to the range $[-1, 1]$, which improves the network's convergence during training. The data was split randomly into three subsets: 70% for training, 15% for validation, and 15% for testing, using a random partitioning method to ensure that the model could generalize to unseen data effectively.

The network was trained using the Levenberg-Marquardt (LM) algorithm [33, 36], which is a combination of the Gauss-Newton method and gradient descent, widely regarded for its efficiency in training feedforward networks. The LM algorithm updates the network's weights by minimizing the error between predicted outputs and actual target values, represented by the Mean Squared Error (MSE) loss function:

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 \tag{4.3}$$

where n is the number of data points, y_i is the actual target value, and $\hat{y}_i$ is the predicted output of the network. The Levenberg-Marquardt update rule for weight adjustments is given by:

$$W_{new} = W_{old} - \left[J^T J + \mu I\right]^{-1} J^T e \tag{4.4}$$

where J is the Jacobian matrix of the network errors, e is the vector of errors, and μ is the damping factor. When μ is large, the update rule behaves like gradient descent, and when μ is small, it behaves like the Gauss-Newton method. The goal is to minimize the MSE by adjusting the weights through multiple iterations, called epochs. In this case, the network converged after 42 epochs, achieving a final MSE of 4.39×10^{-5}, with the gradient reduced to 2.15×10^{-5}. The network used early stopping, where validation checks reached a maximum of 20, ensuring that the network did not overfit the training data.

After training, the performance of the network was evaluated using several metrics. The regression plot, which compares predicted values to actual target values for the training, validation, and testing sets, showed that the predicted outputs closely followed the target values, with a regression coefficient (R^2) close to 1. This coefficient measures the goodness of fit, where:

$$R^2 = 1 - \frac{\sum(y_i - \hat{y}_i)^2}{\sum(y_i - \bar{y})^2}. \tag{4.5}$$

Here, y_i is the actual value, $\hat{y}_i$ is the predicted value, and $\bar{y}$ is the mean of the actual values. A value of $R^2 = 1$ would indicate perfect predictions, and the high R^2 values observed in this study indicate the model's effectiveness. Additionally, the error histogram was plotted to visualize the distribution of prediction errors:

$$\text{Error} = y_i - \hat{y}_i. \tag{4.6}$$

Most of the errors were around zero, and this shows that the majority of the predictions were correct with minimum deviation from the actual values. The error histogram also confirmed that the ANN model had learned the underlying patterns in the data well, with very few large errors. All numerical simulations and graphical plots of the inequalities in this paper were generated using MATLAB. The ANN model, which employed the Levenberg-Marquardt learning algorithm, successfully predicted the left boundary terms of the fractional H-HTIs. The ANN model's success in this regard was validated using various parameters such as Mean Squared Error, regression analysis, and error distribution plot. The final MSE of 4.39×10^{-5} and the low error values in the error histogram, coupled with the high values of R^2, confirm the high predictive power of the ANN model. The ANN model is effective in solving problems related to fractional order inequalities and interval-valued LR-convex functions. Figure 8 shows the schematic representation of the ANN model used in predicting the left boundaries of inequalities (3.11) and (3.29). As seen in this figure, the inputs to this ANN model include fractional orders ϱ and ζ. The inputs then pass through multiple hidden layers before finally passing through the output layer, where the left boundaries of inequalities (3.11) and (3.29) are predicted. More specifically, the left boundary of inequality (3.11), i.e., the midterm of inequality (3.11), and the left boundary of inequality (3.29), i.e., the left term of inequality (3.29), are predicted in this output layer.

Figure 9 shows the error histogram of the ANN model used in predicting inequalities (3.11) and (3.29). As seen in this figure, there are 20 bins in this error histogram. Each bin in this error histogram includes data from the training set, validation set, and test set used in this ANN model. There is also a zero error line in this figure to show how well this ANN model performs in predicting inequalities (3.11) and (3.29).

Figure 10 shows the training set, validation set, and Figure 11 illustrates the training state of the ANN model at epoch 218, showing the gradient (9.1083×10^{-8}), Mu (1×10^{-9}), and validation checks (0), indicating stable training behavior. Figure 12 presents regression plots displaying the ANN model's performance for training, validation, testing, and overall data. The correlation coefficients (R) are nearly 0.99999 across all datasets, indicating excellent predictive accuracy. Figures 13 and 14 illustrate the ANN model's predictive performance for the left boundaries in inequalities (3.11) and (3.29). In Figure 13, the model accurately predicts the left boundary of the midterm in inequality (3.11), with predicted values closely next the actual data points across the entire range. Similarly, Figure 14 depicts the predictions for the left term in inequality (3.29), where the model once again demonstrates excellent accuracy in capturing the underlying patterns. The close alignment between predicted and actual values in both cases showcases the model's strong capability in handling complex fractional inequalities. This consistent and precise performance highlights the ANN model's robustness and efficiency in generating reliable boundary predictions across different scenarios. It is important to note that the ANN results are expected to generalize only within the range of parameter values included in the training dataset. While the network provides accurate predictions inside this domain, its performance outside the trained parameter region is not guaranteed and should be interpreted with caution.

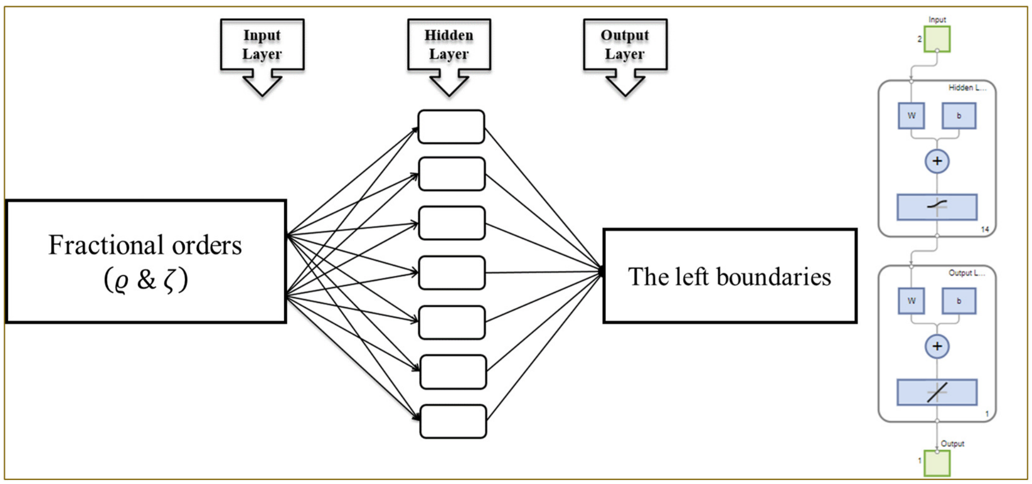

Figure 8. ANN model architecture for predicting inequalities (3.11) and (3.29). The input layer consists of fractional orders ϱ and ζ, and the output predicts the left boundary of mid term of inequality (3.11) and the left boundary of left term of inequality (3.29).

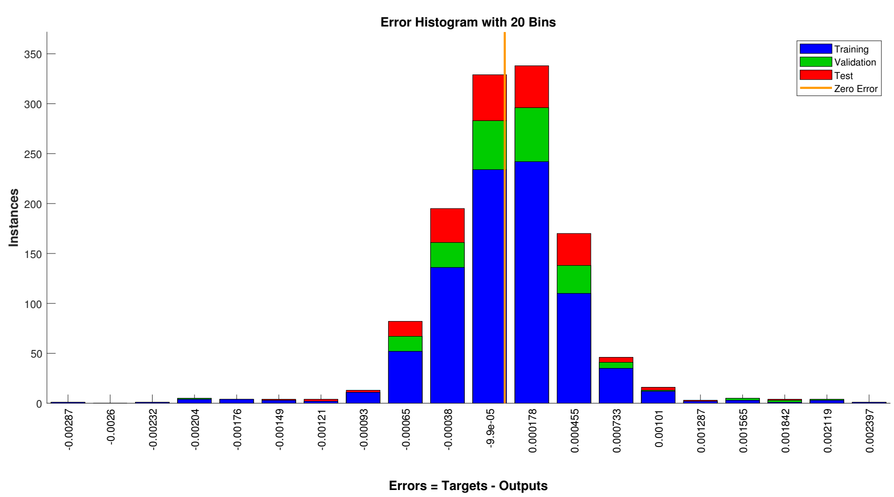

Figure 9. Error histogram of the ANN model predicting inequalities (3.11) and (3.29). The plot illustrates the error distribution across 20 bins, with a zero-error reference line for comparison.

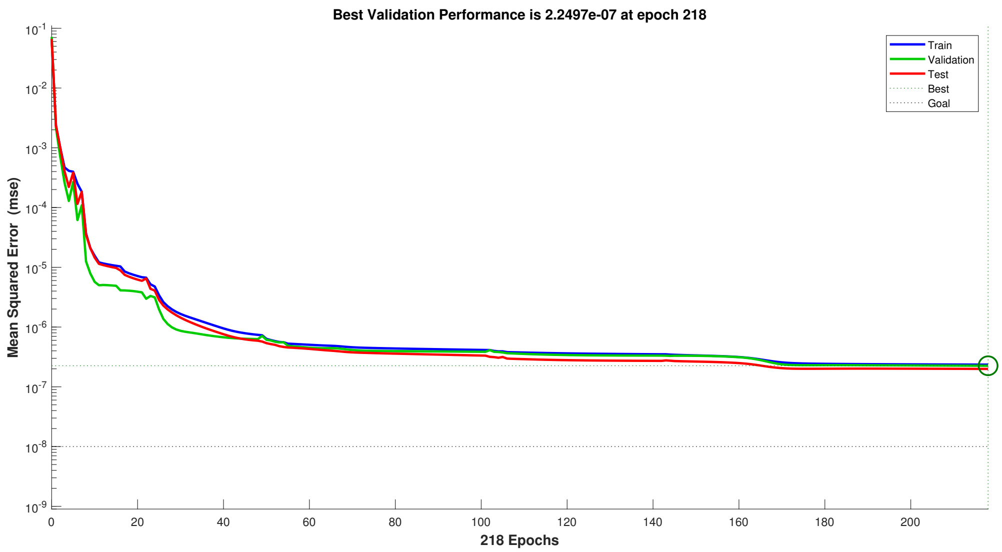

Figure 10. Training, validation, and test performance of the ANN model, showing the mean squared error (MSE) across 336 epochs. The best validation performance of 1.1919×10^{-7} occurs at epoch 316.

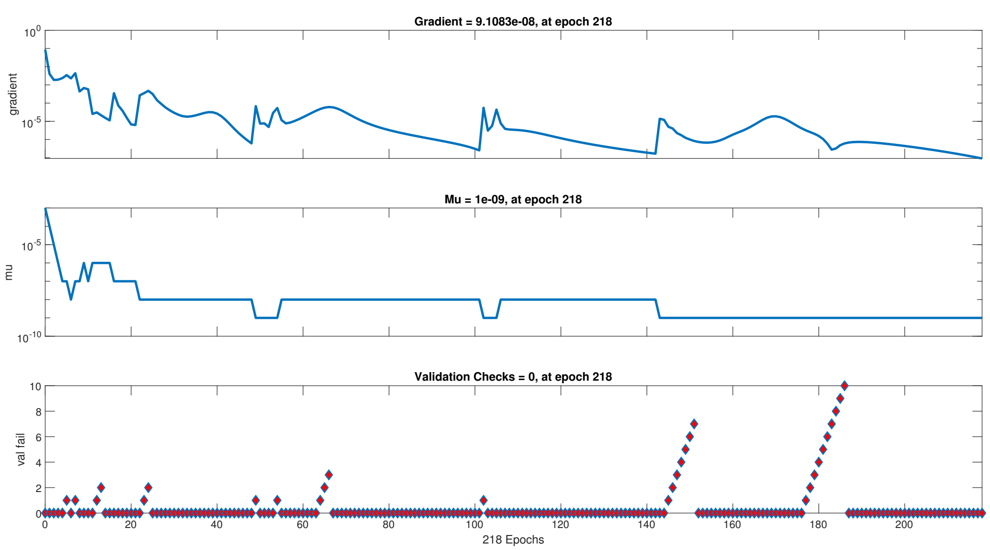

Figure 11. Training state of the ANN model at epoch 218, displaying the gradient (9.1083×10^{-8}), Mu (1×10^{-9}), and validation checks (0).

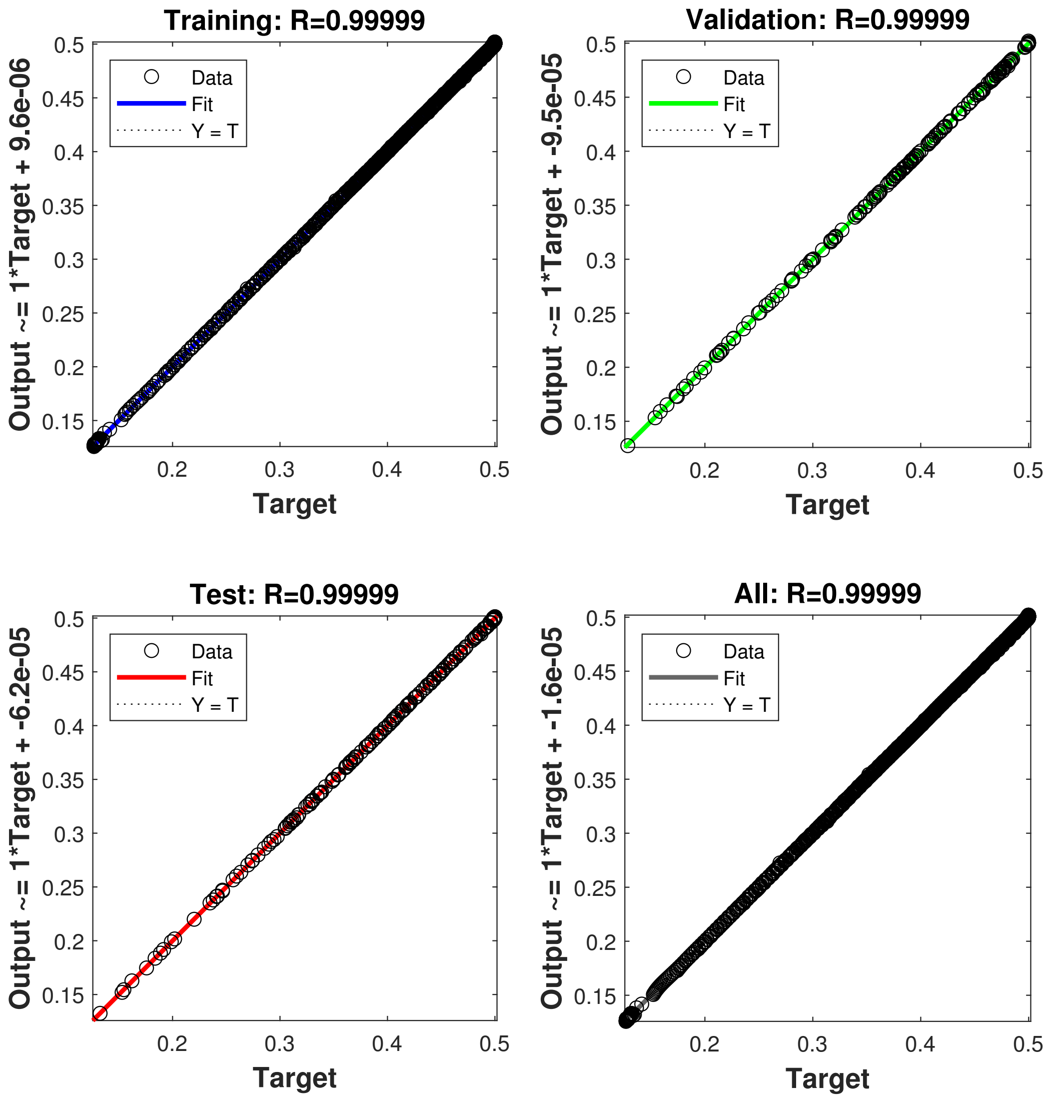

Figure 12. Regression plots showing the ANN model's performance for training, validation, test, and overall data, with correlation coefficients (R) all close to 0.99999.

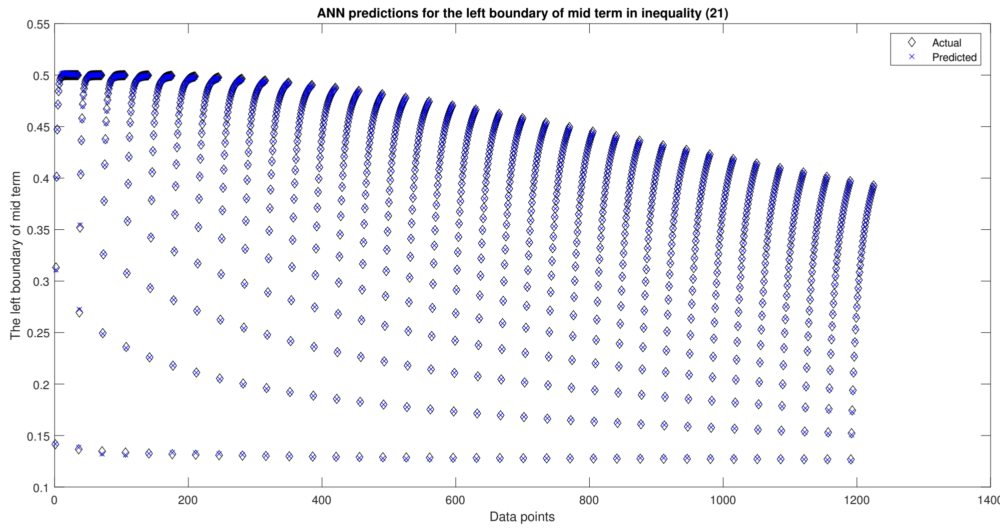

Figure 13. ANN predictions versus actual values for the left boundary of mid term in inequality (3.11), showing data points across the specified range.

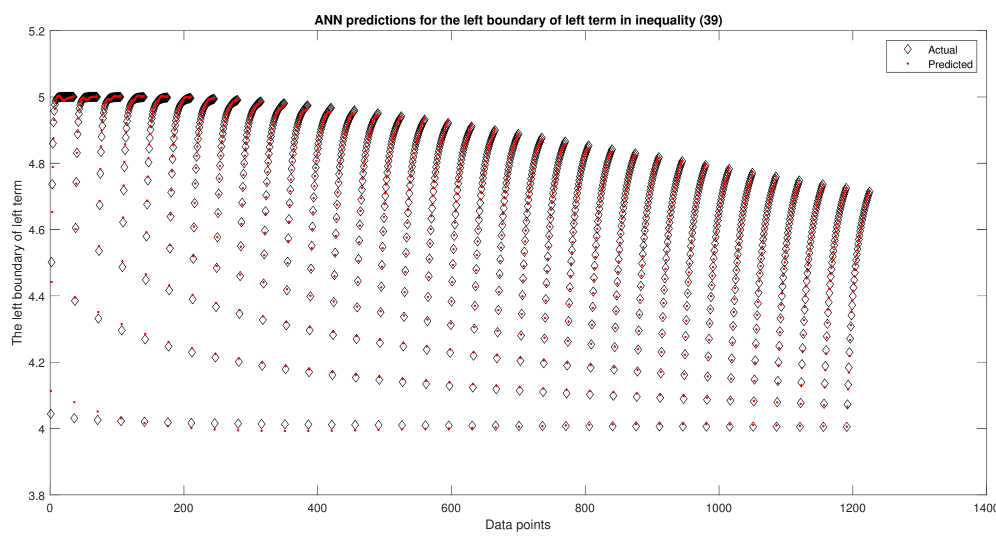

Figure 14. ANN model predictions versus actual values for the left boundary of left term in inequality (3.29), displaying data points across the range.

5. Conclusion

In this paper, we have proposed a novel framework that extends fractional integral inequalities to I-VFs, focusing on LR-convex I-VFs. We unified the classical H-HTIs with the R-L fractional H-HTIs within a single, comprehensive framework. This allowed us to simplify the derivation process by avoiding the need to prove each inequality individually. In addition, we developed new H-HTIs using generalized fractional integrals for LR-convex I-VFs. These inequalities offer significant generalizations of existing inequalities, providing new possibilities for further applications, especially in mathematical analysis, particularly when dealing with uncertainties that often occur in real-world problems. In order to verify the theoretical results, we presented computed examples for the derived inequalities, along with graphical simulations. These examples illustrate the accuracy of the proposed inequalities, their applicability, and their potential applications. Moreover, we introduced ANNs as a computer tool to forecast the boundary values for the derived inequalities. The ANN model successfully predicted the boundary values, providing an additional level of verification for the theoretical results. This is particularly significant, especially when dealing with complex problems that often occur in fractional calculus.

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