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A NOVEL RELAXATION PROJECTION METHOD FOR SOLVING SPLIT EQUALITY PROBLEMS AND ITS APPLICATION IN SIGNAL PROCESSING

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Abstract The split equality problem(SEP) finds significant applications in fields including image reconstruction, language processing, and seismic exploration, all of which often demand real-time processing. Thus, how to effectively enhance the convergence speed of algorithms has long been a core concern for researchers and engineers. To address this challenge, this paper proposes a novel relaxed projection approach for solving the split equality problem in real Hilbert spaces. Specifically, the relaxed projections introduced herein are computed via simple operations at each iteration step. On one hand, the relaxation technique accelerates the algorithm's convergence; on the other hand, it mitigates the computational difficulty associated with general metric projections. Under mild conditions, we analyze the weak convergence of the proposed algorithm. Finally, two numerical experiments, one involving an application to signal recovery and another to image deblurring are conducted to demonstrate the advantages of the proposed method over a recently developed related algorithm.

Keywords Split equality problem, relaxed projection method, signal processing, image reconstruction.

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1. Introduction

This paper assumes that C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The *split feasibility problem* (SFP) is to find a point x satisfying the property:

$$x \in C, Ax \in Q,$$

if such point exists.

The SFP was first introduced by Censor, Elfving [5], which attract many authors' attention due to its applications in signal processing [5] and intensity-modulated radiation therapy [6]. Various algorithms have been invented to solve it see [3, 4, 12, 15, 17, 21, 22, 25–27], etc.

In 2013, Moudafi [13] proposed a new *split equality problem* (SEP): Let H_1, H_2 and H_3 be real Hilbert spaces, $C \subseteq H_1, Q \subseteq H_2$ be two nonempty closed convex sets, and let $A : H_1 \to$

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$H_3, B : H_2 \to H_3$ be two bounded linear operators. Find $x \in C, y \in Q$ satisfying

$$Ax = By. \tag{1.1}$$

When $B = I$, SEP reduces to the well known SFP.

Due to its wide applicability in many fields of applied mathematics, such as signal processing, game theory, image reconstruction, and intensity-modulated radiation therapy, algorithms for solving the SEP continue to receive great attention.

As early as 2014, Moudafi [14] introduced a weakly convergent algorithm for solving the SEP which called alternating CQ algorithm (ACQA).

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases}$$

Here $\{\gamma_k\}$ is a positive nondecreasing sequence, P_C and P_Q are the metric projections onto C and Q, respectively.

In general, calculating the metric projection onto a general closed convex subset is difficult, because it has no closed-form expression. To solve this difficulty, there are various instances of solution algorithms with using relaxed projection methods. It is worth mentioning to very interesting result is Moudafi [13] gave an alternating relaxed alternating CQ-algorithm iterative algorithm (RACQA) for solving the SEP, where C and Q were given as level sets of convex functions

$$C = \{x \in H_1 : c(x) \le 0\} \quad \text{and} \quad Q = \{y \in H_2 : q(y) \le 0\},$$

where c and q are two convex and sub-differentiable functions on H_1 and H_2, respectively. Define two relaxed convex sets C_k and Q_k by

$$C_k = \{x \in H_1 : c(x_k) \le \langle \xi_k, x_k - x\rangle\}$$

and

$$Q_k = \{y \in H_2 : q(y) \le \langle \eta_k, y_k - y\rangle\},$$

where $\xi_k \in \partial c(x_k)$ and $\eta_k \in \partial q(y_k)$. The RACQA is as follows:

$$\begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases}$$

Note that C_k and Q_k are half-spaces, and thus, the corresponding projections are easily calculated. Moudafi [13], proved that the RACQA also converges weakly to a solution of the SEP.

In order to obtain the strong convergence of algorithms for SEP, in the paper [19], the authors used the well-known Tychonov regularization got some algorithms converge strongly to the minimum-norm solution of the SEP. Let Γ denote the solution set of SEP, i.e.,

$$\Gamma = \{(x, y) \in H_1 \times H_2, Ax = By, x \in C, y \in Q\}$$

and assume consistency of SEP so that Γ is closed, convex and nonempty. Let $S = C \times Q$ in $H = H_1 \times H_2$, define $G : H \to H_3$ by $G = [A, -B]$, then the SEP problem can be reformulated as finding $w = (x, y) \in S$ with $Gw = 0$. For each $w \in H$ and $\lambda > 0$, they considered the solution mapping $T : H \longrightarrow H$ of the problem SEP defined in the form:

$$Tw = P_S(I - \lambda G^*G)w. \tag{1.2}$$

In the paper [19], they obtained that $w \in S$ is a solution of the SEP if and only if it is a fixed point of the mapping T. On this basis, they got some algorithms converge strongly to the minimum-norm solution of the SEP. From then on, in order to improve the convergence of the algorithm, various acceleration algorithms have emerged to solve the SEP. All kinds of projection algorithms in [7–9, 13, 14, 16, 18–21, 23, 24] and the references cited therein.

Note that all the algorithms above are one-step methods. Recently, motivated and inspired by an idea for solving a variational inequality, Tian [20] introduced several two-step methods for solving the SEP as follows:

$$\begin{cases} z_k = P_S(w_k - \alpha_k G^*Gw_k), \\ w_{k+1} = P_S(w_k - \alpha_k G^*Gz_k). \end{cases}$$

However, computing two projections onto $C \times Q$, at each iteration k, may affect the efficiency of the extragradient algorithm. To address this difficulty, in the present paper we introduce a new relaxed projection approach for solving the SEP in a real Hilbert space. Our approach approximates the set S in two ways: First with a sequence of half-spaces S_k for the initial projection T, and then with a new half-space H_{w_k} for the subsequent step. Based on this, we introduce a new solution mapping and a relaxed projection algorithm, and analyze its convergence.

This paper is organized as follows. In Section 2, we present some useful definitions, technical lemmas and solution mappings. The new algorithm and its analysis for solving the problem SEP are presented in Section 3. In order to obtain the strong convergence of algorithm for SEP, the Halpern-type relaxed projection algorithm and its analysis for solving the problem SEP are presented in Section 4. In the last section, several numerical simulation experiments are provided to illustrate the efficiency and accuracy of our proposed algorithms.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let S be a nonempty, closed and convex subset of H. We write $w_k \rightharpoonup w$ to indicate that the sequence $\{w_k\}$ converges weakly to w, and $w_k \longrightarrow w$ to indicate that the sequence $\{w_k\}$ converges strongly to w. Given a sequence $\{w_k\}$ in H, $\omega_w(w_k)$ stands for the set of cluster points in the weak topology. Let P_S denote the projection from H onto S, that is,

$$P_S(w) = \min_{x \in S} ||x - w||.$$

Projections are an important tool for our work in this paper. It is well known that P_S is characterized by the following Lemma.

Lemma 2.1. [5] *Given $x \in H$ and $z \in S$. Then $y = P_S x$ if and only if one of the following conditions holds:*

(i) $\langle x - y, z - y \rangle \leq 0$;

(ii) $\langle z - x, y - x \rangle \geq \|y - x\|^2$;

(iii) $\|z - y\|^2 \leq \|z - x\|^2 - \|y - x\|^2$;

(iv) $\langle z - x, z - y \rangle \geq 0$.

Moreover, for $u, v \in H$ and $w \in S$, we have

(i) $\|P_S u - P_S v\| \leq \|u - v\|$;

(ii) $\|P_S u - P_S v\|^2 \leq \langle P_S u - P_S v, u - v \rangle$.

Let $\gamma > 0$, we introduce a new half space as follows:

$$H_w = \{u \in H : \langle w - \lambda G^* G w - T w, u - T w \rangle \leq \gamma \|w - T w\|^2\}. \tag{2.1}$$

By the definition of the projection P_S and (1), we have

$$\langle w - \lambda G^* G w - T w, u - T w \rangle \leq 0, \forall u \in S,$$

and hence

$$\langle w - \lambda G^* G w - T w, u - T w \rangle \leq \gamma \|w - T w\|^2, \forall u \in S.$$

Thus, $S \subseteq H_w$ for all $w \in H$. Note that, H_w is a half-space, and thus, for each $z \in H$, the projection of z onto H_w is presented in an explicit form:

$$P_{H_w} z = \begin{cases} z - \dfrac{\langle w - \lambda G^* G w - T w, z - T w \rangle - \gamma \|w - T w\|^2}{\|w - \lambda G^* G w - T w\|^2} (w - \lambda G^* G w - T w), & \text{if } z \notin H_w, \\ z, & \text{otherwise.} \end{cases}$$

Now we propose a new solution mapping $R : H \longrightarrow H$ for the SEP as follows:

$$R w = P_{H_w}(w - \nu \lambda G^* G(T w)), \tag{2.2}$$

where $\nu > 0$ is a suitable parameter.

The following lemmas present some important properties of the operators T and R that will be needed in the sequel.

Lemma 2.2. *Let parameters ν, λ and γ satisfy the following conditions:*

$$\nu > 0, \lambda \in (0, \min\{\frac{2}{\nu L}, \frac{1}{L}\}), \gamma \in (0, 1 - \lambda L),$$

where L is the spectral radius of the self-adjoint operator $G^* G$. Then, $w^* \in S$ is a solution of the SEP if and only if it is a fixed point of the solution mapping R.

Proof. For each $w \in H$, since $R w \in H_w$, we have

$$\langle w - \lambda G^* G w - T w, R w - T w \rangle \leq \gamma \|w - T w\|^2,$$

it follows that

$$\langle w - T w, R w - T w \rangle - \gamma \|w - T w\|^2 \leq \lambda \langle G^* G w, R w - T w \rangle. \tag{2.3}$$

Using the property of P_{H_w} and the definition of $R w$, we can get

$$\langle R w - w + \nu \lambda G^* G(T w), z - R w \rangle \geq 0, \forall z \in H_w,$$

and then

$$\nu\lambda\langle G^*G(Tw), z - Rw\rangle \geq \langle w - Rw, z - Rw\rangle, \forall z \in H_w. \tag{2.4}$$

According to (2.3), (2.4) we have, for $z \in H_w$,

$$\begin{aligned}
&\langle w - Rw, z - Rw\rangle - \nu\lambda\langle G^*G(Tw), z - Tw\rangle \\
\leq& - \nu\lambda\langle G^*G(Tw), Rw - Tw\rangle \\
=& - \nu\lambda\langle G^*G(Tw) - G^*Gw, Rw - Tw\rangle \\
&- \nu\lambda\langle G^*Gw, Rw - Tw\rangle \\
\leq& \nu\lambda L\|Tw - w\|\|Rw - Tw\| \\
&- \nu(\langle w - Tw, Rw - Tw\rangle - \gamma\|w - Tw\|^2) \\
\leq& \frac{1}{2}\nu\lambda L\|Tw - w\|^2 + \frac{1}{2}\nu\lambda L\|Rw - Tw\|^2 \\
&+ \nu(\langle Tw - w, Rw - Tw\rangle + \nu\gamma\|w - Tw\|^2.
\end{aligned} \tag{2.5}$$

It is easy to see that

$$2\langle w - Rw, z - Rw\rangle = \|w - Rw\|^2 + \|z - Rw\|^2 - \|z - w\|^2,$$

and

$$2\langle Tw - w, Rw - Tw\rangle = \|w - Rw\|^2 - \|Tw - w\|^2 - \|Rw - Tw\|^2.$$

Together with (6) implies that

$$\begin{aligned}
\|Rw - z\|^2 \leq& \|w - z\|^2 - (1 - \nu)\|w - Rw\|^2 \\
&- \nu(1 - 2\gamma - \lambda L)\|w - Tw\|^2 \\
&- \nu(1 - \lambda L)\|Rw - Tw\|^2 \\
&+ 2\nu\lambda\langle G^*G(Tw), z - Tw\rangle, \forall z \in H_w.
\end{aligned} \tag{2.6}$$

Assume that w^* is a fixed point of R. Note that $S \subseteq H_w$ for all w. Substituting $w = w^*$ into (7), we obtain

$$\begin{aligned}
\|w^* - z\|^2 \leq& \|w^* - z\|^2 - (1 - \nu)\|w^* - w^*\|^2 \\
&-\nu(1 - 2\gamma - \lambda L)\|w^* - Tw^*\|^2 \\
&-\nu(1 - \lambda L)\|w^* - Tw^*\|^2 \\
&+2\nu\lambda\langle G^*G(Tw^*), z - Tw^*\rangle, \forall z \in S.
\end{aligned}$$

It follows that

$$\langle G^*G(Tw^*), z - Tw^*\rangle \geq (1 - \gamma - \lambda L)\|w^* - Tw^*\| \geq 0, \forall z \in S. \tag{2.7}$$

And then, $Tw^* \in S$ is a solution of the SEP, by the Lemma 2.5 in [19]. Substituting $z = Tw^*$ into (8), we can get that $Tw^* = w^*$, and then w^* is a solution of the SEP.

On the other hands, assume w^* is a solution of the SEP. Then $Tw^* = w^*$. Substituting $z = w = w^*$ into (8), we have

$$(2 - \nu\lambda L)\|Rw^* - w^*\|^2 \leq 0.$$

It yields that $Rw^* = w^*$. This implies the proof. □

The following three lemmas are two important properties of sequences and are very useful when proving the convergence of sequences.

Lemma 2.3. [1, Lemma 2.39] *Let S be a nonempty subset of H and $\{w_k\} \subseteq H$ such that the following two assumptions hold:*

(i) For all $w \in S$, the limit $\lim_{k\to\infty} \|w_k - w\|$ exists.

(ii) Every sequential weak cluster point of the sequence $\{w_k\}$ is in S. Then, the sequence $\{w_k\}$ converges weakly to a point in S.

Lemma 2.4. [2] *Let S be a nonempty closed convex subset of a real Hilbert space H, and the sequence $\{w_k\}$ be Fejér-monotone with respect to S, i.e., for every $w^* \in S$,*

$$\|w_{k+1} - w^*\| \leq \|w_k - w^*\|, \ \forall k \geq 0.$$

Then the sequence $\{P_S(w_k)\}$ converges strongly to some $w \in S$.

Lemma 2.5. [11] *Assume that ρ_k is a sequence of nonnegative real numbers such that for all k*

$$\rho_{k+1} \leq (1-\alpha_k)\rho_k + \alpha_k\delta_k, \quad \text{and} \quad \rho_{k+1} \leq \rho_k - \eta_k + \tau_k,$$

where $\{\alpha_k\}$ is a sequence in $(0,1)$, $\{\eta_k\}$ is a sequence of nonnegative real numbers, $\{\delta_k\}$ and $\{\tau_k\}$ are two real sequences such that

(i) $\sum_{k\geq 0} \alpha_k = \infty$;

(ii) $\lim_{k\to\infty} \tau_k = 0$;

(iii) $\lim_{k\to\infty} \eta_{k_n} = 0$ implies that $\limsup_{k\to\infty} \delta_{k_n} \leq 0$ for any subsequence $\{k_n\}$ of $\{k\}$.

Then $\lim_{k\to\infty} \rho_k = 0$.

3. Relaxed projection algorithm for SEP and its convergence

In this section, we propose a new relaxed projection algorithm and we obtain the desired convergence for the algorithm. For the SEP, we assume that the closed convex sets C and Q are given by level sets of convex functions

$$C = \{x \in H_1 : c(x) \leq 0\} \ \text{ and } \ Q = \{y \in H_2 : q(y) \leq 0\},$$

where c and q are two convex and sub-differentiable functions on H_1 and H_2, respectively. Define two sets relaxed convex sets C_k and Q_k by

$$C_k = \{x \in H_1 : c(x_k) \leq \langle \xi_k, x_k - x\rangle\} \ \text{ and } \ Q_k = \{y \in H_2 : q(y) \leq \langle \eta_k, y_k - y\rangle\},$$

where $\xi_k \in \partial c(x_k)$ and $\eta_k \in \partial q(y_k)$. It is clearly that $C \subseteq C_k$ and $Q \subseteq Q_k$ for each k. And let $S_k = C_k \times Q_k$, then $S \subseteq S_k$.

First, we impose the following conditions on the parameter sequences for every integer $k \geq 0$:

$$\begin{cases} \nu \in (0, \min\{1, \dfrac{1}{L}\}), \ a \in (0, 1-\nu L), \\ b > 0, \ \lambda_k \in (b, \min\{\dfrac{1}{L}, \sqrt{\dfrac{\nu}{2L}}\}), \ c = \lim\limits_{k\to\infty} \lambda_k, \\ \gamma_k \in (0, \min\{\dfrac{1-\lambda_k^2 L^2}{2}, \ \dfrac{1-\nu L - a}{2}\}). \end{cases}$$

Algorithm 1 Relaxed Projection Algorithm.

Step 0. Choose starting points $w_0 \in H$, $k=0$, $\nu>0$, two positive sequences λ_k and γ_k.
Step 1. Compute the projection

$$u_k = Tw_k = P_{S_k}(w_k - \lambda_k G^*Gw_k).$$

If $u_k = w_k$ then stop. Otherwise, go to Step 2.
Step 2. Compute
$v_k = w_k - \nu\lambda_k G^*G(u_k)$ and the next iterate

$$w_{k+1} = \begin{cases} v_k - \dfrac{d_k}{\|w_k - \lambda_k G^*Gw_k - u_k\|^2}(w_k - \lambda_k G^*Gw_k - u_k), & \text{if } d_k > 0, \\ v_k, & \text{otherwise}, \end{cases}$$

where $d_k = \langle w_k - \lambda_k G^*Gw_k - u_k, v_k - u_k\rangle - \gamma_k\|u_k - w_k\|^2$.
Step 3. Set $k := k+1$ and return to **Step 1**.

Remark 3.1. By the Lemma 2.5 in [19], w^* is a solution of the SEP if and only if it is a fixed point of the mapping T defined by (2). Therefore, if $u_k = w_k$ in Algorithm 1 then $w_k = Tw_k \in S_k$, it follows that $w_k \in S$. In the meantime, $w_k = u_k = P_{S_k}(w_k - \lambda_k G^*Gw_k)$, then w_k is a solution of the SEP. The stopping criterion in Step 1 is valid.

The next lemma is crucial for the proof of our convergent theorem.

Lemma 3.1. *Let $\{w_k\}$ and $\{u_k\}$ be the two sequences generated by Algorithm 1 and let $w^* \in \Gamma$. Then, the following claim holds*

$$\begin{aligned}\|w_{k+1} - w^*\|^2 \le \|w_k - w^*\|^2 - \nu(1 - \nu L - 2\gamma_k)\|w_k - u_k\|^2 \\ -(\nu - \lambda_k^2 L)\|w_{k+1} - u_k\|^2 - (1-\nu)\|w_{k+1} - w_k\|^2.\end{aligned}$$

Proof. According to definition of T, H_w and R in (1.2), (2.1) and (2.2), it is clearly that $u_k = Tw_k$, $w_{k+1} = P_{H_{w_k}}(w_k - \nu\lambda_k G^*G(u_k))$, where

$$H_{w_k} = \{u \in H : \langle w_k - \lambda_k G^*Gw_k - Tw_k, u - Tw_k\rangle \le \gamma_k\|w_k - Tw_k\|^2\}.$$

So $w_{k+1} = Rw_k$.

Note $w^* \in \Gamma$, $Tw_k \in S \subseteq S_k$, we have $\langle G^*Gw^*, Tw_k - w^*\rangle \ge 0$, by the Lemma 2.5 in [19]. Since G^*G is monotone, it follows $\langle G^*GTw_k, Tw_k - w^*\rangle \ge 0$.

And then, using Lemma 2.1,

$$\begin{aligned}\|w_{k+1} - w^*\|^2 &= \|Rw_k - w^*\|^2 \\ &= \|P_{H_{w_k}}(w_k - \nu\lambda_k G^*G(u_k)) - P_{H_{w_k}}w^*\|^2 \\ &\le \|w_k - \nu\lambda_k G^*G(u_k) - w^*\|^2 \\ &\quad - \|w_k - \nu\lambda_k G^*G(u_k) - Rw_k\|^2 \\ &= \|w_k - w^*\|^2 - 2\nu\lambda_k\langle G^*Gu_k, Rw_k - w^*\rangle \\ &\quad - \|w_k - Rw_k\|^2 \\ &= \|w_k - w^*\|^2 - 2\nu\lambda_k\langle G^*Gu_k, u_k - w^*\rangle \\ &\quad - 2\nu\lambda_k\langle G^*Gu_k, Rw_k - u_k\rangle - \|w_k - Rw_k\|^2 \\ &= \|w_k - w^*\|^2 - 2\nu\lambda_k\langle G^*GTw_k, Tw_k - w^*\rangle\end{aligned}$$

$$
\begin{aligned}
&- 2\nu\lambda_k\langle G^*GTw_k, Rw_k - Tw_k\rangle - \|w_k - Rw_k\|^2\\
\le\ & \|w_k - w^*\|^2 - 2\nu\lambda_k\langle G^*GTw_k, Rw_k - Tw_k\rangle\\
&- \|w_k - Rw_k\|^2\\
=\ & \|w_k - w^*\|^2\\
&+ 2\nu\langle w_k - Tw_k - \lambda_k G^*GTw_k, Rw_k - Tw_k\rangle\\
&+ 2\nu\langle Tw_k - w_k, Rw_k - Tw_k\rangle - \|w_k - Rw_k\|^2\\
=\ & \|w_k - w^*\|^2\\
&+ 2\nu\langle w_k - Tw_k - \lambda_k G^*Gw_k, Rw_k - Tw_k\rangle\\
&- (1-\nu)\|w_k - Rw_k\|^2\\
&+ 2\nu\lambda_k\langle G^*Gw_k - G^*GTw_k, Rw_k - Tw_k\rangle\\
&- \nu\|w_k - Tw_k\|^2 - \nu\|Tw_k - Rw_k\|^2.
\end{aligned}\tag{3.1}
$$

Since $Rw_k \in H_{w_k}$, it follows

$$
\langle w_k - \lambda_k G^*Gw_k - Tw_k, Rw_k - Tw_k\rangle \le \gamma_k\|w_k - Tw_k\|^2. \tag{3.2}
$$

Note that

$$
\begin{aligned}
&2\nu\lambda_k\langle G^*Gw_k - G^*GTw_k, Rw_k - Tw_k\rangle\\
\le& 2\nu\lambda_k\|G^*Gw_k - G^*GTw_k\|\|Rw_k - Tw_k\|\\
\le& 2\nu\lambda_k L\|w_k - Tw_k\|\|Rw_k - Tw_k\|\\
\le& L(\nu^2\|w_k - Tw_k\|^2 + \lambda_k^2\|Rw_k - Tw_k\|^2).
\end{aligned}\tag{3.3}
$$

Combining with (3.1), (3.2) and (3.3), we obtain that

$$
\begin{aligned}
\|w_{k+1} - w^*\|^2 \le\ & \|w_k - w^*\|^2 + 2\nu\langle w_k - Tw_k - \lambda_k G^*Gw_k, Rw_k - Tw_k\rangle\\
&-(1-\nu)\|w_k - Rw_k\|^2 - \nu\|w_k - Tw_k\|^2 - \nu\|Tw_k - Rw_k\|^2\\
&+2\nu\lambda_k\langle G^*Gw_k - G^*GTw_k, Rw_k - Tw_k\rangle\\
\le\ & \|w_k - w^*\|^2 + 2\nu\gamma_k\|w_k - Tw_k\|^2\\
&-(1-\nu)\|Rw_k - w_k\|^2 + \nu^2 L\|w_k - Tw_k\|^2\\
&+\lambda_k^2 L\|Rw_k - Tw_k\|^2 - \nu\|w_k - Tw_k\|^2 - \nu\|Tw_k - Rw_k\|^2\\
=\ & \|w_k - w^*\|^2 - \nu(1 - \nu L - 2\gamma_k)\|w_k - Tw_k\|^2\\
&-(\nu - \lambda_k^2 L)\|Rw_k - Tw_k\|^2 - (1-\nu)\|Rw_k - w_k\|^2\\
=\ & \|w_k - w^*\|^2 - \nu(1 - \nu L - 2\gamma_k)\|w_k - u_k\|^2\\
&-(\nu - \lambda_k^2 L)\|w_{k+1} - u_k\|^2 - (1-\nu)\|w_{k+1} - w_k\|^2.
\end{aligned}
$$

□

Theorem 3.1. *The two iteration sequences $\{w_k\}$ and $\{u_k\}$ generated by Algorithm 1 converge weakly to the same solution point $w^* = \lim P_S(w_k)$.*

Proof. Choose $w^* \in \Gamma$. By the assumptions of the parameters, we have

$$
\nu(1 - \nu L - 2\gamma_k) > a\nu,\ \nu - \lambda_k^2 L > 0,\ 1 - \nu > 0,\ \forall k \ge 0.
$$

Using Lemma 3.1,

$$
\|w_{k+1} - w^*\|^2 \le \|w_k - w^*\|^2 - \nu(1 - \nu L - 2\gamma_k)\|w_k - u_k\|^2
$$

$$
\begin{aligned}
&-(\nu-\lambda_k^2L)\|w_{k+1}-u_k\|^2-(1-\nu)\|w_{k+1}-w_k\|^2\\
&\le \|w_k-w^*\|^2-a\nu\|w_k-u_k\|^2,\ \forall k\ge 0.
\end{aligned}
$$

It follows that

$$
a\nu\|w_k-u_k\|^2\le \|w_k-w^*\|^2,\ \forall k\ge 0.
$$

And consequently,

$$
\begin{aligned}
a\nu\|w_{k+1}-u_{k+1}\|^2&\le \|w_{k+1}-w^*\|^2\\
&\le \|w_k-w^*\|^2-a\nu\|w_k-u_k\|^2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
a\nu(\|w_{k+1}-u_{k+1}\|^2+\|w_k-u_k\|^2)&\le \|w_k-w^*\|^2\\
&\le \|w_{k-1}-w^*\|^2-a\nu\|w_{k-1}-u_{k-1}\|^2.
\end{aligned}
$$

It follows that

$$
a\nu\sum_{k=0}^{n}\|w_k-u_k\|^2\le \|w_0-w^*\|^2,\ \forall n\ge 0.
$$

It implies that $\lim_{k\to\infty}\|w_k-u_k\|=0$.

On the other hand, by Lemma 3.1, $\|w_{k+1}-w^*\|^2\le\|w_k-w^*\|^2,\forall k\ge 0$. That is to say $\{w_k\}$ is Fejér-monotone with respect to Γ. And then $\lim_{k\to\infty}\|w_k-w^*\|$ exists. So, the sequences $\{w_k\}$ and $\{u_k\}$ are bounded. Hence, without loss of generality, we may assume that there exists subsequences $\{w_{k_n}\}$ and $\{u_{k_n}\}$ such that $w_{k_n}\rightharpoonup\bar{w}$ and $u_{k_n}\rightharpoonup\bar{w}$. Note that $u_{k_n}=P_S(w_{k_n}-\lambda_{k_n}G^*G)w_{k_n}$, and using the definition of P_S and u_{k_n}, we can get that for each $u\in S$,

$$
\begin{aligned}
0&\le\langle w_{k_n}-\lambda_{k_n}G^*Gw_{k_n},u_{k_n}-u\rangle\\
&=\langle w_{k_n}-u_{k_n},u_{k_n}-u\rangle-\lambda_{k_n}\langle G^*Gw_{k_n},u_{k_n}-u\rangle\\
&=\langle w_{k_n}-u_{k_n},u_{k_n}-u\rangle-\lambda_{k_n}\langle G^*Gw_{k_n},u_{k_n}-w_{k_n}\rangle\\
&\quad-\lambda_{k_n}\langle G^*Gw_{k_n},w_{k_n}-u\rangle.
\end{aligned}
$$

Then,

$$
\lambda_{k_n}\langle G^*Gw_{k_n},w_{k_n}-u\rangle\le\langle w_{k_n}-u_{k_n},u_{k_n}-u\rangle-\lambda_{k_n}\langle G^*Gw_{k_n},u_{k_n}-w_{k_n}\rangle.
$$

By the boundedness of w_k and $\lambda_k\to c$ and $\|w_k-u_k\|\to 0$, we have

$$
\begin{aligned}
c\lim_{n\to\infty}\langle G^*Gw_{k_n},w_{k_n}-u\rangle&\le\lim_{n\to\infty}[\langle w_{k_n}-u_{k_n},u_{k_n}-u\rangle\\
&\quad-\lambda_{k_n}\langle G^*Gw_{k_n},u_{k_n}-w_{k_n}\rangle]\\
&\le\lim_{n\to\infty}[\|w_{k_n}-u_{k_n}\|\|u_{k_n}-u\|\\
&\quad+\lambda_{k_n}\|G^*Gw_{k_n}\|\|u_{k_n}-w_{k_n}\rangle\|]\\
&=0,\quad\forall u\in S.
\end{aligned}
\tag{3.4}
$$

Furthermore,

$$
\langle G^*Gw_{k_n},w_{k_n}-u\rangle=\langle G^*Gw_{k_n}-G^*G\bar{w},w_{k_n}-u\rangle+\langle G^*G\bar{w},w_{k_n}-u\rangle
$$

$$\begin{aligned}&= \langle G^*G(w_{k_n} - \bar{w}), w_{k_n} - \bar{w}\rangle + \langle G^*G\bar{w}, w_{k_n} - u\rangle \\ &\quad + \langle G^*G(w_{k_n} - \bar{w}), \bar{w} - u\rangle \\ &\geq \langle G^*G\bar{w}, w_{k_n} - u\rangle + \langle G^*G(w_{k_n} - \bar{w}), \bar{w} - u\rangle.\end{aligned}$$

Since $w_k \rightharpoonup \bar{w}$, combining with (3.3), we can get that $\langle G^*G\bar{w}, \bar{w} - u\rangle \leq 0, \forall u \in S$. By the Lemma 2.5 in [19], we have $\bar{w} \in \Gamma$. Therefore, it follows from Lemma 2.3 that the sequence w_k converges weakly to $\bar{w}$. Since the sequence w_k is Fejér-monotone under conditions using Lemma 2.4, we see that $P_\Gamma(w_k)$ converges strongly to some $w^* \in \Gamma$. Then, we also have

$$\langle w_k - P_\Gamma w_k, w - P_\Gamma w_k\rangle \leq 0, \forall w \in \Gamma.$$

Taking the limit as $k \to \infty$ and replacing $w = \bar{w}$, we can get

$$\langle \bar{w} - w^*, \bar{w} - w^*\rangle \leq 0.$$

It follows that $\{w_k\}$ converges weakly to $\bar{w} = w^*$. □

4. Halpern-type relaxed projection algorithm

In this section we aim to prove the strong convergence of a Halpern-type algorithm for the SEP. It is well known that Halpern's algorithm [10] has a strong convergence for finding a fixed point of a nonexpansive mapping. Let S be a closed convex subset of a Hilbert space H and let $T : S \longrightarrow S$ be a nonexpansive mapping with fixed points. Let $w \in S$ be fixed. Recall that Halpern's algorithm generates a sequence w_n in S via the recursions

$$w_{k+1} = \alpha_k w + (1 - \alpha_k)Tw_k,$$

where $\{\alpha_k\}$ is a sequence in $[0, 1]$ and satisfies the assumptions:

(i) $\alpha_n \longrightarrow 0$ and $\sum_{k\geq 0} \alpha_k = \infty$;

(ii) either $\sum_{k\geq 0} |\alpha_{k+1} - \alpha_k| < \infty$ or $\frac{\alpha_{k+1}}{\alpha_k} \longrightarrow 1$.

Then the sequence w_n generated by Halpern's algorithm converges in norm to the fixed point of T which is closest to w from all fixed points of T.

We now adapt Halpern's algorithm to treat the SEP. The algorithm introduced below is referred to as a Halpern-type algorithm.

Theorem 4.1. *The iteration sequence $\{w_k\}$ generated by Algorithm 2 converges strongly to the solution point $w^* = P_\Gamma w$.*

Proof. Let $w^* = P_\Gamma w$. Without loss of generality, we assume that the sequence $\{w_k\}$ is infinite, that is, Algorithm 2 does not terminate in a finite number of iterations. By repeating the proof of Lemma 3.1, we have

$$\begin{aligned}\|z_k - w^*\|^2 \leq &\|w_k - w^*\|^2 - \nu(1 - \nu L - 2\gamma_k)\|w_k - u_k\|^2 \\ &- (\nu - \lambda_k^2 L)\|z_k - u_k\|^2 - (1 - \nu)\|z_k - w_k\|^2 \\ \leq &\|w_k - w^*\|^2.\end{aligned} \tag{4.1}$$

Hence, using the convexity of the square of the norm, we obtain

$$\|w_{k+1} - w^*\|^2 = \|\alpha_k(w - w^*) + (1 - \alpha_k)(z_k - w^*)\|^2$$

Algorithm 2 Halpern-type relaxed Projection Algorithm.

Step 0. Choose starting points $w_0 \in H$ and $w \in S$, $k = 0$, $\nu > 0$, two positive sequences λ_k and γ_k.
Step 1. Compute the projection
$$u_k = Tw_k = P_{S_k}(w_k - \lambda_k G^* G w_k).$$
If $u_k = w_k$ then stop. Otherwise, go to Step 2.
Step 2. Compute
$v_k = w_k - \nu\lambda_k G^* G(u_k)$ and the next iterate
$$z_k = \begin{cases} v_k - \dfrac{d_k}{\|w_k - \lambda_k G^* G w_k - u_k\|^2}(w_k - \lambda_k G^* G w_k - u_k), & \text{if } d_k > 0, \\ v_k, & \text{otherwise,} \end{cases}$$
where $d_k = \langle w_k - \lambda_k G^* G w_k - u_k, v_k - u_k\rangle - \gamma_k \|u_k - w_k\|^2$.
Step 3. Compute
$$w_{k+1} = \alpha_k w + (1 - \alpha_k) z_k.$$

Step 4. Set $k := k + 1$ and return to **Step 1**.

$$\begin{aligned} &\le \alpha_k \|w - w^*\|^2 + (1-\alpha_k)\|(z_k - w^*)\|^2 \\ &\le \alpha_k \|w - w^*\|^2 + (1-\alpha_k)\|(w_k - w^*)\|^2 \\ &\le \max\{\|w - w^*\|^2, \|(w_k - w^*)\|^2\}. \end{aligned}$$

It follows that
$$\|w_{k+1} - w^*\|^2 \le \max\{\|w - w^*\|^2, \|(w_0 - w^*)\|^2\},$$
which implies that the sequence $\{w_k\}$ is bounded.

Furthermore,
$$\begin{aligned} \|w_{k+1} - w^*\|^2 &= \|\alpha_k(w - w^*) + (1-\alpha_k)(z_k - w^*)\|^2 \\ &= \alpha_k^2\|w - w^*\|^2 + (1-\alpha_k)^2\|(z_k - w^*)\|^2 \\ &\quad + 2\alpha_k\langle w - w^*, (1-\alpha_k)(z_k - w^*)\rangle \\ &= \alpha_k^2\|w - w^*\|^2 + (1-\alpha_k)^2\|(z_k - w^*)\|^2 \\ &\quad + 2\alpha_k\langle w - w^*, \alpha_k(w - w^*) \\ &\quad + (1-\alpha_k)(z_k - w^*)\rangle - 2\alpha_k^2\|w - w^*\|^2 \\ &\le (1-\alpha_k)^2\|(z_k - w^*)\|^2 + 2\alpha_k\langle w - w^*, w_{k+1} - w^*\rangle \\ &\le (1-\alpha_k)\|(z_k - w^*)\|^2 + 2\alpha_k\langle w - w^*, w_{k+1} - w^*\rangle. \end{aligned} \tag{4.2}$$

Combining with (13), we obtain that
$$\begin{aligned} \|w_{k+1} - w^*\|^2 \le &(1-\alpha_k)\{\|w_k - w^*\|^2 - \nu(1 - \nu L - 2\gamma_k)\|w_k - u_k\|^2 \\ &- (\nu - \lambda_k^2 L)\|z_k - u_k\|^2 - (1-\nu)\|z_k - w_k\|^2\} \\ &+ 2\alpha_k\langle w - w^*, w_{k+1} - w^*\rangle. \end{aligned} \tag{4.3}$$

Since $\alpha_k \to 0$, and $\nu(1 - \nu L - 2\gamma_k) > a\nu, \nu - \lambda_k^2 L > \frac{\nu}{2}, 1 - \nu > 0, \forall k \ge 0$. Without loss of generality, we may assume that there exists $\varepsilon > 0$, such that
$$(1-\alpha_k)\min\{\nu(1 - \nu L - 2\gamma_k), \nu - \lambda_k^2 L, 1 - \nu\} > \varepsilon.$$

It follows from (15) that

$$\begin{aligned} &\|w_{k+1} - w^*\|^2 - \|w_k - w^*\|^2 + \alpha_k\|w_k - w^*\|^2 \\ &+ \varepsilon(\|w_k - u_k\|^2 + \|z_k - u_k\|^2 + \|z_k - w_k\|^2) \\ \leq &2\alpha_k\langle w - w^*, w_{k+1} - w^*\rangle. \end{aligned} \tag{4.4}$$

And we have the following two inequalities

$$\|w_{k+1} - w^*\|^2 \leq (1 - \alpha_k)\|(w_k - w^*)\|^2 + \alpha_k\delta_k,$$

and

$$\|w_{k+1} - w^*\|^2 \leq \|(w_k - w^*)\|^2 - \eta_k + \alpha_k M,$$

where $2\|w - w^*\|\|w_{k+1} - w^*\| \leq M$,

$$\eta_k = \varepsilon(\|w_k - u_k\|^2 + \|z_k - u_k\|^2 + \|z_k - w_k\|^2) \text{ and } \delta_k = 2\langle w - w^*, w_{k+1} - w^*\rangle.$$

Set $\rho_k = \|w_k - w^*\|^2$, according to Lemma 2.5, it remains to prove that for any subsequence $\{k_n\}$ of $\{k\}$,

$$\eta_{k_n} \to 0 \Longrightarrow \limsup_{n\to\infty} \delta_{k_n} \leq 0.$$

Assume that $\eta_{k_n} \to 0$, we have $\|w_{k_n} - u_{k_n}\| \to 0$,$\|z_{k_n} - u_{k_n}\| \to 0$ and $\|z_{k_n} - w_{k_n}\| \to 0$. Then, as in the proof of Theorem 3.1, we can obtain that the set of the weak limit points of the sequence $\{w_{k_n}\}$ is in Γ.

Moreover,

$$\|w_{k_n+1} - w_{k_n}\| \leq \alpha_k\|w - w_{k_n}\| + (1 - \alpha_k)\|w_{k_n} - z_{k_n}\| \longrightarrow 0.$$

And then, recalling $w^* = P_\Gamma w$, we can deduce that

$$\begin{aligned} \limsup_{n\to\infty} \delta_{k_n} &= \lim_{n\to\infty} 2\langle w - w^*, w_{k_n+1} - w^*\rangle \\ &= \lim_{n\to\infty} 2\langle w - w^*, w_{k_n} - w^*\rangle \\ &\leq \max_{u\in\omega_w(w_{k_n})} \langle w - P_\Gamma w, u - P_\Gamma w\rangle \\ &\leq 0. \end{aligned}$$

Applying finally Lemma 2.5, we can conclude that $\|w_k - w^*\| \longrightarrow 0$. □

5. Numerical experiments

In this section, we provide some numerical results to illustrate the effectiveness of Relaxed Projection Algorithm. The algorithms are implemented in MATLAB2015b running on the HP ZBook 14u G4, Intel(R) Core(TM) i7-7500U CPU @ 2.70GHz and 16GB RAM.

5.1. Example 5.1

We consider the Problem SEP under the following hypotheses: Let $H_1 = \mathbb{R}^2$, $H_2 = \mathbb{R}^2$, $H_3 = \mathbb{R}^3$ be endowed with the norm

$$\|x\| =: \langle x, x\rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{2} x_i^2\right)^{\frac{1}{2}}, \ \forall x \in \mathbb{R}^2$$

where the inner product

$$\langle x, y\rangle = \sum_{i=1}^{2} x_i y_i, \ \forall x, \ y \in \mathbb{R}^2.$$

Let C, and Q be defined as $C = \{x \in \mathbb{R}^2 : \|x\| \leq 15\}$, $Q = \{y \in \mathbb{R}^2 : \|y\| \leq 15\}$. Let the mapping $A : \mathbb{R}^2 \to \mathbb{R}^3$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$Ax = A(x_1, x_2)^T = (x_1, x_2, 0)^T, \ \forall x \in C,$$

and

$$By = B(y_1, y_2)^T = (y_1, y_2, 0)^T, \ \forall y \in Q,$$

respectively.

Clearly, C and Q are nonempty, closed and convex subsets of $\mathbb{R}^2$. Also, A and B are bounded linear operator, respectively. The control parameters are detailed in Table 2. The choices of the starting points $x_0, \ y_0$ are generated randomly in $\mathbb{R}^2$ as follows:

Case 1. $x_0 = \text{rand}(2, 1), \ y_0 = \text{rand}(2, 1)$.

Case 2. $x_0 = 2 * \text{rand}(2, 1), \ y_0 = 3 * \text{rand}(2, 1)$.

Case 3. $x_0 = 5 * \text{rand}(2, 1), \ y_0 = 4 * \text{rand}(2, 1)$.

We compare the performance of our Proposed Algorithms 1 and 2 with the ACQA and RACQA algorithms by Moudafi. The stopping criterion used for our computations is $\text{TOL}_k = \|w_{k+1} - w_k\| < 10^{-9}$. We plot the graphs of errors against the number of iterations in each case. The figures and the numerical results are shown in Figure 1 and Table 1, respectively.

Table 1. Numerical results for Example 5.1.

	Case 1		Case 2		Case 3	
	Iter	CPU Time	Iter	CPU Time	Iter	CPU Time
Proposed Alg1	383	0.0836	451	0.1042	440	0.0962
Proposed Alg2	1023	0.1654	929	0.3022	550	0.0835
Moudafi ACQA	1939	0.3022	2328	0.3501	2266	0.3822
Moudafi RACQA	1418	0.2581	1695	0.3146	1651	0.2986

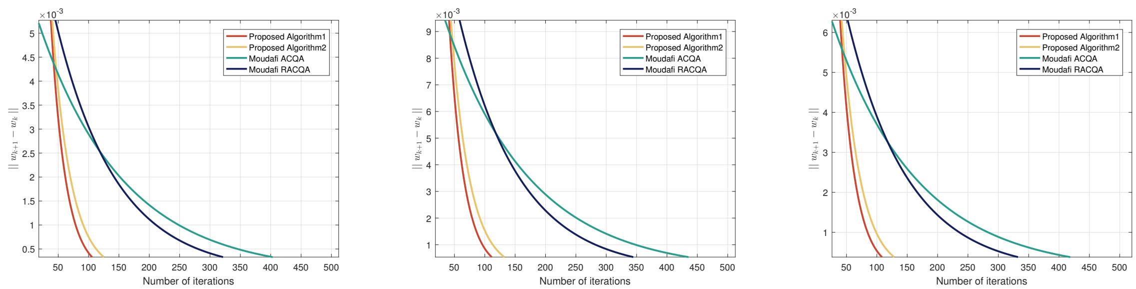
5.2. Example 5.2

Here, we are interested in the following linear model, which is popular in the image deblurring field:

$$Y = AX^* + \epsilon,$$

Table 2. Control parameters for Examples 5.1.

Moudafi ACQA	$\gamma_k = 1/280$					
Moudafi RACQA	$\gamma_k = 1/280$					
Proposed Alg1	v=0.25	b=0.1	a=0.25	$\lambda_k = b - (1/2^{k+5})$	$\gamma_k = 1/100$	
Proposed Alg2	v=0.25	b=0.1	a=0.25	$\lambda_k = b - 1/(1/2^{k+5})$	$\gamma_k = 1/100$	$\alpha_k = 1/(k * 10^7)$



(a) Case 1 with TOL=10^{-9}. (b) Case 2 with TOL=10^{-9}. (c) Case 3 with TOL=10^{-9}.

Figure 1. From left to right: Case 1 to 3.

where Y is the degraded image obtained by the action of the blurring matrix A on the original image X^* followed by the addition of the Gaussian noise ϵ. For a typical grayscale image of dimension M pixels wide by N pixels height, each pixel value has integer values in the range $[0, 255]$. Thus, here we are considering the Hilbert space $\mathbb{R}^D$ endowed with the standard Frobenius norm $\|\cdot\|_F$, where $D = m \times n$. A common technique for estimating an approximation of X is to recast the deblurring problem as the following convex minimization problem:

$$\min_{x \in \mathcal{S}} \frac{1}{2}\|AX - Y\|_F^2, \tag{5.1}$$

where $\mathcal{S} = [0, 255]^D$. Using Section 3, we deduce that (5.1) is approximately equal to the following pseudo SEP:

$$\text{find} \quad X \in \mathcal{S}, \text{ such that } AX \approx BY,$$

where $B = I$ is the identity Operator, $H_1 = H_2 = H_3 = \mathcal{S}$. Therefore, our result can be applied to solving this problem. We apply Algorithms 1 and 2 to recover the original image X^* from the blurred image Y and compare the performance with algorithms by Modafi, ACQA and RACQA. In particular, when $H_1 = H_2 = H_3 = S$, the algorithm ACQA is equivalent to the algorithm RACQA. The quality of the restored image is measured by the Signal-to-Noise ratio (SNR).

SNR as follows:

$$\text{SNR} := 20 \log_{10} \frac{\|X^*\|_F}{\|X - X^*\|_F},$$

where X^* is the original image and X is the restored image. It is known that the larger the SNR, the better the quality of the restored image. The initial values for our experiments are $X_0 = 0 \in \mathbb{R}^D$.

The color test images (see Figure 2) for our experiments are Car.tif, Door.tif. Each test image is degraded by a Gaussian 9×9 blurring kernel with standard deviation 4, Circle blurring model with radius 6, Motion blurring model with length 20, Theta 30 (see Figure 3). And the blurred matrices are denoted as M_{Gauss}, M_{Circle}, and M_{Motion} respectively.

Substitute the blurred images into the algorithms and perform iterative solution. The control parameters of algorithms are detailed in Table 2. Set the number of iterations as $k = 200$, 800, and 3200 respectively. The original, blurred, and restored images by each of the algorithms are shown in Figure 2 to Figure 5. The SNR of the restored images are listed in Table 3 to Table 5.

(a) Original car. (b) Original door.

Figure 2. Original images.

(a) Be blurred by the M_{Gauss}. (b) Be blurred by the M_{Circle}. (c) Be blurred by the M_{Motion}.

(d) Be blurred by the M_{Gauss}. (e) Be blurred by the M_{Circle}. (f) Be blurred by the M_{Motion}.

Figure 3. Images that have been blurred by the matrices M_{Gauss}, M_{Circle}, and M_{Motion} respectively.

Regarding the numerical experiments for the example 1 and 2, the ensuing observational outcomes are as follows.

(i) Table 1 and Figure 1 illustrate that, under specific parameter settings, the two algorithms proposed in this study require fewer iterations and less computational time compared

(a) Restored image by Proposed Algorithm 1 with Figure 3(a).

(b) Restored image by Proposed Algorithm 2 with Figure 3(a).

(c) Restored image by Proposed Algorithm ACQA/RACQA with Figure 3(a).

(d) Restored image by Proposed Algorithm 1 with Figure 3(b).

(e) Restored image by Proposed Algorithm 2 with Figure 3(b).

(f) Restored image by Proposed Algorithm ACQA/RACQA with Figure 3(b).

(g) Restored image by Proposed Algorithm 1 with Figure 3(c).

(h) Restored image by Proposed Algorithm 2 with Figure 3(c).

(i) Restored image by Proposed Algorithm ACQA/RACQA with Figure 3(c).

Figure 4. Images that have been restored by the Proposed Algorithm 1, Algorithm 2 and Modafi ACQA/RACQA, where the oringinal image is Car.tif and all the algorithm's iteration number k=3200.

to the Modafi ACQA and RACQA algorithms. The data disparity between them is significant and cannot be overlooked. Researchers are encouraged to conduct experiments with different parameter configurations. The parameter values provided herein are merely one approach to demonstrate the remarkable efficiency of our algorithms. There are additional parameter values that researchers can explore independently.

(ii) In Figures 4 and 5, and Tables 3-5, our algorithms demonstrate superior SNR values compared to Moudafi's ACQA and RACQA algorithms, indicating better restoration performance.

(a) Restored image by Proposed Algorithm 1 with Figure 3(d).

(b) Restored image by Proposed Algorithm 2 with Figure 3(d).

(c) Restored image by Proposed Algorithm ACQA/RACQA with Figure 3(d).

(d) Restored image by Proposed Algorithm 1 with Figure 3(e).

(e) Restored image by Proposed Algorithm 2 with Figure 3(e).

(f) Restored image by Proposed Algorithm ACQA/RACQA with Figure 3(e).

(g) Restored image by Proposed Algorithm 1 with Figure 3(f).

(h) Restored image by Proposed Algorithm 2 with Figure 3(f).

(i) Restored image by Proposed Algorithm ACQA/RACQA with Figure 3(f).

Figure 5. Images that have been restored by the Proposed Algorithm 1, Algorithm 2 and Modafi ACQA/RACQA, where the oringinal image is Door.tif and all the algorithm's iteration number k=3200.

6. Conclusion

In this paper, we introduce two RPM (Relaxed Projection Method) algorithms designed to tackle the split equality problem, and we prove their weak and strong convergence properties. Additionally, we have incorporated a diverse range of techniques into the algorithms to improve their practicality and accelerate the iteration process. The relaxation technique not only expedites the algorithm's convergence but also addresses the issue of the computational complexity associated with general metric projections. The weak convergence of the proposed algorithms is analyzed under relatively mild conditions.

Table 3. Numerical comparison of SNR (dB) values of Proposed Algorithm 1, Algorithm 2, and Moudafi ACQA/RACQA with Gauss blurring.

Images	k	Proposed Alg1	Proposed Alg2	Moudafi ACQA/RACQA
Car.tif (512 × 384 × 3)	200	14.157	13.0359	10.3082
	800	15.2956	14.6786	12.1327
	3200	16.9968	16.0785	14.8169
Door.tif (512 × 384 × 3)	200	16.4014	13.8449	12.7124
	800	16.4014	13.8449	12.7124
	3200	19.0667	16.1744	15.4847

Table 4. Numerical comparison of SNR (dB) values of Algorithm 1, Algorithm 2, and Moudafi ACQA/RACQA with Circle blurring.

Images	k	Proposed Alg1	Proposed Alg2	Moudafi ACQA/RACQA
Car.tif (512 × 384 × 3)	200	14.1392	13.6668	11.3729
	800	15.271	14.4089	12.852
	3200	16.1609	15.8614	13.0834
Door.tif (512 × 384 × 3)	200	14.9837	13.996	12.3729
	800	15.158	14.086	13.4916
	3200	17.988	15.6812	14.828

Table 5. Numerical comparison of SNR (dB) values of Algorithm 1, Algorithm 2, and Moudafi ACQA/RACQA with Motion blurring.

Images	k	Proposed Alg1	Proposed Alg2	Moudafi ACQA/RACQA
Car.tif (512 × 384 × 3)	200	16.7198	13.3592	12.1998
	800	17.4335	15.0476	14.1314
	3200	19.7581	17.2899	15.0937
Door.tif (512 × 384 × 3)	200	16.6129	12.9887	11.7236
	800	17.6545	14.8296	13.8532
	3200	19.996	17.1693	15.9281

Finally, through numerical experiments on the split equality problem in finite-dimensional spaces and image restoration tasks, we demonstrate the superior effectiveness of our algorithms.

Ethical approval. Not applicable.

Availability of supporting data. The datasets generated during and/or analyzed during the current study are available from the corresponding author.

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