Journal of Applied Analysis and Computation
Volume 16, Number 5, October 2026, 2771–2778
Website:http://www.jaac-online.com
DOI:10.11948/20250352

A NOTE ON A CLASS OF DISCRETE LARGE- AND SUB-CENTRE SYSTEMS*

Yanglan Ou[1,†], Hongfeng Ren[2,†], Guang Zhang[2] and Qiru Wang[3]

Abstract For different mappings, there are various methods and techniques to obtain the normal forms and calculate the center manifolds of a system, with the computations being highly complicated. Unfortunately, all the results obtained are merely of a local nature. Recently, a discrete logistic-lottery system with large- and sub-centres was studied. The authors established a flip bifurcation theorem and provided some numerical simulations. Thus, they demonstrated that the lottery competition is indeed driven by the logistic growth of the dominant species. In this note, we have once again considered that system and theoretically proved that all its stable periodic solutions are induced by the main equation. In particular, the method is remarkably straightforward, and the logistic model can be generalized to other recursive equations, such as those describing the weak Allee effect, the strong Allee effect, the Ricker model, or the Caspari-Watson function.

Keywords Large- and sub-centre, discrete system, periodic solution, chaos.

MSC(2010) 39A10, 39A28, 39A30.

1. Introduction

Hierarchical structures are the norm in spatial systems. Globally, we see this in groupings such as the G20, G7, APEC, the European Union, and ASEAN. At the national level within China, prominent examples frequently reported include the Beijing-Tianjin-Hebei region, the Guangdong-Hong Kong-Macao Greater Bay Area, and the Yangtze River Delta. Hutton [11] identified service industries as increasingly significant drivers of urban development in the Asia-Pacific and proposed a conceptual and analytical framework for scholarly investigation in this domain. Cattaneo et al. [5] provided a review article that introduced the concept of catchment areas differentiated along an urban-rural continuum to better capture these interconnections. Meanwhile, Mercure in [18] developed a core theoretical framework for innovation and the diffusion of new socio-technical regimes, based on an age-structured demographic theory of technological change.

†The corresponding authors.
[1]School of Pharmacy, Guangzhou Xinhua University, 19 Huamei Road, Tianhe District, Guangzhou 510520, Guangdong, China
[2]Zhujiang College, South China Agricultural University, Guangzhou 510900, Guangdong, China
[3]School of Mathematics (Zhuhai), Sun Yat-Sen University, Zhuhai 519082, Guangdong, China
*The authors were supported by Doctoral guidance program of Guangzhou Xinhua University, the Characteristic Innovation Projects of Ordinary Universities in Guangdong [Nos. 2024KTSCX098, 2025KTSCX242], School level Scientific Research Project of the Zhujiang College of South China Agricultural University under Grant number [Nos. 2024ZJKYC004, 2024ZJKYC068].
Email: ownerounan@163.com(Y. Ou), hf.ren@163.com(H. Ren), qd_gzhang@126.com(G. Zhang), mcswqr@mail.sysu.edu.cn(Q. Wang)

Indeed, Nijkamp and Reggiani [19–21] have considered the discrete system of the form

$$\begin{cases} u_{t+1} = au_t(1-u_t), \\ v_{t+1} = rv_t(1-bu_t-v_t), \end{cases} \tag{1.1}$$

where u represents the size of the large centre and v the size of the sub-centre, $0 < a \leq 4$, $r > 0$ and $0 < b < 1$. The complex behavior of system (1.1) had also been numerically investigated by Nijkamp and Reggiani [19–21]. Similarly, such systems are also considered in mathematical biology, for example, Xue et al. [24] proposed an almost periodic discrete two-species Lotka-Volterra commensalism system with delays, and the dynamics of the microbial parasitism, see Xu [23].

In [19–21], the central position of the large centre u is not clear. In other words, we do not know the contribution of u and v for the complex behavior of (1.1). Recently, we considered the dynamical behaviors for the discrete large- and sub-centre system of the form

$$\begin{cases} x_{t+1} = ax_t\left(1-x_t\right), \\ y_{t+1} = \dfrac{(1+b)\,y_t}{1+b\left(x_t+cy_t\right)}, \end{cases} \tag{1.2}$$

and find that all dynamical behaviors of the first equation and the second equation for (1.2) are consistent such as stability, bifurcation, and chaos, where $b, c > 0$ and $1 < a \leq 4$ (see [6]). For different mappings, there are various methods and techniques to obtain the normal forms and calculate the center manifolds of a system, with the computations being highly complicated. Unfortunately, all the results obtained are merely of a local nature. In the following, we will give the main result (see Theorem 2 in [6]).

Theorem 1.1. *For $a = 3$, system (1.2) undergoes a flip bifurcation and the bifurcated 2-periodic points are partially stable.*

In view of Theorem 1.1, system (1.2) has a stable 2-periodic points for $a \in (3, 3+\varepsilon)$, where ε is a sufficiently small positive number. However, the other stable periodic solutions are only seen by numerical simulations, see Figures 4-6 in [6]. In particular, obtaining the flip bifurcation theorem is difficult when the first equation of (1.2) is replaced by other different recurrence sequences.

It is well known that the fixed points 0 and $(a-1)/a$ of the discrete equation

$$x_{t+1} = ax_t\left(1-x_t\right), \tag{1.3}$$

is stable for $a \in (0,1)$ and $a \in (1,3)$, respectively. When $3 < a < a^* = 1+2\sqrt{2}$, there exists $3 = a_1 < a_2 < \cdots < a_n < \cdots < a^*$ such that the equation has a unique stable 2^n-periodic solution for any n. When $a^* < a \leq 4$, the equation is chaotic [1, 2, 9, 15–17, 22, 25]. Existence and stability are important [3, 8, 12], in this note, we will prove that the dynamical behaviors of the sub-centre y are exactly driven by the large centre x. That is, all stable periodic solutions of the second equation in (1.2) can be obtained by its first equation. In particular, the first equation of (1.2) can be replaced by other recurrence sequences, our method is also valid.

2. Model and the main results

In fact, we can consider a more general discrete large- and sub-centre system of the form

$$\begin{cases} x_{t+1} = ax_t\left(1-x_t\right), \\ y_{t+1} = \dfrac{y_t}{b+cx_t+dy_t}, \end{cases} \tag{2.1}$$

where the parameters a, b, c and d satisfy the following assumptions:

(H1) $0 < a < a^* = 1 + 2\sqrt{2}$;

(H2) $b > 0$, $c > 0$, $d > 0$, and $b + c < 1$;

(H3) $b > 0$, $c < 0$, $d > 0$, $b + c > 0$, and $b < 1$.

At this time, the second equation of (2.1) can be rewritten by

$$\frac{1}{y_{t+1}} = (b + cx_t)\,\frac{1}{y_t} + d,$$

or a more general linear difference equation of the form

$$z_{t+1} = (b + cx_t)\,z_t + d, \tag{2.2}$$

where $z_t = 1/y_t$.

A typical linear homogeneous first-order equation is given by

$$w_{t+1} = p_t w_t, \quad t \geq 0, \tag{2.3}$$

and the associated nonhomogeneous equation is given by

$$v_{t+1} = p_t v_t + q_t, \quad t \geq 0, \tag{2.4}$$

where in both equations it is assumed that $p_t \neq 0$, and p_t and q_t are real-valued sequences defined for $t \geq 0$. For any initial value w_0, we have

$$w_1 = p_0 w_0,\ w_2 = p_1 w_1 = p_1 p_0 w_0,\ \cdots,$$

and

$$w_t = \left[\prod_{i=0}^{t-1} p_i\right] w_0. \tag{2.5}$$

Now, we assume that

$$v_t = \left[\prod_{i=0}^{t-1} p_i\right] u_t$$

is a solution of (2.4), then

$$\left[\prod_{i=0}^{t} p_i\right] u_{t+1} - \left[\prod_{i=0}^{t} p_i\right] u_t = q_t,$$

$$u_{t+1} - u_t = q_t \prod_{i=0}^{t} p_i^{-1}$$

or

$$v_t = v_0 \left[\prod_{i=0}^{t-1} p_i\right] + \prod_{j=0}^{t-1} p_j \sum_{i=0}^{t-1} q_i \prod_{j=0}^{i} p_j^{-1}, \tag{2.6}$$

also see Elaydi [7].

When p and q are ω-periodic, we assume that

$$v_\omega^* = v_0^* \prod_{i=0}^{\omega-1} p_i + \prod_{i=0}^{\omega-1} p_i \sum_{i=0}^{\omega-1} q_i \prod_{j=0}^{i} p_j^{-1},$$

for some v_0^*, which implies that

$$v_0^* = \frac{\alpha\beta}{1-\alpha}, \tag{2.7}$$

when

$$\alpha \triangleq \prod_{j=0}^{\omega-1} p_j \neq 0, \tag{2.8}$$

where

$$\beta \triangleq \sum_{i=0}^{\omega-1} q_i \prod_{j=0}^{i} p_j^{-1}. \tag{2.9}$$

In this case, we can obtain a unique ω-periodic solution

$$v_t^* = \prod_{i=0}^{t-1} p_i \left\{ \frac{\alpha\beta}{1-\alpha} + \sum_{i=0}^{t-1} q_i \prod_{j=0}^{i} p_j^{-1} \right\}. \tag{2.10}$$

Clearly, we have

$$\lim_{t\to\infty} |v_t - v_t^*| = |v_0 - v_0^*| \lim_{t\to\infty} \prod_{i=0}^{t-1} |p_i| = 0, \tag{2.11}$$

when $|\alpha| < 1$.

We can summarize the above discussion as the following result.

Lemma 2.1. *Assume that p_t and q_t are ω-periodic with $|\alpha| < 1$, then, (2.4) has a unique ω-periodic solution v_t^* which is globally attractive.*

When $0 < a < a^*$, let x_t^* be the unique 2^n-periodic solution of (1.3).

$$y_{t+1} = \frac{y_t}{b + cx_t^* + dy_t},$$

or

$$\frac{1}{y_{t+1}} = (b + cx_t^*) \frac{1}{y_t} + d. \tag{2.12}$$

We assume that b and d are positive. If $c > 0$, we let

$$b + cx_t^* < b + c < 1.$$

When $c < 0$, assume that

$$0 < b + c < b + cx_t^* < b.$$

Theorem 2.1. *When $0 < a < a^*$, let x_t^* be the unique 2^n-periodic solution of (1.3). If b, c and d are positive, assume $b + c < 1$, or b and d are positive, c is negative, $b + c > 0$, and $b < 1$. Then, the unique 2^n-periodic solution of (2.1) is partially stable.*

Proof. In view of Lemma 2.1, the existence is clear. From the second equation of (2.1), we have

$$\frac{1}{y_{t+1}} = (b + cx_t) \frac{1}{y_t} + d,$$

$$\lim_{t\to\infty} \left| \frac{1}{y_t} - \frac{1}{y_t^*} \right| = \left| \frac{1}{y_0} - \frac{1}{y_0^*} \right| \lim_{t\to\infty} \prod_{i=0}^{t-1} |b + cx_t| = 0. \tag{2.13}$$

The proof is complete. □

3. Conclusions and some remarks

By using the linearized transform, the sub-centre equation is transformed into a linear equation. At this time, its general solution can be given. Furthermore, we prove that the dynamics of the sub-centre equation are exactly driven by the one-dimensional large-centre map. The following conclusions are obtained.

(i) For a general linear equation (2.4), assume that p_t and q_t are ω-periodic with

$$0 < \prod_{i=0}^{\omega-1} |p_i| < 1,$$

then, (2.4) exists a unique ω-periodic solution y_t^* which is globally attractive. See Lemma 2.1.

(ii) When all conditions of Theorem 2.1 hold, the dynamics of the sub-centre are governed by the logistic model, with the large centre acting as the driving component. In view of the flip bifurcation Theorem 2 in Du et al. [6], we have known that the 2-periodic solution of the second equation of (2.1) is stable when a is sufficiently close to $a_1 = 3$. However, Theorem 2.1 implies that the two-periodic solution is stable when $a \in (a_1, a_2)$, where a_2 is the bifurcation point of 4-periodic solution.

(iii) It is well known that there exists $3 = a_1 < a_2 < \cdots < a_n < \cdots < a^* = 1 + 2\sqrt{2} \approx 3.828$ such that the logistic equation has a unique stable 2^n-periodic solution for $a \in (a_{n-1}, a_n)$. In view of Theorem 2.1, we have proved that the corresponding 2^n-periodic solution of the second equation of (2.1) is also stable.

Remark 3.1. Our method is also applicable to the discrete two-species Lottery-Ricker competition model investigated in [13] and [14]. In fact, our approach can be extended to a more general form of the discrete large-centre equation as

$$x_{t+1} = f(x_t), \tag{3.1}$$

where the function $f(x)$ may be the logistic function

$$f(x) = x\left(1 - \frac{x}{K}\right), \ K > 0,$$

the weak Allee effect

$$f(x) = x\left(1 - \frac{x}{K}\right)\frac{x}{x+\theta}, \ K, \theta > 0,$$

the strong Allee effect Freedman [10], the Ricker model [13, 14, 17]

$$f(x) = x \exp r(1-x), \ r > 0,$$

or the Caspari-Watson function

$$f(x) = x + \frac{s_h x (1-x)(x - s_f/s_h)}{s_h x^2 - (s_f + s_h) x + 1},$$

where $s_h, s_f \in (0,1)$ with $s_f < s_h$ [4] and [26], etc.

Remark 3.2. Theorem 2.1 is also valid for the stable fixed points.

Example 3.1. When $f(x)$ is the Caspari-Watson function, we know that equation (3.1) has three fixed points 0, 1 and s_f/s_h, where 0 and 1 are partially stable, and s_f/s_h is unstable. When b, c and d are positive, we easily prove that equation

$$y_{t+1} = \frac{y_t}{b + cx_t + dy_t},$$

has the corresponding fixed points

$$\frac{1-b}{d}, \; \frac{1-(b+c)}{d} \text{ and } \frac{1-(b+cs_f/s_h)}{d},$$

when $b + c < 1$. In view of (2.13), we easily prove that the fixed points $(0, (1-b)/d)$ and $(1, (1-b-c)/d)$ are partially stable. In any case, the fixed point

$$\left(\frac{s_f}{s_h}, \frac{1-(b+cs_f/s_h)}{b}\right)$$

is also unstable. When $b + c \geq 1$, the fixed point $(1, (1-b-c)/d)$ vanishes, thus, the condition is sharp. Assume that (x^*, y^*) is a fixed point of system

$$\begin{cases} x_{t+1} = x_t + \dfrac{s_h x_t (1-x_t)(x_t - s_f/s_h)}{s_h x_t^2 - (s_f + s_h) x_t + 1}, \\ y_{t+1} = \dfrac{y_t}{b + cx_t + dy_t}. \end{cases}$$

In fact, we can also prove its stability and instability by using the roots of the characteristic equation of Jacobian matrix at (x^*, y^*), see Elaydi [7]. However, the calculation of the Jacobian matrix and its eigenvalues is relatively tedious.

Remark 3.3. The upper bound of that constant interval may be greater than 1. At this time, we can choose that c is small enough such that $0 < b + cx_t < 1$. For example, the Ricker model [13, 14, 17]

$$f(x) = x \exp r(1-x), \; r > 0.$$

Remark 3.4. When the delay occurs in the sub-centre equation, our method is also valid. For example, consider

$$\begin{cases} x_{t+1} = ax_t (1-x_t), \\ y_{t+1} = \dfrac{y_t}{b + cx_{t-\tau} + dy_t}, \end{cases} \tag{3.2}$$

where τ is a positive integer and the other parameters are same above. Similarly, we have

$$\frac{1}{y_{t+1}} = (b + cx_{t-\tau}) \frac{1}{y_t} + d.$$

Thus, the similar results can be obtained.

Remark 3.5. When the sub-centre equations are two or more, our method is invalid. It will be considered in the future.

Acknowledgements

The authors wish to express appreciation to the anonymous referees for their critical input and actionable suggestions on the manuscript. This work was also supported by P. R. China Guangdong Engineering Technology Research Center for Artificial Intelligence Empowered Media Technologies and Applications.

References

[1] G. L. Baker and J. P. Gollub, *Chaotic Dynamics: An Introduction*, Cambridge University Press, Cambridge, 1990.

[2] W. J. Baumol and J. Benhabib, *Chaos: Significance, mechanism and economic applications*, J. Econ. Perspect., 1989, 3(1), 77–105.

[3] I. Bula and A. SĪle, *About a system of piecewise linear difference equations with many periodic solutions*, in: S. Olaru, et al. (eds.), Difference Equations, Discrete Dynamical Systems and Applications, Springer Proceedings in Mathematics & Statistics, 2024, 444, 29–50.

[4] E. Caspari and G. S. Watson, *On the evolutionary importance of cytoplasmic sterility in mosquitoes*, Evolution, 1959, 13, 568–570.

[5] A. Cattaneo, et al., *Economic and social development along the urban–rural continuum: New opportunities to inform policy*, World Development, 2022, 157, 105941.

[6] B. B. Du, C. J. Wu, G. Zhang and X. L. Zhou, *Dynamical behaviors of a discrete two-dimensional competitive system exactly driven by the large centre*, J. Appl. Anal. Comput., 2024, 14(5), 2822–2844.

[7] S. Elaydi, *An Introduction to Difference Equations, Third Edition*, Springer Science+Business Media, Inc., 2005.

[8] E. M. Elsayed, J. G. AL-Juaid and H. Malaikah, *On the dynamical behaviors of a quadratic difference equation of order three*, Eur. J. Math. Appl., 2023, 3, Article ID 1.

[9] M. Frank and T. Stengos, *Chaotic dynamics in economic time-series*, J. Econ. Surv., 1988, 2(2), 103–133.

[10] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, 1980.

[11] T. A. Hutton, *Service industries, globalization, and urban restructuring within the Asia-Pacific: New development trajectories and planning responses*, Progress in Planning, 2004, 61, 1–74.

[12] J. G. AL-Juaid, *On the periodicity solutions of five systems of rational systems of difference equations of order five*, Eur. J. Pure and Appl. Math., 2024, 17(4), 3254–3267.

[13] Y. Kang, *Pre-images of invariant sets of a discrete-time two-species competition model*, J. Differ. Equ. Appl., 2012, 18(10), 1709–1733.

[14] Y. Kang and H. Smith, *Global dynamics of a discrete two-species Lottery-Ricker competition model*, J. Biol. Dynam., 2012, 6(2), 358–376.

[15] D. Kelsey, *The economics of chaos or the chaos of economics*, Oxford Economic Papers, 1988, 40, 1–3.

[16] T. Y. Li and J. A. Yorke, *Period three implies chaos*, Am. Math. Mon., 1975, 82(10), 985–992.

[17] R. M. May, *Simple mathematical models with very complicated dynamics*, Nature, 1976, 261, 459–467.

[18] J. F. Mercure, *An age structured demographic theory of technological change*, J. Evol. Econ., 2015, 25(4), 787–820.

[19] P. Nijkamp and A. Reggiani, *Spatial competition and ecologically based socioeconomic models*, Socio-Spatial Dynamics, 1992, 3(2), 89–109.

[20] P. Nijkamp and A. Reggiani, *Space-Time Dynamics, Spatial Competition and the Theory of Chaos, Structure and Change in the Space-Economv (T.R. Lakshmanan and P. Nijkamp, eds.)*, Springer, Berlin, 1993.

[21] P. Nijkamp and A. Reggiani, *Non-linear evolution of dynamic spatial systems: The relevance of chaos and ecologically-based models*, Reg. Sci. Urban Econ., 1995, 25, 183–210.

[22] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak and V. V. Fedorenko, *Dynamies of One-Dimensional Maps*, Naukova Dumka, Kiev, 1989.

[23] P. Xu, *Dynamics of microbial competition, commensalism, and cooperation and its implications for coculture and microbiome engineering*, Biotechnology and Bioengineering, 2021, 118, 199–209.

[24] Y. Xue, X. Xie, F. Chen and R. Han, *Almost periodic solution of a discrete commensalism system*, Discrete Dyn. Nat. Soc., 2015, 2015, Article ID 295483, 11 pp.

[25] G. Zhang, D. M. Jiang and S. S. Cheng, *3-periodic traveling wave solutions for a dynamical coupled map lattice*, Nonlinear Dyn., 2007, 50, 235–247.

[26] B. Zheng and J. S. Yu, *Wolbachia spread dynamics in mosquito populations in cyclic environments*, J. Differ. Equ. Appl., 2024, 30(2), 252–268.

Received November 2025; Accepted April 2026; Available online May 2026.