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STRONGLY LACUNARY $\mathcal{I}_2$-CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS

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Abstract In this paper, firstly, we define the concepts of strongly lacunary convergence, strongly lacunary $\mathcal{I}$-convergence and strongly lacunary $\mathcal{I}^*$-convergence of double sequence of functions, and also we investigate the relations between them. Then, we define the concepts of strongly lacunary Cauchy sequence, strongly lacunary $\mathcal{I}_2$-Cauchy sequence and and strongly lacunary $\mathcal{I}_2^*$-Cauchy sequence of functions, and also we investigate the relations among strongly lacunary $\mathcal{I}_2$-convergence, strongly lacunary $\mathcal{I}_2$-Cauchy sequence and strongly lacunary $\mathcal{I}_2^*$-Cauchy sequence.

Keywords Lacunary sequence, double sequences of functions, $\mathcal{I}_2$-convergence, $\mathcal{I}_2$-Cauchy sequence.

MSC(2010) 40A30, 40A35.

1. Introduction and definitions

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [18] and Schoenberg [27]. The concept of $\mathcal{I}$-convergence in a metric space, which is a generalized from statistical convergence, was introduced by Kostyrko et al. [21]. Nabiev [23] studied on $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence with some properties. Recently, Das et al. [6] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal. Debnath [7] studied the notion of lacunary ideal convergence in intuitionistic fuzzy normed linear spaces as a variant of the notion of ideal convergence. Tripathy et al. [30] introduced the concept of $\mathcal{I}$-lacunary convergent sequences. Dündar et al. [14, 15] introduced the concepts of lacunary $\mathcal{I}$-convergence and lacunary $\mathcal{I}$-Cauchy sequences of functions. Akın and Dündar [1] investigated the concepts of lacunary $\mathcal{I}_2^*$-convergence and lacunary $\mathcal{I}_2^*$-Cauchy sequence. Dündar and Altay [10, 11] introduced ideal convergence and ideal Cauchy double sequences in the linear metric space and they investigated some characteristics and between relations. Dündar et al. [17] studied strongly $\mathcal{I}_2$-lacunary convergence and $\mathcal{I}_2$-lacunary Cauchy double sequences of sets. Hazarika [20] studied the lacunary ideal convergence for double sequences. Dündar and Altay [12, 13] studied $\mathcal{I}_2$-convergence for double sequences of functions. Dündar [8] scolayed strongly lacunary $\mathcal{I}$-convergence of sequences of functions.

Now, we recall some basic concepts and definitions (see [2–5, 7, 9–11, 14–16, 19, 21, 23, 25–33]).

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

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(i) $\emptyset \in \mathcal{I}$, (ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$, (iii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter if and only if

(i) $\emptyset \notin \mathcal{F}$, (ii) If $A, B \in F$, then $A \cap B \in \mathcal{F}$, (iii) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.

$\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$, then the set $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$ is a filter in $\mathbb{N}$, called the filter associated with $\mathcal{I}$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \cdots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$. It is evident that a strongly admissible ideal is admissible also.

Throughout the paper, we let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty$$

as $r \to \infty$. Throughout this paper, we take $\theta = \{k_r\}$ be a lacunary sequence and also the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Throughout the paper, we take $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$, $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ be two sequences of functions and f, g be two functions on a set $S \subset \mathbb{R}$. Also, we take convergent instead of pointwise convergent.

A sequence $\{f_n\}$ is strongly lacunary convergent to f on $S \subset \mathbb{R}$, if for each $x \in S$

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} |f_n(x) - f(x)| = 0.$$

In this case, we write $f_n \to f[\mathcal{N}_\theta]$.

A sequence $\{f_n\}$ is strongly lacunary $\mathcal{I}$-convergent to f on $S \subset \mathbb{R}$, if for every $\varepsilon > 0$ and each $x \in S$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |f_n(x) - f(x)| \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $f_n \to f[\mathcal{I}_\theta]$.

A sequence $\{f_n\}$ is strongly lacunary $\mathcal{I}^*$-convergent to f on $S \subset \mathbb{R}$ if there exists a set $M = \{m_1 < m_2 < \cdots < m_i < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_i \in I_r\} \in \mathcal{F}(\mathcal{I})$ and for each $x \in S$ we have

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{m_i \in I_r} |f_{m_i}(x) - f(x)| = 0.$$

In this case, we write $f_n \to f[\mathcal{I}^*_\theta]$.

A sequence $\{f_n\}$ is a strongly lacunary Cauchy sequence on $S \subset \mathbb{R}$ if there exists a subsequence $\{f_{n'(r)}\}$ of $\{f_n\}$ such that $\lim\limits_{r\to\infty} f_{n'(r)}(x) = f(x)$ for each $x \in S$, where $n'(r) \in I_r$ for each r, and for every $\varepsilon > 0$

$$\frac{1}{h_r} \sum_{n \in I_r} \left| f_n(x) - f_{n'(r)}(x) \right| < \varepsilon,$$

for all $r > r_0 = r_0(\varepsilon, x)$.

A sequence $\{f_n\}$ is a strongly lacunary $\mathcal{I}$-Cauchy sequence on $S \subset \mathbb{R}$ if there exists a subsequence $\{f_{n'(r)}\}$ of $\{f_n\}$ such that $\lim\limits_{r\to\infty} f_{n'(r)}(x) = f(x)$ for each $x \in S$, where $n'(r) \in I_r$ for each r, and for every $\varepsilon > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} \left| f_n(x) - f_{n'(r)}(x) \right| \geq \varepsilon \right\} \in \mathcal{I}.$$

A sequence $\{f_n\}$ is strongly lacunary $\mathcal{I}^*$-Cauchy sequence on $S \subset \mathbb{R}$ if there exists a set $M' = \{r \in \mathbb{N} : m_i \in I_r\} \in \mathcal{F}(\mathcal{I})$ for the set $M = \{m_1 < m_2 < \cdots < m_i < \cdots\} \subset \mathbb{N}$ such that there exists a subsequence $\{f_{n'(r)}\}$ of $\{f_n\}$ such that $\lim\limits_{r\to\infty} f_{n'(r)}(x) = f(x)$ for each $x \in S$, where $n'(r) \in I_r$ for each $r \in M'$, and

$$\lim_{\substack{r\to\infty \\ (r \in M')}} \frac{1}{h_r} \sum_{m_i \in I_r} \left| f_{m_i}(x) - f_{n'(r)}(x) \right| = 0.$$

The double sequence $\theta_2 = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ and } j_0 = 0,\ \bar{h}_u = j_u - j_{u-1} \to \infty \ as\ r, u \to \infty.$$

We use following notations in the sequel:

$$k_{ru} = k_r j_u,\ \ h_{ru} = h_r \bar{h}_u,\ \ I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \ \text{ and } \ j_{u-1} < j \leq j_u\},$$
$$q_r = \frac{k_r}{k_{r-1}} \ \text{ and } \ q_u = \frac{j_u}{j_{u-1}}.$$

After that, throughout the paper, we take $\theta_2 = \{(k_r, j_u)\})$ be any double lacunary sequence, $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{nm}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ and $\{g_{nm}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ be two double sequences of functions and f, g be two functions on a set $S \subset \mathbb{R}$. Also, we take convergent instead of pointwise convergent.

A double sequence $\{f_{mn}\}$ is lacunary convergent ($\mathcal{N}_{\theta_2}$-convergent) to f on $S \subset \mathbb{R}$, if for each $x \in S$

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} f_{mn}(x) = f(x).$$

A double sequence $\{f_{mn}\}$ is lacunary $\mathcal{I}_2$-convergent to f on $S \subset \mathbb{R}$, if for every $\varepsilon > 0$ and each $x \in S$

$$\left\{ (r, u) \in \mathbb{N}^2 : \left| \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} f_{mn}(x) - f(x) \right| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $f_{mn} \to f(\mathcal{I}_{\theta_2})$.

A double sequence $\{f_{mn}\}$ is lacunary $\mathcal{I}_2^*$-convergent to f on $S \subset \mathbb{R}$ if and only if there exists a set $G = \{(m,n) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (m,n) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ and for each $x \in S$ we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} f_{mn}(x) = f(x).$$

In this case, we write $f_{mn} \to f(\mathcal{I}^*_{\theta_2})$.

A double sequence $\{f_{mn}\}$ is lacunary Cauchy sequence on $S \subset \mathbb{R}$ if there exists a subsequence $\left\{f_{m'(r)n'(u)}\right\}$ of $\{f_{mn}\}$ such that $\lim\limits_{r,u\to\infty} f_{m'(r)n'(u)}(x) = f(x)$ for each $x \in S$, where $(m'(r), n'(u)) \in I_{ru}$ for each (r,u), and for every $\varepsilon > 0$

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(m,n)\,\in I_{ru}} (f_{mn}(x) - f_{m'(r)n'(u)}(x)) = 0,$$

for all $r > r_0 = r_0(\varepsilon, x), u > u_0 = u_0(\varepsilon, x)$.

A double sequence $\{f_{mn}\}$ is lacunary $\mathcal{I}_2$-Cauchy sequence on $S \subset \mathbb{R}$ if there exists a subsequence $\left\{f_{m'(r)n'(u)}\right\}$ of $\{f_{mn}\}$ such that $\lim\limits_{r,u\to\infty} f_{m'(r)n'(u)}(x) = f(x)$ for each $x \in S$, where $(m'(r), n'(u)) \in I_{ru}$ for each (r,u), and for every $\varepsilon > 0$

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \left|\frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} (f_{mn}(x) - f_{m'(r)n'(u)}(x))\right| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

A double sequence $\{f_{mn}\}$ is lacunary $\mathcal{I}_2^*$-Cauchy sequence on $S \subset \mathbb{R}$ if there exists a set $G = \{(m,n) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (m,n) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$, there exists a subsequence $\left\{f_{m'(r)n'(u)}\right\}$ of $\{f_{mn}\}$ such that $\lim\limits_{r,u\to\infty} f_{m'(r)n'(u)}(x) = f(x)$ for each $x \in S$, where $(m'(r), n'(u)) \in I_{ru}$ for each $(r,u) \in G'$, and we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} \left(f_{mn}(x) - f_{m'(r)n'(u)}(x)\right) = 0.$$

Lemma 1.1. [10] *Let $\{P_k\}_{k=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}^2$ such that $P_k \in F(\mathcal{I}_2)$ for each k, where $\mathcal{F}(\mathcal{I}_2)$ is a filter associate with a strongly admissible ideal $\mathcal{I}_2$ by $(AP2)$. Therefore, there exists a set $P \subset \mathbb{N}^2$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and the set $P \backslash P_k$ is finite for all k.*

2. Main results

Definition 2.1. A double sequence $\{f_{mn}\}$ is strongly lacunary convergent to f on $S \subset \mathbb{R}$, if for each $x \in S$

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| = 0.$$

In this case, we write $f_{mn} \to f[\mathcal{N}_{\theta_2}]$.

Theorem 2.1. *If a double sequence $\{f_{mn}\}$ is strongly lacunary convergent to f, then it is lacunary convergent to f on $S \subset \mathbb{R}$.*

Proof. Let a double sequence $\{f_{mn}\}$ is strongly lacunary convergent to f on $S \subset \mathbb{R}$. Then, for every $\varepsilon > 0$ and each $x \in S$ there is a $r_0 = r_0(\varepsilon, x), u_0 = u_0(\varepsilon, x) \in \mathbb{N}$ such that for all $r > r_0$ and $u > u_0$ we have

$$\frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| < \varepsilon.$$

Therefore, for every $\varepsilon > 0$ and each $x \in S$ we have

$$\left| \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} f_{mn}(x) - f(x) \right| \le \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| < \varepsilon,$$

for all $r > r_0$ and $u > u_0$, so $\{f_{mn}\}$ is lacunary convergent to f on $S \subset \mathbb{R}$. □

Definition 2.2. A double sequence $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-convergent to f on $S \subset \mathbb{R}$, if for every $\varepsilon > 0$ and each $x \in S$

$$\left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| \ge \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $f_{mn} \to f[\mathcal{I}_{\theta_2}]$.

Theorem 2.2. *If a double sequence $\{f_{mn}\}$ is strongly lacunary convergent to f, then $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-convergent to f on $S \subset \mathbb{R}$.*

Proof. Let a double sequence $\{f_{mn}\}$ is strongly lacunary convergent to f on $S \subset \mathbb{R}$. Then, for every $\varepsilon > 0$ and each $x \in S$ there is a $r_0 = r_0(\varepsilon, x), u_0 = u_0(\varepsilon, x) \in \mathbb{N}$ such that for all $r > r_0$ and $u > u_0$ we have

$$\frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| < \varepsilon.$$

Then, for every $\varepsilon > 0$ and each $x \in S$

$$\begin{aligned} A(\varepsilon) = &\left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| \ge \varepsilon \right\} \\ &\subseteq \{(1,1),(1,2),(2,1),(2,2),\cdots,(r_0,u_0)\}. \end{aligned}$$

Since $\mathcal{I}_2$ is an admissible ideal, we have

$$\{(1,1),(1,2),(2,1),(2,2),\cdots,(r_0,u_0)\} \in \mathcal{I}_2 \text{ and so } A(\varepsilon) \in \mathcal{I}_2.$$

Hence, we have $f_{mn} \to f[\mathcal{I}_{\theta_2}]$. □

Definition 2.3. A double sequence $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2^*$-convergent to f on $S \subset \mathbb{R}$ if there exists a set $G = \{(m,n) \in \mathbb{N} \times \mathbb{N}\}$ such that for the set

$$G' = \{(r,u) \in \mathbb{N} \times \mathbb{N} : (m,n) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$$

and for each $x \in S$ we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| = 0.$$

In this case, we write $f_{mn} \to f[\mathcal{I}_{\theta_2}^*]$.

Theorem 2.3. *If $f_{mn} \to f[\mathcal{I}^*_{\theta_2}]$, then $f_{mn} \to f[\mathcal{I}_{\theta_2}]$ on $S \subset \mathbb{R}$.*

Proof. Let $f_{mn} \to f[\mathcal{I}^*_{\theta_2}]$ on $S \subset \mathbb{R}$. Then, there exists a set $G = \{(m,n) \in \mathbb{N} \times \mathbb{N}\}$ such that for the set $G' = \{(r,u) \in \mathbb{N} \times \mathbb{N} : (m,n) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ $(i.e., H = \mathbb{N} \times \mathbb{N} \backslash G' \in \mathcal{I}_2)$ and for every $\varepsilon > 0$ and each $x \in S$ there are $r_0 = r_0(\varepsilon, x), u_0 = u_0(\varepsilon, x) \in \mathbb{N}$ such that for all $r > r_0, u > u_0$ we have

$$\frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} |f_{mn}(x) - f(x)| < \varepsilon, \ ((r,u) \in G').$$

Then, for every $\varepsilon > 0$ and each $x \in S$

$$\begin{aligned} A(\varepsilon) &= \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} |f_{mn}(x) - f(x)| \geq \varepsilon \right\} \\ &\subseteq H \cup \left[G' \cap \big((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,u_0\}) \big) \right]. \end{aligned}$$

Since $\mathcal{I}_2$ is an admissible ideal, we have

$$H \cup \left[G' \cap \big((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,u_0\}) \big) \right] \in \mathcal{I}_2,$$

and so, $A(\varepsilon) \in \mathcal{I}_2$. Hence, $f_{mn} \to f[\mathcal{I}_{\theta_2}]$. □

Theorem 2.4. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a admissible ideal with property $(AP2)$. If $f_{mn} \to f[\mathcal{I}_{\theta_2}]$, then $f_{mn} \to f[\mathcal{I}^*_{\theta_2}]$ on $S \subset \mathbb{R}$.*

Proof. Assume that $f_{mn} \to f[\mathcal{I}_{\theta_2}]$ on $S \subset \mathbb{R}$. Then, for every $\varepsilon > 0$ and each $x \in S$,

$$P(\varepsilon) = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} |f_{mn}(x) - f(x)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

Put

$$P_1 = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} |f_{mn}(x) - f(x)| \geq 1 \right\}$$

and

$$P_i = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{i} \leq \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} |f_{mn}(x) - f(x)| < \frac{1}{i-1} \right\},$$

for $i \geq 2$ and $i \in \mathbb{N}$. It is clear that $P_i \cap P_k = \emptyset$ for $i \neq k$ and $P_i \in \mathcal{I}_2$ for each $i \in \mathbb{N}$. By property $(AP2)$ there is a sequence $\{U_k\}_{k \in \mathbb{N}}$ such that $P_k \Delta U_k$ is a finite set for each $k \in \mathbb{N}$ and

$$U = \bigcup_{k=1}^{\infty} U_k \in \mathcal{I}_2.$$

We prove that, for each $x \in S$ and $G' = \mathbb{N} \times \mathbb{N} \backslash U \in \mathcal{F}(\mathcal{I}_2)$

$$\lim_{\substack{r,u \to \infty \\ ((r,u) \in G')}} \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} |f_{mn}(x) - f(x)| = 0.$$

Let $\delta > 0$ be given. Choose $v \in \mathbb{N}$ such that $\dfrac{1}{v} < \delta$. Then, for each $x \in S$

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| \geq \delta\right\} \subset \bigcup_{k=1}^{v-1} P_k.$$

Since $P_k \Delta U_k$ is a finite set for $k \in \{1, 2, \cdots, v-1\}$, there exists $r_0, u_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{k=1}^{v-1} P_k\right) \cap \left\{(r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq u_0\right\} = \left(\bigcup_{k=1}^{v-1} U_k\right) \cap \left\{(r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq u_0\right\}.$$

If $r \geq r_0, u \geq u_0$ and $(r,u) \notin U$, then

$$(r,u) \notin \bigcup_{k=1}^{v-1} U_k \text{ and so } (r,u) \notin \bigcup_{k=1}^{v-1} P_k.$$

For each $x \in S$ we have

$$\frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| < \frac{1}{v} < \delta.$$

This implies that for each $x \in S$

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} |f_{mn}(x) - f(x)| = 0.$$

Hence, we have $f_{mn} \to f[\mathcal{I}^*_{\theta_2}]$. □

Definition 2.4. A double sequence $\{f_{mn}\}$ is a strongly lacunary Cauchy sequence on $S \subset \mathbb{R}$, there exists a subsequence $\left\{f_{m'(r)n'(u)}\right\}$ of $\{f_{mn}\}$ such that

$$\lim_{r,u\to\infty} f_{m'(r)n'(u)}(x) = f(x)$$

for each $x \in S$, where $(m'(r), n'(u)) \in I_{ru}$ for each (r,u), and for every $\varepsilon > 0$

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} \left|f_{mn}(x) - f_{m'(r)n'(u)}(x)\right| < \varepsilon.$$

Theorem 2.5. *If a double sequence $\{f_{mn}\}$ is strongly lacunary Cauchy sequence, then it is lacunary Cauchy sequence on $S \subset \mathbb{R}$.*

Proof. The proof can be easily seen using the same technique used in the proof of Theorem 2.1. □

Definition 2.5. A double sequence $\{f_{mn}\}$ is a strongly lacunary $\mathcal{I}_2$-Cauchy sequence on $S \subset \mathbb{R}$there exists a subsequence $\left\{f_{m'(r)n'(u)}\right\}$ of $\{f_{mn}\}$ such that

$$\lim_{r,u\to\infty} f_{m'(r)n'(u)}(x) = f(x)$$

for each $x \in S$, where $(m'(r), n'(u)) \in I_{ru}$ for each (r,u), and for every $\varepsilon > 0$

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} \left|f_{mn}(x) - f_{m'(r)n'(u)}(x)\right| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Theorem 2.6. *A sequence $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-convergent if and only if $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-Cauchy sequence on $S \subset \mathbb{R}$.*

Proof. Assume that a double sequence $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-convergent to f on $S \subset \mathbb{R}$. Then, for every $\varepsilon > 0$ and each $x \in S$. Now, for each $s, t \in \mathbb{N}$ and each $x \in S$ let

$$U^{(s,t)} = \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| < \frac{1}{st}\right\}.$$

Hence, for each for each $s, t \in \mathbb{N}$ and each $x \in S$, $U^{(s+1,t+1)} \subseteq U^{(s,t)}$ and

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{|U^{(s,t)} \cap I_{ru}|}{h_{ru}} \geq \frac{1}{st}\right\} \in \mathcal{I}_2.$$

We select $m(1), n(1)$ such that $r \geq m(1)$, $u \geq n(1)$ implies

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{|U^{(1,1)} \cap I_{ru}|}{h_{ru}} < 1\right\} \notin \mathcal{I}_2.$$

Next we choose $m(2) > m(1)$, $n(2) > n(1)$ so that $r \geq m(2)$, $u \geq n(2)$ implies

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{|U^{(2,2)} \cap I_{ru}|}{h_{ru}} < 1\right\} \notin \mathcal{I}_2.$$

In this way, we can select $m(i+1) > m(i)$, $n(j+1) > n(j)$ so that $r > m(i+1), u > n(j+1)$ implies $U^{(i+1,j+1)} \cap I_{ru} \neq \emptyset$. Then, for each (r,u) satisfying $m(1) \leq r < m(2), n(1) \leq u < n(2))$, select $(m'(r), n'(u)) \in I_{ru} \cap U^{(i,j)}$, i.e., for each $x \in S$

$$|f_{mn}(x) - f_{m'(r)n'(u)}(x)| < \frac{1}{ij}$$

and so we have $(m'(r), n'(u)) \in I_{ru}$ for each $(r,u) \in I_{ru}$, and $\lim\limits_{r,u \to \infty} f_{m'(r)n'(u)}(x) = f(x)$. Consequently, for every $\varepsilon > 0$ and each $x \in S$, we have

$$\begin{aligned}
&\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} \left|f_{mn}(x) - f_{m'(r)n'(u)}(x)\right| \geq \varepsilon\right\} \\
\subseteq &\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} |f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \\
\bigcup &\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m'(r),n'(u)) \in I_{ru}} \left|f_{m'(r)n'(u)}(x) - f(x)\right| \geq \frac{\varepsilon}{2}\right\}
\end{aligned}$$

that is,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} \left|f_{mn}(x) - f_{m'(r)n'(u)}(x)\right| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

This means that $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-Cauchy sequence on $S \subset \mathbb{R}$.

Conversely, assume that $\{f_{mn}\}$ is a strongly lacunary $\mathcal{I}_2$-Cauchy sequence on $S \subset \mathbb{R}$. for every $\varepsilon > 0$ and each $x \in S$, we have

$$\begin{aligned}
&\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} |f_{mn}(x) - f(x)| \geq \varepsilon\right\} \\
\subseteq &\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} \left|f_{mn}(x) - f_{m'(r)n'(u)}(x)\right| \geq \frac{\varepsilon}{2}\right\} \\
\bigcup &\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m'(r),n'(u)) \in I_{ru}} \left|f_{m'(r)n'(u)}(x) - f(x)\right| \geq \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

From here it is understood that $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-convergent on $S \subset \mathbb{R}$. □

Definition 2.6. A double sequence $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2^*$-Cauchy sequence on $S \subset \mathbb{R}$ if there exists a set $G = \{(m,n) \in \mathbb{N}^2\}$ such that for the set

$$G' = \{(r,u) \in \mathbb{N}^2 : (m,n) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2),$$

there exists a subsequence $\{f_{m'(r)n'(u)}\}$ of $\{f_{mn}\}$ such that

$$\lim_{r,u \to \infty} f_{m'(r)n'(u)}(x) = f(x)$$

for each $x \in S$, where $(m'(r), n'(u)) \in I_{ru}$ for each $(r,u) \in G'$, and we have

$$\lim_{\substack{r,u \to \infty \\ (r,u \in G')}} \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} \left|f_{mn}(x) - f_{m'(r)n'(u)}(x)\right| = 0.$$

Theorem 2.7. *If a double sequence $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2^*$-Cauchy sequence, then $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-Cauchy sequence on $S \subset \mathbb{R}$.*

Proof. Let $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2^*$-Cauchy sequence on $S \subset \mathbb{R}$. Then, there exists a set $G = \{(m,n) \in \mathbb{N} \times \mathbb{N}\}$ such that for the set $G' = \{(r,u) \in \mathbb{N} \times \mathbb{N} : (m,n) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$, there exists a subsequence $\{f_{m'(r)n'(u)}\}$ of $\{f_{mn}\}$ such that $\lim_{r,u \to \infty} f_{m'(r)n'(u)}(x) = f(x)$ for each $x \in S$, where $(m'(r), n'(u)) \in I_{ru}$ for each $(r,u) \in G'$, for every $\varepsilon > 0$

$$\frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} \left|f_{mn}(x) - f_{m'(r)n'(u)}(x)\right| < \varepsilon, \quad ((r,u) \in G')$$

for all $r > r_0 = r_0(\varepsilon, x)$, $u > u_0 = u_0(\varepsilon, x)$. Now, let $H = \mathbb{N}^2 \setminus G' \in \mathcal{I}_2$. Then, for every $\varepsilon > 0$, $(r,u) \in G'$ and each $x \in S$,

$$\begin{aligned}
A(\varepsilon) = &\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n) \in I_{ru}} \left|f_{mn}(x) - f_{m'(r)n'(u)}(x)\right| \geq \varepsilon\right\} \\
&\subset H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,u_0\})\right)\right].
\end{aligned}$$

Since $\mathcal{I}_2$ is an admissible ideal, we have

$$H \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,u_0\})\right)\right] \in \mathcal{I}_2.$$

and so, $A(\varepsilon) \in \mathcal{I}_2$. Hence, $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-Cauchy sequence on $S \subset \mathbb{R}$. □

Theorem 2.8. *Let $\mathcal{I}_2$ admissible ideal with property $(AP2)$. If a double sequence $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-Cauchy sequence, then $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2^*$-Cauchy sequence on $S \subset \mathbb{R}$.*

Proof. Assume that $\{f_{mn}\}$ is strongly lacunary $\mathcal{I}_2$-Cauchy sequence on $S \subset \mathbb{R}$. Then, there exists a subsequence $\{f_{m'(r)n'(u)}\}$ of $\{f_{mn}\}$ such that $\lim_{r\to\infty} f_{m'(r)n'(u)}(x) = f(x)$ for each $x \in S$, where $(m'(r), n'(u)) \in I_{ru}$ for each (r,u) and for every $\varepsilon > 0$

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} \left| f_{mn}(x) - f_{m'(r)n'(u)}(x) \right| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

Let

$$P_i = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} \left| f_{mn}(x) - f_{m_i(r)n_i(u)}(x) \right| < \frac{1}{i} \right\},\ i = 1, 2, \dots,$$

where $m_i(r) = m'(r)\left(\frac{1}{i}\right)$ and $n_i(u) = n'(u)\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I}_2)$ for $i = 1, 2, \cdots$. Since $\mathcal{I}_2$ has the $(AP2)$ property, then by Lemma 1.1 there exists a set $P = \{(m,n) \in \mathbb{N} \times \mathbb{N}\}$ such that for the set $P' = \{(r,u) \in \mathbb{N} \times \mathbb{N} : (m,n) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$,and $P' \setminus P_i$ is finite for all i. Now for each $x \in S$, we show that

$$\lim_{\substack{(r,u)\to\infty \\ ((r,u)\in P')}} \frac{1}{h_{ru}} \sum_{(m,n),(s,t)\in I_{ru}} \left| f_{mn}(x) - f_{m'(r)n'(u)}(x) \right| = 0.$$

To prove this let $\varepsilon > 0$, $j \in \mathbb{N}$ such that $j > \dfrac{2}{\varepsilon}$. If $(r,u) \in P$ then $P' \setminus P_j$ is a finite set, so there exists $r_0 = r_0(j)$ and $u_0 = u_0(j)$ such that $(r,u) \in P_j$ for all $r > r_0(j)$ and $u > u_0(j)$. Therefore, for all $r > r_0(j)$ and $u > u_0(j)$ and each $x \in S$

$$\frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} \left| f_{mn}(x) - f_{m_j(r)n_j(u)}(x) \right| < \frac{1}{j}$$

and

$$\frac{1}{h_{ru}} \sum_{(m'(r),n'(u))\in I_{ru}} \left| f_{m'(r)n'(u)}(x) - f_{m_j(r)n_j(u)}(x) \right| < \frac{1}{j}.$$

Hence, for all $r > r_0(j)$ and $u > u_0(j)$ and each $x \in S$ it follows that

$$\begin{aligned}
\frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} \left| f_{mn}(x) - f_{m'(r)n'(u)}(x)) \right| &\leq \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} \left| f_{mn}(x) - f_{m_j(r)n_j(u)}(x) \right| \\
&\quad + \frac{1}{h_{ru}} \sum_{(m'(r),n'(u))\in I_{ru}} \left| f_{m'(r)n'(u)}(x) - f_{m_j(r)n_j(u)}(x) \right| \\
&< \frac{1}{j} + \frac{1}{j} \\
&< \varepsilon.
\end{aligned}$$

Thus, for any $\varepsilon > 0$ and each $x \in S$ there exists $r_0 = r_0(\varepsilon, x)$ and $u_0 = u_0(\varepsilon, x)$ such that for all $r > r_0$, $u > u_0$ and $(r, u) \in P' \in \mathcal{F}(\mathcal{I}_2)$

$$\frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} \left| f_{mn}(x) - f_{m'(r)n'(u)}(x)) \right| < \varepsilon.$$

This shows that the sequence $\{f_{mn}\}$ is lacunary $\mathcal{I}_2^*$-Cauchy sequence on $S \subset \mathbb{R}$. $\square$

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