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# ASYMPTOTIC STABILITY AND UNIQUENESS OF CRITICAL TRAVELING WAVES IN A REACTION-DIFFUSION MODEL WITH NONLOCAL DELAY

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**Abstract** We study the stability and uniqueness of critical traveling waves for nonlocal delayed reaction-diffusion model for a single species with age structure. By introducing a suitable solution space, using the anti-weighted energy method and the nonlinear Halanay's inequality, we prove the exponential stability of critical traveling waves when the initial perturbations are suitably small in a weighted norm. As a corollary of stability result, we obtain the uniqueness of critical traveling waves up to translations. Meanwhile, we apply our result to the nonlocal delayed diffusive Nicholson's blowflies equation in population dynamics. Our results provide significant insight into theoretical research and practical application since the spreading speeds of the traveling waves in the applications of biological sciences usually are the critical speed.

**Keywords** Delayed reaction-diffusion equation, nonlocal effect, critical wave, exponential stability.

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## 1. Introduction

In the field of applied mathematics, the reaction-diffusion equations with time-delay and spatially nonlocal term have been proposed in population biology since this types of equations are believed to be more realistic than the usual kind of reaction-diffusion equations for certain population dynamics [1–3, 8, 9, 12, 14]. In this paper, we consider the following reaction-diffusion equation with nonlocal delay

$$\begin{cases} v_t(t,x) = Dv_{xx}(t,x) - dv(t,x) \\ \qquad\qquad +\varepsilon \displaystyle\int_{-\infty}^{\infty} b(v(t-r,x-y))f_\alpha(y)dy,\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v(s,x) = v_0(s,x),\ (s,x) \in [-r,0] \times \mathbb{R}, \end{cases} \tag{1.1}$$

which was derived as a model for describing the population dynamics of a single species with age structure [34]. In (1.1), $v(t,x)$ denotes the total population of mature species after the mature age $r > 0$ at time $t$ and location $x$, $D > 0$ represents the spatial diffusion rate of the mature species, $\alpha > 0$ is the total amount of diffusion for the immature species and satisfies $\alpha \le rD$, $d > 0$ is the death rate coefficient, the time-delay $r > 0$ is the time taken from birth to maturity,

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$\varepsilon > 0$ refers to the survival rate of the species in time $r$ period and reflects the impact of the death rate of the immature population, $b(v)$ denotes the birth rate function and $f_\alpha(y)$ is the heat kernel in the form of

$$f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{y^2}{4\alpha}} \text{ with } \int_{-\infty}^{\infty} f_\alpha(y)dy = 1.$$

To be biologically feasible, we suppose that

(H1) Eq. (1.1) has two stationary solutions: $v_- = 0$ is unstable and $v_+ = K > 0$ is stable, i.e., $b(0) = 0$, $\varepsilon b(v_+) = dv_+$, $\varepsilon b'(0) - d > 0$ and $\varepsilon b'(v_+) - d < 0$;

(H2) $b(v)$ is a non-negative, $C^2$-smooth function and satisfies $|b'(v)| < b'(0)$ for any $v \in (0, \infty)$;

(H3) there exists a $v_* \in (0, v_+)$ such that $b(v)$ is increasing on $[0, v_*]$ and decreasing on $[v_*, \infty)$.

As far as we know, the nonlocal delayed diffusive Nicholson's blowflies equation [11, 17, 18, 26, 28, 31, 34, 36, 38], i.e., Eq. (1.1) with

$$b(v) = pve^{-av}, \ a > 0, \ p > 0, \tag{1.2}$$

is a typical example satisfying (H1)-(H3). Indeed, (1.1) with (1.2) has two stationary solutions: $v_- = 0$ is an unstable node and $v_+ = \frac{1}{a} \ln \frac{\varepsilon p}{d}$ is a stable node. By elementary calculations, we get $b'(v) = pe^{-av}(1 - av)$ and $b''(v) = -ape^{-av}(2 - av)$. When $\frac{\varepsilon p}{d} > e$, $b(v)$ attains its maximum at $v_* := \frac{1}{a} \in (0, v_+)$. Clearly, one can see that $b'(v) = pe^{-av}(1 - av)$ is decreasing on $(0, \frac{2}{a})$ and $b'(v) = pe^{-av}(1 - av)$ is increasing on $(\frac{2}{a}, \infty)$. It is easy to check that $b'(0) = p$, $|b'(\frac{2}{a})| = pe^{-2}$ and $|b'(\infty)| = 0$. Hence, $|b'(v)| < b'(0)$ for any $v \in (0, \infty)$.

Noting the fact that

$$\lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} b(v(t - r, x - y)) f_\alpha(y)dy = b(v(t - r, x)),$$

we obtain the reaction-diffusion equation with local delay [13, 21, 25, 29, 35]

$$\begin{cases} v_t(t, x) = Dv_{xx}(t, x) - dv(t, x) + \varepsilon b(v(t - r, x)), \ (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v(s, x) = v_0(s, x), \ (s, x) \in [-r, 0] \times \mathbb{R}. \end{cases} \tag{1.3}$$

In the present paper, our aim is to explore the stability and uniqueness of the critical traveling wave solutions of (1.1), when $b(v)$ is non-monotone. A traveling wave solution in (1.1) is a special solution of the form $v(t, x) = \phi(x+ct) \geq 0$, which can be viewed as a heteroclinic orbit connecting two steady states $v_\pm$ at infinity. Substituting $v(t, x) = \phi(z)$, $z = x + ct$ into (1.1) yields

$$\begin{cases} c\phi'(z) = D\phi''(z) - d\phi(z) + \varepsilon \displaystyle\int_{-\infty}^{\infty} b(\phi(z - y - cr)) f_\alpha(y)dy, \\ \phi(\pm\infty) = v_\pm, \end{cases} \tag{1.4}$$

where $c > 0$ is the wave speed.

Now let us briefly review some results and methods concerning the stability of non-critical traveling waves for equations (1.1) and (1.3). When $b(v)$ is monotone for $v \in [0, v_+]$, the stability of traveling wavefronts has been extensively studied in the literature and many methods have been developed for this issue. By a spectral analysis method, Schaaf [32] first investigated the linearized stability of traveling wavefronts for time-delayed monostable equations which include (1.3). Another effective method is the comparison principle coupled with the squeezing technique developed by Chen [4], which has been further used by many investigators for various monostable equations [5,23,37]. In addition to these methods, Mei et al. [26,28] applied the weighted energy method together with comparison principle to obtain nonlinear stability of traveling wavefronts for monostable equation (1.1). After that, Lv et al. [22] and Wu et al. [39] extended the same method [26, 28] to more general time-delayed reaction-diffusion equations with monostability. When $b(v)$ is non-monotone on $[0, v_+]$, the situation is totally different. Because equations (1.1) and (1.3) are all lack of monotonicity and the traveling waves are non-monotone. In such case, comparison principle cannot be applied any more. However, the weighted energy method doesn't require the monotonicity of the equations and works for any non-monotone equations if it possesses some viscosity or a damping or relaxation effect [6,15,24]. Based on these concerns, Wu et al. [40] first obtained asymptotic stability for non-monotone traveling waves of (1.3) with (1.2) under the additional assumptions that $\frac{\varepsilon p}{d} \approx e$ and the wave speed is sufficiently large. To overcome these restrictions, Lin et al. [20] further established the stability of traveling waves for both $e < \frac{\varepsilon p}{d} \leq e^2$ and $\frac{\varepsilon p}{d} > e^2$ by the weighted energy method and the nonlinear Halanay's inequality. Particularly, they [20] constructed weight function different from previous works where the weight functions are selected to be greater than one.

For the stability of critical traveling waves for equations (1.1) and (1.3), it seems that little has been done for this challenging case except for [7, 27, 30], because the analytical approaches for this case are quite limited until now. It is well known that the stability of the critical waves is vital in the study of biological invasions because the critical wave speed is also the spreading speed for all solutions with initial data having compact supports, see [19,33,41] and the references therein. When $b(v)$ is monotone for $v \in [0, v_+]$, Mei et al. [27] established the global stability of critical traveling wavefronts for monostable equation (1.1) with optimal convergent rates by using the Fourier transform and Green's function method combined with energy estimates. Mei et al. [30] extended the same idea [27] to obtain the global stability of critical traveling wavefronts to nonlocal Fisher-KPP equations with optimal convergent rates. When $b(v)$ is non-monotone on $[0, v_+]$, the critical traveling waves are non-monotone, thus their stability analysis is much more difficult. Recently, Chern et al. [7] derived the exponential stability of non-monotone critical traveling waves for local delayed equation (1.3) by employing the anti-weighted energy method and the nonlinear Halanay's inequality.

To our best knowledge, the stability of the critical traveling waves for nonlocal delayed equation (1.1) has not been investigated yet. It is a much challenging problem for the sake of difficulties cased by lacking of monotonicity both of equation (1.1) and traveling waves. Motivated by [7], we will prove the stability of non-monotone critical traveling waves for (1.1) by appealing to the anti-weighted energy method and the nonlinear Halanay's inequality. It is necessary to point out that model (1.1) and the views of analysis have three differences with that in [7]. Firstly, model (1.1) is more complicated than that in [7], because we consider the reaction-diffusion equation with nonlocal delayed effects while the authors in [7] studied the one with local delayed effects. Secondly, we introduce a solution space (see (2.13)) which is different from that in [7]. The benefit of this solution space is that we do not need to estimate the $H^3$-norm of $\sqrt{w(z)}u(t,z)$ to guarantee $\sqrt{w(z)}u(t,z)$ ($w(z)$ is the weight function (2.12) and $u(t,z)$ is the perturbed solution of (2.7)) and its first and second order derivatives to vanish at far fields in the proof of exponential decay of $H^1$-norm for $\sqrt{w(z)}u(t,z)$ (see Remarks 4.1 and 4.2). Lastly, in our newly introduced solution space, we prove the local existence and uniqueness of perturbed problem by the iteration technique and Banach fixed point theorem, while the authors [7] obtained the similar result by extending the mild solution in time of $r$-unit step by step.

The rest of this paper is organized as follows. In Section 2, we show the preliminaries and reformulate equation (1.1) to the perturbed equation (2.7) around a given critical traveling wave. Then we state our main stability and uniqueness theorems of the critical traveling waves for equation (1.1) and give an application to the nonlocal delayed diffusive Nicholson's blowflies equation. In Section 3, we obtain the local existence and uniqueness of the perturbed solution by the iteration technique and Banach fixed point theorem. In Section 4, employing the anti-weighted energy method and the nonlinear Halanay's inequality, we establish priori estimates of the perturbed solution. In Section 5, we establish the uniqueness of critical traveling waves of (1.1) by the stability theorem.

**Notations.** We denote a generic constant by $C > 0$ and specific constants by $C_i > 0$ ($i = 1, 2, \cdots$). Let $I$ be an interval such as $I = \mathbb{R}$. $L^2(I)$ is the space of the square integrable functions defined on $I$ and $H^k(I)$ ($k \geq 0$) is the Sobolev space of the $L^2(I)$-functions $f(x)$ defined on the interval $I$ whose derivatives $\frac{d^i}{dx^i} f$ ($i = 1, \cdots, k$) also belong to $L^2(I)$. Let $T > 0$ be a number and $\mathbf{B}$ be a Banach space. We denote by $C([0, T]; \mathbf{B})$ the space of the $\mathbf{B}$-valued continuous functions on $[0, T]$ and $L^2([0, T]; \mathbf{B})$ is the space of the $\mathbf{B}$-valued $L^2$-functions on $[0, T]$.

## 2. Preliminaries and main results

In this section, we state the stability and uniqueness theorems of the critical traveling waves of (1.1) and apply them to nonlocal delayed diffusive Nicholson's blowflies equation (1.1) with (1.2). The proof of stability theorem will be carried out in Sections 3, 4 and 5.

For our purpose, we recall some results concerning the Cauchy problem and traveling waves of (1.1) with assumptions (H1)-(H3).

(i) So et al. [34] obtained the existence, uniqueness and positiveness of the solution to the Cauchy problem in (1.1).

(ii) Fang et al. [10] proved that there is a unique critical speed $c_* > 0$ such that for each $c \geq c_*$, (1.1) has a unique (up to a constant shift) traveling wave solution $\phi(x + ct)$ satisfying $\phi(-\infty) = 0$ and $\phi(\infty) = K$.

(iii) Since $\phi(z) \to v_-$ as $z \to -\infty$, we expect that $\phi(z)$ is close to a function $v(z)$ which satisfies the linearized equation of (1.4) around $v_-$ as $z \to -\infty$

$$cv'(z) - Dv''(z) + dv(z) = \varepsilon b'(0) \int_{-\infty}^{\infty} v(z - y - cr) f_\alpha(y) dy, \ v(-\infty) = 0. \tag{2.1}$$

Heuristically, we expect $v(z) = Ce^{\lambda z} \to 0$ as $z \to -\infty$ for some eigenvalue $\lambda > 0$. Substituting $v(z) = Ce^{\lambda z}$ into (2.1) yields

$$c\lambda - D\lambda^2 + d = \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda cr}. \tag{2.2}$$

For convenience, we denote

$$F_c(\lambda) := c\lambda - D\lambda^2 + d \text{ and } G_c(\lambda) := \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda cr}.$$

As stated in [38] that, for each $r \geq 0$, there exists a unique $c_* = c_*(r) > 0$ at which two graphs of $F_c$ and $G_c$ are tangent at $\lambda_*$. This means that $(c_*, \lambda_*)$ are determined by $F_{c_*}(\lambda_*) = G_{c_*}(\lambda_*)$ and $F'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*)$, i.e.,

$$\begin{cases} c_*\lambda_* - D\lambda_*^2 + d = \varepsilon b'(0)e^{\alpha\lambda_*^2 - \lambda_* c_* r}, \\ c_* - 2D\lambda_* = \varepsilon b'(0)(2\alpha\lambda_* - c_* r)e^{\alpha\lambda_*^2 - \lambda_* c_* r}. \end{cases} \tag{2.3}$$

Consequently, (a) for $c > c_*$, Eq. (2.2) has two different positive roots $\lambda_1 < \lambda_2$; (b) for $c = c_*$, Eq. (2.2) has multiple root $\lambda_1 = \lambda_2 = \lambda_* > 0$; (c) for $c < c_*$, Eq. (2.2) has no positive roots. Moreover,

$$\begin{cases} \phi(z) = Ce^{\lambda_1 z} \to 0, \text{ as } z \to -\infty \text{ for } c > c_*, \\ \phi(z) = C\big|z\big|e^{\lambda_* z} \to 0, \text{ as } z \to -\infty \text{ for } c = c_*. \end{cases} \tag{2.4}$$

Now we present the profile of the critical traveling wave $\phi(z)$ of (1.1).

**Lemma 2.1.** *Let $\phi(x + c_*t) = \phi(z)$, $z = x + c_*t$ be a critical traveling wave of (1.1), then there exists a constant $A > 0$ such that $\big|\phi(z)\big| + \big|\phi'(z)\big| \leq A$ on $\mathbb{R}$ and $\lim_{z\to\pm\infty} \phi'(z) = 0$.*

**Proof.** Firstly, the boundedness of $\phi(z)$ follows from (ii) aforementioned. Secondly, set

$$\rho_1 = \frac{c_* - \sqrt{c_*^2 + 4Dd}}{2D} \text{ and } \rho_2 = \frac{c_* + \sqrt{c_*^2 + 4Dd}}{2D}.$$

By (1.4) and the boundedness of $\phi(z)$, we have

$$\phi(z) = \frac{1}{D(\rho_2 - \rho_1)}\Big[\int_{-\infty}^{z} e^{\rho_1(z-s)}H(\phi)(s)ds + \int_{z}^{\infty} e^{\rho_2(z-s)}H(\phi)(s)ds\Big], \tag{2.5}$$

where $H(\phi)(s) = \varepsilon \int_{\mathbb{R}} b(\phi(s - y - c_*r))f_\alpha(y)dy$.

Differentiating (2.5) with respect to $z$ yields

$$\phi'(z) = \frac{1}{D(\rho_2 - \rho_1)}\Big[\int_{-\infty}^{z} \rho_1 e^{\rho_1(z-s)}H(\phi)(s)ds + \int_{z}^{\infty} \rho_2 e^{\rho_2(z-s)}H(\phi)(s)ds\Big]. \tag{2.6}$$

Since $D(\rho_2 - \rho_1) = \sqrt{c_*^2 + 4Dd} > 2\sqrt{Dd}$ and boundedness of $\phi(z)$, we obtain

$$\big|\phi'(z)\big| < \frac{1}{\sqrt{Dd}} \max_{s\in\mathbb{R}} \big|H(\phi)(s)\big|, \ z \in \mathbb{R}.$$

Finally, applying L'Hôpital's rule in (2.6) gives $\lim_{z\to\pm\infty} \phi'(z) = 0$. □

To state the asymptotic stability result, we reformulate (1.1) to a perturbed problem around the critical traveling wave solution. Let $\phi(x+c_*t) = \phi(z)$, $z = x+c_*t$ be a given critical traveling wave solution of (1.1) and define

$$u(t, z) := v(t, x) - \phi(x + c_*t) \text{ and } u_0(s, z) := v_0(s, x) - \phi(x + c_*s).$$

Then we infer from (1.1) and (1.4) that $u(t, z)$ satisfies

$$\begin{cases} \quad u_t + c_*u_z - Du_{zz} + du \\ = \varepsilon \displaystyle\int_{-\infty}^{\infty} P(u(t - r, z - y - c_*r))f_\alpha(y)dy, \ (t, z) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(s, z) = u_0(s, z), \ (s, z) \in [-r, 0] \times \mathbb{R}, \end{cases} \tag{2.7}$$

where

$$P(u) := b(\phi + u) - b(\phi) \tag{2.8}$$

with $\phi = \phi(z - y - c_* r)$ and $u = u(t - r, z - y - c_* r)$.

Moreover, (2.7) and (2.8) are equivalent to

$$\begin{cases} u_t + c_* u_z - D u_{zz} + du \\ -\varepsilon \displaystyle\int_{-\infty}^{\infty} b'(\phi(z - y - c_* r)) u(t - r, z - y - c_* r) f_\alpha(y) dy \\ = \varepsilon \displaystyle\int_{-\infty}^{\infty} G(u(t - r, z - y - c_* r)) f_\alpha(y) dy, \ (t, z) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(s, z) = u_0(s, z), \ (s, z) \in [-r, 0] \times \mathbb{R}, \end{cases} \tag{2.9}$$

where

$$G(u) := b(\phi + u) - b(\phi) - b'(\phi) u \tag{2.10}$$

with $\phi = \phi(z - y - c_* r)$ and $u = u(t - r, z - y - c_* r)$. By Taylor's theorem, we have

$$\big|G(u)\big| \le C\big|u\big|^2. \tag{2.11}$$

As in [7, 26], we define a weight function

$$w(z) := e^{-2\lambda_* z}, \quad z \in \mathbb{R}, \tag{2.12}$$

where $\lambda_* > 0$ is defined in (2.3). Notice that $\lim_{z \to -\infty} w(z) = \infty$ and $\lim_{z \to \infty} w(z) = 0$.

For $0 \le T \le \infty$, we define the solution space for (2.7) as follows

$$\begin{aligned} X(-r, T) := \Big\{ u(t, z) \Big| \ & u(t, z) \in C_{\text{unif}}[-r, T], \sqrt{w(z)} u(t, z) \in C([-r, T]; \\ & H^1(\mathbb{R})) \cap L^2([-r, T]; H^1(\mathbb{R})) \Big\}, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} C_{\text{unif}}[-r, T] := \Big\{ & u(t, z) \in C([-r, T] \times \mathbb{R}) \text{ such that } \lim_{z \to \infty} u(t, z) \text{ exists uniformly} \\ & \text{in } t \in [-r, T] \text{ and } \lim_{z \to \infty} \partial_z^k u(t, z) = \lim_{z \to \pm\infty} \sqrt{w(z)} u(t, z) = 0 \\ & \text{exist uniformly with respect to } t \in [-r, T] \Big\} \end{aligned}$$

for $k = 1, 2$. Let $X(-r, T)$ be endowed with the norm

$$M_u^2(T) = \sup_{t \in [-r, T]} \Big[ \big\|u(t)\big\|_C^2 + \big\|\sqrt{w} u(t)\big\|_{H^1}^2 \Big]. \tag{2.14}$$

Now we state the stability theorem as follows.

**Theorem 2.1.** (Stability) *Suppose that (H1)-(H3) hold, $u_0(s, z) \in X(-r, 0)$ and*

$$\max_{s \in [-r, 0]} \big\|u_0(s)\big\|_C^2 + \big\|\sqrt{w} u_0(0)\big\|_{H^1}^2 + \int_{-r}^0 \big\|\sqrt{w} u_0(s)\big\|_{H^1}^2 ds \le \delta_0^2$$

*for a small constant $\delta_0 > 0$. Let $\phi(x + c_* t)$ be the unique critical traveling wave of (1.1). Then* *(i) for any* $r > 0$ *and* $d \geq \varepsilon |b'(v_+)|$*, the solution* $u(t,z)$ *of (2.7) exists uniquely and globally in* $X(-r,\infty)$*, which satisfies*

$$\sup_{z\in\mathbb{R}} |u(t,z)| \leq Ce^{-\mu t}, \quad t > 0, \tag{2.15}$$

*i.e.,*

$$\sup_{x\in\mathbb{R}} |v(t,x) - \phi(x + c_* t)| \leq Ce^{-\mu t}, \quad t > 0 \tag{2.16}$$

*for some constant* $\mu > 0$*;*

*(ii) for* $0 < r < \bar{r}$ *and* $d < \varepsilon |b'(v_+)|$*, where*

$$\bar{r} := \frac{\pi - \arctan(\sqrt{[\varepsilon|b'(v_+)|]^2 - d^2}/d)}{\sqrt{[\varepsilon|b'(v_+)|]^2 - d^2}}, \tag{2.17}$$

*the solution* $u(t,z)$ *of (2.7) exists uniquely and globally in* $X(-r,\infty)$*, which satisfies (2.15).*

**Proof.** The global existence is based on Propositions 3.1, 4.1 (in Sections 3 and 4 stated below) and the standard continuity extension method [28, 29]. The stability result is a direct consequence of Proposition 4.1 for $t \in (0,\infty)$. □

By Theorem 2.1, we can establish the uniqueness of critical traveling waves of (1.1).

**Corollary 2.1.** (Uniqueness) *Suppose that (H1)-(H3) hold and let either* $d \geq \varepsilon |b'(v_+)|$ *for any* $r > 0$ *or* $d < \varepsilon |b'(v_+)|$ *for* $0 < r < \bar{r}$*, where* $\bar{r}$ *is defined in (2.17). Then any critical traveling wave solutions* $\phi(x + c_* t)$ *of (1.1) are unique up to translations.*

Next we apply Theorem 2.1 to nonlocal delayed diffusive Nicholson's blowflies equation (1.1) with (1.2).

**Theorem 2.2.** (Application) *Let* $\phi(x + c_* t)$ *be the unique critical traveling wave solution of (1.1)-(1.2),* $u_0(s,z) \in X(-r,0)$ *and*

$$\max_{s\in[-r,0]} \|u_0(s)\|_C^2 + \|\sqrt{w}u_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\sqrt{w}u_0(s)\|_{H^1}^2 ds \leq \delta_0^2$$

*for a small constant* $\delta_0 > 0$*. Then*

*(i) for any* $r > 0$ *and* $e < \frac{\varepsilon p}{d} \leq e^2$*, the solution* $u(t,z)$ *of (2.7) exists uniquely and globally in* $X(-r,\infty)$*, which satisfies (2.15) for some constant* $\mu > 0$ *and any* $t > 0$*;*

*(ii) for any* $0 < r < \bar{r}$ *and* $\frac{\varepsilon p}{d} > e^2$*, where*

$$\bar{r} := \frac{\pi - \arctan\sqrt{\ln\frac{\varepsilon p}{d}(\ln\frac{\varepsilon p}{d} - 2)}}{d\sqrt{\ln\frac{\varepsilon p}{d}(\ln\frac{\varepsilon p}{d} - 2)}},$$

*the solution* $u(t,z)$ *of (2.7) exists uniquely and globally in* $X(-r,\infty)$*, which satisfies (2.15) for some constant* $\mu > 0$ *and any* $t > 0$*.*

## 3. Local existence and uniqueness of the perturbed solution

This section provides the proof of the local existence and uniqueness of perturbed solution of (2.7), which is the basis for proving Theorem 2.1.

**Proposition 3.1.** (Local existence and uniqueness) *Assume that (H1)-(H3) hold and let $d \geq \varepsilon\left|b'(v_+)\right|$ for any $r > 0$ or $d < \varepsilon\left|b'(v_+)\right|$ for $0 < r < \bar{r}$. When $u_0(s,z) \in X(-r,0)$ and*

$$\max_{s\in[-r,0]} \left\|u_0(s)\right\|_C^2 + \left\|\sqrt{w}u_0(0)\right\|_{H^1}^2 + \int_{-r}^0 \left\|\sqrt{w}u_0(s)\right\|_{H^1}^2 ds \leq \delta_0^2$$

*for a small constant $\delta_0 > 0$, there exists a small $0 < t_0 = t_0(\delta_0) \ll 1$ such that the local solution $u(t,z)$ of (2.7) exists uniquely on $[-r,t_0]$ and satisfies $u(t,z) \in X(-r,t_0)$ and $M_u(t_0) \leq CM_{u_0}(0)$ for some constant $C > 0$.*

**Proof.** The local existence and uniqueness of the solution can be proved by the iteration technique, elementary energy method and Banach fixed point theorem. We will divide the proof into four steps.

**Step 1.** This step is devoted to constructing the iteration scheme for equation (2.7). Let $u_{(0)}(s,z) := u_0(s,z) \in X(-r,0) \subseteq X(-r,t_0)$ and define the iteration $u_{(n+1)} := \mathcal{F}(u_{(n)})$ $(n \in \mathbb{N})$ by

$$\begin{cases} \quad [u_{(n+1)}]_t + c_*[u_{(n+1)}]_z - D[u_{(n+1)}]_{zz} + du_{(n+1)} \\ = \varepsilon \displaystyle\int_{-\infty}^{\infty} P(u_{(n)}(t-r, z-y-c_*r)) f_\alpha(y) dy, \\ u_{(n+1)}(s,z) = u_0(s,z),\ (s,z) \in [-r,0]\times\mathbb{R}, \end{cases} \tag{3.1}$$

where

$$P(u_{(n)}) = b(\phi + u_{(n)}) - b(\phi) \tag{3.2}$$

with $\phi = \phi(z-y-c_*r)$ and $u_{(n)} = u_{(n)}(t-r, z-y-c_*r)$.

**Step 2.** This step is devoted to proving the implication $u_{(n)}(t,z) \in C_{\text{unif}}[-r,t_0] \Rightarrow u_{(n+1)}(t,z) \in C_{\text{unif}}[-r,t_0]$. Note that (3.1) can be rewritten as follows

$$\begin{aligned} u_{(n+1)}(t,z) =& e^{-dt}\int_{-\infty}^{\infty}\Gamma(\eta,t)u_0(0,z-\eta)d\eta + \varepsilon\int_0^t e^{-d(t-s)}\int_{-\infty}^{\infty}\Gamma(\eta,t-s) \\ &\times \int_{-\infty}^{\infty} P(u_{(n)}(s-r,z-\eta-y-c_*r))f_\alpha(y)dyd\eta ds, \end{aligned} \tag{3.3}$$

where $\Gamma(\eta,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{(\eta+c_*t)^2}{4Dt}}$. Assume that $u_{(n)}(t,z) \in C_{\text{unif}}[-r,t_0]$, then

$$u_{(n)}(t,z) \in C([-r,t_0]\times\mathbb{R}), \tag{3.4}$$

$$\lim_{z\to\infty} u_{(n)}(t,z) =: u_{(n,\infty)}(t) \tag{3.5}$$

and

$$\lim_{z\to\infty} \partial_z^k u_{(n)}(t,z) = \lim_{z\to\pm\infty} \sqrt{w(z)}u_{(n)}(t,z) = 0 \tag{3.6}$$

exist uniformly with respect to $t \in [-r,t_0]$ for $k = 1,2$.

It follows from (3.2)-(3.4) and (H2) that

$$u_{(n+1)}(t,z) \in C([-r,t_0]\times\mathbb{R}) \tag{3.7}$$

and

$$\|u_{(n+1)}(t)\|_C \le C\|u_0(0)\|_C + Ct_0 \sup_{t\in[-r,t_0]} \|u_{(n)}(t)\|_C,\ t\in[0,t_0]. \tag{3.8}$$

Furthermore, integration by parts, we have from (3.3) that, for $k=1,2$,

$$\begin{aligned}
&\partial_z^k u_{(n+1)}(t,z)\\
&= e^{-dt}\int_{-\infty}^{\infty}\Gamma(\eta,t)\partial_z^k u_0(0,z-\eta)d\eta + \varepsilon\int_0^t e^{-d(t-s)}\int_{-\infty}^{\infty}\Gamma(\eta,t-s)\\
&\quad\times\int_{-\infty}^{\infty}\partial_z^k P(u_{(n)}(s-r,z-\eta-y-c_*r))f_\alpha(y)dyd\eta ds\\
&= (-1)^k e^{-dt}\int_{-\infty}^{\infty}\Gamma(\eta,t)\partial_\eta^k u_0(0,z-\eta)d\eta + (-1)^k\varepsilon\int_0^t e^{-d(t-s)}\\
&\quad\times\int_{-\infty}^{\infty}\Gamma(\eta,t-s)\int_{-\infty}^{\infty}\partial_\eta^k P(u_{(n)}(s-r,z-\eta-y-c_*r))f_\alpha(y)dyd\eta ds\\
&= (-1)^{2k} e^{-dt}\int_{-\infty}^{\infty}\partial_\eta^k\Gamma(\eta,t)u_0(0,z-\eta)d\eta + (-1)^{2k}\varepsilon\int_0^t e^{-d(t-s)}\\
&\quad\times\int_{-\infty}^{\infty}\partial_\eta^k\Gamma(\eta,t-s)\int_{-\infty}^{\infty}P(u_{(n)}(s-r,z-\eta-y-c_*r))f_\alpha(y)dyd\eta ds\\
&= e^{-dt}\int_{-\infty}^{\infty}\partial_\eta^k\Gamma(\eta,t)u_0(0,z-\eta)d\eta + \varepsilon\int_0^t e^{-d(t-s)}\int_{-\infty}^{\infty}\partial_\eta^k\Gamma(\eta,t-s)\\
&\quad\times\int_{-\infty}^{\infty}P(u_{(n)}(s-r,z-\eta-y-c_*r))f_\alpha(y)dyd\eta ds.
\end{aligned}\tag{3.9}$$

Next we will prove $u_{(n+1)}(t,z)\in C_{\mathrm{unif}}[-r,t_0]$. Using (3.3), (3.5), $u_0\in X(-r,t_0)$ and

$$\int_{-\infty}^{\infty}\Gamma(\eta,t-s)d\eta = \int_{-\infty}^{\infty}f_\alpha(y)dy = 1, \tag{3.10}$$

we get that

$$\begin{aligned}
&\lim_{z\to\infty} u_{(n+1)}(t,z)\\
&= e^{-dt}\int_{-\infty}^{\infty}\Gamma(\eta,t)\lim_{z\to\infty}u_0(0,z-\eta)d\eta + \varepsilon\int_0^t e^{-d(t-s)}\int_{-\infty}^{\infty}\Gamma(\eta,t-s)\\
&\quad\times\int_{-\infty}^{\infty}\lim_{z\to\infty}P(u_{(n)}(s-r,z-\eta-y-c_*r))f_\alpha(y)dyd\eta ds\\
&= u_{(0,\infty)}(0)e^{-dt}\int_{-\infty}^{\infty}\Gamma(\eta,t)d\eta + \varepsilon\int_0^t e^{-d(t-s)}\\
&\quad\times P(u_{(n,\infty)}(s-r))\int_{-\infty}^{\infty}\Gamma(\eta,t-s)\int_{-\infty}^{\infty}f_\alpha(y)dyd\eta ds\\
&= u_{(0,\infty)}(0)e^{-dt} + \varepsilon\int_0^t e^{-d(t-s)}P(u_{(n,\infty)}(s-r))ds
\end{aligned}$$

$$=: u_{(n+1,\infty)}(t) \tag{3.11}$$

exists uniformly with respect to $t \in [-r, t_0]$.

In view of (3.5), (3.9), (3.10), $u_0 \in X(-r, t_0)$ and the facts that

$$\Gamma(\eta, t)|_{\eta=\pm\infty} = 0, \quad \partial_\eta \Gamma(\eta, t)|_{\eta=\pm\infty} = 0, \tag{3.12}$$

we obtain for $k = 1, 2$ that

$$\begin{aligned}
&\lim_{z\to\infty} \partial_z^k u_{(n+1)}(t, z) \\
&= e^{-dt} \int_{-\infty}^{\infty} \partial_\eta^k \Gamma(\eta, t) \lim_{z\to\infty} u_0(0, z-\eta) d\eta + \varepsilon \int_0^t e^{-d(t-s)} \int_{-\infty}^{\infty} \partial_\eta^k \Gamma(\eta, t-s) \\
&\quad \times \int_{-\infty}^{\infty} \lim_{z\to\infty} P(u_{(n)}(s-r, z-\eta-y-c_* r)) f_\alpha(y) dy d\eta ds \\
&= u_{(0,\infty)}(0) e^{-dt} \int_{-\infty}^{\infty} \partial_\eta^k \Gamma(\eta, t) d\eta \\
&\quad + \varepsilon \int_0^t e^{-d(t-s)} P(u_{(n,\infty)}(s-r)) \int_{-\infty}^{\infty} f_\alpha(y) dy \int_{-\infty}^{\infty} \partial_\eta^k \Gamma(\eta, t-s) d\eta ds \\
&= 0.
\end{aligned} \tag{3.13}$$

By the mean value theorem and (H2), we have

$$\begin{aligned}
&\left|P(u_{(n)}(s-r, z-\eta-y-c_* r)))\right| \\
&= \left|b(\phi(z-\eta-y-c_* r) + u_{(n)}(s-r, z-\eta-y-c_* r)) - b(\phi(z-\eta-y-c_* r))\right| \\
&= \left|b'(\tilde{\phi}_1) u_{(n)}(s-r, z-\eta-y-c_* r)\right| \\
&< b'(0)\left|u_{(n)}(s-r, z-\eta-y-c_* r)\right|,
\end{aligned} \tag{3.14}$$

where $\tilde{\phi}_1 = \phi(z-\eta-y-c_* r) + \theta u_{(n)}(s-r, z-\eta-y-c_* r)$ for some $0 < \theta < 1$.

Using (3.3), (3.6), (3.14) and $u_0 \in X(-r, t_0)$, we deduce

$$\begin{aligned}
&\lim_{z\to\pm\infty} \left|\sqrt{w(z)} u_{(n+1)}(t, z)\right| \\
&= \left| e^{-dt} \int_{-\infty}^{\infty} \Gamma(\eta, t) \lim_{z\to\pm\infty} \left[\sqrt{w(z)} u_0(0, z-\eta)\right] d\eta + \varepsilon \int_0^t e^{-d(t-s)} \int_{-\infty}^{\infty} \Gamma(\eta, t-s) \right. \\
&\quad \left. \times \int_{-\infty}^{\infty} \lim_{z\to\pm\infty} \left[\sqrt{w(z)} P(u_{(n)}(s-r, z-\eta-y-c_* r))\right] f_\alpha(y) dy d\eta ds \right| \\
&\le \varepsilon \int_0^t e^{-d(t-s)} \int_{-\infty}^{\infty} \Gamma(\eta, t-s) \\
&\quad \times \int_{-\infty}^{\infty} b'(0) \lim_{z\to\pm\infty} \left|\sqrt{w(z)} u_{(n)}(s-r, z-\eta-y-c_* r)\right| f_\alpha(y) dy d\eta ds \\
&= 0,
\end{aligned} \tag{3.15}$$

which implies that

$$\lim_{z\to\pm\infty} \sqrt{w(z)} u_{(n+1)}(t, z) = 0 \tag{3.16}$$

exists uniformly with respect to $t \in [-r, t_0]$.

Moreover, by (3.6), (3.9), (3.12), (3.14) and $u_0 \in X(-r, t_0)$, we obtain

$$\begin{aligned}
&\lim_{z\to\pm\infty} \left| \sqrt{w(z)}\partial_z^k u_{(n+1)}(t,z) \right| \\
&\le e^{-dt}\left| \int_{-\infty}^{\infty} \partial_\eta^k \Gamma(\eta,t) \lim_{z\to\pm\infty} \left[\sqrt{w(z)}u_0(0,z-\eta)\right] d\eta \right| \\
&\quad + \varepsilon \left| \int_0^t e^{-d(t-s)} \int_{-\infty}^{\infty} \partial_\eta^k \Gamma(\eta, t-s) \right. \\
&\quad \times \left. \int_{-\infty}^{\infty} b'(0) \lim_{z\to\pm\infty} \left[\sqrt{w(z)}u_{(n)}(s-r, z-\eta-y-c_* r)\right] f_\alpha(y) dy d\eta ds \right| \\
&= 0 \text{ for } k = 1, 2,
\end{aligned} \tag{3.17}$$

which yields

$$\lim_{z\to\pm\infty} \sqrt{w(z)}\partial_z^k u_{(n+1)}(t,z) = 0 \tag{3.18}$$

exists uniformly with respect to $t \in [-r, t_0]$ for $k = 1, 2$.

Thus we conclude from (3.7), (3.11), (3.13) and (3.16) that $u_{(n+1)}(t,z) \in C_{\text{unif}}[-r, t_0]$.

**Step 3.** This step is devoted to obtaining the implication

$$\begin{aligned}
&\sqrt{w}u_{(n)}(t,z) \in C([-r,t_0]; H^1(\mathbb{R})) \cap L^2([-r,t_0]; H^1(\mathbb{R})) \\
\Rightarrow &\sqrt{w}u_{(n+1)}(t,z) \in C([-r,t_0]; H^1(\mathbb{R})) \cap L^2([-r,t_0]; H^1(\mathbb{R})).
\end{aligned}$$

Assuming $\sqrt{w}u_{(n)}(t,z) \in C([-r,t_0]; H^1(\mathbb{R})) \cap L^2([-r,t_0]; H^1(\mathbb{R}))$, i.e.,

$$\left\| \sqrt{w}u_{(n)}(t) \right\|_{H^1} < \infty \tag{3.19}$$

and

$$\int_{-r}^t \left\| \sqrt{w}u_{(n)}(s) \right\|_{H^1}^2 ds < \infty \text{ for } t \in [-r, t_0], \tag{3.20}$$

we are going to prove $\sqrt{w}u_{(n+1)}(t,z) \in C([-r,t_0]; H^1(\mathbb{R})) \cap L^2([-r,t_0]; H^1(\mathbb{R}))$, i.e.,

$$\left\| \sqrt{w}u_{(n+1)}(t) \right\|_{H^1} < \infty \tag{3.21}$$

and

$$\int_{-r}^t \left\| \sqrt{w}u_{(n+1)}(s) \right\|_{H^1}^2 ds < \infty \text{ for } t \in [-r, t_0]. \tag{3.22}$$

We will derive these results by the following two lemmas.

**Lemma 3.1.** *It holds that*

$$\begin{aligned}
&\left\| \sqrt{w}u_{(n+1)}(t) \right\|_{L^2}^2 + \int_0^t \left\| \sqrt{w}u_{(n+1)}(s) \right\|_{L^2}^2 ds \\
&\le C\left[ \left\| \sqrt{w}u_0(0) \right\|_{L^2}^2 + \int_{-r}^0 \left\| \sqrt{w}u_0(s) \right\|_{L^2}^2 ds + \int_0^t \left\| \sqrt{w}u_{(n)}(s) \right\|_{L^2}^2 ds \right]
\end{aligned} \tag{3.23}$$

*for $t \in [0, t_0]$ and some constant $C > 0$.*

**Proof.** Multiplying (3.1) by $2w(z)u_{(n+1)}(t,z)$ and noting that $w(z)=e^{-2\lambda_* z}$, we have

$$\begin{aligned}
&\left[wu_{(n+1)}^2\right]_t+\left\{c_*wu_{(n+1)}^2-2Dwu_{(n+1)}[u_{(n+1)}]_z\right\}_z+2c_*\lambda_*wu_{(n+1)}^2\\
&-4D\lambda_*wu_{(n+1)}[u_{(n+1)}]_z+2Dw[u_{(n+1)}]_z^2+2dwu_{(n+1)}^2\\
=&\,2\varepsilon wu_{(n+1)}\int_{-\infty}^{\infty}P(u_{(n)}(t-r,z-y-c_*r))f_\alpha(y)dy. \qquad (3.24)
\end{aligned}$$

Using the Cauchy-Schwarz inequality ($2ab\le a^2+b^2$) yields

$$\left|4D\lambda_*wu_{(n+1)}[u_{(n+1)}]_z\right|\le 2Dw[u_{(n+1)}]_z^2+2D\lambda_*^2wu_{(n+1)}^2. \qquad (3.25)$$

Then it follows from (3.24) and (3.25) that

$$\begin{aligned}
&\left[wu_{(n+1)}^2\right]_t+\left\{c_*wu_{(n+1)}^2-2Dwu_{(n+1)}[u_{(n+1)}]_z\right\}_z+2(c_*\lambda_*-D\lambda_*^2+d)wu_{(n+1)}^2\\
\le&\,2\varepsilon wu_{(n+1)}\int_{-\infty}^{\infty}P(u_{(n)}(t-r,z-y-c_*r))f_\alpha(y)dy. \qquad (3.26)
\end{aligned}$$

Using (3.14) and the Cauchy-Schwarz inequality ($ab\le \eta a^2+\frac{1}{4\eta}b^2,\eta>0$) yields

$$\begin{aligned}
&2\varepsilon w\left|u_{(n+1)}(t,z)P(u_{(n)}(t-r,z-y-c_*r))\right|\\
&<2\varepsilon b'(0)w\left|u_{(n+1)}(t,z)u_{(n)}(t-r,z-y-c_*r)\right|\\
&\le 2\varepsilon b'(0)\delta_1wu_{(n+1)}^2(t,z)+\frac{\varepsilon b'(0)}{2\delta_1}wu_{(n)}^2(t-r,z-y-c_*r), \qquad (3.27)
\end{aligned}$$

where $\delta_1>0$ will be determined later.

Integrating (3.26) over $\mathbb{R}\times[0,t]$ with respect to $z$ and $t$, and noting the vanishing term at far fields, i.e.,

$$\left\{c_*wu_{(n+1)}^2-2Dwu_{(n+1)}[u_{(n+1)}]_z\right\}\Big|_{z=-\infty}^{\infty}=0,$$

because of $\sqrt{w}u_{(n+1)}|_{z=\pm\infty}=0$ (by (3.16)) and $\sqrt{w}[u_{(n+1)}]_z|_{z=\pm\infty}=0$ (by (3.18)), substituting (3.27) into the resultant inequality yield

$$\begin{aligned}
&\left\|\sqrt{w}u_{(n+1)}(t)\right\|_{L^2}^2+2[c_*\lambda_*-D\lambda_*^2+d-\varepsilon b'(0)\delta_1]\int_0^t\left\|\sqrt{w}u_{(n+1)}(s)\right\|_{L^2}^2ds\\
\le&\left\|\sqrt{w}u_0(0)\right\|_{L^2}^2+\frac{\varepsilon b'(0)}{2\delta_1}\int_0^t\int_{-\infty}^{\infty}w(z)\int_{-\infty}^{\infty}u_{(n)}^2(s-r,z-y-c_*r)f_\alpha(y)dydzds. \qquad (3.28)
\end{aligned}$$

Using (2.3) and choosing $\delta_1=\frac{1}{2}e^{\alpha\lambda_*^2-\lambda_*c_*r}$, then (3.28) is reduced to

$$\begin{aligned}
&\left\|\sqrt{w}u_{(n+1)}(t)\right\|_{L^2}^2+\varepsilon b'(0)e^{\alpha\lambda_*^2-\lambda_*c_*r}\int_0^t\left\|\sqrt{w}u_{(n+1)}(s)\right\|_{L^2}^2ds\\
\le&\left\|\sqrt{w}u_0(0)\right\|_{L^2}^2+\varepsilon b'(0)e^{-\alpha\lambda_*^2+\lambda_*c_*r}\\
&\times\int_0^t\int_{-\infty}^{\infty}w(z)\int_{-\infty}^{\infty}u_{(n)}^2(s-r,z-y-c_*r)f_\alpha(y)dydzds. \qquad (3.29)
\end{aligned}$$

By changing variables $y \to y$, $z - y - c_* r \to z$, $s - r \to s$ and using the fact that

$$\int_{-\infty}^{\infty} f_\alpha(y) \frac{w(z+y+c_*r)}{w(z)} dy = e^{4\alpha\lambda_*^2 - 2\lambda_* c_* r}, \tag{3.30}$$

one can estimate the last term on the right-hand side of (3.29) that

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} w(z) \int_{-\infty}^{\infty} u_{(n)}^2(s-r, z-y-c_*r) f_\alpha(y) dy dz ds \\
=& \int_{-r}^{t-r} \int_{-\infty}^{\infty} w(z+y+c_*r) \int_{-\infty}^{\infty} u_{(n)}^2(s,z) f_\alpha(y) dy dz ds \\
=& \int_0^{t-r} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{w(z+y+c_*r)}{w(z)} f_\alpha(y) dy \right] w(z) u_{(n)}^2(s,z) dz ds \\
&+ \int_{-r}^{0} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{w(z+y+c_*r)}{w(z)} f_\alpha(y) dy \right] w(z) u_0^2(s,z) dz ds \\
\le& e^{4\alpha\lambda_*^2 - 2\lambda_* c_* r} \left[ \int_0^t \int_{-\infty}^{\infty} w(z) u_{(n)}^2(s,z) dz ds + \int_{-r}^{0} \int_{-\infty}^{\infty} w(z) u_0^2(s,z) dz ds \right].
\end{aligned} \tag{3.31}$$

Substituting (3.31) into (3.29) yields

$$\begin{aligned}
&\left\|\sqrt{w} u_{(n+1)}(t)\right\|_{L^2}^2 + \int_0^t \left\|\sqrt{w} u_{(n+1)}(s)\right\|_{L^2}^2 ds \\
\le& C\left[ \left\|\sqrt{w} u_0(0)\right\|_{L^2}^2 + \int_{-r}^{0} \left\|\sqrt{w} u_0(s)\right\|_{L^2}^2 ds + \int_0^t \left\|\sqrt{w} u_{(n)}(s)\right\|_{L^2}^2 ds \right]
\end{aligned} \tag{3.32}$$

for $t \in [0, t_0]$ and some constant $C > 0$. The proof is completed. □

**Lemma 3.2.** *It holds that*

$$\begin{aligned}
&\left\|\sqrt{w} [u_{(n+1)}]_z(t)\right\|_{L^2}^2 + \int_0^t \left\|\sqrt{w} [u_{(n+1)}]_z(s)\right\|_{L^2}^2 ds \\
\le& C \Big\{ \left\|\sqrt{w} [u_0]_z(0)\right\|_{L^2}^2 + \int_0^t \left\|\sqrt{w} [u_{(n)}]_z(s)\right\|_{L^2}^2 ds + \int_{-r}^{0} \left\|\sqrt{w} (u_0)_z(s)\right\|_{L^2}^2 ds \\
&+ \int_0^t \left\|\sqrt{w} u_{(n)}(s)\right\|_{L^2}^2 ds + \int_{-r}^{0} \left\|\sqrt{w} u_0(s)\right\|_{L^2}^2 ds \Big\}
\end{aligned} \tag{3.33}$$

*for $t \in [0, t_0]$ and some constant $C > 0$.*

**Proof.** Differentiating (3.1) with respect to $z$, multiplying the resultant equation by $2w(z) \times [u_{(n+1)}]_z(t,z)$ and noting that $w(z) = e^{-2\lambda_* z}$, we obtain

$$\begin{aligned}
&\Big\{ w[u_{(n+1)}]_z^2 \Big\}_t + \Big\{ c_* w[u_{(n+1)}]_z^2 - 2Dw[u_{(n+1)}]_z [u_{(n+1)}]_{zz} \Big\}_z \\
&\quad + 2(c_*\lambda_* + d) w[u_{(n+1)}]_z^2 - 4D\lambda_* w[u_{(n+1)}]_z [u_{(n+1)}]_{zz} + 2Dw[u_{(n+1)}]_{zz}^2 \\
=& 2\varepsilon w[u_{(n+1)}]_z \int_{-\infty}^{\infty} \Big[ P_1(u_{(n)}(t-r, z-y-c_*r)) \\
&\quad + P_2(u_{(n)}(t-r, z-y-c_*r)) \Big] f_\alpha(y) dy,
\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}P_1(u_{(n)}(t-r,z-y-c_*r)) =&\Big[b'(\phi(z-y-c_*r)+u_{(n)}(t-r,z-y-c_*r))\\&-b'(\phi(z-y-c_*r))\Big]\phi'(z-y-c_*r)\end{aligned}$$

and

$$\begin{aligned}P_2(u_{(n)}(t-r,z-y-c_*r)) =&b'(\phi(z-y-c_*r)+u_{(n)}(t-r,z-y-c_*r))\\&\times [u_{(n)}]_z(t-r,z-y-c_*r).\end{aligned}$$

Using the Cauchy-Schwarz inequality ($2ab \le a^2+b^2$) yields

$$\left|4D\lambda_* w[u_{(n+1)}]_z[u_{(n+1)}]_{zz}\right| \le 2Dw[u_{(n+1)}]_{zz}^2 + 2D\lambda_*^2 w[u_{(n+1)}]_z^2. \tag{3.35}$$

Then we infer from (3.34) and (3.35) that

$$\begin{aligned}&\left\{w[u_{(n+1)}]_z^2\right\}_t + \left\{c_* w[u_{(n+1)}]_z^2 - 2Dw[u_{(n+1)}]_z[u_{(n+1)}]_{zz}\right\}_z\\&+2(c_*\lambda_* - D\lambda_*^2 + d)w[u_{(n+1)}]_z^2\\ \le& 2\varepsilon w[u_{(n+1)}]_z \int_{-\infty}^{\infty}\Big[P_1(u_{(n)}(t-r,z-y-c_*r))\\&+P_2(u_{(n)}(t-r,z-y-c_*r))\Big]f_\alpha(y)dy.\end{aligned} \tag{3.36}$$

Integrating (3.36) over $\mathbb{R}\times[0,t]$ with respect to $z$ and $t$, and noting the second term on the left-hand side vanishes at infinite, i.e.,

$$\left\{c_* w[u_{(n+1)}]_z^2 - 2Dw[u_{(n+1)}]_z[u_{(n+1)}]_{zz}\right\}\Big|_{z=-\infty}^{\infty} = 0,$$

because of $\sqrt{w}[u_{(n+1)}]_z|_{z=\pm\infty} = \sqrt{w}[u_{(n+1)}]_{zz}|_{z=\pm\infty} = 0$ (by (3.18)), we further have

$$\begin{aligned}&\left\|\sqrt{w}[u_{(n+1)}]_z(t)\right\|_{L^2}^2 + 2(c_*\lambda_* - D\lambda_*^2 + d)\int_0^t\int_{-\infty}^{\infty} w[u_{(n+1)}]_z^2 dzds\\ \le& \left\|\sqrt{w}(u_0)_z(0)\right\|_{L^2}^2 + 2\varepsilon\int_0^t\int_{-\infty}^{\infty} w(z)[u_{(n+1)}]_z(s,z)\int_{-\infty}^{\infty}\Big[P_1(u_{(n)}(s-r,z-y-c_*r))\\&+P_2(u_{(n)}(s-r,z-y-c_*r))\Big]f_\alpha(y)dydzds.\end{aligned} \tag{3.37}$$

By the mean value theorem, Cauchy-Schwarz inequality ($ab \le \eta a^2 + \frac{1}{4\eta}b^2, \eta > 0$), (H2) and Lemma 2.1, we get

$$\begin{aligned}&2\varepsilon w(z)\Big|[u_{(n+1)}]_z(s,z)P_1(u_{(n)}(s-r,z-y-c_*r))\Big|\\ =& 2\varepsilon w(z)\Big|[u_{(n+1)}]_z(s,z)\Big[b'(\phi(z-y-c_*r)+u_{(n)}(s-r,z-y-c_*r))\end{aligned}$$

$$
\begin{aligned}
&\quad - b'(\phi(z-y-c_*r))\Big]\phi'(z-y-c_*r)\Big| \\
&\le 2\varepsilon A b''(\tilde{\phi}_2) w(z)\Big|[u_{(n+1)}]_z(s,z)u_{(n)}(s-r,z-y-c_*r)\Big| \\
&\le 2\varepsilon A C_1\delta_2 w(z)[u_{(n+1)}]_z^2(s,z) + \frac{\varepsilon A C_1}{2\delta_2} w(z)u_{(n)}^2(s-r,z-y-c_*r)
\end{aligned} \tag{3.38}
$$

and

$$
\begin{aligned}
&\quad 2\varepsilon w(z)\Big|[u_{(n+1)}]_z(s,z)P_2(u_{(n)}(s-r,z-y-c_*r))\Big| \\
&= 2\varepsilon\Big|w(z)[u_{(n+1)}]_z(s,z)b'(\phi(z-c_*r) \\
&\quad + u_{(n)}(s-r,z-y-c_*r))[u_{(n)}]_z(s-r,z-y-c_*r)\Big| \\
&\le 2\varepsilon b'(0)\delta_3 w(z)[u_{(n+1)}]_z^2(s,z) + \frac{\varepsilon b'(0)}{2\delta_3} w(z)[u_{(n)}]_z^2(s-r,z-y-c_*r),
\end{aligned} \tag{3.39}
$$

where $C_1 > 0$ is some constant, $A$ is defined in Lemma 2.1, $\tilde{\phi}_2 = \phi(z-y-c_*r) + \sigma u_n(s-r,z-y-c_*r)$ for some $0<\sigma<1$ and $\delta_2, \delta_3 > 0$ will be determined later.

Substituting (3.38) and (3.39) into (3.37) yields

$$
\begin{aligned}
&\quad \big\|\sqrt{w}[u_{(n+1)}]_z(t)\big\|_{L^2}^2 + 2\Big[c_*\lambda_* - D\lambda_*^2 + d - \varepsilon b'(0)\delta_3 - \varepsilon A C_1\delta_2\Big] \\
&\quad \times \int_0^t \big\|\sqrt{w}[u_{(n+1)}]_z(s)\big\|_{L^2}^2 ds \\
&\le \big\|\sqrt{w}(u_0)_z(0)\big\|_{L^2}^2 + \frac{\varepsilon b'(0)}{2\delta_3}\int_0^t\int_{-\infty}^{\infty} w(z) \\
&\quad \times \int_{-\infty}^{\infty}[u_{(n)}]_z^2(s-r,z-y-c_*r)f_\alpha(y)dydzds \\
&\quad + \frac{\varepsilon A C_1}{2\delta_2}\int_0^t\int_{-\infty}^{\infty} w(z)\int_{-\infty}^{\infty} u_{(n)}^2(s-r,z-y-c_*r)f_\alpha(y)dydzds.
\end{aligned} \tag{3.40}
$$

Using (2.3) and choosing $0<\delta_2\ll 1$ and $\delta_3 = \frac{1}{2}e^{\alpha\lambda_*^2-\lambda_*c_*r}$, inequality (3.40) is reduced to

$$
\begin{aligned}
&\quad \big\|\sqrt{w}[u_{(n+1)}]_z(t)\big\|_{L^2}^2 + \int_0^t \big\|\sqrt{w}[u_{(n+1)}]_z(s)\big\|_{L^2}^2 ds \\
&\le C\Big\{\big\|\sqrt{w}[u_0]_z(0)\big\|_{L^2}^2 + \int_0^t\int_{-\infty}^{\infty} w(z)\int_{-\infty}^{\infty}[u_{(n)}]_z^2(s-r,z-y-c_*r)f_\alpha(y)dydzds \\
&\quad + \int_0^t\int_{-\infty}^{\infty} w(z)\int_{-\infty}^{\infty} u_{(n)}^2(s-r,z-y-c_*r)f_\alpha(y)dydzds\Big\}.
\end{aligned} \tag{3.41}
$$

By changing variables $y\to y$, $z-y-c_*r\to z$, $s-r\to s$, using (3.30) and similar to (3.31), we derive

$$
\int_0^t\int_{-\infty}^{\infty} w(z)\int_{-\infty}^{\infty}[u_{(n)}]_z^2(s-r,z-y-c_*r)f_\alpha(y)dydzds
$$

$$
\begin{aligned}
&+\int_0^t\int_{-\infty}^{\infty}w(z)\int_{-\infty}^{\infty}u_{(n)}^2(s-r,z-y-c_*r)f_\alpha(y)dydzds\\
&\le C\bigg\{\int_0^t\big\|\sqrt{w}[u_{(n)}]_z(s)\big\|_{L^2}^2ds+\int_{-r}^0\big\|\sqrt{w}[u_0]_z(s)\big\|_{L^2}^2ds\\
&\quad+\int_0^t\big\|\sqrt{w}u_{(n)}(s)\big\|_{L^2}^2ds+\int_{-r}^0\big\|\sqrt{w}u_0(s)\big\|_{L^2}^2ds\bigg\}. \qquad (3.42)
\end{aligned}
$$

Inserting (3.42) into (3.41) yields

$$
\begin{aligned}
&\big\|\sqrt{w}[u_{(n+1)}]_z(t)\big\|_{L^2}^2+\int_0^t\big\|\sqrt{w}[u_{(n+1)}]_z(s)\big\|_{L^2}^2ds\\
&\le C\bigg\{\big\|\sqrt{w}[u_0]_z(0)\big\|_{L^2}^2+\int_0^t\big\|\sqrt{w}[u_{(n)}]_z(s)\big\|_{L^2}^2ds+\int_{-r}^0\big\|\sqrt{w}[u_0]_z(s)\big\|_{L^2}^2ds\\
&\quad+\int_0^t\big\|\sqrt{w}u_{(n)}(s)\big\|_{L^2}^2ds+\int_{-r}^0\big\|\sqrt{w}u_0(s)\big\|_{L^2}^2ds\bigg\}
\end{aligned}
$$

for $t\in[0,t_0]$ and some constant $C>0$. This finishes the proof. □

Noting the facts that

$$
\begin{aligned}
\big\|\sqrt{w}u_{(n+1)}\big\|_{H^1}^2&=\big\|\sqrt{w}u_{(n+1)}\big\|_{L^2}^2+\big\|[\sqrt{w}u_{(n+1)}]_z\big\|_{L^2}^2\\
&=\big\|\sqrt{w}u_{(n+1)}\big\|_{L^2}^2+\big\|\sqrt{w}[u_{(n+1)}]_z-\lambda_*\sqrt{w}u_{(n+1)}\big\|_{L^2}^2\\
&\le C\bigg\{\big\|\sqrt{w}u_{(n+1)}\big\|_{L^2}^2+\big\|\sqrt{w}[u_{(n+1)}]_z\big\|_{L^2}^2\bigg\} \qquad (3.43)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|\sqrt{w}u_{(n)}\big\|_{L^2}^2+\big\|\sqrt{w}[u_{(n)}]_z\big\|_{L^2}^2\\
&=\big\|\sqrt{w}u_{(n)}\big\|_{L^2}^2+\big\|\sqrt{w}[u_{(n)}]_z-\lambda_*\sqrt{w}u_{(n)}+\lambda_*\sqrt{w}u_{(n)}\big\|_{L^2}^2\\
&=\big\|\sqrt{w}u_{(n)}\big\|_{L^2}^2+\big\|[\sqrt{w}u_{(n)}]_z+\lambda_*\sqrt{w}u_{(n)}\big\|_{L^2}^2\\
&\le\big\|\sqrt{w}u_{(n)}\big\|_{L^2}^2+2\big\|[\sqrt{w}u_{(n)}]_z\big\|_{L^2}^2+2\lambda_*^2\big\|\sqrt{w}u_{(n)}\big\|_{L^2}^2\\
&\le C\big\|\sqrt{w}u_{(n)}\big\|_{H^1}^2\quad\text{for } n\in\mathbb{N}. \qquad (3.44)
\end{aligned}
$$

Combining Lemma 3.1, Lemma 3.2, (3.43) and (3.44), we obtain the following energy estimate

$$
\begin{aligned}
&\big\|\sqrt{w}u_{(n+1)}(t)\big\|_{H^1}^2+\int_0^t\big\|\sqrt{w}u_{(n+1)}(s)\big\|_{H^1}^2ds\\
&\le C\bigg[\big\|\sqrt{w}u_0(0)\big\|_{H^1}^2+\int_{-r}^0\big\|\sqrt{w}u_0(s)\big\|_{H^1}^2ds+\int_0^t\big\|\sqrt{w}u_{(n)}(s)\big\|_{H^1}^2ds\bigg]\\
&\le C\bigg[\big\|\sqrt{w}u_0(0)\big\|_{H^1}^2+\int_{-r}^0\big\|\sqrt{w}u_0(s)\big\|_{H^1}^2ds+t_0\sup_{t\in[-r,t_0]}\big\|\sqrt{w}u_{(n)}(t)\big\|_{H^1}^2\bigg] \qquad (3.45)
\end{aligned}
$$

for $t\in[0,t_0]$ and some constant $C>0$.

Thus from (3.19), (3.20) and the assumption $u_0\in X(-r,t_0)$, we conclude that (3.21) and (3.22) hold, i.e.,

$$\sqrt{w}u_{(n+1)}(t,z)\in C([-r,t_0];H^1(\mathbb{R}))\cap L^2([-r,t_0];H^1(\mathbb{R})).$$

**Step 4.** This step is devoted to completing the proof of Proposition 3.1.

Combining (3.8) and (3.45), we conclude that

$$M^2_{u_{(n+1)}}(t_0) \le C\Big[\max_{s\in[-r,0]} \big\|u_0(s)\big\|^2_C + \big\|\sqrt{w}u_0(0)\big\|^2_{H^1} + \int_{-r}^0 \big\|\sqrt{w}u_0(s)\big\|^2_{H^1} ds\Big] + Ct_0 M^2_{u_{(n)}}(t_0). \tag{3.46}$$

Thus, we can see that $\mathcal{F}$ defined in (3.1) is a contraction mapping from $X(-r,t_0)$ to itself if $\max_{s\in[-r,0]} \|u_0(s)\|^2_C + \|\sqrt{w}u_0(0)\|^2_{H^1} + \int_{-r}^0 \|\sqrt{w}u_0(s)\|^2_{H^1} ds$ is suitably small and $0 < t_0 \ll 1$. Hence Banach fixed point theorem implies that (3.1) admits a unique local solution $u(t,z) \in X(-r,t_0)$. Since the convergence $\lim_{n\to\infty} u_{(n)}(t,z) = u(t,z)$ is uniform for $(t,z) \in [-r,t_0]\times\mathbb{R}$ and $u_{(n)}(t,z) \in C_{\text{unif}}[-r,t_0]$, we can guarantee $u(t,z) \in C_{\text{unif}}[-r,t_0]$. For this unique solution $u(t,z) \in X(-r,t_0)$, it is easy to see from (3.46) that $M_u(t_0) \le CM_{u_0}(0)$ for some constant $C > 0$. □

**Remark 3.1.** (i) By the definitions of the weight function (2.12) and the solution space (2.13), we know that $\lim_{z\to-\infty} \sqrt{w(z)}u(t,z) = 0$ and $\lim_{z\to-\infty} w(z) = \infty$. This is enough to justify that $\lim_{z\to-\infty} u(t,z) = 0$ exists uniformly with respect to $t \in [-r,t_0]$, so one does not need to require that $\lim_{z\to-\infty} u(t,z)$ exists uniformly with respect to $t \in [-r,t_0]$ in the solution space $X(-r,t_0)$.

(ii) Compared with the solution space defined in [7], our solution space (2.13) requires that the function $u(t,z)$ satisfies additional condition

$$\lim_{z\to\pm\infty} \sqrt{w(z)}u(t,z) = 0. \tag{3.47}$$

From the proof of Proposition 3.1, one can see that this space setting has helped us obtain from (3.18) that

$$\lim_{z\to\pm\infty} \sqrt{w(z)}u_z(t,z) = \lim_{z\to\pm\infty} \sqrt{w(z)}u_{zz}(t,z) = 0, \tag{3.48}$$

which will be used to ensure (4.11) and (4.28) hold.

## 4. A priori estimate

In this section, we establish the a priori estimate of the perturbed solution (2.7) by the anti-weighted energy method and the nonlinear Halanay's inequality.

**Proposition 4.1.** (A priori estimate) *Assume that (H1)-(H3) hold and $d \ge |b'(v_+)|$ for any $r > 0$ or $d < |b'(v_+)|$ for $0 < r < \bar{r}$. Let $u(t,z) \in X(-r,t_0)$ be a local solution for (2.7) with constant $t_0 > 0$, then there exist constants $C > 0$ and $\mu > 0$ independent of $t_0$ such that, when $M_u(t_0) \ll 1$,*

$$\big\|u(t)\big\|^2_C + \big\|\sqrt{w}u(t)\big\|^2_{H^1} \le Ce^{-2\mu t}\Big[\max_{s\in[-r,0]} \big\|u_0(s)\big\|^2_C + \big\|\sqrt{w}u_0(0)\big\|^2_{H^1} + \int_{-r}^0 \big\|\sqrt{w}u_0(s)\big\|^2_{H^1} ds\Big] \tag{4.1}$$

*hold for $t \in [0,t_0]$.*

First of all, we shift $u(t,z)$ to $u(t,z+x_0)$ by the constant $x_0$, which will be determined in Lemma 4.6 and make the following transformation

$$U(t,z)=\sqrt{w(z)}u(t,z+x_0)=e^{-\lambda_* z}u(t,z+x_0). \tag{4.2}$$

The transformation (4.2) is inspired by Huang et al. [16]. Substituting $u(t,z+x_0)=[w(z)]^{-\frac{1}{2}}U(t,z)$ into (2.9) yields the following equation for the new unknown $U(t,z)$

$$\begin{cases} \quad U_t-DU_{zz}+k_1U_z+k_2U \\ \quad -\varepsilon e^{-\lambda_* c_* r}\displaystyle\int_{-\infty}^{\infty} b'(\phi(z-y-c_*r+x_0))U(t-r,z-y-c_*r)e^{-\lambda_* y}f_\alpha(y)dy \\ =\varepsilon\displaystyle\int_{-\infty}^{\infty}\tilde{G}(U(t-r,z-y-c_*r))f_\alpha(y)dy, \quad (t,z)\in\mathbb{R}^+\times\mathbb{R}, \\ U(s,z)=\sqrt{w(z)}u(s,z+x_0)=:U_0(s,z),\ (s,z)\in[-r,0]\times\mathbb{R}, \end{cases} \tag{4.3}$$

where

$$k_1:=c_*-2D\lambda_*, \qquad k_2:=c_*\lambda_*-D\lambda_*^2+d \tag{4.4}$$

and

$$\tilde{G}(U(t-r,z-y-c_*r))=e^{-\lambda_* z}G(u(t-r,z-y-c_*r+x_0)), \tag{4.5}$$

which satisfies (by Taylor' theorem)

$$\begin{aligned} \left|\tilde{G}(U(t-r,z-y-c_*r))\right| &\le Ce^{-\lambda_* z}\left|u(t-r,z-y-c_*r+x_0)\right|^2 \\ &=Ce^{\lambda_* z-2\lambda_* y-2\lambda_* c_* r}\left|U(t-r,z-y-c_*r)\right|^2 \\ &=\frac{C}{\sqrt{w(z)}}e^{-2\lambda_* y-2\lambda_* c_* r}\left|U(t-r,z-y-c_*r)\right|^2. \end{aligned} \tag{4.6}$$

Next we will prove the a priori estimate (4.1) by several lemmas.

**Lemma 4.1.** *It holds that*

$$\begin{aligned} &e^{2\mu_1 t}\left\|U(t)\right\|_{L^2}^2+2D\int_0^t e^{2\mu_1 s}\left\|U_z(s)\right\|_{L^2}^2ds+\int_0^t\int_{-\infty}^{\infty}A_{\mu_1}(z)e^{2\mu_1 s}\left|U(s,z)\right|^2dzds \\ &\le\left\|U_0(0)\right\|_{L^2}^2+C\int_{-r}^0\left\|U_0(s)\right\|_{L^2}^2ds+CM_u(t_0)\int_0^t\int_{-\infty}^{\infty}e^{2\mu_1 s}\left|U(s,z)\right|^2dzds, \end{aligned} \tag{4.7}$$

*where*

$$A_{\mu_1}(z):=B(z)-2\mu_1-\varepsilon e^{\alpha\lambda_*^2-\lambda_* c_* r}(e^{2\mu_1 r}-1)\left|b'(\phi(z+x_0))\right| \tag{4.8}$$

*and*

$$B(z):=2\varepsilon e^{\alpha\lambda_*^2-\lambda_* c_* r}\left[b'(0)-\left|b'(\phi(z+x_0))\right|\right]. \tag{4.9}$$

**Proof.** Multiplying (4.3) by $2e^{2\mu_1 t}U(t,z)$, where $\mu_1>0$ is a constant and will be specified later, we obtain

$$\begin{aligned} &\left(e^{2\mu_1 t}U^2\right)_t+e^{2\mu_1 t}\left(k_1U^2-2DUU_z\right)_z+2De^{2\mu_1 t}U_z^2+2(k_2-\mu_1)e^{2\mu_1 t}U^2 \\ &-2\varepsilon e^{-\lambda_* c_* r}e^{2\mu_1 t}\int_{-\infty}^{\infty}b'(\phi(z-y-c_*r+x_0)) \end{aligned}$$

$$\begin{aligned}&\times e^{-\lambda_* y}U(t,z)U(t-r,z-y-c_*r)f_\alpha(y)dy\\ &=2\varepsilon e^{2\mu_1 t}\int_{-\infty}^{\infty}U(t,z)\tilde{G}(U(t-r,z-y-c_*r))f_\alpha(y)dy.\end{aligned} \tag{4.10}$$

Integrating (4.10) over $\mathbb{R}\times[0,t]$ with respect to $z$ and $t$, and noting the vanishing term at far fields, i.e.,

$$\big(k_1U^2-2DUU_z\big)\Big|_{z=-\infty}^{\infty}=0, \tag{4.11}$$

due to $u(t,z)\in X(-r,t_0)$ implies $U|_{z=\pm\infty}=\sqrt{w}u|_{z=\pm\infty}=0$ and $U_z|_{z=\pm\infty}=(-\lambda_*\sqrt{w}u+\sqrt{w}u_z)|_{z=\pm\infty}=0$ (by (3.47) and (3.48)), we have

$$\begin{aligned}&e^{2\mu_1 t}\big\|U(t)\big\|_{L^2}^2+2D\int_0^t e^{2\mu_1 s}\big\|U_z(s)\big\|_{L^2}^2ds+2\int_0^t\int_{-\infty}^{\infty}(k_2-\mu_1)e^{2\mu_1 s}U^2(s,z)dzds\\ &-2\varepsilon e^{-\lambda_*c_*r}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}b'(\phi(z-y-c_*r+x_0))\\ &\times e^{-\lambda_*y}e^{2\mu_1 s}U(s,z)U(s-r,z-y-c_*r)f_\alpha(y)dydzds\\ =&\big\|U_0(0)\big\|_{L^2}^2+2\varepsilon\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2\mu_1 s}U(s,z)\tilde{G}(U(s-r,z-y-c_*r))f_\alpha(y)dydzds.\end{aligned} \tag{4.12}$$

We now turn to estimate the last term on the left-hand side of (4.12). By making the change of variables $y\to y$, $z-y-c_*r\to z$, $s-r\to s$ and using the fact that

$$\int_{-\infty}^{\infty}e^{-\lambda_*y}f_\alpha(y)dy=e^{\alpha\lambda_*^2}, \tag{4.13}$$

one can get that

$$\begin{aligned}&\Big|\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2\mu_1 s}b'(\phi(z-y-c_*r+x_0))U^2(s-r,z-y-c_*r)e^{-\lambda_*y}f_\alpha(y)dydzds\Big|\\ =&\Big|\int_{-r}^{t-r}\int_{-\infty}^{\infty}e^{2\mu_1(s+r)}b'(\phi(z+x_0))U^2(s,z)\Big[\int_{-\infty}^{\infty}e^{-\lambda_*y}f_\alpha(y)dy\Big]dzds\Big|\\ \le&\, e^{\alpha\lambda_*^2+2\mu_1 r}\Big[\int_0^t\int_{-\infty}^{\infty}e^{2\mu_1 s}\big|b'(\phi(z+x_0))\big|U^2(s,z)dzds\\ &+\int_{-r}^0\int_{-\infty}^{\infty}e^{2\mu_1 s}\big|b'(\phi(z+x_0))\big|U_0^2(s,z)dzds\Big].\end{aligned} \tag{4.14}$$

Using the Cauchy-Schwarz inequality ($2ab\le a^2+b^2$), (4.13), (4.14) and (H2), one can estimate the last term on the left-hand side of (4.12) as follows

$$\begin{aligned}&2\varepsilon e^{-\lambda_*c_*r}\Big|\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2\mu_1 s}b'(\phi(z-y-c_*r+x_0))\\ &\times U(s,z)U(s-r,z-y-c_*r)e^{-\lambda_*y}f_\alpha(y)dydzds\Big|\\ \le&\,\varepsilon e^{-\lambda_*c_*r}\Big|\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2\mu_1 s}b'(\phi(z-y-c_*r+x_0))\end{aligned}$$

$$
\begin{aligned}
&\times \left[U^2(s,z) + U^2(s-r, z-y-c_*r)\right] e^{-\lambda_* y} f_\alpha(y) dydzds \Bigg| \\
&\le \varepsilon e^{-\lambda_* c_* r} \left[ \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu_1 s} \big|b'(\phi(z-y-c_*r+x_0))\big| U^2(s,z) e^{-\lambda_* y} f_\alpha(y) dydzds \right. \\
&\quad + \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu_1 s} \big|b'(\phi(z-y-c_*r+x_0))\big| \\
&\quad \left. \times U^2(s-r, z-y-c_*r) e^{-\lambda_* y} f_\alpha(y) dydzds \right] \\
&\le \varepsilon e^{-\lambda_* c_* r} \left[ \int_0^t \int_{-\infty}^{\infty} e^{2\mu_1 s} \int_{-\infty}^{\infty} \big|b'(\phi(z-y-c_*r+x_0))\big| e^{-\lambda_* y} f_\alpha(y) dy U^2(s,z) dzds \right. \\
&\quad + e^{\alpha\lambda_*^2 + 2\mu_1 r} \int_0^t \int_{-\infty}^{\infty} e^{2\mu_1 s} \big|b'(\phi(z+x_0))\big| U^2(s,z) dzds \\
&\quad \left. + e^{\alpha\lambda_*^2 + 2\mu_1 r} \int_{-r}^0 \int_{-\infty}^{\infty} e^{2\mu_1 s} \big|b'(\phi(z+x_0))\big| U_0^2(s,z) dzds \right] \\
&\le \varepsilon b'(0) e^{\alpha\lambda_*^2 - \lambda_* c_* r} \int_0^t \int_{-\infty}^{\infty} e^{2\mu_1 s} U^2(s,z) dzds \\
&\quad + \varepsilon e^{\alpha\lambda_*^2 - \lambda_* c_* r + 2\mu_1 r} \int_0^t \int_{-\infty}^{\infty} e^{2\mu_1 s} \big|b'(\phi(z+x_0))\big| \big|U(s,z)\big|^2 dzds \\
&\quad + C \int_{-r}^0 \int_{-\infty}^{\infty} \big|U_0(s,z)\big|^2 dzds.
\end{aligned} \tag{4.15}
$$

Using (4.6) and the fact that

$$
\big|u(t, z+x_0)\big| = \frac{\big|U(t,z)\big|}{\sqrt{w(z)}} \le C M_u(t_0), \quad (t,z) \in [0,t_0] \times \mathbb{R}, \tag{4.16}
$$

we can estimate the second term on the right-hand side of (4.12) as follows

$$
\begin{aligned}
&2\varepsilon \left| \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu_1 s} U(s,z) \tilde{G}(U(s-r, z-y-c_*r)) f_\alpha(y) dydzds \right| \\
&\le 2C\varepsilon \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{2\mu_1 s}}{\sqrt{w(z)}} \big|U(s,z)\big| \\
&\quad \times e^{-2\lambda_* y - 2\lambda_* c_* r} \big|U(s-r, z-y-c_*r)\big|^2 f_\alpha(y) dydzds \\
&\le 2C\varepsilon M_u(t_0) \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu_1 s} \\
&\quad \times e^{-2\lambda_* y - 2\lambda_* c_* r} \big|U(s-r, z-y-c_*r)\big|^2 f_\alpha(y) dydzds.
\end{aligned} \tag{4.17}
$$

Making the change of variables $y \to y$, $z - y - c_* r \to z$, $s - r \to s$ and using (3.30), we can estimate that

$$
\begin{aligned}
&2C\varepsilon M_u(t_0) \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu_1 s} e^{-2\lambda_* y - 2\lambda_* c_* r} \big|U(s-r, z-y-c_*r)\big|^2 f_\alpha(y) dydzds \\
&= 2C\varepsilon M_u(t_0) \int_{-r}^{t-r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu_1 (s+r)} e^{-2\lambda_* y - 2\lambda_* c_* r} \big|U(s,z)\big|^2 f_\alpha(y) dydzds
\end{aligned}
$$

$$
\begin{aligned}
&= 2C\varepsilon M_u(t_0)\int_{-r}^{t-r}\int_{-\infty}^{\infty} e^{2\mu_1(s+r)}e^{4\alpha\lambda_*^2-2\lambda_*c_*r}\big|U(s,z)\big|^2dzds\\
&\le 2C\varepsilon M_u(t_0)\int_{0}^{t}\int_{-\infty}^{\infty} e^{2\mu_1(s+r)}e^{4\alpha\lambda_*^2-2\lambda_*c_*r}\big|U(s,z)\big|^2dzds\\
&\quad + 2C\varepsilon M_u(t_0)\int_{-r}^{0}\int_{-\infty}^{\infty} e^{2\mu_1(s+r)}e^{4\alpha\lambda_*^2-2\lambda_*c_*r}\big|U_0(s,z)\big|^2dzds\\
&\le CM_u(t_0)\int_{0}^{t}\int_{-\infty}^{\infty} e^{2\mu_1 s}\big|U(s,z)\big|^2dzds + C\int_{-r}^{0}\int_{-\infty}^{\infty}\big|U_0(s,z)\big|^2dzds. \qquad (4.18)
\end{aligned}
$$

Plugging (4.18) into (4.17) yields

$$
\begin{aligned}
&2\varepsilon\Big|\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\mu_1 s}U(s,z)\tilde{G}(U(s-r,z-y-c_*r))f_\alpha(y)dydzds\Big|\\
&\le CM_u(t_0)\int_{0}^{t}\int_{-\infty}^{\infty} e^{2\mu_1 s}\big|U(s,z)\big|^2dzds + C\int_{-r}^{0}\int_{-\infty}^{\infty}\big|U_0(s,z)\big|^2dzds. \qquad (4.19)
\end{aligned}
$$

It follows from (4.12), (4.15) and (4.19) that

$$
\begin{aligned}
&e^{2\mu_1 t}\big\|U(t)\big\|_{L^2}^2 + 2D\int_0^t e^{2\mu_1 s}\big\|U_z(s)\big\|_{L^2}^2ds + \int_0^t\int_{-\infty}^{\infty} A_{\mu_1}(z)e^{2\mu_1 s}\big|U(s,z)\big|^2dzds\\
&\le \big\|U_0(0)\big\|_{L^2}^2 + C\int_{-r}^{0}\big\|U_0(s)\big\|_{L^2}^2ds + CM_u(t_0)\int_0^t\int_{-\infty}^{\infty} e^{2\mu_1 s}\big|U(s,z)\big|^2dzds, \qquad (4.20)
\end{aligned}
$$

where $A_{\mu_1}(z)$ is defined in (4.8) and we have used (4.4) and $k_2 = c_*\lambda_* - D\lambda_*^2 + d = \varepsilon b'(0)e^{\alpha\lambda_*^2-\lambda_*c_*r}$. The proof is finished. □

**Remark 4.1.** (i) In [7, 20], the authors used the boundedness of $H^1$-norm of $U \in X(-r,t_0)$ to guarantee $U$ vanishes at infinite (see equation (3.15) in [7] and Proposition 2.2 in [20]). However, they didn't obtain the boundedness of $H^2$-norm of $U$ to ensure $U_z$ vanish at infinite or directly prove $U_z$ vanishes at infinite, such that (4.11) can hold.

(ii) Under our setting of the solution space $X(-r,t_0)$ (see (2.13), (3.47) and (3.48)), one can see that, to ensure (4.11) hold, we only require $U = \sqrt{w}u \in H^1(\mathbb{R})$ and do not need to estimate the $H^2$-norm of $U$. Actually, it seems difficult to obtain the boundedness of $H^2$-norm of $U$.

Based on Lemma 4.1, we now establish the crucial lemma.

**Lemma 4.2.** *There exists a sufficiently small $\mu_1 > 0$ such that*

$$A_{\mu_1}(z) \ge C_2 > 0 \qquad (4.21)$$

*for some positive constant $C_2$.*

**Proof.** Using (H2), we estimate $A_{\mu_1}(z)$ as follows

$$
\begin{aligned}
A_{\mu_1}(z) &= 2\varepsilon e^{\alpha\lambda_*^2-\lambda_*c_*r}\Big[b'(0) - \big|b'(\phi(z+x_0))\big|\Big] - 2\mu_1\\
&\quad - \varepsilon e^{\alpha\lambda_*^2-\lambda_*c_*r}(e^{2\mu_1 r}-1)\big|b'(\phi(z+x_0))\big|
\end{aligned}
$$

$$
\begin{aligned}
&\geq 2\varepsilon e^{\alpha\lambda_*^2-\lambda_* c_* r}\left[b'(0)-\max_{v\in(0,M]}\left|b'(v)\right|\right]-2\mu_1\\
&\quad -\varepsilon e^{\alpha\lambda_*^2-\lambda_* c_* r}(e^{2\mu_1 r}-1)\left|b'(\phi(z+x_0))\right|\\
&\geq C_3-\left[2\mu_1+\varepsilon e^{\alpha\lambda_*^2-\lambda_* c_* r}(e^{2\mu_1 r}-1)b'(0)\right]\\
&:= C_2\\
&> 0 \quad \text{for } 0<\mu_1<\mu_2,
\end{aligned}
\tag{4.22}
$$

where $\mu_2$ is the unique positive solution of the equation

$$
C_3-\left[2\mu+\varepsilon e^{\alpha\lambda_*^2-\lambda_* c_* r}(e^{2\mu r}-1)b'(0)\right]=0 \tag{4.23}
$$

and

$$
C_3 := 2\varepsilon e^{\alpha\lambda_*^2-\lambda_* c_* r}\left[b'(0)-\max_{v\in(0,M]}\left|b'(v)\right|\right]>0.
$$

The proof is completed. □

**Lemma 4.3.** *Let $U(t,z)\in X(-r,t_0)$, then*

$$
\left\|U(t)\right\|_{L^2}^2 \leq C\left[\left\|U_0(0)\right\|_{L^2}^2+\int_{-r}^0\left\|U_0(s)\right\|_{L^2}^2 ds\right]e^{-2\mu_1 t}, \tag{4.24}
$$

*provided $M_u(t_0)\ll 1$, where $\mu_1$ and $C$ are positive constants.*

**Proof.** Combing Lemma 4.1 and Lemma 4.2, i.e., substituting (4.21) into (4.7) yields

$$
\begin{aligned}
&e^{2\mu_1 t}\left\|U(t)\right\|_{L^2}^2+2D\int_0^t e^{2\mu_1 s}\left\|U_z(s)\right\|_{L^2}^2 ds\\
&\quad +[C_2-CM_u(t_0)]\int_0^t\int_{-\infty}^{\infty} e^{2\mu_1 s}\left|U(s,z)\right|^2 dzds\\
&\leq C\left[\left\|U_0(0)\right\|_{L^2}^2+\int_{-r}^0\left\|U_0(s)\right\|_{L^2}^2 ds\right].
\end{aligned}
\tag{4.25}
$$

Dropping the positive terms $\int_0^t\int_{\mathbb{R}} e^{2\mu_1 s}|U(s,z)|^2 dzds$ and $\int_0^t e^{2\mu_1 s}\|U_z(s)\|_{L^2}^2 ds$ in (4.25), since $C_2-CM_u(t_0)>C$ for $M_u(t_0)\ll 1$ and $D>0$, inequality (4.25) is reduced to

$$
e^{2\mu_1 t}\left\|U(t)\right\|_{L^2}^2 \leq C\left[\left\|U_0(0)\right\|_{L^2}^2+\int_{-r}^0\left\|U_0(s)\right\|_{L^2}^2 ds\right], \tag{4.26}
$$

which gives

$$
\left\|U(t)\right\|_{L^2}^2 \leq C\left[\left\|U_0(0)\right\|_{L^2}^2+\int_{-r}^0\left\|U_0(s)\right\|_{L^2}^2 ds\right]e^{-2\mu_1 t}.
$$

This ends the proof. □

**Lemma 4.4.** *Let $U(t,z)\in X(-r,t_0)$, then*

$$
\left\|U_z(t)\right\|_{L^2}^2 \leq C\left[\left\|[U_0]_z(0)\right\|_{L^2}^2+\int_{-r}^0\left\|[U_0]_z(s)\right\|_{L^2}^2 ds\right]e^{-2\mu_1 t}, \tag{4.27}
$$

*provided $M_u(t_0)\ll 1$.*

**Proof.** Differentiating equation (4.3) with respect to $z$ and multiplying it by $2e^{2\mu_1 t}U_z$, then integrating the resultant equation with respect to $z$ and $t$ over $\mathbb{R}\times[0,t]$, one can similarly prove (4.27) provided $M_u(t_0)\ll 1$. The details is omitted here. □

**Remark 4.2.** Indeed, in the proof of (4.27), we have used the fact that

$$\left(c_* U_z^2 - 2DU_zU_{zz}\right)\Big|_{z=-\infty}^{\infty} = 0, \tag{4.28}$$

which is also guaranteed by (3.47) and (3.48). Hence we do not need to estimate the $H^3$-norm of $U=\sqrt{w}u$ to ensure (4.28) hold.

Combining (4.24) and (4.27), we have established the following energy estimate.

**Lemma 4.5.** *Let $U(t,z)\in X(-r,t_0)$, then*

$$\|U(t)\|_{H^1}^2 \le C\delta^2 e^{-2\mu_1 t}, \tag{4.29}$$

*i.e.,*

$$\|\sqrt{w}u(t)\|_{H^1}^2 \le C\delta^2 e^{-2\mu_1 t}, \tag{4.30}$$

*provided $M_u(t_0)\ll 1$, where*

$$\delta^2 := \|\sqrt{w}u_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\sqrt{w}u_0(s)\|_{H^1}^2 ds. \tag{4.31}$$

Next let $u(t,z)\in X(-r,t_0)$ be the unique local solution of the Cauchy problem (2.7) guaranteed by Proposition 3.1, then $u(t,z)\in C_{\text{unif}}[-r,t_0]$. Based on this, we derive the exponential decay of $u(t,z)$ when $z$ is sufficiently close to $\infty$.

**Lemma 4.6.** *There exist a large number $x_0\gg 1$ (independent of $t$) and a number $\mu_3>0$ such that*

*(i) for any $r>0$ and $d\ge \varepsilon|b'(v_+)|$,*

$$\|u(t)\|_{L^\infty[x_0,\infty)} \le Ce^{-\mu_3 t}\|u_0\|_{L^\infty([-r,0]\times\mathbb{R})}, \tag{4.32}$$

*provided $M_u(t_0)\ll 1$;*

*(ii) for $0<r<\bar r$ and $d<\varepsilon|b'(v_+)|$, where $\bar r$ is defined in (2.17), the exponential decay (4.32) holds for $M_u(t_0)\ll 1$.*

**Proof.** Since $u(t,z)\in X(-r,t_0)$, we obtain that $\lim_{z\to\infty} u(t,z)$ and $\lim_{z\to\infty} u_z(t,z) = \lim_{z\to\infty} u_{zz}(t,z) = 0$ exist uniformly with respect to $t\in[-r,t_0]$. Denote $w(t):=u(t,\infty)$ and $w_0(s):=u_0(s,\infty)$ for $s\in[-r,0]$. Taking $z\to\infty$ to (2.9), we have

$$\begin{cases} w'(t)+dw(t)-\varepsilon b'(v_+)w(t-r)=\varepsilon G(w(t-r)), \\ w(s)=w_0(s),\ s\in[-r,0]. \end{cases} \tag{4.33}$$

Applying the nonlinear Halanay's inequality [7, 16, 20], we obtain that: For any $r>0$ and $d\ge\varepsilon|b'(v_+)|$,

$$|w(t)| \le C\|w_0\|_{L^\infty(-r,0)} e^{-\mu_3 t} \tag{4.34}$$

for some $0 < \mu_3 < d$, provided $M_u(t_0) \ll 1$; for $0 < r < \bar{r}$ and $d < \varepsilon|b'(v_+)|$, where $\bar{r}$ is defined in (2.17), the decay estimate (4.34) holds for $M_u(t_0) \ll 1$.

Multiplying (2.9) by $e^{dt}$ leads to

$$\begin{aligned}&(e^{dt}u)_t + c_* e^{dt} u_z - De^{dt}u_{zz}\\&- \varepsilon e^{dt}\int_{-\infty}^{\infty} b'(\phi(z-y-c_*r))u(t-r, z-y-c_*r)f_\alpha(y)dy\\=\ &\varepsilon e^{dt}\int_{-\infty}^{\infty} G(u(t-r, z-y-c_*r))f_\alpha(y)dy.\end{aligned} \tag{4.35}$$

Integrating (4.35) with respect to $t$ over $[0,t]$ yields

$$\begin{aligned}u(t,z) = e^{-dt}\Big[&u_0(0,z) - c_*\int_0^t e^{ds}u_z(s,z)ds + D\int_0^t e^{ds}u_{zz}(s,z)ds\\&+ \varepsilon\int_0^t e^{ds}\int_{-\infty}^{\infty} b'(\phi(z-y-c_*r))u(s-r,z-y-c_*r)f_\alpha(y)dyds\\&+ \varepsilon\int_0^t e^{ds}\int_{-\infty}^{\infty} G(u(s-r,z-y-c_*r))f_\alpha(y)dyds\Big].\end{aligned}$$

Then we deduce for $0 < \mu_3 < d$ that

$$\begin{aligned}e^{\mu_3 t}u(t,z) = e^{-(d-\mu_3)t}\Big[&u_0(0,z) - c_*\int_0^t e^{ds}u_z(s,z)ds + D\int_0^t e^{ds}u_{zz}(s,z)ds\\&+ \varepsilon\int_0^t e^{ds}\int_{-\infty}^{\infty} b'(\phi(z-y-c_*r))u(s-r,z-y-c_*r)f_\alpha(y)dyds\\&+ \varepsilon\int_0^t e^{ds}\int_{-\infty}^{\infty} G(u(s-r,z-y-c_*r))f_\alpha(y)dyds\Big].\end{aligned} \tag{4.36}$$

Passing to the limits in (4.36) as $z \to \infty$, noting all these limits exist uniformly in $t$ and applying the facts $|G(w)| \le Cw^2$ (by (2.11)), (H2) and the decay estimate (4.34) for $w(t)$, we obtain

$$\begin{aligned}&\lim_{z\to\infty}\left|e^{\mu_3 t}u(t,z)\right|\\&= e^{-(d-\mu_3)t}\Big|\lim_{z\to\infty}u_0(0,z) - c_*\int_0^t e^{ds}\lim_{z\to\infty}u_z(s,z)ds + D\int_0^t e^{ds}\lim_{z\to\infty}u_{zz}(s,z)ds\\&\quad+ \varepsilon\int_0^t e^{ds}\int_{-\infty}^{\infty}\lim_{z\to\infty}\Big[b'(\phi(z-y-c_*r))u(s-r,z-y-c_*r))\Big]f_\alpha(y)dyds\\&\quad+ \varepsilon\int_0^t e^{ds}\int_{-\infty}^{\infty}\lim_{z\to\infty}G(u(s-r,z-y-c_*r))f_\alpha(y)dyds\Big|\\&= e^{-(d-\mu_3)t}\left|w_0(0) + \varepsilon b'(v_+)\int_0^t e^{ds}w(s-r)ds + \varepsilon\int_0^t e^{ds}G(w(s-r))ds\right|\\&\le e^{-(d-\mu_3)t}\Big[\big|w_0(0)\big| + \varepsilon b'(0)\int_0^t e^{ds}\big|w(s-r)\big|ds + C\varepsilon\int_0^t e^{ds}\big|w(s-r)\big|^2 ds\Big]\\&\le e^{-(d-\mu_3)t}\Big[\big|w_0(0)\big| + C\varepsilon b'(0)\int_0^t e^{(d-\mu_3)s}e^{\mu_3 r}ds + C\varepsilon\int_0^t e^{ds}e^{-2\mu_3(s-r)}ds\Big]\end{aligned}$$

$$
\begin{aligned}
&\le e^{-(d-\mu_3)t}\big|w_0(0)\big| + C\varepsilon b'(0)e^{\mu_3 r}\left|\frac{1}{d-\mu_3} - \frac{e^{-(d-\mu_3)t}}{d-\mu_3}\right| \\
&\quad + C\varepsilon e^{2\mu_3 r}\left|\frac{e^{-\mu_3 t}}{d-2\mu_3} - \frac{e^{-(d-\mu_3)t}}{d-2\mu_3}\right| \\
&\le C \quad \text{for } t \ge 0.
\end{aligned}
\tag{4.37}
$$

Hence there exists a number $x_0 \gg 1$ independent of $t$, such that when $x \ge x_0$,

$$
\sup_{z\in[x_0,\infty)} \big|u(t,z)\big| \le Ce^{-\mu_3 t}\|u_0\|_{L^\infty([-r,0]\times\mathbb{R})}, \quad t \in \mathbb{R}^+,
\tag{4.38}
$$

either for any $r > 0$ and $d \ge \varepsilon|b'(v_+)|$ or $0 < r < \bar{r}$ and $d < \varepsilon|b'(v_+)|$. This finishes the proof. □

From (4.29) and by Sobolev's inequality $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, we obtain

$$
\big\|U(t,z)\big\|_C \le C\big\|U(t)\big\|_{H^1} \le C\delta e^{-\mu_1 t}.
$$

Notice that $U(t,z) = \sqrt{w(z)}u(t,z+x_0) = e^{-\lambda_* z}u(t,z+x_0)$ and $\sqrt{w(z)} = e^{-\lambda_* z} \ge 1$ for $z \in (-\infty, 0]$, we then get

$$
\sup_{z\in(-\infty,0]} \big|u(t,z+x_0)\big| \le C\delta e^{-\mu_1 t}.
$$

This proves the following estimate for the unshifted $u(t,z)$ as follows

$$
\sup_{z\in(-\infty,x_0]} \big|u(t,z)\big| \le C\delta e^{-\mu_1 t},
\tag{4.39}
$$

provided $M_u(t_0) \ll 1$. Finally, combining (4.38) and (4.39), we obtain the following lemma.

**Lemma 4.7.** *For $0 < \mu \le \min\{\mu_1, \mu_3\}$ and $M_u(t_0) \ll 1$, it holds that:*

*(i) for any $r > 0$ and $d \ge \varepsilon|b'(v_+)|$,*

$$
\big\|u(t)\big\|_{L^\infty(\mathbb{R})} \le C\delta e^{-\mu t};
$$

*(ii) for $0 < r < \bar{r}$ and $d < \varepsilon|b'(v_+)|$, where $\bar{r}$ is defined in (2.17),*

$$
\big\|u(t)\big\|_{L^\infty(\mathbb{R})} \le C\delta e^{-\mu t}
$$

*for $t \in [0, t_0]$, $\delta$ is defined in (4.31) and some constant $C > 0$.*

**Proof of Proposition 4.1.** Combining Lemma 4.5 and Lemma 4.7, we obtain the a priori estimate (4.1), i.e.,

$$
\begin{aligned}
\big\|u(t)\big\|_C^2 + \big\|\sqrt{w}u(t)\big\|_{H^1}^2 \le &Ce^{-2\mu t}\Big[\max_{s\in[-r,0]} \big\|u_0(s)\big\|_C^2 + \big\|\sqrt{w}u_0(0)\big\|_{H^1}^2 \\
&+ \int_{-r}^0 \big\|\sqrt{w}u_0(s)\big\|_{H^1}^2 ds\Big]
\end{aligned}
$$

for $t \in [0, t_0]$ and some constant $C > 0$, where $\mu$ is taken to satisfy $0 < \mu \le \min\{\mu_1, \mu_3\}$. □

## 5. Uniqueness of critical traveling waves

In this section, we prove the uniqueness of critical traveling wave solutions of (1.1), i.e., Corollary 2.1. Suppose that $\phi_1(x + c_*t)$ and $\phi_2(x + c_*t)$ are two distinct critical traveling wave solutions of (1.1), which admit the same exponential decay at minus infinity:

$$\phi_1(z) = C_1|z|e^{\lambda_* z} \quad \text{as } z \to -\infty$$

and

$$\phi_2(z) = C_2|z|e^{\lambda_* z} \quad \text{as } z \to -\infty$$

for some constants $C_1, C_2 > 0$, where $\lambda_* > 0$ defined in (2.3). Shift $\phi_2(z)$ to $\phi_2(z + x_*)$ with some constant shift $x_*$. Then we obtain

$$\phi_2(z + x_*) = C_2|z + x_*|e^{\lambda_*(z+x_*)} \to C_1|z|e^{\lambda_* z} \quad \text{as } z \to -\infty$$

by an appropriate selection $x_*$. Therefore, we get

$$|\phi_2(z + x_*) - \phi_1(z)| = o(|z|e^{\lambda_* z}) \quad \text{as } z \to -\infty,$$

which implies that

$$\sqrt{w(z)}[\phi_2(z + x_*) - \phi_1(z)] = e^{-\lambda_* z}[\phi_2(z + x_*) - \phi_1(z)] \in C(\mathbb{R}) \cap H^1(\mathbb{R}).$$

Taking the initial value of (1.1) by

$$v_0(s, x) = \phi_2(x + c_*s + x_*), \quad (s, x) \in [-r, 0] \times \mathbb{R}. \tag{5.1}$$

Clearly, the solution of (1.1) with (5.1) is $\phi_2(x + c_*t + x_*)$. By Theorem 2.1, we have that either $d \geq \varepsilon\big|b'(v_+)\big|$ for any $r > 0$ or $d < \varepsilon\big|b'(v_+)\big|$ for $0 < r < \bar{r}$,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |\phi_2(x + c_*t + x_*) - \phi_1(x + c_*t)| = 0.$$

In other words, $\phi_2(x + c_*t + x_*) = \phi_1(x + c_*t)$ for any $x \in \mathbb{R}$ if $t \gg 1$. Consequently, we obtain the uniqueness of critical traveling waves up to a constant shift.

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