Journal of Applied Analysis and Computation
Volume 16, Number 6, December 2026, 2841–2850
Website:http://www.jaac-online.com
DOI:10.11948/20260023

INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR DIFFERENTIABLE GT-CONVEX FUNCTIONS

Ying Wu[1], Yu-Mei Bai[1] and Feng Qi[2,†]

Abstract In the paper, we establish some new inequalities of the Hermite–Hadamard type for differentiable GT-convex functions.

Keywords GT-convex function, integral inequality, Hermite–Hadamard type inequality.

MSC(2010) 26A51, 26D15, 26E60, 41A55.

1. Introduction

We first recite some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

The famous Hermite–Hadamard integral inequality for convex functions and some of its diverse generalizations can be recalled as follows.

Theorem 1.1. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}.$$

In [7], Tunç and Yildrim defined the MT-convexity as follows.

Definition 1.2 ([7, Definition 2]). Let $I \subseteq \mathbb{R}$ be an interval. A nonnegative function $f : I \to \mathbb{R}_0 = [0, \infty)$ is said to be MT-convex if the inequality

$$f(tx + (1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y)$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

In [4–6], some integral inequalities for MT-convex functions were established.

In [2, 9] the concepts of the HT-convex and GT-convex functions was innovated below, respectively.

[†]The corresponding author.
[1]College of Mathematical Sciences, Inner Mongolia Minzu University, Inner Mongolia, China
[2]17709 Sabal Court, University Village, Dallas, TX 75252-8024, USA
E-mail: wuying19800920@qq.com(Y. Wu), baiym20082025@126.com(Y.-M. Bai),
qifeng618@gmail.com(F. Qi)

Definition 1.3 ([9, Definition 2.1]). Let $I \subseteq \mathbb{R}_+ = (0, \infty)$ is an interval. A function $f : I \to \mathbb{R}$ is called GT-convex if the inequality

$$f\big(x^t y^{1-t}\big) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y)$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

Definition 1.4 ([2, Definition 3.1]). Let $I \subseteq \mathbb{R} \setminus \{0\}$ be a nonempty interval. A function f is called HT-convex on I if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le \frac{\sqrt{1-t}}{2\sqrt{t}} f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}} f(y)$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

In recent decades, a lot of inequalities of the Hermite–Hadamard type for various kinds of the MT-type convex functions have been established. We recite some of them as follows.

Theorem 1.2 ([7, Theorem 2]). *Let $I \subseteq \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}_0$ is MT-convex on I, $a, b \in I$ with $a < b$, and $f \in L_1([a, b])$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \,\mathrm{d}x \quad \text{and} \quad \frac{2}{b-a} \int_a^b \tau(x) f(x) \,\mathrm{d}x \le \frac{f(a) + f(b)}{2},$$

where

$$\tau(x) = \frac{2\sqrt{(b-x)(x-a)}}{b-a}, \quad x \in [a, b].$$

Theorem 1.3 ([9, Theorem 3.1]). *Suppose that $I \subseteq \mathbb{R}_+$ is an interval. If $f : I \to \mathbb{R}_+$ is GT-convex on I, $a, b \in I$ with $a < b$, and $f \in L_1([a, b])$, then*

$$f\big(\sqrt{ab}\big) \le \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \,\mathrm{d}x \le \frac{\pi}{4}[f(a) + f(b)].$$

Theorem 1.4 ([2, Theorem 4.1]). *Suppose that $I \subseteq \mathbb{R} \setminus \{0\}$ is an interval. If $f : I \to \mathbb{R}_0$ is HT-convex on I, $a, b \in I$ with $a < b$, and $f \in L_1([a, b])$, then*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \,\mathrm{d}x \le \frac{\pi}{4}[f(a) + f(b)].$$

For recent developments, please refer to the papers [3, 8, 10], for example.

In this paper, we will establish some new integral inequalities of the Hermite–Hadamard type for differentiable GT-convex functions.

2. Several lemmas

For $u, v \in \mathbb{R}_+$, the weighted geometric mean is

$$G_t(u, v) = u^t v^{1-t}, \quad t \in [0, 1]. \tag{2.1}$$

Lemma 2.1 ([11, Lemma 2.1]). *Suppose that $I \subseteq \mathbb{R}_+$ is an interval. Let $f : I \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a,b])$ and $x \in [a,b]$, then*

$$\begin{aligned}f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}u = \frac{(\ln x - \ln a)^2}{\ln b - \ln a}\int_0^1 (1-t)G_t(a,x)f'(G_t(a,x))\,\mathrm{d}t \\ - \frac{(\ln b - \ln x)^2}{\ln b - \ln a}\int_0^1 tG_t(x,b)f'(G_t(x,b))\,\mathrm{d}t.\end{aligned}$$

Lemma 2.2. *Let $r, s > -1$ and $u, v \in \mathbb{R}_+$ with $u \le v$. Then*

$$\begin{aligned}H_{r,s}(u,v) &= \int_0^1 t^r(1-t)^s G_t(u,v)\,\mathrm{d}t \\ &= \begin{cases} v\Gamma(r+1)\Gamma(s+1)\,{}_1F_1\left(r+1, r+s+2, \ln\dfrac{u}{v}\right), & u < v, \\ uB(r+1, s+1), & u = v, \end{cases}\end{aligned}$$

where $G_t(u,v)$ is defined by (2.1) and

$$\begin{aligned}&\Gamma(z) = \int_0^\infty t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \quad \Re(z) > 0, \\ &B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t, \quad \Re(\alpha), \Re(\beta) > 0, \\ &{}_1F_1(\varrho;\rho;z) = \sum_{k=0}^\infty \frac{(\varrho)_k}{(\rho)_k}\frac{z^k}{k!}, \quad \varrho \in \mathbb{C}, \quad \rho \in \mathbb{C}\setminus\{0,-1,-2,\dots\}, \quad |z| < 1,\end{aligned} \tag{2.2}$$

stand for the Euler gamma function, the Euler beta function, and the Kummer confluent hypergeometric function, respectively.

Proof. The proof is straightforward. □

In Lemma 2.2, taking $r = 1$ and $s = 0$ leads to:

Lemma 2.3. *For $u, v \in \mathbb{R}_+$,*

$$H_{1,0}(u,v) = \int_0^1 tG_t(u,v)\,\mathrm{d}t = \begin{cases} \dfrac{L(u,v) - u}{\ln v - \ln u}, & u \ne v, \\ \dfrac{u}{2}, & u = v, \end{cases}$$

where $G_t(u,v)$ is defined in (2.1) and the logarithmic mean $L(u,v)$ is defined by

$$L(u,v) = \begin{cases} \dfrac{u - v}{\ln u - \ln v}, & u \ne v, \\ u, & u = v. \end{cases} \tag{2.3}$$

In Lemma 2.2, taking $(r,s) = \left(\frac{1}{2}, \frac{1}{2}\right)$, $(r,s) = \left(-\frac{1}{2}, \frac{3}{2}\right)$, and $(r,s) = \left(\frac{3}{2}, -\frac{1}{2}\right)$ results in

Lemma 2.4. *For $u, v \in \mathbb{R}_+$ with $u \le v$,*

$$H_{1/2,1/2}(u,v) = \int_0^1 \sqrt{t(1-t)}\,G_t(u,v)\,\mathrm{d}t$$

$$
\begin{aligned}
&= \begin{cases} \dfrac{\pi\sqrt{uv}}{2(\ln u - \ln v)} J_1\left(\ln\sqrt{\dfrac{u}{v}}\right), & u<v,\\ \dfrac{\pi u}{8}, & u=v, \end{cases}\\
H_{-1/2,3/2}(u,v) &= \int_0^1 \sqrt{\frac{(1-t)^3}{t}}\, G_t(u,v)\,\mathrm{d}t\\
&= \begin{cases} \dfrac{\pi}{2}\sqrt{uv}\left[J_0\left(\ln\sqrt{\dfrac{u}{v}}\right) - \dfrac{\ln\frac{u}{v}+1}{\ln\frac{u}{v}} J_1\left(\ln\sqrt{\dfrac{u}{v}}\right)\right], & u<v,\\ \dfrac{3\pi u}{8}, & u=v, \end{cases}\\
H_{3/2,-1/2}(u,v) &= \int_0^1 \sqrt{\frac{t^3}{1-t}}\, G_t(u,v)\,\mathrm{d}t\\
&= \begin{cases} \dfrac{\pi}{2}\sqrt{uv}\left[J_0\left(\ln\sqrt{\dfrac{u}{v}}\right) - \dfrac{1-\ln\frac{u}{v}}{\ln\frac{u}{v}} J_1\left(\ln\sqrt{\dfrac{u}{v}}\right)\right], & u<v,\\ \dfrac{3\pi u}{8}, & u=v, \end{cases}
\end{aligned}
$$

where $G_t(u,v)$ is defined in (2.1) *and the Bessel function of the first kind $J_\nu(x)$ is defined* [1, Chapters 9 and 10] *by*

$$
J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{2n+\nu}, \quad \nu\in\mathbb{C}\setminus\{-1,-2,\dots\}, \quad z\in\mathbb{C}.
$$

In Lemma 2.2, putting $(r,s)=\left(\frac{1}{2},-\frac{1}{2}\right)$ and $(r,s)=\left(-\frac{1}{2},\frac{1}{2}\right)$ gives

Lemma 2.5. *Let $u,v\in\mathbb{R}_+$ with $u\le v$, then*

$$
\begin{aligned}
H_{1/2,-1/2}(u,v) &= T(u,v)\\
&= \int_0^1 \sqrt{\frac{t}{1-t}}\, G_t(u,v)\,\mathrm{d}t\\
&= \begin{cases} \dfrac{\pi}{2}\sqrt{uv}\left[J_0\left(\ln\sqrt{\dfrac{u}{v}}\right) + J_1\left(\ln\sqrt{\dfrac{u}{v}}\right)\right], & u<v,\\ \dfrac{\pi u}{2}, & u=v, \end{cases}\\
H_{-1/2,1/2}(u,v) &= T(u,v)\\
&= \int_0^1 \sqrt{\frac{1-t}{t}}\, G_t(u,v)\,\mathrm{d}t\\
&= \begin{cases} \dfrac{\pi}{2}\sqrt{uv}\left[J_0\left(\ln\sqrt{\dfrac{u}{v}}\right) - J_1\left(\ln\sqrt{\dfrac{u}{v}}\right)\right], & u<v,\\ \dfrac{\pi u}{2}, & u=v, \end{cases}
\end{aligned}
$$

where $G_t(u,v)$ is defined in (2.1) *and $J_\nu(x)$ is defined in Lemma 2.4.*

3. Integral inequalities of Hermite–Hadamard type

We now start out to establish some new integral inequalities of the Hermite–Hadamard type for differentiable GT-convex functions.

Theorem 3.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ for $q \ge 1$ is GT-convex on $[a, b]$ and $x \in [a, b]$, then*

$$\begin{aligned}
&\left|f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}u\right| \\
\le&\frac{(\ln x - \ln a)^2}{2^{1/q}(\ln b - \ln a)}[H_{0,1}(a,x)]^{1-1/q}\big[H_{1/2,1/2}(a,x)|f'(a)|^q + H_{-1/2,3/2}(a,x)|f'(x)|^q\big]^{1/q} \\
&+ \frac{(\ln b - \ln x)^2}{2^{1/q}(\ln b - \ln a)}[H_{1,0}(x,b)]^{1-1/q}\big[H_{3/2,-1/2}(x,b)|f'(x)|^q + H_{1/2,1/2}(x,b)|f'(b)|^q\big]^{1/q},
\end{aligned} \tag{3.1}$$

where $H_{1,0}(u, v)$ is defined in Lemma 2.3, and $H_{1/2,1/2}(u, v)$, $H_{-1/2,3/2}(u, v)$, and $H_{3/2,-1/2}(u, v)$ are defined in Lemma 2.4.

Proof. For $u, v \in [a, b]$ with $u \le v$, we denote

$$D(u,v) = \frac{(\ln v - \ln u)^2}{\ln b - \ln a}. \tag{3.2}$$

From Lemma 2.1 and Hölder's integral inequality, it follows that

$$\begin{aligned}
&\left|f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}u\right| \\
&\le D(a,x)\int_0^1 (1-t)G_t(a,x)|f'(G_t(a,x))|\,\mathrm{d}t + D(x,b)\int_0^1 tG_t(x,b)|f'(G_t(x,b))|\,\mathrm{d}t \\
&\le D(a,x)\left[\int_0^1 (1-t)G_t(a,x)\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 (1-t)G_t(a,x)|f'(G_t(a,x))|^q\,\mathrm{d}t\right]^{1/q} \\
&\quad + D(x,b)\left[\int_0^1 tG_t(x,b)\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 tG_t(x,b)|f'(G_t(x,b))|^q\,\mathrm{d}t\right]^{1/q}.
\end{aligned} \tag{3.3}$$

Using Lemma 2.3, we obtain

$$\int_0^1 (1-t)G_t(a,x)\,\mathrm{d}t = H_{0,1}(a,x) \quad \text{and} \quad \int_0^1 tG_t(x,b)\,\mathrm{d}t = H_{1,0}(x,b). \tag{3.4}$$

By the GT-convexity of f on $[a, b]$ and Lemma 2.4, we acquire

$$\begin{aligned}
\int_0^1 (1-t)G_t(a,x)|f'(G_t(a,x))|^q\,\mathrm{d}t &\le \int_0^1 (1-t)G_t(a,x)\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^q + \frac{\sqrt{1-t}}{2\sqrt{t}}|f'(x)|^q\right]\mathrm{d}t \\
&= \frac{1}{2}\big[H_{1/2,1/2}(a,x)|f'(a)|^q + H_{-1/2,3/2}(a,x)|f'(x)|^q\big]
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\int_0^1 tG_t(x,b)|f'(G_t(x,b))|^q\,\mathrm{d}t &\le \int_0^1 tG_t(x,b)\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q + \frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right]\mathrm{d}t \\
&= \frac{1}{2}\big[H_{3/2,-1/2}(x,b)|f'(x)|^q + H_{1/2,1/2}(x,b)|f'(b)|^q\big].
\end{aligned} \tag{3.6}$$

Substituting (3.4) to (3.6) into (3.3) yields (3.1). The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ for $q \ge 1$ is GT-convex on $[a, b]$, then*

$$\begin{aligned}
&\left|f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}u\right| \\
\le&\frac{(\ln x - \ln a)^2}{2(\ln b - \ln a)}\bigl[H_{1/2,1/2}(a^q, x^q)|f'(a)|^q + H_{-1/2,3/2}(a^q, x^q)|f'(x)|^q\bigr]^{1/q} \\
&+ \frac{(\ln b - \ln x)^2}{2(\ln b - \ln a)}\bigl[H_{3/2,-1/2}(x^q, b^q)|f'(x)|^q + H_{1/2,1/2}(x^q, b^q)|f'(b)|^q\bigr]^{1/q}
\end{aligned} \tag{3.7}$$

for $x \in [a, b]$, where $H_{1/2,1/2}(u, v)$, $H_{-1/2,3/2}(u, v)$, and $H_{3/2,-1/2}(u, v)$ are defined in Lemma 2.4.

Proof. Using Lemma 2.1, Hölder's integral inequality, and the notation (3.2), we obtain

$$\begin{aligned}
&\left|f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}u\right| \\
&\le D(a, x)\int_0^1 (1 - t)G_t(a, x)|f'(G_t(a, x))|\,\mathrm{d}t + D(x, b)\int_0^1 tG_t(x, b)|f'(G_t(x, b))|\,\mathrm{d}t \\
&\le D(a, x)\left[\int_0^1 (1 - t)\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 (1 - t)[G_t(a, x)]^q|f'(G_t(a, x))|^q\,\mathrm{d}t\right]^{1/q} \\
&\quad + D(x, b)\left(\int_0^1 t\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1 t[G_t(x, b)]^q|f'(G_t(x, b))|^q\,\mathrm{d}t\right]^{1/q}.
\end{aligned}$$

From the inequalities (3.5) and (3.6), it follows that

$$\begin{aligned}
&\int_0^1 (1 - t)[G_t(a, x)]^q|f'(G_t(a, x))|^q\,\mathrm{d}t \\
\le&\int_0^1 (1 - t)G_t(a^q, x^q)\left[\frac{\sqrt{t}}{2\sqrt{1 - t}}|f'(a)|^q + \frac{\sqrt{1 - t}}{2\sqrt{t}}|f'(x)|^q\right]\mathrm{d}t \\
=&\frac{1}{2}\bigl[H_{1/2,1/2}(a^q, x^q)|f'(a)|^q + H_{-1/2,3/2}(a^q, x^q)|f'(x)|^q\bigr]
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
\int_0^1 t[G_t(x, b)]^q|f'(G_t(x, b))|^q\,\mathrm{d}t &\le \int_0^1 tG_t(x^q, b^q)\left[\frac{\sqrt{t}}{2\sqrt{1 - t}}|f'(x)|^q + \frac{\sqrt{1 - t}}{2\sqrt{t}}|f'(b)|^q\right]\mathrm{d}t \\
&= \frac{1}{2}\bigl[H_{3/2,-1/2}(x^q, b^q)|f'(x)|^q + H_{1/2,1/2}(x^q, b^q)|f'(b)|^q\bigr].
\end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3) and simplifying yield the inequality (3.7). The proof of Theorem 3.2 is thus completed. □

Theorem 3.3. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ for $q > 1$ is GT-convex on $[a, b]$, then*

$$\begin{aligned}
&\left|f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}u\right| \\
\le&\frac{(\ln x - \ln a)^2}{\ln b - \ln a}\left[H_{0,\frac{q-\ell}{q-1}}\left(a^{\frac{q-\hbar}{q-1}}, x^{\frac{q-\hbar}{q-1}}\right)\right]^{1-1/q}
\end{aligned}$$

$$\times\left[\frac{H_{1/2,(2\ell-1)/2}\big(a^{\hbar},x^{\hbar}\big)|f'(a)|^q+H_{-1/2,(2\ell+1)/2}\big(a^{\hbar},x^{\hbar}\big)|f'(x)|^q}{2}\right]^{1/q}$$
$$+\frac{(\ln b-\ln x)^2}{\ln b-\ln a}\left[H_{\frac{q-\ell}{q-1},0}\Big(x^{\frac{q-\hbar}{q-1}},b^{\frac{q-\hbar}{q-1}}\Big)\right]^{1-1/q}$$
$$\times\left[\frac{H_{(2\ell+1)/2,-1/2}\big(x^{\hbar},b^{\hbar}\big)|f'(x)|^q+H_{(2\ell-1)/2,1/2}\big(x^{\hbar},b^{\hbar}\big)|f'(b)|^q}{2}\right]^{1/q} \tag{3.10}$$

for $q>\ell$, $\hbar\ge 0$, and $x\in[a,b]$, where $H_{r,s}(u,v)$ are defined in Lemma 2.2.

Proof. Using Lemma 2.1, Hölder's integral inequality, and the notation (3.2), we derive

$$\begin{aligned}&\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}u\right|\\
\le&D(a,x)\int_0^1(1-t)G_t(a,x)|f'(G_t(a,x))|\,\mathrm{d}t+D(x,b)\int_0^1 tG_t(x,b)|f'(G_t(x,b))|\,\mathrm{d}t\\
\le&D(a,x)\left[\int_0^1(1-t)^{\frac{q-\ell}{q-1}}[G_t(a,x)]^{\frac{q-\hbar}{q-1}}\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1(1-t)^{\ell}[G_t(a,x)]^{\hbar}|f'(G_t(a,x))|^q\,\mathrm{d}t\right]^{1/q}\\
&+D(x,b)\left[\int_0^1 t^{\frac{q-\ell}{q-1}}[G_t(x,b)]^{\frac{q-\hbar}{q-1}}\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 t^{\ell}[G_t(x,b)]^{\hbar}|f'(G_t(x,b))|^q\,\mathrm{d}t\right]^{1/q}.\end{aligned} \tag{3.11}$$

Applying Lemma 2.2, we obtain

$$\int_0^1(1-t)^{\frac{q-\ell}{q-1}}[G_t(a,x)]^{\frac{q-\hbar}{q-1}}\,\mathrm{d}t=H_{0,\frac{q-\ell}{q-1}}\Big(a^{\frac{q-\hbar}{q-1}},x^{\frac{q-\hbar}{q-1}}\Big) \tag{3.12}$$

and

$$\int_0^1 t^{\frac{q-\ell}{q-1}}[G_t(x,b)]^{\frac{q-\hbar}{q-1}}\,\mathrm{d}t=H_{\frac{q-\ell}{q-1},0}\Big(x^{\frac{q-\hbar}{q-1}},b^{\frac{q-\hbar}{q-1}}\Big).$$

Using the GT-convexity of f, we derive

$$\begin{aligned}&\int_0^1(1-t)^{\ell}[G_t(a,x)]^{\hbar}|f'(G_t(a,x))|^q\,\mathrm{d}t\\
\le&\int_0^1(1-t)^{\ell}G_t\big(a^{\hbar},x^{\hbar}\big)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(x)|^q\right)\mathrm{d}t\\
=&\frac12\Big[H_{1/2,(2\ell-1)/2}\big(a^{\hbar},x^{\hbar}\big)|f'(a)|^q+H_{-1/2,(2\ell+1)/2}\big(a^{\hbar},x^{\hbar}\big)|f'(x)|^q\Big]\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 t^{\ell}[G_t(x,b)]^{\hbar}|f'(G_t(x,b))|^q\,\mathrm{d}t\\
\le&\int_0^1 t^{\ell}G_t\big(x^{\hbar},b^{\hbar}\big)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)\mathrm{d}t\\
=&\frac12\left[H_{(2\ell+1)/2,-1/2}\big(x^{\hbar},b^{\hbar}\big)|f'(x)|^q+H_{(2\ell-1)/2,1/2}\big(x^{\hbar},b^{\hbar}\big)|f'(b)|^q\right].\end{aligned} \tag{3.13}$$

Substituting the equalities and inequalities between (3.12) and (3.13) into the inequality (3.11) leads to the inequality (3.10). The proof of Theorem 3.3 is thus completed. □

In Theorem 3.3, taking $\ell=\hbar=0$ leads to:

Theorem 3.4. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ is GT-convex on $[a, b]$ for $q, p > 1$ with $p = \frac{q}{q-1}$, then*

$$\begin{aligned}&\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}u\right|\\ \le&\frac{(\ln x-\ln a)^2}{\ln b-\ln a}[H_{0,p}(a^p,x^p)]^{1/p}\left[\frac{\pi(|f'(x)|^q+|f'(a)|^q)}{4}\right]^{1/q}\\ &+\frac{(\ln b-\ln x)^2}{\ln b-\ln a}[H_{p,0}(x^p,b^p)]^{1/p}\left[\frac{\pi(|f'(x)|^q+|f'(b)|^q)}{4}\right]^{1/q}\end{aligned}\tag{3.14}$$

for $x \in [a, b]$, where $H_{r,s}(u, v)$ are defined in Lemma 2.2.

Proof. Similar to the proof in (3.11), we acquire

$$\begin{aligned}\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}u\right|\le D(a,x)&\left[\int_0^1[(1-t)G_t(a,x)]^p\,\mathrm{d}t\right]^{1/p}\left[\int_0^1|f'(G_t(a,x))|^q\,\mathrm{d}t\right]^{1/q}\\ &+D(x,b)\left[\int_0^1[tG_t(x,b)]^p\,\mathrm{d}t\right]^{1/p}\left[\int_0^1|f'(G_t(x,b))|^q\,\mathrm{d}t\right]^{1/q}.\end{aligned}$$

Applying Lemma 2.2, we derive

$$\begin{aligned}&\int_0^1[(1-t)G_t(a,x)]^p\,\mathrm{d}t=H_{0,p}(a^p,x^p), &&\int_0^1[tG_t(x,b)]^p\,\mathrm{d}t=H_{p,0}(x^p,b^p),\\ &\int_0^1|f'(G_t(a,x))|^q\,\mathrm{d}t\le\frac{\pi}{4}[|f'(a)|^q+|f'(x)|^q], &&\int_0^1|f'(G_t(x,b))|^q\,\mathrm{d}t\le\frac{\pi}{4}[|f'(x)|^q+|f'(b)|^q].\end{aligned}$$

From the above inequalities and integral equations, we arrive at the inequality (3.14). The proof of Theorem 3.4 is thus completed. □

Putting $\ell = 0$ and $\hbar = q$ in Theorem 3.3 for $q > 1$ demonstrates.

Theorem 3.5. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ is GT-convex on $[a, b]$ for $q, p > 1$ with $p = \frac{q}{q-1}$, then*

$$\begin{aligned}&\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}u\right|\\ \le&\frac{(\ln x-\ln a)^2}{(p+1)^{1/p}(\ln b-\ln a)}\left[\frac{H_{1/2,-1/2}(a^q,x^q)|f'(a)|^q+H_{-1/2,1/2}(a^q,x^q)|f'(x)|^q}{2}\right]^{1/q}\\ &+\frac{(\ln b-\ln x)^2}{(p+1)^{1/p}(\ln b-\ln a)}\left[\frac{H_{1/2,-1/2}(x^q,b^q)|f'(x)|^q+H_{-1/2,1/2}(x^q,b^q)|f'(b)|^q}{2}\right]^{1/q}\end{aligned}\tag{3.15}$$

for $x \in [a, b]$, where $T(u, v)$ is defined as in Lemma 2.5.

Proof. Using Lemma 2.1, Hölder's integral inequality, and the notation (3.2), we derive

$$\begin{aligned}&\left|f(x)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}\,\mathrm{d}u\right|\\ \le&D(a,x)\left[\int_0^1(1-t)^p\,\mathrm{d}t\right]^{1/p}\left[\int_0^1[(G_t(a,x)]^q|f'(G_t(a,x))|^q\,\mathrm{d}t\right]^{1/q}\end{aligned}$$

$$+ D(x,b)\left[\int_0^1 t^p \,\mathrm{d}t\right]^{1/p}\left[\int_0^1 [G_t(x,b)]^p|f'(G_t(x,b))|^q \,\mathrm{d}t\right]^{1/q}. \tag{3.16}$$

It is easy to see that

$$\int_0^1 t^p \,\mathrm{d}t = \int_0^1 (1-t)^p \,\mathrm{d}t = \frac{1}{p+1}. \tag{3.17}$$

From the GT-convexity of f and Lemma 2.5, we acquire

$$\begin{aligned}\int_0^1 \big[G_t(a,x)\big]^q|f'(G_t(a,x))|^q \,\mathrm{d}t &\le \int_0^1 G_t(a^q,x^q)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^q + \frac{\sqrt{1-t}}{2\sqrt{t}}|f'(x)|^q\right)\mathrm{d}t \\ &= \frac{1}{2}\big[H_{1/2,-1/2}(a^q,x^q)|f'(a)|^q + H_{-1/2,1/2}(a^q,x^q)|f'(x)|^q\big]\end{aligned}$$

and

$$\int_0^1 \big[G_t(x,b)\big]^q|f'(G_t(x,b))|^q \,\mathrm{d}t \le \frac{1}{2}\big[H_{1/2,-1/2}(x^q,b^q)|f'(x)|^q + H_{-1/2,1/2}(x^q,b^q)|f'(b)|^q\big]. \tag{3.18}$$

Substituting three inequalities between (3.17) to (3.18) into the inequality (3.16) arrives at the inequality (3.15). The proof of Theorem 3.5 is thus completed. □

Theorem 3.6. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a,b])$. If $|f'|^q$ is GT-convex on $[a,b]$ for $q, p > 1$ with $p = \frac{q}{q-1}$, then*

$$\begin{aligned}&\left|f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}u\right| \\ &\le \frac{(\ln x - \ln a)^2}{\ln b - \ln a}[L(a^p,x^p)]^{1/p}\left[\frac{B\big(\frac{3}{2},q+\frac{1}{2}\big)|f'(a)|^q + B\big(\frac{1}{2},q+\frac{3}{2}\big)|f'(x)|^q}{2}\right]^{1/q} \\ &\quad + \frac{(\ln b - \ln x)^2}{\ln b - \ln a}[L(x^p,b^p)]^{1/p}\left[\frac{B\big(\frac{1}{2},q+\frac{3}{2}\big)|f'(x)|^q + B\big(\frac{3}{2},q+\frac{1}{2}\big)|f'(b)|^q}{2}\right]^{1/q}\end{aligned}$$

for $x \in [a,b]$, where $L(u,v)$ is logarithmic mean (2.3) and $B(\alpha,\beta)$ is the classical Euler beta function (2.2).

Proof. Similar to the proofs of the above theorems, we obtain

$$\begin{aligned}&\left|f(x) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}\,\mathrm{d}u\right| \\ \le& D(a,x)\left[\int_0^1 [(G_t(a,x)]^p \,\mathrm{d}t\right]^{1/p}\left[\int_0^1 (1-t)^q|f'(G_t(a,x))|^q \,\mathrm{d}t\right]^{1/q} \\ &+ D(x,b)\left[\int_0^1 [G_t(x,b)]^p \,\mathrm{d}t\right]^{1/p}\left[\int_0^1 t^q|f'(G_t(x,b))|^q \,\mathrm{d}t\right]^{1/q}.\end{aligned} \tag{3.19}$$

Straightforward computation gives

$$\int_0^1 [G_t(a,x)]^p \,\mathrm{d}t = L(a^p,x^p) \quad \text{and} \quad \int_0^1 [G_t(x,b)]^p \,\mathrm{d}t = L(x^p,b^p).$$

By the GT-convexity of f, we obatin

$$\begin{aligned}\int_0^1 (1-t)^q |f'(G_t(a,x))|^q \,\mathrm{d}t &\le \int_0^1 (1-t)^q \left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^q + \frac{\sqrt{1-t}}{2\sqrt{t}}|f'(x)|^q\right) \mathrm{d}t \\ &= \frac{1}{2}B\left(\frac{3}{2}, q+\frac{1}{2}\right)|f'(a)|^q + \frac{1}{2}B\left(\frac{1}{2}, q+\frac{3}{2}\right)|f'(x)|^q\end{aligned}$$

and

$$\int_0^1 t^q |f'(G_t(x,b))|^q \,\mathrm{d}t \le \frac{1}{2}B\left(\frac{1}{2}, q+\frac{3}{2}\right)|f'(x)|^q + \frac{1}{2}B\left(\frac{3}{2}, q+\frac{1}{2}\right)|f'(b)|^q. \tag{3.20}$$

Combining the inequalities and equalities between (3.19) and (3.20) yields required result. Theorem 3.6 is thus proved. □

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series, **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.

[2] S.-P. Bai, S.-H. Wang and F. Qi, *On HT-convexity and Hadamard-type inequalities*, J. Inequal. Appl., 2020, 3, 12 pp. DOI: 10.1186/s13660-019-2276-3.

[3] C.-Y. He, A. Wan and B.-N. Guo, *Hermite–Hadamard type inequalities for harmonic-arithmetic extended (s_1, m_1)-(s_2, m_2) coordinated convex functions*, AIMS Math., 2023, 8(7), 17027–17037. DOI: 10.3934/math.2023869.

[4] S. Jin, A. Wan and B.-N. Guo, *Some new integral inequalities of the Simpson type for MT-convex functions*, Adv. Theory Nonlinear Anal. Appl., 2022, 6(2), 168–172. DOI: 10.31197/atnaa.1003964.

[5] H. Mei, A. Wan and B.-N. Guo, *Coordinated MT-(s_1, s_2)-convex functions and their integral inequalities of Hermite–Hadamard type*, J. Math., 2021, Art. ID 5586377, 10 pp. DOI: 10.1155/2021/5586377.

[6] J. Park, *Hermite-Hadamard-like type inequalities for twice differentiable MT-convex functions*, Appl. Math. Sci., 2015, 9, 5235–5250.

[7] M. Tunç and H. Yildirim, *On MT-convexity*, arXiv:1205.5453. DOI: 10.48550/arXiv.1205.5453.

[8] J.-Y. Wang, X.-M. Liu, H.-P. Yin and B.-N. Guo, *Some new integral inequalities of Hermite–Hadamard type for functions whose third derivatives are (α, s, m)-convex*, Miskolc Math. Notes, 2025, 26(2), 1079–1088. DOI: 10.18514/MMN.2025.4756.

[9] S.-H. Wang, X.-W. Sun and B.-N. Guo, *On GT-convexity and related integral inequalities*, AIMS Math., 2020, 5(4), 3952–3965. DOI: 10.3934/math.2020255.

[10] Y. Wang, X.-M. Liu and B.-N. Guo, *Several integral inequalities of the Hermite–Hadamard type for s-(β, F)-convex functions*, ScienceAsia, 2023, 49(2), 200–204. DOI: 10.2306/scienceasia1513-1874.2022.136.

[11] B.-Y. Xi and F. Qi, *Some inequalities of Hermite–Hadamard type for geometrically P-convex functions*, Adv. Stud. Contemp. Math. (Kyungshang), 2016, 26(1), 211–220.

Received January 2026; Accepted May 2026; Available online May 2026.