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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A COUPLED SYSTEM OF HILFER-HADAMARD SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTI-POINT FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

Tian Dong[1], Jiqiang Jiang[1,†], Guoqiang Chen[2], Yixue Zhang[2] and Na Li[2]

Abstract In this paper, we investigate the existence and uniqueness of solutions for a coupled system of Hilfer-Hadamard sequential fractional differential equations with multi-point Riemann-Liouville fractional integral boundary value conditions. The existence of solutions is derived by applying Leray-Schauder alternative theorem, while the existence and uniqueness of solutions is established using the Banach fixed point theorem. In the end, two examples are included to illustrate the applicability of the results.

Keywords Coupled system, Hilfer-Hadamard fractional derivative, sequential fractional, boundary value problem, existence and uniqueness.

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1. Introduction

Many researchers are interested in developing the concept of fractional calculus and studying differential equations because of its extensive applications in scientific and technical phenomena, such as immune system with memory [11], co-infection of malaria and HIV/AIDS [9], chaos [47], chaotic systems [12], synchronization [48], dynamical networks [43], neural networks [5], economic model [30], financial economics [13], compressible system with fractional viscous term [34], etc.

As the mathematical models associated with physical phenomena contain initial and boundary value problems, many researchers have shown a keen interest in developing the theoretical aspects of such problems, for example, see the books [2, 4, 21], and a series of research articles [17, 18, 22, 23, 36–38, 40, 44, 50, 52]. The occurrence of systems of fractional differential equations in the mathematical models of physical problems [10, 33, 42] motivated many researchers to explore the qualitative aspects of such systems, for example, see [19, 24, 25].

The study of existence and uniqueness of solutions of fractional differential equations are typically established using fixed point theorems, which are foundational tools in functional analysis, see the monographs [8, 31, 35]. In recent years, there are many works settling with the existence and uniqueness of the solutions of boundary value problems for fractional differential

[†]The corresponding author.
[1]School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China
[2]Dongying Vocational College, Dongying 257091, Shandong, China
Email: 13898143296@163.com(T. Dong), qfjjq@163.com(J. Jiang), 252221190@qq.com(G. Chen), 2404609696@qq.com(Y. Zhang), 35007498@qq.com(N. Li)

equations by means of techniques of nonlinear analysis (fixed point theorem, Leray-Schauder theory, and so forth), see for example [1, 3, 7, 26–28, 39, 41, 45, 46, 49, 51]. Nowadays, Among the various applications of FDEs, coupled systems of sequential fractional differential equations with boundary value problems have also garnered significant attention due to their ability to describe complex interactions between multiple components within a system.

In [14], Feng and Zhang considered the following a coupled system of nonlinear fractional differential equations with three-point boundary conditions

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), D^{p}v(t)), \quad t \in (0,1), \\ D^{\beta}v(t) = g(t, u(t), D^{q}v(t)), \quad t \in (0,1), \\ u(0) = 0, u(1) = \gamma u(\eta), v(0) = 0, v(1) = \gamma v(\eta), \end{cases}$$

where $1 < \alpha, \beta < 2, p, q, \gamma > 0, 0 < \eta < 1, \alpha - q \geq 1, \beta - p \geq 1, \gamma\eta^{\alpha-1}, \gamma\eta^{\beta-1} < 1$, $D^{\chi}(\chi = \alpha, \beta, p, q)$ is the Riemann-Liouville fractional derivative and $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. The authors derived the existence of solutions by using the Schauder fixed point theorem.

In [6], Aljoudi et al investigated a coupled system of Hadamard type sequential fractional differential equation with coupled strip conditions given by

$$\begin{cases} (D^{q} + kD^{q-1})u(t) = f(t, u(t), v(t), D^{\alpha}v(t)), \quad k > 0, 1 < q \leq 2, 0 < \alpha < 1, \\ (D^{p} + kD^{p-1})v(t) = g(t, u(t), v(t), D^{\delta}u(t)), \quad 1 < p \leq 2, 0 < \delta < 1, \\ u(1) = 0, u(e) = I^{\gamma}v(\eta) = \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_1^{\eta} (\log \frac{\eta}{s})^{\gamma-1} \frac{v(s)}{s} ds, \quad \gamma > 0, 1 < \eta < e, \\ v(1) = 0, v(e) = I^{\beta}u(\zeta) = \dfrac{1}{\Gamma(\beta)} \displaystyle\int_1^{\zeta} (\log \frac{\zeta}{s})^{\beta-1} \frac{u(s)}{s} ds, \quad \beta > 0, 1 < \zeta < e, \end{cases}$$

where $D^{(\cdot)}$ and $I^{(\cdot)}$ denote the Hadamard fractional derivative and Hadamard fractional integral, and $f, g : [1, e] \times \mathbb{R}^3 \to \mathbb{R}$ are given continuous functions. The nonlinearities in the coupled system of equations depend on the unknown functions as well as their lower order fractional derivatives. The authors obtained the existence and uniqueness results by employing the Leray-Schauder alternative and Banach's contraction mapping principle.

In [29], the authors studied a coupled system of sequential Hilfer-Hadamard fractional differential equations

$$\begin{cases} \left({}_{H}D_{1^+}^{\alpha_1, \beta_1} + k_1 {}_{H}D_{1^+}^{\alpha_1 - 1, \beta_1} \right) u(t) = f(t, u(t), v(t)), \quad 1 < \alpha_1 \leq 2, \quad 0 < \beta_1 \leq 1, t \in [1, e], \\ \left({}_{H}D_{1^+}^{\alpha_2, \beta_2} + k_2 {}_{H}D_{1^+}^{\alpha_2 - 1, \beta_2} \right) v(t) = g(t, u(t), v(t)), \quad 1 < \alpha_2 \leq 2, \quad 0 < \beta_2 \leq 1, t \in [1, e], \end{cases}$$

with four-point coupled boundary conditions

$$\begin{aligned} u(1) = 0, u(e) = \lambda v(\theta), \quad 1 < \theta < e, \\ v(1) = 0, v(e) = \mu u(\eta), \quad 1 < \eta < e, \end{aligned}$$

where ${}_{H}D_{1^+}^{\alpha_i, \beta_i}$ denotes the Hilfer-Hadamard fractional derivatives of order $\alpha_i \in (1, 2]$ and type $\beta_i \in [0, 1]$ for $i \in \{1, 2\}, k_1, k_2 \in \mathbb{R}^+, f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and λ and μ are real constants. The authors in [29] established both the uniqueness and presence of solutions through the application of Banach's contraction mapping principle and the Leray-Schauder alternative theorem.

In [32], the authors studied a coupled system of sequential Hilfer-hadamard fractional differential equations with multi-point Riemann-Liouville fractional integral boundary conditions:

$$\begin{cases} \left({}_HD_{1^+}^{\alpha_1,\beta_1} + k_1 {}_HD_{1^+}^{\alpha_1-1,\beta_1}\right)u(t) = f(t,u(t),v(t)), \quad t\in[1,e], \\ \left({}_HD_{1^+}^{\alpha_2,\beta_2} + k_2 {}_HD_{1^+}^{\alpha_2-1,\beta_2}\right)v(t) = g(t,u(t),v(t)), \quad t\in[1,e], \\ u(1)=0, u(e) = \sum_{i=1}^{n}\lambda_i I_{1^+}^{\delta_i}v(\theta_i) = \sum_{i=1}^{n}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}v(s)ds, \\ v(1)=0, v(e) = \sum_{j=1}^{n}\mu_j I_{1^+}^{\sigma_j}u(\eta_j) = \sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}u(s)ds, \end{cases}$$

where ${}_HD_{1^+}^{\alpha_i,\beta_i}$ denotes the Hilfer-Hadamard fractional derivatives of order $\alpha_i\in(1,2]$ and type $\beta_i\in[0,1]$ for $i\in\{1,2\}$, $f,g:[1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions. The existence of solutions is proved using Krasnosel'skii fixed point theorem, while the existence and uniqueness of solutions is established using the Banach fixed point theorem.

Motivated by the above papers, in this article, we consider the existence of solutions for the following a coupled system of Hilfer-Hadamard sequential fractional differential equations

$$\begin{cases} \left({}_HD_{1^+}^{\alpha_1,\beta_1} + k_1 {}_HD_{1^+}^{\alpha_1-1,\beta_1}\right)u(t) = f(t,u(t),v(t),D^\delta v(t)), \quad 0<\delta<1, t\in[1,e], \\ \left({}_HD_{1^+}^{\alpha_2,\beta_2} + k_2 {}_HD_{1^+}^{\alpha_2-1,\beta_2}\right)v(t) = g(t,u(t),v(t),D^\gamma u(t)), \quad 0<\gamma<1, t\in[1,e], \end{cases} \tag{1.1}$$

with multi-point fractional integral boundary conditions

$$\begin{cases} u(1)=0, u(e) = \sum_{i=1}^{m}\lambda_i I_{1^+}^{\delta_i}v(\theta_i) = \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}v(s)ds, \\ v(1)=0, v(e) = \sum_{j=1}^{n}\mu_j I_{1^+}^{\sigma_j}u(\eta_j) = \sum_{j=1}^{n}\frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}u(s)ds, \end{cases} \tag{1.2}$$

where ${}_HD_{1^+}^{\alpha_i,\beta_i}$ denotes the Hilfer-Hadamard fractional derivatives of order $\alpha_i\in(1,2]$ and type $\beta_i\in[0,1]$ for $i\in\{1,2\}$, D^γ, D^δ are the Hadamard fractional derivatives, $k_1,k_2\in\mathbb{R}^+, f,g:[1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions. I^{δ_i} and I^{σ_j} are the Riemann-Liouville fractional integral of positive order θ_i and $\eta_j\in(1,e)$, and $\lambda_i,\mu_j(1\le i\le m, 1\le j\le n)$ are real constants.

The novelty of our problem (1.1)-(1.2) consists of the consideration of the system of Hilfer-Hadamard fractional differential equations with derivatives of diverse orders and the presence of fractional derivative terms in the nonlinearities of the system, this different from [29, 32]. The important highlight of this research are the boundary conditions of the problem, which are coupled strip boundary conditions, multi-point Riemann-Liouville fractional integral boundary conditions that make it applicable to a broader class of mathematical models than that of [6, 14, 29]. The existence and uniqueness of solutions are established using fixed point theorems such as Leray-Schauder alternative theorem and the Banach fixed point theorem. All these remarks lead to the conclusion that problem (1.1)-(1.2) are novel in the context of the systems of coupled Hilfer-Hadamard boundary value problems.

The rest of the paper is organized as follows. In Section 2, we present some useful preliminaries and lemmas that are to be used to prove our main results. We also appear some important theorems to demonstrate our main problems. In Section 3, by using Leray-Schauder alternative theorem and Banach contraction mapping principle to obtain the existence and uniqueness results. In Section 4, two examples are given to illustrate the result of our theoretical results.

2. Preliminaries and lemmas

In this section, for the convenience of reader, we introduce some notations and lemmas that will be used in the proof of our main results.

Definition 2.1 ([20]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : [a, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a.$$

Definition 2.2 ([20]). The Hadamard fractional integral of order $\alpha > 0$ for a function $f : [a, \infty) \to \mathbb{R}$ is defined as

$${}_HI^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}, \quad t > a,$$

where $[\alpha]$ denotes the integer part of the real number α, $\log(\cdot) = \log_e(\cdot)$.

Definition 2.3 ([20]). The Hadamard fractional derivatives of order $\alpha > 0$, for a function $f : [a, \infty) \to \mathbb{R}$ can be written as

$${}_HD^{\alpha}_{a^+} f(t) = \delta^n ({}_HI^{n-\alpha}_{a^+} f(t)), \quad n-1 < \alpha < n, \quad n = [\alpha] + 1,$$

where $\delta^n = (t\frac{d}{dt})^n$.

Definition 2.4 ([16]). The Hilfer-Hadamard fractional derivative of order α and type β of f is defined as

$$\begin{aligned}({}_HD^{\alpha,\beta}_{a^+} f)(t) &= ({}_HI^{\beta(n-\alpha)}_{a^+} \delta^n_H I^{(n-\alpha)(1-\beta)}_{a^+} f)(t)\\ &= ({}_HI^{\alpha(n-\alpha)}_{a^+} \delta^n_H I^{n-\gamma}_{a^+} f)(t)\\ &= ({}_HI^{\beta(n-\alpha)}_{a^+} {}_HD^{\gamma}_{a^+} f)(t),\end{aligned}$$

where ${}_HI^{(\cdot)}_{a^+}$, $\gamma = \alpha + n\beta - \alpha\beta$ and ${}_HD^{(\cdot)}_{a^+}$ are the Hadamard fractional integral and derivative.

Lemma 2.1 ([16]). *Let $\alpha > 0$, $n = [\alpha] + 1$ and $0 < a < b < \infty$. If $f \in L^1[a, b]$ and $({}_HI^{n-\alpha}_{a^+} f(t)) \in AC^n_\delta[a, b]$, then*

$$({}_HI^{\alpha}_{a^+} {}_HD^{\alpha}_{a^+} f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_HI^{n-\alpha}_{a^+} f))(a)}{\Gamma(\alpha - j)} (\log \frac{t}{a})^{\alpha-j-1}.$$

Lemma 2.2 ([16]). *Let $\alpha > 0$, $0 \le \beta \le 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n-1 < \gamma \le n$, $n = [\alpha] + 1$ and $0 < a < b < \infty$. If $f \in L^1[a, b]$ and $({}_HI^{n-\gamma}_{a^+} f(t)) \in AC^n_\delta[a, b]$, then*

$${}_HI^{\alpha}_{a^+} ({}_HD^{\alpha,\beta}_{a^+} f)(t) = {}_H I^{\gamma}_{a^+} ({}_HD^{\gamma}_{a^+} f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_HI^{n-\gamma}_{a^+} f))(a)}{\Gamma(\gamma - j)} (\log \frac{t}{a})^{\gamma-j-1}.$$

Lemma 2.3. *Let* $h_1, h_2 \in C([1,e],\mathbb{R})$, *and* $\Delta = 1 - AB \neq 0$,

$$A = \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} (\log s)^{\gamma_2 - 1} ds, \;\; B = \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} (\log s)^{\gamma_1 - 1} ds.$$

Then, $u, v \in C([1,e],\mathbb{R})$ *are solutions of the Hilfer-Hadamard sequential fractional differential equations*

$$\begin{cases} \left({}_H D_{1^+}^{\alpha_1,\beta_1} + k_1 {}_H D_{1^+}^{\alpha_1-1,\beta_1}\right) u(t) = h_1(t), \quad 1 < \alpha_1 \leq 2, \quad 0 < \beta_1 \leq 1, t \in [1,e], \\ \left({}_H D_{1^+}^{\alpha_2,\beta_2} + k_2 {}_H D_{1^+}^{\alpha_2-1,\beta_2}\right) v(t) = h_2(t), \quad 1 < \alpha_2 \leq 2, \quad 0 < \beta_2 \leq 1, t \in [1,e], \end{cases} \tag{2.1}$$

if and only if

$$\begin{aligned} u(t) =& \frac{(\log t)^{\gamma_1 - 1}}{\Delta} \Bigg\{ \Bigg[k_1 \int_1^e u(s) \frac{ds}{s} - \frac{1}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha_1 - 1} h_1(s) \frac{ds}{s} \\ & - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} \left(k_2 \int_1^s v(r) \frac{dr}{r} - \frac{1}{\Gamma(\alpha_2)} \int_1^s \left(\log \frac{s}{r}\right)^{\alpha_2 - 1} h_2(r) \frac{dr}{r} \right) ds \Bigg] \\ & + A \Bigg[k_2 \int_1^e v(s) \frac{ds}{s} - \frac{1}{\Gamma(\alpha_2)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha_2 - 1} h_2(s) \frac{ds}{s} \\ & - \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} \left(k_1 \int_1^s u(r) \frac{dr}{r} - \frac{1}{\Gamma(\alpha_1)} \int_1^s \left(\log \frac{s}{r}\right)^{\alpha_1 - 1} h_1(r) \frac{dr}{r} \right) ds \Bigg] \Bigg\} \\ & - k_1 \int_1^t u(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha_1)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_1 - 1} h_1(s) \frac{ds}{s}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} v(t) =& \frac{(\log t)^{\gamma_2 - 1}}{\Delta} \Bigg\{ \Bigg[k_2 \int_1^e v(s) \frac{ds}{s} - \frac{1}{\Gamma(\alpha_2)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha_2 - 1} h_2(s) \frac{ds}{s} \\ & - \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1} \left(k_1 \int_1^s u(r) \frac{dr}{r} - \frac{1}{\Gamma(\alpha_1)} \int_1^s \left(\log \frac{s}{r}\right)^{\alpha_1 - 1} h_1(r) \frac{dr}{r} \right) ds \Bigg] \\ & + B \Bigg[k_1 \int_1^e u(s) \frac{ds}{s} - \frac{1}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha_1 - 1} h_1(s) \frac{ds}{s} \\ & - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} \left(k_2 \int_1^s v(r) \frac{dr}{r} - \frac{1}{\Gamma(\alpha_2)} \int_1^s \left(\log \frac{s}{r}\right)^{\alpha_2 - 1} h_2(r) \frac{dr}{r} \right) ds \Bigg] \Bigg\} \\ & - k_2 \int_1^t v(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha_2)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_2 - 1} h_2(s) \frac{ds}{s}, \end{aligned} \tag{2.3}$$

where $\gamma_i = \alpha_i + 2\beta_i - \alpha_i \beta_i$ *for* $i \in \{1,2\}$.

Proof. Solving (2.1), we obtain

$$\left({}_H I_{1^+}^{\alpha_1} {}_H D_{1^+}^{\alpha_1,\beta_1} + k_1 {}_H I_{1^+}^{\alpha_1} {}_H D_{1^+}^{\alpha_1-1,\beta_1}\right) u(t) = {}_H I_{1^+}^{\alpha_1} h_1(t).$$

By Lemma 2.2, we have

$$u(t) - \sum_{j=0}^{1} \frac{(\delta^{(2-j-1)}({}_H I_{1^+}^{2-\gamma_1} u))(1)}{\Gamma(\gamma_1 - j)} (\log t)^{\gamma_1 - j - 1} + k_1 {}_H I_{1^+}^{\alpha_1} {}_H D_{1^+}^{\alpha_1-1,\beta_1} u(t) = {}_H I_{1^+}^{\alpha_1} h_1(t). \tag{2.4}$$

From Eq. (2.4), by Definition 2.4 one has

$$\begin{aligned} &u(t) - \frac{(\delta_H I_{1^+}^{2-\gamma_1} u)(1)}{\Gamma(\gamma_1)}(\log t)^{\gamma_1 - 1} - \frac{({}_H I_{1^+}^{2-\gamma_1} u)(1)}{\Gamma(\gamma_1 - 1)}(\log t)^{\gamma_1 - 2} \\ &+ k_1 {}_H I_{1^+}^1 ({}_H I_{1^+}^{\gamma_1 - 1} {}_H D_{1^+}^{\gamma_1 - 1}) u(t) = {}_H I_{1^+}^{\alpha_1} h_1(t), \end{aligned} \tag{2.5}$$

where $\gamma_1, \gamma_2 \in (1,2]$ for $i \in \{1,2\}$. Then, by Lemma 2.1 we get

$$\begin{aligned} &u(t) - \frac{(\delta_H I_{1^+}^{2-\gamma_1} u)(1)}{\Gamma(\gamma_1)}(\log t)^{\gamma_1 - 1} - \frac{({}_H I_{1^+}^{2-\gamma_1} u)(1)}{\Gamma(\gamma_1 - 1)}(\log t)^{\gamma_1 - 2} \\ &+ k_1 {}_H I_{1^+}^1 \left(u(t) - \frac{({}_H I_{1^+}^{2-\gamma_1} u)(1)}{\Gamma(\gamma_1 - 1)}(\log t)^{\gamma_1 - 2} \right) = {}_H I_{1^+}^{\alpha_1} h_1(t). \end{aligned} \tag{2.6}$$

Equation (2.6) can be written as

$$\begin{aligned} u(t) =& c_0(\log t)^{\gamma_1 - 1} + c_1\left((\log t)^{\gamma_1 - 2} + k_1 \int_1^t (\log s)^{\gamma_1 - 2} \frac{ds}{s} \right) - k_1 \int_1^t u(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha_1)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha_1 - 1} h_1(s) \frac{ds}{s}, \end{aligned} \tag{2.7}$$

where c_0, c_1 are arbitrary constants. In a similar way, we have

$$\begin{aligned} v(t) =& d_0(\log t)^{\gamma_2 - 1} + d_1\left((\log t)^{\gamma_2 - 2} + k_2 \int_1^t (\log s)^{\gamma_2 - 2} \frac{ds}{s} \right) - k_2 \int_1^t v(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha_2)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha_2 - 1} h_2(s) \frac{ds}{s}, \end{aligned} \tag{2.8}$$

where d_0, d_1 are arbitrary constants. Using the first boundary conditions of (1.2), $u(1) = 0$, and $v(1) = 0$ together with Eqs. (2.7) and (2.8), yield $c_1 = 0$ and $d_1 = 0$, respectively. Eqs. (2.7) and (2.8) become

$$u(t) = c_0(\log t)^{\gamma_1 - 1} - k_1 \int_1^t u(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha_1)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha_1 - 1} h_1(s) \frac{ds}{s}, \tag{2.9}$$

and

$$v(t) = d_0(\log t)^{\gamma_2 - 1} - k_2 \int_1^t v(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha_2)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha_2 - 1} h_2(s) \frac{ds}{s}. \tag{2.10}$$

Next, the second boundary conditions of (1.2) together with Eqs. (2.9) and (2.10) yield

$$\begin{aligned} &c_0 - k_1 \int_1^e u(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha_1 - 1} h_1(s) \frac{ds}{s} \\ &= \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} \left(d_0(\log s)^{\gamma_2 - 1} - k_2 \int_1^s v(r) \frac{dr}{r} \right. \\ &\left. + \frac{1}{\Gamma(\alpha_2)} \int_1^s \left(\log \frac{s}{r} \right)^{\alpha_2 - 1} h_2(r) \frac{dr}{r} \right) ds \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&d_0 - k_2 \int_1^e v(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha_2)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2 - 1} h_2(s)\frac{ds}{s}\\ =&\sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1}\Bigg(c_0(\log s)^{\gamma_1 - 1} - k_1 \int_1^s u(r)\frac{dr}{r}\\ &+ \frac{1}{\Gamma(\alpha_1)} \int_1^s \left(\log\frac{s}{r}\right)^{\alpha_1-1} h_1(r)\frac{dr}{r}\Bigg)ds.\end{aligned} \tag{2.12}$$

Note that $\Delta = 1 - AB \neq 0$, thus the system (2.11) and (2.12) has a unique solution for c_0 and d_0. By Cramer's rule and simple calculations, it follows that

$$\begin{aligned}c_0 =&\frac{1}{\Delta}\Bigg[k_1 \int_1^e u(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha_1)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1} h_1(s)\frac{ds}{s}\\ &- \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1}\left(k_2 \int_1^s v(r)\frac{dr}{r} - \Gamma(\alpha_2) \int_1^s \left(\log\frac{s}{r}\right)^{\alpha_2-1} h_2(r)\frac{dr}{r}\right)ds\Bigg]\\ &+ \frac{A}{\Delta}\Bigg[k_2 \int_1^e v(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha_2)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1} h_2(s)\frac{ds}{s}\\ &- \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1}\left(k_1 \int_1^s u(r)\frac{dr}{r} - \Gamma(\alpha_1) \int_1^s \left(\log\frac{s}{r}\right)^{\alpha_1-1} h_1(r)\frac{dr}{r}\right)ds\Bigg],\\ d_0 =&\frac{1}{\Delta}\Bigg[k_2 \int_1^e v(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha_2)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1} h_2(s)\frac{ds}{s}\\ &- \sum_{j=1}^{n} \frac{\mu_j}{\Gamma(\sigma_j)} \int_1^{\eta_j} (\eta_j - s)^{\sigma_j - 1}\left(k_1 \int_1^s u(r)\frac{dr}{r} - \Gamma(\alpha_1) \int_1^s \left(\log\frac{s}{r}\right)^{\alpha_1-1} h_1(r)\frac{dr}{r}\right)ds\Bigg]\\ &+ \frac{B}{\Delta}\Bigg[k_1 \int_1^e u(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha_1)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1} h_1(s)\frac{ds}{s}\\ &- \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1}\left(k_2 \int_1^s v(r)\frac{dr}{r} - \Gamma(\alpha_2) \int_1^s \left(\log\frac{s}{r}\right)^{\alpha_2-1} h_2(r)\frac{dr}{r}\right)ds\Bigg].\end{aligned}$$

Substituting the values of c_0, d_0 back into Eqs. (2.9) and (2.10), we obtain solution (2.2) and (2.3). □

Lemma 2.4 ([15]). (Leray-Schauder alternative theorem) *Let F be a completely continuous operator. Let $\varepsilon(F) = \{x \in E : x = \lambda F(x),$ for some $0 < \lambda < 1\}$. Then either the set $\varepsilon(F)$ is unbounded or F has at least one fixed point.*

3. Main results

Let $X = \{u : u \in C([1, e], \mathbb{R})$ and $D^\gamma u \in C([1, e], \mathbb{R})\}$, $Y = \{v : v \in C([1, e], \mathbb{R})$ and $D^\delta v \in C([1, e], \mathbb{R})\}$ with the norm

$$\|u\|_X = \|u\| + \|D^\gamma u\| = \sup_{t\in[1,e]} |u| + \sup_{t\in[1,e]} |D^\gamma u|,$$

$$\|v\|_Y = \|v\| + \|D^\delta v\| = \sup_{t\in[1,e]} |v| + \sup_{t\in[1,e]} |D^\delta v|.$$

Observe that $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are the Banach spaces. In turn, the product space $(X \times Y, \|.\|_{X\times Y})$ is a Banach spaces with the norm

$$\|(u,v)\|_{X\times Y} = \|u\|_X + \|v\|_Y,\ \ for\ (u,v) \in X \times Y.$$

Define an operator $F : X \times Y \to X \times Y$ by

$$F(u,v)(t) = (F_1(u,v)(t), F_2(u,v)(t)), \tag{3.1}$$

where

$$\begin{aligned}
F_1(u,v)(t) = &\frac{(\log t)^{\gamma_1-1}}{\Delta}\Bigg\{\Bigg[k_1\int_1^e u(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1} f(s,u(s),v(s),D^\delta v(s))\frac{ds}{s}\\
&-\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(k_2\int_1^s v(r)\frac{dr}{r}\\
&-\frac{1}{\Gamma(\alpha_2)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_2-1} g(r,u(r),v(r),D^\gamma u(r))\frac{dr}{r}\Bigg)ds\Bigg]\\
&+A\Bigg[k_2\int_1^e v(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1} g(s,u(s),v(s),D^\gamma u(s))\frac{ds}{s}\\
&-\sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Bigg(k_1\int_1^s u(r)\frac{dr}{r}\\
&-\frac{1}{\Gamma(\alpha_1)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_1-1} f(r,u(r),v(r),D^\delta v(r))\frac{dr}{r}\Bigg)ds\Bigg]\Bigg\}\\
&-k_1\int_1^t u(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1} f(s,u(s),v(s),D^\delta v(s))\frac{ds}{s},
\end{aligned}\tag{3.2}$$

$$\begin{aligned}
F_2(u,v)(t) = &\frac{(\log t)^{\gamma_2-1}}{\Delta}\Bigg\{\Bigg[k_2\int_1^e v(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1} g(s,u(s),v(s),D^\gamma u(s))\frac{ds}{s}\\
&-\sum_{j=1}^n \frac{\mu_j}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Bigg(k_1\int_1^s u(r)\frac{dr}{r}\\
&-\frac{1}{\Gamma(\alpha_1)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_1-1} f(r,u(r),v(r),D^\delta v(r))\frac{dr}{r}\Bigg)ds\Bigg]\\
&+B\Bigg[k_1\int_1^e u(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1} f(s,u(s),v(s),D^\delta v(s))\frac{ds}{s}\\
&-\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(k_2\int_1^s v(r)\frac{dr}{r}\\
&-\frac{1}{\Gamma(\alpha_2)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_2-1} g(r,u(r),v(r),D^\gamma u(r))\frac{dr}{r}\Bigg)ds\Bigg]\Bigg\}\\
&-k_2\int_1^t v(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha_2)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_2-1} g(s,u(s),v(s),D^\gamma u(s))\frac{ds}{s}.
\end{aligned}\tag{3.3}$$

Next, we give the following assumptions, which will be used in the rest of this paper.

($\mathbf{H}_1$) There exist real contants $m_i, n_i \geq 0 (i = 1, 2, 3)$ and $m_0 > 0, n_0 > 0$, such that for all $t \in [1, e], x_i \in \mathbb{R}(i = 1, 2, 3)$

$$
\begin{aligned}
|f(t, x_1, x_2, x_3)| &\leq m_0 + m_1|x_1| + m_2|x_2| + m_3|x_3|,\\
|g(t, x_1, x_2, x_3)| &\leq n_0 + n_1|x_1| + n_2|x_2| + n_3|x_3|.
\end{aligned}
$$

($\mathbf{H}_2$) There exist positive constants L and $\bar{L}$, such that for all $t \in [1, e], x_i, y_i, z_i \in \mathbb{R}(i = 1, 2)$

$$
\begin{aligned}
|f(t, x_1, y_1, z_3) - f(t, x_2, y_2, z_3)| &\leq L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|),\\
|g(t, x_1, y_1, z_3) - g(t, x_2, y_2, z_3)| &\leq \bar{L}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|).
\end{aligned}
$$

We use the following notations in the proofs:

$$\omega = \sum_{i=1}^{m} |\lambda_i| \frac{(\theta_i - 1)^{\delta_i}}{\Gamma(\delta_i + 1)}, \quad \varpi = \sum_{j=1}^{n} |\mu_j| \frac{(\eta_j - 1)^{\sigma_j}}{\Gamma(\sigma_j + 1)}, \tag{3.4}$$

$$M = \frac{1}{|\Delta|}[1 + \omega\varpi + |\Delta|], \quad W_1 = \frac{2\omega}{|\Delta|}, \quad W_2 = \frac{2\varpi}{|\Delta|}. \tag{3.5}$$

Note that $A \leq \omega$ and $B \leq \varpi$. And

$$
\begin{aligned}
\omega_1 =& \left(1 + \frac{1}{\Gamma(1-\gamma)}\right)\left(\frac{m_0 M}{\Gamma(\alpha_1 + 1)} + \frac{n_0 W_1}{\Gamma(\alpha_2 + 1)}\right)\\
&+ \left(1 + \frac{1}{\Gamma(1-\delta)}\right)\left(\frac{n_0 M}{\Gamma(\alpha_2 + 1)} + \frac{m_0 W_2}{\Gamma(\alpha_1 + 1)}\right),
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\omega_2 =& \left(1 + \frac{1}{\Gamma(1-\gamma)}\right)\left(k_1 M + \frac{m_1 M}{\Gamma(\alpha_1 + 1)} + \frac{\max\{n_1, n_3\} W_1}{\Gamma(\alpha_2 + 1)}\right)\\
&+ \left(1 + \frac{1}{\Gamma(1-\delta)}\right)\left(k_1 W_2 + \frac{\max\{n_1, n_3\} M}{\Gamma(\alpha_2 + 1)} + \frac{m_1 W_2}{\Gamma(\alpha_1 + 1)}\right),
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\omega_3 =& \left(1 + \frac{1}{\Gamma(1-\gamma)}\right)\left(k_2 W_1 + \frac{\max\{m_2, m_3\} M}{\Gamma(\alpha_1 + 1)} + \frac{n_2 W_1}{\Gamma(\alpha_2 + 1)}\right)\\
&+ \left(1 + \frac{1}{\Gamma(1-\delta)}\right)\left(k_2 M + \frac{n_2 M}{\Gamma(\alpha_2 + 1)} + \frac{\max\{m_2, m_3\} W_2}{\Gamma(\alpha_1 + 1)}\right).
\end{aligned} \tag{3.8}
$$

Theorem 3.1. *Assume that functions $f, g : [1, e] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous and (H_1) hold. Then the system* (1.1)-(1.2) *has at least one solution $(u^*, v^*) \in X \times Y$.*

Proof. We will prove that F, defined by (3.1), has a fixed point. We divide the proof into two steps.

Step I. We show that the operator $F : X \times Y \to X \times Y$, defined by (3.1), is completely continuous.

First the continuity of the functions f and g implies that the operator F is continuous.

Now, we show that F is compact. Let $B_R = \{(u, v) \in X \times Y : \|(u, v)\|_{X\times Y} \leq R\}$ be a bounded subset of $X \times Y$. By (H_1) for $(u, v) \in B_R$, we have

$$
\begin{aligned}
|f(t, u(t), v(t), D^\delta v(t))| &\leq m_0 + m_1|u(t)| + m_2|v(t)| + m_3|D^\delta v(t)|\\
&\leq m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y
\end{aligned}
$$

$$
\begin{aligned}
&\le m_0 + \max\{m_1, m_2, m_3\}\|(u,v)\|_{X\times Y}\\
&:= L_1,
\end{aligned}
$$

and similarly

$$|g(t,u(t),v(t),D^\gamma u(t))| \le n_0 + \max\{n_1,n_2,n_3\}\|(u,v)\|_{X\times Y} := L_2.$$

Then, we have

$$
\begin{aligned}
&|F_1(u,v)(t)|\\
\le& \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\int_1^e |u(s)|\frac{ds}{s} + \frac{1}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}|f(s,u(s),v(s),D^\delta v(s))|\frac{ds}{s}\\
&+\sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\bigg(k_2\int_1^s |v(r)|\frac{dr}{r}\\
&+\frac{1}{\Gamma(\alpha_2)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_2-1}|g(r,u(r),v(r),D^\gamma u(r))|\frac{dr}{r}\bigg)ds\Bigg]\\
&+A\Bigg[k_2\int_1^e |v(s)|\frac{ds}{s} + \frac{1}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}|g(s,u(s),v(s),D^\gamma u(s))|\frac{ds}{s}\\
&-\sum_{j=1}^n \frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\bigg(k_1\int_1^s |u(r)|\frac{dr}{r}\\
&+\frac{1}{\Gamma(\alpha_1)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_1-1}|f(r,u(r),v(r),D^\delta v(r))|\frac{dr}{r}\bigg)ds\Bigg]\Bigg\}\\
&+k_1\int_1^t |u(s)|\frac{ds}{s} + \frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}|f(s,u(s),v(s),D^\delta v(s))|\frac{ds}{s}\\
\le& \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\|u\| + \frac{L_1}{\Gamma(\alpha_1+1)} + \sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\bigg(k_2\|v\|\log s + \frac{L_2(\log s)^{\alpha_2}}{\Gamma(\alpha_2+1)}\bigg)ds\Bigg]\\
&+\omega\Bigg[k_2\|v\| + \frac{L_2}{\Gamma(\alpha_2+1)} + \sum_{j=1}^n\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\bigg(k_1\|u\|\log s + \frac{L_1(\log s)^{\alpha_1}}{\Gamma(\alpha_1+1)}\bigg)ds\Bigg]\Bigg\}\\
&+k_1\|u\|\log t + \frac{L_1(\log t)^{\alpha_1}}{\Gamma(\alpha_1+1)}\\
\le& \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)} + \omega\bigg(k_2\rho + \frac{L_2}{\Gamma(\alpha_2+1)}\bigg)\Bigg]\\
&+\omega\Bigg[k_2\rho + \frac{L_2}{\Gamma(\alpha_2+1)} + \varpi\bigg(k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)}\bigg)\Bigg]\Bigg\} + k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)}\\
=& \bigg(k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)}\bigg)M + \bigg(k_2\rho + \frac{L_2}{\Gamma(\alpha_2+1)}\bigg)W_1,
\end{aligned}
$$

which, upon taking the norm for $t\in[1,e]$, yields

$$\|F_1(u,v)(t)\| \le \bigg(k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)}\bigg)M + \bigg(k_2\rho + \frac{L_2}{\Gamma(\alpha_2+1)}\bigg)W_1,$$

By definition of Hadamard fractional derivative with $0<\delta<1$, we get

$$|D^\gamma F_1(u,v)(t)| \le \frac{1}{\Gamma(1-\gamma)}\bigg(t\frac{d}{dt}\bigg)\int_1^t\left(\log\frac{t}{s}\right)^{-\delta}|F_1(u,v)(t)|\frac{ds}{s}$$

$$\leq \frac{1}{\Gamma(1-\gamma)}\left(\left(k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)}\right)M + \left(k_2\rho + \frac{L_2}{\Gamma(\alpha_2+1)}\right)W_1\right).$$

Hence,

$$\begin{aligned}\|F_1(u,v)\|_X &= \|F_1(u,v)\| + \|D^\gamma F_1(u,v)\| \\ &\leq \left(1+\frac{1}{\Gamma(1-\gamma)}\right)\left(\left(k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)}\right)M + \left(k_2\rho + \frac{L_2}{\Gamma(\alpha_2+1)}\right)W_1\right).\end{aligned} \tag{3.9}$$

Similarly, we can get

$$|F_2(u,v)(t)| \leq \left(k_2\rho + \frac{L_2}{\Gamma(\alpha_2+1)}\right)M + \left(k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)}\right)W_2,$$

and

$$|D^\delta F_2(u,v)(t)| \leq \frac{1}{\Gamma(1-\delta)}\left(\left(k_2\rho + \frac{L_2}{\Gamma(\alpha_2+1)}\right)M + \left(k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)}\right)W_2\right).$$

In consequence, we get

$$\begin{aligned}\|F_2(u,v)\|_Y &= \|F_2(u,v)\| + \|D^\delta F_2(u,v)\| \\ &\leq \left(1+\frac{1}{\Gamma(1-\delta)}\right)\left(\left(k_2\rho + \frac{L_2}{\Gamma(\alpha_2+1)}\right)M + \left(k_1\rho + \frac{L_1}{\Gamma(\alpha_1+1)}\right)W_2\right).\end{aligned} \tag{3.10}$$

Hence, from (3.11) and (3.12), we infer that F_1 and F_2 are uniformly bounded, which implies that the operator F is uniformly bounded.

Finally, we show that F is equicontinuous.

Let $t_1, t_2 \in [1,e]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}&|F_1(u,v)(t_2) - F_1(u,v)(t_1)| \\ \leq&\frac{|(\log t_2)^{\gamma_1-1} - (\log t_1)^{\gamma_1-1}|}{|\Delta|}\Bigg\{\Bigg[k_1\int_1^e |u(s)|\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}|f(s,u(s),v(s),D^\delta v(s))|\frac{ds}{s} \\ &+ \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(k_2\int_1^s|v(r)|\frac{dr}{r} \\ &+ \frac{1}{\Gamma(\alpha_2)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_2-1}|g(r,u(r),v(r),D^\gamma u(r))|\frac{dr}{r}\Bigg)ds\Bigg] \\ &+ A\Bigg[k_2\int_1^e|v(s)|\frac{ds}{s} + \frac{1}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}|g(s,u(s),v(s),D^\gamma u(s))|\frac{ds}{s} \\ &- \sum_{j=1}^n \frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Bigg(k_1\int_1^s|u(r)|\frac{dr}{r} \\ &+ \frac{1}{\Gamma(\alpha_1)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_1-1}|f(r,u(r),v(r),D^\delta v(r))|\frac{dr}{r}\Bigg)ds\Bigg]\Bigg\} + k_1\int_{t_1}^{t_2}|u(s)|\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha_1)}\Bigg|\int_1^{t_1}\left(\left(\log\frac{t_2}{s}\right)^{\alpha_1-1} - \left(\log\frac{t_1}{s}\right)^{\alpha_1-1}\right)|f(s,u(s),v(s),D^\delta v(s))|\frac{ds}{s}\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha_1-1}|f(s,u(s),v(s),D^\delta v(s))|\frac{ds}{s}\Bigg|\\
\le&\frac{|(\log t_2)^{\gamma_1-1}-(\log t_1)^{\gamma_1-1}|}{|\Delta|}\Bigg\{\Bigg[k_1\|u\|+\frac{L_1}{\Gamma(\alpha_1+1)}\\
&+\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\left(k_2\|v\|\log s+\frac{L_2(\log s)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)ds\Bigg]+\omega\Bigg[k_2\|v\|+\frac{L_2}{\Gamma(\alpha_2+1)}\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\left(k_1\|u\|\log s+\frac{L_1(\log s)^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)ds\Bigg]\Bigg\}+k_1\|u\|\int_{t_1}^{t_2}\frac{ds}{s}\\
&+\frac{L_1}{\Gamma(\alpha_1)}\Bigg|\int_1^{t_1}\left[\left(\left(\log\frac{t_2}{s}\right)^{\alpha_1-1}-\left(\log\frac{t_1}{s}\right)^{\alpha_1-1}\right)\right]\frac{ds}{s}+\int_{t_1}^{t_2}(\log\frac{t_2}{s})^{\alpha_1-1}\frac{ds}{s}\Bigg|\\
\le&\frac{|(\log t_2)^{\gamma_1-1}-(\log t_1)^{\gamma_1-1}|}{|\Delta|}\Bigg\{\Bigg[k_1R+\frac{L_1}{\Gamma(\alpha_1+1)}+\omega\left(k_2R+\frac{L_2}{\Gamma(\alpha_2+1)}\right)\Bigg]\\
&+\omega\Bigg[k_2R+\frac{L_2}{\Gamma(\alpha_2+1)}+\varpi\left(k_1R+\frac{L_1}{\Gamma(\alpha_1+1)}\right)\Bigg]\Bigg\}\\
&+k_1R+\frac{L_1}{\Gamma(\alpha_1+1)}[(\log t_2)^{\alpha_1}-(\log t_1)^{\alpha_1}],
\end{aligned}
$$

and

$$
\begin{aligned}
&|D^\gamma F_1(u,v)(t_2)-D^\gamma F_1(u,v)(t_1)|\\
\le&\frac{1}{\Gamma(1-\gamma)}\left|\left(t_2\frac{d}{dt_2}\right)\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{-\gamma}F_1(u,v)(s)\frac{ds}{s}-\left(t_1\frac{d}{dt_1}\right)\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{-\gamma}F_1(u,v)(s)\frac{ds}{s}\right|\\
\le&\frac{|F_1(u,v)|}{\Gamma(1-\gamma)}\Bigg\{t_2\frac{d}{dt_2}\int_1^{t}\left|\left(\log\frac{t_2}{s}\right)^{-\gamma}-\left(\log\frac{t_1}{s}\right)^{-\gamma}\right|\frac{ds}{s}+\left|t_2\frac{d}{dt_2}-t_1\frac{d}{dt_1}\right|\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{-\gamma}\frac{ds}{s}\\
&+t_2\frac{d}{dt_2}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{-\gamma}\frac{ds}{s}\Bigg\}.
\end{aligned}
$$

Hence, we get

$$|F_1(u,v)(t_2)-F_1(u,v)(t_1)|\to 0 \textit{ and } |D^\gamma F_1(u,v)(t_2)-D^\gamma F_1(u,v)(t_1)|\to 0,\quad t_2\to t_1.$$

In the same way, we have

$$|F_2(u,v)(t_2)-F_2(u,v)(t_1)|\to 0 \textit{ and } |D^\delta F_2(u,v)(t_2)-D^\delta F_2(u,v)(t_1)|\to 0,\quad t_2\to t_1.$$

Hence, F_1 and F_2 are equicontinuous and so the operator F is equicontinuous. Therefore, by Arzelá-Ascoli theorem, we deduce that the operator F is completely continuous.

Step II. We define a set $\varepsilon(F)=\{(u,v)\in X\times Y|(u,v)=\kappa F(u,v), 0\le\kappa\le 1\}$ and show that it is bounded. Let $(u,v)\in\varepsilon(F)$, then $(u,v)=\kappa F(u,v)$. For any $t\in[1,e]$, we have $u(t)=\kappa F_1(u,v)(t), v(t)=\kappa F_2(u,v)(t)$. Then, by (H_1), we obtain

$$
\begin{aligned}
&|u(t)|\\
\le&|F_1(u,v)(t)|\\
\le&\frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\int_1^e|u(s)|\frac{ds}{s}+\frac{1}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}|f(s,u(s),v(s),D^\delta v(s))|\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\bigg(k_2\int_1^s|v(r)|\frac{dr}{r}\\
&+\frac{1}{\Gamma(\alpha_2)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_2-1}|g(r,u(r),v(r),D^{\gamma}u(r))|\frac{dr}{r}\bigg)ds\bigg]\\
&+A\bigg[k_2\int_1^e|v(s)|\frac{ds}{s}+\frac{1}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}|g(s,u(s),v(s),D^{\gamma}u(s))|\frac{ds}{s}\\
&-\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\bigg(k_1\int_1^s|u(r)|\frac{dr}{r}\\
&+\frac{1}{\Gamma(\alpha_1)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_1-1}|f(r,u(r),v(r),D^{\delta}v(r))|\frac{dr}{r}\bigg)ds\bigg]\bigg\}\\
&+k_1\int_1^t|u(s)|\frac{ds}{s}+\frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}|f(s,u(s),v(s),D^{\delta}v(s))|\frac{ds}{s}\\
\le&\frac{1}{|\Delta|}\bigg\{\bigg[k_1\int_1^e|u(s)|\frac{ds}{s}+\frac{1}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}(m_0+m_1|u(t)|+m_2|v(t)|+m_3|D^{\delta}v(t)|)\frac{ds}{s}\\
&+\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\bigg(k_2\int_1^s|v(r)|\frac{dr}{r}\\
&+\frac{1}{\Gamma(\alpha_2)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_2-1}(n_0+n_1|u(t)|+n_2|v(t)|+n_3|D^{\gamma}u(t)|)\frac{dr}{r}\bigg)ds\bigg]\\
&+A\bigg[k_2\int_1^e|v(s)|\frac{ds}{s}+\frac{1}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}(n_0+n_1|u(t)|+n_2|v(t)|+n_3|D^{\gamma}u(t)|)\frac{ds}{s}\\
&-\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\bigg(k_1\int_1^s|u(r)|\frac{dr}{r}\\
&+\frac{1}{\Gamma(\alpha_1)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_1-1}(m_0+m_1|u(t)|+m_2|v(t)|+m_3|D^{\delta}v(t)|)\frac{dr}{r}\bigg)ds\bigg]\bigg\}\\
&+k_1\int_1^t|u(s)|\frac{ds}{s}+\frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}(m_0+m_1|u(t)|+m_2|v(t)|+m_3|D^{\delta}v(t)|)\frac{ds}{s}\\
\le&\frac{1}{|\Delta|}\bigg\{\bigg[k_1\|u\|_X+\frac{m_0+m_1\|u\|_X+\max\{m_2,m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)}\\
&+\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\bigg(k_2\|v\|_Y\log s\\
&+\frac{(n_0+\max\{n_1,n_3\}\|u\|_X+n_2\|v\|_Y)(\log s)^{\alpha_2}}{\Gamma(\alpha_2+1)}\bigg)ds\bigg]\\
&+\omega\bigg[k_2\|v\|_Y+\frac{n_0+\max\{n_1,n_3\}\|u\|_X+n_2\|v\|_Y}{\Gamma(\alpha_2+1)}\\
&+\sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\bigg(k_1\|u\|_X\log s\\
&+\frac{(m_0+m_1\|u\|_X+\max\{m_2,m_3\}\|v\|_Y)(\log s)^{\alpha_1}}{\Gamma(\alpha_1+1)}\bigg)ds\bigg]\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ k_1\|u\| \log t + \frac{(m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y)(\log t)^{\alpha_1}}{\Gamma(\alpha_1+1)} \\
\le & \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\|u\|_X + \frac{m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)} \\
&+ \omega\Bigg(k_2\|v\|_Y + \frac{n_0 + \max\{n_1, n_3\}\|u\|_X + n_2\|v\|_Y}{\Gamma(\alpha_2+1)}\Bigg)\Bigg] \\
&+ \omega\Bigg[k_2\|v\|_Y + \frac{n_0 + \max\{n_1, n_3\}\|u\|_X + n_2\|v\|_Y}{\Gamma(\alpha_2+1)} \\
&+ \varpi\Bigg(k_1\|u\|_X + \frac{m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)}\Bigg)\Bigg]\Bigg\} \\
&+ k_1\|u\|_X + \frac{m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)} \\
= & \Bigg(k_1\|u\|_X + \frac{m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)}\Bigg)M \\
&+ \Bigg(k_2\|v\|_Y + \frac{n_0 + \max\{n_1, n_3\}\|u\|_X + n_2\|v\|_Y}{\Gamma(\alpha_2+1)}\Bigg)W_1,
\end{aligned}
$$

which, on taking the norm for $t \in [1, e]$, yields

$$
\begin{aligned}
\|u\| \le & \Bigg(k_1\|u\|_X + \frac{m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)}\Bigg)M \\
&+ \Bigg(k_2\|v\|_Y + \frac{n_0 + \max\{n_1, n_3\}\|u\|_X + n_2\|v\|_Y}{\Gamma(\alpha_2+1)}\Bigg)W_1.
\end{aligned}
$$

Also

$$
\begin{aligned}
\|D^\gamma u\| \le & \frac{1}{\Gamma(1-\gamma)}\Bigg[\Bigg(k_1\|u\|_X + \frac{m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)}\Bigg)M \\
&+ \Bigg(k_2\|v\|_Y + \frac{n_0 + \max\{n_1, n_3\}\|u\|_X + n_2\|v\|_Y}{\Gamma(\alpha_2+1)}\Bigg)W_1\Bigg].
\end{aligned}
$$

In consequence, we get

$$
\begin{aligned}
\|u\|_X = & \|u\| + \|D^\gamma u\| \\
\le & \Bigg(1 + \frac{1}{\Gamma(1-\gamma)}\Bigg)\Bigg[\Bigg(k_1\|u\|_X + \frac{m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)}\Bigg)M \\
&+ \Bigg(k_2\|v\|_Y + \frac{n_0 + \max\{n_1, n_3\}\|u\|_X + n_2\|v\|_Y}{\Gamma(\alpha_2+1)}\Bigg)W_1\Bigg].
\end{aligned} \tag{3.11}
$$

Similarly, we can obtain

$$
\begin{aligned}
\|v\|_Y = & \|v\| + \|D^\gamma v\| \\
\le & \Bigg(1 + \frac{1}{\Gamma(1-\delta)}\Bigg)\Bigg[\Bigg(k_2\|v\|_Y + \frac{n_0 + n_2\|v\|_Y + \max\{n_1, n_3\}\|u\|_X}{\Gamma(\alpha_2+1)}\Bigg)M \\
&+ \Bigg(k_1\|u\|_X + \frac{m_0 + m_1\|u\|_X + \max\{m_2, m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)}\Bigg)W_2\Bigg].
\end{aligned} \tag{3.12}
$$

Taking into account the estimates (3.11) and (3.12) together with (3.6)-(3.8), we have

$$\|(u, v)\|_{X\times Y}$$

$$
\begin{aligned}
&=\|u\|_X+\|v\|_Y\\
&\le\left(1+\frac{1}{\Gamma(1-\gamma)}\right)\Bigg[\left(k_1\|u\|_X+\frac{m_0+m_1\|u\|_X+\max\{m_2,m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)}\right)M\\
&\quad+\left(k_2\|v\|_Y+\frac{n_0+\max\{n_1,n_3\}\|u\|_X+n_2\|v\|_Y}{\Gamma(\alpha_2+1)}\right)W_1\Bigg]\\
&\quad+\left(1+\frac{1}{\Gamma(1-\delta)}\right)\Bigg[\left(k_2\|v\|_Y+\frac{n_0+n_2\|v\|_Y+\max\{n_1,n_3\}\|u\|_X}{\Gamma(\alpha_2+1)}\right)M\\
&\quad+\left(k_1\|u\|_X+\frac{m_0+m_1\|u\|_X+\max\{m_2,m_3\}\|v\|_Y}{\Gamma(\alpha_1+1)}\right)W_2\Bigg]\\
&=\left(1+\frac{1}{\Gamma(1-\gamma)}\right)\left(\frac{m_0M}{\Gamma(\alpha_1+1)}+\frac{n_0W_1}{\Gamma(\alpha_2+1)}\right)+\left(1+\frac{1}{\Gamma(1-\delta)}\right)\left(\frac{n_0M}{\Gamma(\alpha_2+1)}+\frac{m_0W_2}{\Gamma(\alpha_1+1)}\right)\\
&\quad+\|u\|_X\Bigg[\left(1+\frac{1}{\Gamma(1-\gamma)}\right)\left(k_1M+\frac{m_1M}{\Gamma(\alpha_1+1)}+\frac{\max\{n_1,n_3\}W_1}{\Gamma(\alpha_2+1)}\right)\\
&\quad+\left(1+\frac{1}{\Gamma(1-\delta)}\right)\left(k_1W_2+\frac{\max\{n_1,n_3\}M}{\Gamma(\alpha_2+1)}+\frac{m_1W_2}{\Gamma(\alpha_1+1)}\right)\Bigg]\\
&\quad+\|v\|_Y\Bigg[\left(1+\frac{1}{\Gamma(1-\gamma)}\right)\left(k_2W_1+\frac{\max\{m_2,m_3\}M}{\Gamma(\alpha_1+1)}+\frac{n_2W_1}{\Gamma(\alpha_2+1)}\right)\\
&\quad+\left(1+\frac{1}{\Gamma(1-\delta)}\right)\left(k_2M+\frac{n_2M}{\Gamma(\alpha_2+1)}+\frac{\max\{m_2,m_3\}W_2}{\Gamma(\alpha_1+1)}\right)\Bigg]\\
&\le\omega_1+\max\{\omega_2,\omega_3\}\|(u,v)\|_{X\times Y},
\end{aligned}
$$

which, in view of $\|(u,v)\|_{X\times Y}=\|u\|_X+\|v\|_Y$, yields

$$\|(u,v)\|_{X\times Y}\le\frac{\omega_1}{1-\max\{\omega_2,\omega_3\}}.$$

Therefore the set $\varepsilon(F)$ is bounded. By Lemma 2.4, we get the operator F has at least one fixed point, i.e. there exists $(u^*,v^*)\in X\times Y$ such that $(u^*,v^*)=F(u^*,v^*)$. Thus, the problem (1.1)-(1.2) has at least one solution $(u^*,v^*)\in X\times Y$. □

Theorem 3.2. *Assume that* (H_2) *holds. Then the system* (1.1)-(1.2) *has a unique solution* $(u^*,v^*)\in X\times Y$, *provided that*

$$
\begin{aligned}
\Xi:=&\,k_1\left(\frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}M+\frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}W_2\right)+k_2\left(\frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}M+\frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}W_1\right)\\
&+\frac{L}{\Gamma(\alpha_1+1)}\left(\frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}M+\frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}W_2\right)\\
&+\frac{\bar{L}}{\Gamma(\alpha_2+1)}\left(\frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}M+\frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}W_1\right)\\
<&\,1.
\end{aligned}
\tag{3.13}
$$

Proof. We will use the Banach fixed point theorem to prove F has a unique fixed point. Let us define

$$N_1:=\max_{t\in[1,e]}|f(t,0,0,0)|<\infty \text{ and } N_2:=\max_{t\in[1,e]}|g(t,0,0,0)|<\infty.$$

By (H_2), we get

$$|f(t,u(t),v(t),D^\delta v(t))|\le|f(t,u(t),v(t),D^\delta v(t))-f(t,0,0,0)|+|f(t,0,0,0)|$$

$$\begin{aligned} &\leq L(\|u\| + \|v\| + \|D^{\delta} v\|) + N_1 \\ &= L\|(u,v)\| + N_1, \\ |g(t,u(t),v(t),D^{\gamma}u(t))| &\leq |g(t,u(t),v(t),D^{\gamma}u(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\ &\leq \bar{L}(\|u\| + \|v\| + \|D^{\gamma} u\|) + N_2 \\ &= \bar{L}\|(u,v)\| + N_2. \end{aligned}$$

We consider $B_R = \{(u,v) \in X \times Y : \|(u,v)\|_{X\times Y} \leq \mathbb{R}\}$ with

$$R \geq \frac{\frac{N_1}{\Gamma(\alpha_1+1)}\left(\frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}M + \frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}W_2\right) + \frac{N_2}{\Gamma(\alpha_2+1)}\left(\frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}M + \frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}W_1\right)}{1-\Xi}.$$

Step I. First, we show that $F(B_R) \subset B_R$. Let $(u,v) \in B_R$, for $t \in [1,e]$, we have

$$\begin{aligned} &|F_1(u,v)(t)| \\ \leq &\frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\int_1^e |u(s)|\frac{ds}{s} + \frac{1}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1} |f(s,u(s),v(s),D^{\delta}v(s))|\frac{ds}{s} \\ &+ \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(k_2\int_1^s |v(r)|\frac{dr}{r} \\ &+ \frac{1}{\Gamma(\alpha_2)}\int_1^s \left(\log\frac{s}{r}\right)^{\alpha_2-1}|g(r,u(r),v(r),D^{\gamma}u(r))|\frac{dr}{r}\Bigg)ds\Bigg] \\ &+ A\Bigg[k_2\int_1^e |v(s)|\frac{ds}{s} + \frac{1}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}|g(s,u(s),v(s),D^{\gamma}u(s))|\frac{ds}{s} \\ &- \sum_{j=1}^n \frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Bigg(k_1\int_1^s |u(r)|\frac{dr}{r} \\ &+ \frac{1}{\Gamma(\alpha_1)}\int_1^s \left(\log\frac{s}{r}\right)^{\alpha_1-1}|f(r,u(r),v(r),D^{\delta}v(r))|\frac{dr}{r}\Bigg)ds\Bigg]\Bigg\} \\ &+ k_1\int_1^t |u(s)|\frac{ds}{s} + \frac{1}{\Gamma(\alpha_1)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}|f(s,u(s),v(s),D^{\delta}v(s))|\frac{ds}{s} \\ \leq &\frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\|u\| + \frac{L(\|u\| + \|v\| + \|D^{\gamma}u\|) + N_1}{\Gamma(\alpha_1+1)} \\ &+ \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(k_2\|v\|\log s + \frac{(\bar{L}(\|u\| + \|v\| + \|D^{\gamma}u\|) + N_2)(\log s)^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg)ds\Bigg] \\ &+ \omega\Bigg[k_2\|v\| + \frac{\bar{L}(\|u\| + \|v\| + \|D^{\gamma}u\|) + N_2}{\Gamma(\alpha_2+1)} \\ &+ \sum_{j=1}^n \frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j - s)^{\sigma_j-1}\Bigg(k_1\|u\|\log s + \frac{(L(\|u\| + \|v\| + \|D^{\gamma}u\|) + N_1)(\log s)^{\alpha_1}}{\Gamma(\alpha_1+1)}\Bigg)ds\Bigg]\Bigg\} \\ &+ k_1\|u\|\log t + \frac{(L(\|u\| + \|v\| + \|D^{\gamma}u\|) + N_1)(\log t)^{\alpha_1}}{\Gamma(\alpha_1+1)} \\ \leq &\frac{1}{|\Delta|}\Bigg\{\Bigg[k_1 R + \frac{LR + N_1}{\Gamma(\alpha_1+1)} + \omega\Bigg(k_2 R + \frac{\bar{L}R + N_2}{\Gamma(\alpha_2+1)}\Bigg)\Bigg] \\ &+ \omega\Bigg[k_2 R + \frac{\bar{L}R + N_2}{\Gamma(\alpha_2+1)} + \varpi\Bigg(k_1 R + \frac{LR + N_1}{\Gamma(\alpha_1+1)}\Bigg)\Bigg]\Bigg\} + k_1 R + \frac{LR + N_1}{\Gamma(\alpha_1+1)} \end{aligned}$$

$$= \frac{1}{|\Delta|}\left(k_1 R + \frac{LR+N_1}{\Gamma(\alpha_1+1)}\right)[1+\omega\varpi+|\Delta|] + \frac{1}{|\Delta|}\left(k_2 R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}\right)[2\omega]$$
$$= k_1 MR + \frac{LR+N_1}{\Gamma(\alpha_1+1)}M + k_2 W_1 R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}W_1,$$

and

$$\begin{aligned}|D^\gamma F_1(u,v)(t)| \leq & \frac{1}{\Gamma(1-\gamma)}\left(t\frac{d}{dt}\right)\int_1^t \left(\log\frac{t}{s}\right)^{-\delta}\frac{|F_1(u,v)(t)|}{s}ds \\ \leq & \frac{1}{\Gamma(1-\gamma)}\left(k_1 MR + \frac{LR+N_1}{\Gamma(\alpha_1+1)}M + k_2 W_1 R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}W_1\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|F_1(u,v)(t)\|_X =& \|F_1(u,v)(t)\| + \|D^\gamma F_1(u,v)(t)\| \\ \leq & (1+\frac{1}{\Gamma(1-\gamma)})\left(k_1 MR + k_2 W_1 R + \frac{LR+N_1}{\Gamma(\alpha_1+1)}M + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}W_1\right).\end{aligned}$$

In similar ways, we obtain

$$\begin{aligned}|F_2(u,v)(t)| &\leq k_2 MR + k_1 W_2 R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}M + \frac{LR+N_1}{\Gamma(\alpha_1+1)}W_2, \\ |D^\delta F_2(u,v)(t)| &\leq \frac{1}{\Gamma(1-\delta)}\left(k_2 MR + k_1 W_2 R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}M + \frac{LR+N_1}{\Gamma(\alpha_1+1)}W_2\right).\end{aligned}$$

In consequence, we get

$$\begin{aligned}\|F_2(u,v)(t)\|_Y =& \|F_2(u,v)(t)\| + \|D^\gamma F_2(u,v)(t)\| \\ \leq & (1+\frac{1}{\Gamma(1-\delta)})\left(k_2 MR + k_1 W_2 R + \frac{\bar{L}R+N_2}{\Gamma(\alpha_2+1)}M + \frac{LR+N_1}{\Gamma(\alpha_1+1)}W_2\right).\end{aligned}$$

Hence, from (3.13), we get

$$\|F(u,v)(t)\|_{X\times Y} = \|F_1(u,v)(t)\|_X + \|F_2(u,v)(t)\|_Y \leq R,$$

which implies $F(B_R) \subset B_R$.

Step II. We will show that the operator F is a contraction. Let $(u_1,v_1),(u_2,v_2) \in X\times Y$. Then, by (H_2), for any $t\in[1,e]$, we have

$$\begin{aligned}&|F_1(u_2,v_2)(t) - F_1(u_1,v_1)(t)| \\ \leq & \frac{1}{|\Delta|}\Bigg\{\Bigg[k_1\int_1^e |u_2(s)-u_1(s)|\frac{ds}{s} \\ & + \frac{1}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}|f(s,u_2(s),v_2(s),D^\delta v_2(s)) - f(s,u_1(s),v_1(s),D^\delta v_1(s))|\frac{ds}{s} \\ & + \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(k_2\int_1^s |v_2(r)-v_1(r)|\frac{dr}{r} \\ & + \frac{1}{\Gamma(\alpha_2)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_2-1}|g(r,u_2(r),v_2(r),D^\gamma u_2(r)) - g(r,u_1(r),v_1(r),D^\gamma u_1(r))|\frac{dr}{r}\Bigg)ds\Bigg]\end{aligned}$$

$$
\begin{aligned}
&+ A\bigg[k_2\int_1^e |v_2(s)-v_1(s)|\frac{ds}{s} + \frac{1}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}|g(s,u_2(s),v_2(s),D^\gamma u_2(s))\\
&- g(s,u_1(s),v_1(s),D^\gamma u_1(s))|\frac{ds}{s} - \sum_{j=1}^{n}\frac{|\mu_j|}{\Gamma(\sigma_j)}\int_1^{\eta_j}(\eta_j-s)^{\sigma_j-1}\bigg(k_1\int_1^s |u_2(r)-u_1(r)|\frac{dr}{r}\\
&+ \frac{1}{\Gamma(\alpha_1)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha_1-1}|f(r,u_2(r),v_2(r),D^\delta v_2(r)) - f(r,u_1(r),v_1(r),D^\delta v_1(r))|\frac{dr}{r}\bigg)ds\bigg]\bigg\}\\
&+ k_1\int_1^t |u_2(s)-u_1(s)|\frac{ds}{s}\\
&+ \frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}|f(s,u_2(s),v_2(s),D^\delta v_2(s)) - f(s,u_1(s),v_1(s),D^\delta v_1(s))|\frac{ds}{s}\\
\le&\frac{1}{|\Delta|}\bigg\{\bigg[k_1|u_2-u_1| + \frac{L(|u_2-u_1|+|v_2-v_1|+|D^\gamma u_2 - D^\gamma u_1|)}{\Gamma(\alpha_1+1)}\\
&+ \sum_{i=1}^{m}\frac{|\lambda_i|(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i)}\bigg(k_2|v_2-v_1| + \frac{\bar{L}(|u_2-u_1|+|v_2-v_1|+|D^\gamma u_2 - D^\gamma u_1|)}{\Gamma(\alpha_2+1)}\bigg)\bigg]\\
&+ \omega\bigg[k_2|v_2-v_1| + \frac{\bar{L}(|u_2-u_1|+|v_2-v_1|+|D^\gamma u_2 - D^\gamma u_1|)}{\Gamma(\alpha_2+1)}\\
&+ \sum_{j=1}^{n}\frac{|\mu_j|(\eta_j-1)^{\sigma_j}}{\Gamma(\sigma_j)}\bigg(k_1|u_2-u_1| + \frac{L(|u_2-u_1|+|v_2-v_1|+|D^\gamma u_2 - D^\gamma u_1|)}{\Gamma(\alpha_1+1)}\bigg)\bigg]\bigg\}\\
&+ k_1|u_2-u_1| + \frac{L(|u_2-u_1|+|v_2-v_1|+|D^\gamma u_2 - D^\gamma u_1|)}{\Gamma(\alpha_1+1)}\\
\le&\frac{1}{|\Delta|}\bigg\{\bigg[k_1\|u_2-u_1\|_X + \frac{L(\|u_2-u_1\|_X+\|v_2-v_1\|_Y)}{\Gamma(\alpha_1+1)}\\
&+ \sum_{i=1}^{m}\frac{|\lambda_i|(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i)}\bigg(k_2\|v_2-v_1\|_Y + \frac{\bar{L}(\|u_2-u_1\|_X+\|v_2-v_1\|_Y)}{\Gamma(\alpha_2+1)}\bigg)\bigg]\\
&+ \omega\bigg[k_2\|v_2-v_1\|_Y + \frac{\bar{L}(\|u_2-u_1\|_X+\|v_2-v_1\|_Y)}{\Gamma(\alpha_2+1)}\\
&+ \sum_{j=1}^{n}\frac{|\mu_j|(\eta_j-1)^{\sigma_j}}{\Gamma(\sigma_j)}\bigg(k_1\|u_2-u_1\|_X + \frac{L(\|u_2-u_1\|_X+\|v_2-v_1\|_Y)}{\Gamma(\alpha_1+1)}\bigg)\bigg]\bigg\}\\
&+ k_1\|u_2-u_1\|_X + \frac{L(\|u_2-u_1\|_X+\|v_2-v_1\|_Y)}{\Gamma(\alpha_1+1)}\\
=&M\bigg(k_1\|u_2-u_1\|_X + \frac{L(\|u_2-u_1\|_X+\|v_2-v_1\|_Y)}{\Gamma(\alpha_1+1)}\bigg)\\
&+ W_1\bigg(k_2\|v_2-v_1\|_Y + \frac{\bar{L}(\|u_2-u_1\|_X+\|v_2-v_1\|_Y)}{\Gamma(\alpha_2+1)}\bigg)\\
\le&\bigg[k_1M + \frac{ML}{\Gamma(\alpha_1+1)} + k_2W_1 + \frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\bigg](\|u_2-u_1\|_X + \|v_2-v_1\|_Y).
\end{aligned}
$$

Also we have

$$|D^\gamma F_1(u_2,v_2)(t) - D^\gamma F_1(u_1,v_1)(t)|$$

$$\le \frac{1}{\Gamma(1-\gamma)}\left(t\frac{d}{dt}\right)\int_1^t \left(\log\frac{t}{s}\right)^{-\delta}|F_1(u_2,v_2)(t)-F_1(u_1,v_1)(t)|\frac{ds}{s}$$
$$\le \frac{1}{\Gamma(1-\gamma)}\left[k_1M+\frac{ML}{\Gamma(\alpha_1+1)}+k_2W_1+\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\right](\|u_2-u_1\|_X+\|v_2-v_1\|_Y).$$

From the above inequalities, we get

$$\begin{aligned}&\|F_1(u_2,v_2)(t)-F_1(u_1,v_1)(t)\|_X\\ =&\|F_1(u_2,v_2)(t)-F_1(u_1,v_1)(t)\|+\|D^\gamma F_1(u_2,v_2)(t)-D^\gamma F_1(u_1,v_1)(t)\|\\ \le&\left(1+\frac{1}{\Gamma(1-\gamma)}\right)\left[k_1M+\frac{ML}{\Gamma(\alpha_1+1)}+k_2W_1+\frac{W_1\bar{L}}{\Gamma(\alpha_2+1)}\right](\|u_2-u_1\|_X+\|v_2-v_1\|_Y).\end{aligned} \tag{3.14}$$

Similarly, we can find that

$$\begin{aligned}&\|F_2(u_2,v_2)(t)-F_2(u_1,v_1)(t)\|_Y\\ \le&\left(1+\frac{1}{\Gamma(1-\delta)}\right)\left[k_2M+\frac{M\bar{L}}{\Gamma(\alpha_2+1)}+k_1W_2+\frac{W_2L}{\Gamma(\alpha_1+1)}\right](\|u_2-u_1\|_X+\|v_2-v_1\|_Y).\end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\begin{aligned}\|F(u_2,v_2)-F(u_1,v_1)\|_{X\times Y}&=\|F_1(u_2,v_2)-F_1(u_1,v_1)\|_X+\|F_2(u_2,v_2)-F_2(u_1,v_1)\|_Y\\ &\le \Xi(\|u_2-u_1\|_X+\|v_2-v_1\|_Y).\end{aligned}$$

By (3.13), it shows the operator F is a contraction. Therefore, the boundary value problem (1.1)-(1.2) has a unique solution $(u^*,v^*)\in X\times Y$. □

4. Applications

In this section, we give two examples to illustrate our main results.

Example 4.1. Consider the following system

$$\begin{cases}({}_HD^{\frac{3}{2},\frac{1}{2}}+\frac{1}{5}{}_HD^{\frac{1}{2},\frac{1}{4}})u(t)\\ =\dfrac{|u(t)|}{\log t+50}+\dfrac{|v(t)|}{(t+4)^2(4+|v(t)|)}+\dfrac{1}{(t+3)^3}+\dfrac{\sin(D^{\frac{1}{2}}v(t))}{60}, \quad t\in[1,e],\\ ({}_HD^{2,\frac{1}{3}}+\frac{1}{10}{}_HD^{1,\frac{1}{3}})v(t)=\dfrac{|u(t)|}{10\sqrt{(t+24)}}+\dfrac{\sin(\pi v(t))}{70\pi}+\dfrac{1}{60}+\dfrac{\cos(D^{\frac{1}{3}}u(t))}{(t+7)^2}, \quad t\in[1,e],\\ u(1)=0, u(e)=3I^{\frac{7}{2}}v(\frac{4}{3})+13I^{\frac{5}{2}}v(\frac{7}{6}), v(1)=0, v(e)=\dfrac{1}{10}I^{\frac{7}{4}}u(2)+7I^{\frac{5}{2}}u(\frac{5}{3}),\end{cases} \tag{4.1}$$

where $\alpha_1=\frac{3}{2},\alpha_2=2,\beta_1=\frac{1}{2},\beta_2=\frac{1}{3},\delta=\frac{1}{2},\gamma=\frac{1}{3},\gamma_1=\frac{7}{4},\gamma_2=\frac{7}{3},k_1=\frac{1}{5},k_2=\frac{1}{10},\lambda_1=3,\lambda_2=13,\delta_1=\frac{7}{2},\delta_2=\frac{5}{2},\theta_1=\frac{4}{3},\theta_2=\frac{7}{6},\mu_1=\frac{1}{10},\mu_2=7,\sigma_1=\frac{7}{4},\sigma_2=\frac{5}{2},\eta_1=2,\eta_2=\frac{5}{3}$. We see that (H_1) holds, since

$$\begin{aligned}|f(t,u,v,D^\delta v)|&\le \frac{1}{64}+\frac{|u(t)|}{50}+\frac{|v(t)|}{100}+\frac{|D^\delta v(t)|}{60},\\ |g(t,u,v,D^\gamma u)|&\le \frac{1}{60}+\frac{|u(t)|}{50}+\frac{|v(t)|}{70}+\frac{|D^\gamma u(t)|}{64},\end{aligned}$$

with

$$m_0 = \frac{1}{64}, m_1 = \frac{1}{50}, m_2 = \frac{1}{100}, m_3 = \frac{1}{60}, n_0 = \frac{1}{60}, n_1 = \frac{1}{50}, n_2 = \frac{1}{70}, n_3 = \frac{1}{64}.$$

Using the given data, we find that $M = 2.0421, W_1 = 0.0998$, and $W_2 = 1.6545$. Thus, all the conditions of Theorem 3.1 are satisfied. Therefore, Hilfer-Hadamard boundary value problem (4.1) has at least one solution $(u^*, v^*) \in X \times Y$.

Example 4.2. We consider the following Hilfer-Hadamard boundary value problem

$$\begin{cases} ({}_HD^{\frac{5}{4},\frac{1}{2}} + \frac{4}{55}{}_HD^{\frac{1}{4},\frac{1}{2}})u(t) = \frac{(2+\log t)|u(t)|}{80} + \frac{|v(t)|}{\sqrt{99+t^2}(5+|v(t)|)} + \frac{\sin(D^{\frac{1}{2}}v(t))}{40e^t}, \quad t \in [1,e], \\ ({}_HD^{\frac{3}{2},1} + \frac{1}{26}{}_HD^{\frac{1}{2},1})v(t) = \frac{\sin u(t)}{(6+t)^2} + \frac{|v(t)|}{(4+t)^3(1+|v(t)|)} + \frac{\cos(D^{\frac{1}{3}}u(t))}{3t^2+50}, \quad t \in [1,e], \\ u(1) = 0, u(e) = \frac{5}{9}I^{\frac{5}{2}}v(2) + 6I^{\frac{3}{2}}v(\frac{3}{2}), v(1) = 0, v(e) = 5I^{\frac{19}{7}}u(\frac{5}{3}) + \frac{1}{21}I^{\frac{7}{2}}u(\frac{3}{2}), \end{cases} \tag{4.2}$$

where $\alpha_1 = \frac{5}{4}, \alpha_2 = \frac{3}{2}, \beta_1 = \frac{1}{2}, \beta_2 = 1, \delta = \frac{1}{2}, \gamma = \frac{1}{3}, \gamma_1 = \frac{13}{8}, \gamma_2 = 2, k_1 = \frac{4}{55}, k_2 = \frac{1}{26}, \lambda_1 = \frac{5}{9}, \lambda_2 = 6, \delta_1 = \frac{5}{2}, \delta_2 = \frac{3}{2}, \theta_1 = 2, \theta_2 = \frac{3}{2}, \mu_1 = 5, \mu_2 = \frac{1}{21}, \sigma_1 = \frac{19}{7}, \sigma_2 = \frac{7}{2}, \eta_1 = \frac{5}{3}, \eta_2 = \frac{3}{2}$, and

$$\begin{aligned} f(t,u,v,D^\delta v) &= \frac{(2+\log t)|u|}{80} + \frac{|v|}{\sqrt{99+t^2}(5+|v|)} + \frac{\sin(D^{\frac{1}{2}}v)}{40e^t}, \\ g(t,u,v,D^\gamma u) &= \frac{\sin u}{(6+t)^2} + \frac{|v|}{(4+t)^3(1+|v|)} + \frac{\cos(D^{\frac{1}{3}}u)}{3t^2+50}. \end{aligned}$$

We see that hypothesis (H_2) holds, because, for any $u_i, v_i \in \mathbb{R}$ for $i = 1, 2$, one has

$$\begin{aligned} |f(t,u_1,u_2,u_3) - f(t,v_1,v_2,v_3)| &\le \frac{1}{40}(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|), \\ |g(t,u_1,u_2,u_3) - g(t,v_1,v_2,v_3)| &\le \frac{1}{49}(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|), \end{aligned}$$

with $L = \frac{1}{40}$ and $\bar{L} = \frac{1}{49}$. We find that $M = 2.7575$,$W_1 = 3.6622$, and $W_2 = 0.8156$. Therefore, we have

$$\begin{aligned} \Xi := & k_1\Big(\frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}M + \frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}W_2\Big) + k_2\Big(\frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}M + \frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}W_1\Big) \\ & + \frac{L}{\Gamma(\alpha_1+1)}\Big(\frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}M + \frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}W_2\Big) \\ & + \frac{\bar{L}}{\Gamma(\alpha_2+1)}\Big(\frac{1+\Gamma(1-\delta)}{\Gamma(1-\delta)}M + \frac{1+\Gamma(1-\gamma)}{\Gamma(1-\gamma)}W_1\Big) \\ < & 1. \end{aligned}$$

Thus, all the conditions of Theorem 3.2 are satisfied. Therefore, Hilfer-Hadamard boundary value problem (4.2) has a unique solution $(u^*, v^*) \in X \times Y$.

5. Conclusion

We have utilized the Banach contraction mapping theorem to establish both the existence and uniqueness of the solution while applied Leray-Schauder alternative fixed point theorem to obtain the existence of solution to a class of coupled system of Hilfer-Hadamard sequential fractional differential equations with multi-point Riemann-Liouville fractional integral boundary value conditions. Hilfer-Hadamard fractional differential system with derivatives of diverse orders and the presence of fractional derivative terms in the nonlinearities. Consequently, we have presented a set of sufficient conditions that demonstrate the existence and uniqueness of solutions for a boundary value problem with multi-point Riemann-Liouville fractional integral boundary value conditions. Additionally, two examples have been verified to validate the derived results.

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