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UNIFORM ANTI-MAXIMUM PRINCIPLE FOR A CLASS OF STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS OF SECOND ORDER DIFFERENCE EQUATIONS*

Rui Wang[1], Yanqiong Lu[2] and Ruyun Ma[1,2,†]

Abstract We are concerned with the following Sturm-Liouville boundary value problem of second order difference equations

$$\begin{cases} -D^2u(t-1) - \lambda u(t) = f(t), \quad t \in [1,T]_{\mathbb{Z}}, \\ au(0) - Du(0) = 0, \quad bu(T+1) + Du(T) = 0, \end{cases}$$

where $a, b \in [0,\infty)$, D denotes the forward difference operator defined by $Du(t) = u(t+1) - u(t)$, $T \geq 3$ is a fixed positive integer, $[1,T]_{\mathbb{Z}} := \{1, 2, \cdots, T\}$, $f : [1,T]_{\mathbb{Z}} \to [0,\infty)$, and $\lambda \in \mathbb{R}$ is a parameter. We establish a uniform anti-maximum principle for such problem. As an application of the main result, we obtain the existence of solutions to a class of nonlinear problems via the method of lower and upper solutions and fixed point theorem.

Keywords Second-order operator, difference equations, uniform anti-maximum principle, principal eigenvalue.

MSC(2010) 39A05, 39A12, 39A60, 39A70.

1. Introduction

The maximum principle and anti-maximum principle are important tools for qualitative analysis of solutions to boundary value problems for differential equations. They can determine the sign of solutions, prove uniqueness, and provide a *priori* estimates, offering effective support for both theoretical analysis and engineering applications. For more details, see [3,7,8,13,21,25] and the references therein.

As a pioneering work, Clément and Pelletier [6] investigated anti-maximum principle for a class of second-order boundary value problems. In the one-dimensional case, they established uniform anti-maximum principle for boundary value problem of second-order ordinary differential equations as follows.

Theorem A. (Theorem 3, [6]) Suppose $u \in W^{2,1}(a,b)$ satisfies

$$\begin{cases} -pu'' + qu + \nu u - \mu u \geq 0, \qquad \text{a.e. } t \in (a,b), \\ cu(a) - u'(a) = 0, \quad du(b) + u'(b) = 0, \end{cases} \tag{1.1}$$

†The corresponding author.
[1]School of Mathematics and Statistics, Xidian University, Xi'an, Shaanxi 710071, China
[2]Department of Mathematics, Northwest Normal University, Lanzhou, Gansu 730070, China
*The authors were supported by National Natural Science Foundation of China (12461036, 12361040).
Email: wr393023294@163.com(R. Wang), luyq8610@126.com(Y. Lu), ryma@xidian.edu.cn(R. Ma)

where $p \in C([a,b],(0,\infty))$, q, $\nu \in L^\infty(a,b)$ and c, $d \in [0,\infty)$ are real numbers. Then there exists a $\delta > 0$ such that if $\mu_0 < \mu < \mu_0 + \delta$, either $u(t) < 0$ for all $t \in [a,b]$ or $u(t) \equiv 0$ on $[a,b]$. Here μ_0 is the first eigenvalue of the linear problem

$$\begin{cases} -pu'' + qu + \nu u - \mu u = 0, & \text{a.e. } t \in (a,b), \\ cu(a) - u'(a) = 0, \quad du(b) + u'(b) = 0. \end{cases}$$

It is worth noting that 'uniform' means that the δ in Theorem A is independent of f.

Difference equations can not only be regarded as numerical approximations for solutions of differential equations, but have also attracted considerable attention in practical applications due to numerous discrete models [1, 15, 26]. Compared with differential equations, studying difference equations often encounters new challenges, for example:

(1) The logistic model

$$\frac{du}{dt} = ru\left(1 - \frac{u}{K}\right)$$

admits only monotone solutions. However, chaos may occur for its discrete analog

$$Dy(j) = ry(j)\left(1 - \frac{y(j)}{K}\right),$$

see [9].

(2) For $p > 1$, the nonlinear problem

$$-u'' = u^p, \ t \in (0,1), \qquad u(0) = u(1) = 0$$

has a unique positive solution. However, whether its discrete counterpart

$$-D^2y(j-1) = y^p(j), \ j \in [1,T]_{\mathbb{Z}}, \qquad y(0) = y(T+1) = 0$$

has a unique positive solution is still open.

(3) The positive solutions of

$$-u'' = f(t,u), \ t \in (-1,1), \qquad u(-1) = u(1) = 0 \tag{1.2}$$

are symmetric when $f \geq 0$, see [12]. However, the positive solutions of its discrete counterpart

$$-D^2y(j-1) = f(t,y), \ j \in [-T,T]_{\mathbb{Z}}, \qquad y(-(T+1)) = y(T+1) = 0 \tag{1.3}$$

are not symmetric at all, see [17].

(4) If u is a solution of (1.2) and $u(\tau) = \min\limits_{t \in [0,1]} u(t)$, then $u'(\tau) = 0$. However, if y is a solution of (1.3) and $y(\tau) = \min\limits_{t \in [1,T]_{\mathbb{Z}}} y(t)$, then one only has $Dy(\tau) \geq 0$.

(5) Spurious solutions may arise in some discrete formulations of differential equations, see [26].

The maximum principle and anti-maximum principle of boundary value problems for differential equations have been extensively studied, see [5, 10, 14, 16, 20] and the references therein.

However, there is relatively little research on the anti-maximum principle for boundary value problems of difference equations. Potential challenges include constructing an appropriate function space in the discrete setting and clarifying whether δ depends on f. Accordingly, it is natural to ask whether a (uniform) anti-maximum principle holds for such problems.

It is the purpose of this paper to give an affirmative answer. More precisely, we establish a uniform anti-maximum principle for the following Sturm-Liouville boundary value problem of second order difference equations

$$\begin{cases} -D^2u(t-1) - \lambda u(t) = f(t), \quad t \in [1,T]_{\mathbb{Z}}, \\ au(0) - Du(0) = 0, \quad bu(T+1) + Du(T) = 0, \end{cases} \tag{1.4}$$

where $a, b \in [0, \infty)$, D denotes the forward difference operator defined by $Du(t) = u(t+1) - u(t)$, $T \geq 3$ is a fixed positive integer, $[1,T]_{\mathbb{Z}} = \{1, 2, \cdots, T\}$, $f : [1,T]_{\mathbb{Z}} \to [0, \infty)$, and $\lambda \in \mathbb{R}$ is a parameter.

Let λ_1 denote the principal eigenvalue of linear eigenvalue problem associated with (1.4). The main result is as follows.

Theorem 1.1. *Let $a, b \in [0, \infty)$, $\lambda \in \mathbb{R}$ and $f : [1,T]_{\mathbb{Z}} \to [0, \infty)$. There exists $\delta > 0$ such that for all $f \geq 0$ and $f \not\equiv 0$, if $\lambda_1 < \lambda < \lambda_1 + \delta$, then any solution u of problem* (1.4) *satisfies*

$$u(t) < 0, \qquad \forall\ t \in [0, T+1]_{\mathbb{Z}}.$$

Remark 1.1. By properly defining the function space X as shown below, we obtain a discrete analogue of Theorem A.

Remark 1.2. For related recent results on the anti-maximum principle for various differential operators under diverse boundary conditions, see Del Pezzo and Quaas [24], Bobkov, Drábek and Ilyasov [4], Albouy and Ureña [2] and Eloe and Neugebauer [11], etc.

The rest parts are organized as follows: In Section 2 we state some preliminaries including the existence of principal eigenvalue and the maximum principle. Section 3 provides the proof of Theorem 1.1. Finally, we present some applications of the main result in Section 4.

2. Preliminaries

Denote

$$\mathbb{T} = \big\{1, 2, \cdots, T\big\}, \quad \hat{\mathbb{T}} = \big\{0, 1, \cdots, T+1\big\}.$$

Let

$$Y := \Big\{(u(0), u(1), \cdots, u(T), u(T+1)) \;\Big|\; u : \hat{\mathbb{T}} \to \mathbb{R}\Big\}.$$

Then Y is a Banach space under the norm

$$\|u\|_Y = \sum_{t=1}^{T} u(t).$$

Noting that Y is also a Hilbert space under the inner product

$$\langle u, v\rangle = \sum_{t=1}^{T} u(t)v(t), \qquad u, v \in Y.$$

Note that the spectral structure of the linear eigenvalue problem

$$\begin{cases} -D^2u(t-1) = \lambda u(t), \quad t \in \mathbb{T}, \\ au(0) - Du(0) = 0, \quad bu(T+1) + Du(T) = 0 \end{cases} \tag{2.1}$$

has been extensively studied, see [18,19,23] and the references therein. As an immediate consequence, we have the following.

Lemma 2.1 (Theorems 7.2, 7.3, 7.6, [19])**.** *The boundary value problem* (2.1) *has* T *real and simple eigenvalues, which can be ordered as*

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots < \lambda_T.$$

Moreover, the eigenfunction ϕ_k *corresponding to the eigenvalue* λ_k *has exactly* $k-1$ *simple generalized zeros.*

Proposition 2.1. *Let* $\omega > -\lambda_1$ *and* $e : \hat{\mathbb{T}} \to \mathbb{R}$ *be the unique solution of linear problem*

$$\begin{cases} -D^2 u(t-1) + \omega u(t) = 1, \quad t \in \mathbb{T}, \\ au(0) - Du(0) = 0, \quad bu(T+1) + Du(T) = 0. \end{cases} \tag{2.2}$$

Then

$$e(t) > 0, \qquad t \in \hat{\mathbb{T}}.$$

Proof. From [19], Green function $G(t,s)$ of problem (2.2) satisfies

$$G(t,s) \geq 0, \qquad t \in \hat{\mathbb{T}},$$

i.e.,

$$e(t) = \sum_{t=1}^{T} G(t,s) \geq 0, \qquad t \in \hat{\mathbb{T}}.$$

When $-\lambda_1 < \omega < 0$,

$$-D^2 e(t-1) = 1 - \omega e(t) \geq 0, \qquad t \in \hat{\mathbb{T}},$$

this means that e is concave and

$$D^2 e(t-1) = e(t+1) - 2e(t) + e(t-1) \leq 0,$$

i.e.,

$$e(t) > 0, \qquad t \in \mathbb{T}.$$

Suppose on the contrary that $e(0) = 0$, then from boundary conditions in (2.2) we have

$$e(1) = 0.$$

Let $t = 1$ in equation of (2.2), we can see that

$$D^2 e(0) = e(2) - 2e(1) + e(0) = -1.$$

This means that

$$e(2) = -1.$$

This contradicts with $e(t) > 0,\ t \in \mathbb{T}$.

When $0 \leq \omega < \infty$, let $j_0 \in \hat{\mathbb{T}}$ be the final zero of $e(t)$, then $e(j_0) = 0$ and

$$e(j_0 + 1) + e(j_0 - 1) > 0.$$

On the other hand,

$$-D^2e(j_0-1)=1\geq 0,$$

thus

$$D^2e(j_0-1)=e(j_0+1)+e(j_0-1)\leq 0.$$

This is a contradiction. □

Let

$$X:=\Big\{u\in Y \;\Big|\; u \text{ satisfies boundary conditions in (1.4)}\Big\}$$

and there exists constant $\gamma_u\geq 0$ such that

$$-\gamma_u e\leq u\leq \gamma_u e, \qquad t\in\hat{\mathbb{T}},$$

where e is defined in Proposition 2.1. Define the norm of X by

$$\|u\|_e=\inf\Big\{\gamma_u\geq 0 \;\Big|\; -\gamma_u e\leq u\leq \gamma_u e\Big\}.$$

It is easy to check that X is a linear space. In fact, for any $\alpha,\beta\in\mathbb{R}$ and $u,v\in X$, we have

$$\begin{aligned} &au(0)-Du(0)=0, && bu(T+1)+Du(T)=0,\\ &av(0)-Dv(0)=0, && bv(T+1)+Dv(T)=0,\\ &-\gamma_u e\leq u\leq \gamma_u e, && -\gamma_v e\leq v\leq \gamma_v e \end{aligned}$$

for some $\gamma_u,\gamma_v\in[0,\infty)$. Let $z:=\alpha u+\beta v$. Then

$$\begin{aligned} &az(0)-Dz(0)=0, \qquad bz(T+1)+Dz(T)=0,\\ &-(|\alpha|\gamma_u+|\beta|\gamma_v)e\leq z\leq (|\alpha|\gamma_u+|\beta|\gamma_v)e. \end{aligned}$$

Therefore, $z\in X$. Moreover, X is a Banach space in the metric given by this norm $\|\cdot\|_e$, see also [22].

Define the operator $A:X\to Y$,

$$Au:=-D^2u(t-1), \qquad u\in X. \tag{2.3}$$

Let

$$P:=\big\{u\in X \;\big|\; u\geq 0\big\}.$$

Its interior will be denoted by $\mathring{P}$. By the fact $e>0$ in $\hat{\mathbb{T}}$, it is easy to check that

$$\mathring{P}=\big\{u\in P \;\big|\; u>0\big\}.$$

Since X,Y are finite dimensional spaces, $P\subset X\subset Y$ and the operator $A+\omega I:X\to Y$ is an isomorphism. Then $T=(A+\omega I)^{-1}:Y\to X$ is well-defined. By Proposition 2.1, we have $T_e=T\mid_{P\setminus\{0\}}:P\setminus\{0\}\to\mathring{P}$ is well-defined and compact. By the Krein-Rutman theorem, we have the following.

Lemma 2.2. *T_e is a strongly positive operator with the following properties.*

(i) The spectral radius $r(T_e)$ is positive.

(ii) $r(T_e)$ is a simple eigenvalue of T_e with corresponding eigenfunction $\phi_1\in P$, and there is no other eigenvalue with eigenfunction in $\mathring{P}$.

(iii) $r(T_e)$ *is a simple eigenvalue of* T_e^* *with a strictly positive eigenfunction* ψ_1.

(iv) For every $g \in P \setminus \{0\}$, *the equation* $\mu u - Tu = g$ *has exactly one solution in* $\mathring{P}$ *if* $\mu > r(T_e)$.

Set $r(T_e) = \mu_1$. For the operator $T_e : P \setminus \{0\} \to \mathring{P}$,

$$\begin{aligned}
T_e\phi_1 &= \mu_1\phi_1, \\
(A + \omega I)\phi_1 &= \frac{1}{\mu_1}\phi_1, \\
A\phi_1 &= \left(\frac{1}{\mu_1} - \omega\right)\phi_1,
\end{aligned}$$

i.e., $\lambda_1 := \frac{1}{\mu_1} - \omega$ is the only eigenvalue of $A : X \to Y$ that admits a positive eigenfunction. That is to say that λ_1 is the principal eigenvalue of A.

For $f \in Y$, we say that f is positive if $f(t) \geq 0$ in $\mathbb{T}$ and $f(t) > 0$ on some subsets of $\mathbb{T}$. And problem (1.4) can be equivalently expressed as an operator equation

$$Au - \lambda u = f. \tag{2.4}$$

The maximum principle applicable to problem (2.4) is stated as follows.

Proposition 2.2. *Let* $u \in X$ *be a solution of the problem* (2.4) *with* $\lambda < \lambda_1$. *If* $f \geq 0$ *and* $f \not\equiv 0$, *then*

$$u(t) > 0, \qquad \forall\, t \in \hat{\mathbb{T}}.$$

Proof. The equation (2.4) can be written as

$$(A + \omega I)u - (\lambda + \omega)u = f,$$

applying T to the above equation gives

$$u - (\lambda + \omega)Tu = Tf,$$

i.e.,

$$\mu u - Tu = \mu Tf,$$

where $\mu = \frac{1}{\lambda+\omega}$. If $f > 0$, then

$$(A + \omega I)(Tf) = f > 0,$$

by Proposition 2.1,

$$Tf \in P \setminus \{0\}.$$

If $\lambda < \lambda_1$, then

$$\mu = \frac{1}{\lambda + \omega} > \frac{1}{\lambda_1 + \omega} = \mu_1 > 0,$$

and hence

$$\mu Tf \in P \setminus \{0\}.$$

The result now follows from Lemma 2.2(iv). □

3. The proof of Theorem 1.1

For problem (1.4), we prove the main result of this paper: The uniform anti-maximum principle holds when the parameter λ lies in the right neighborhood of the principal eigenvalue.

The proof of Theorem 1.1. From Lemma 2.2, we see that λ_1 is the principal eigenvalue of A, and $\phi_1 \in Y$ is its associated eigenfunction satisfying $\|\phi_1\|_Y = 1$. From [19, Definition 7.4-7.5, Theorem 7.1], it can be concluded that the operator corresponding to

$$\begin{cases} -D^2u(t-1) - \lambda u(t) = 0, \quad t \in \mathbb{T}, \\ au(0) - Du(0) = 0, \quad bu(T+1) + Du(T) = 0 \end{cases}$$

is self-adjoint, so $\langle f, \phi_1 \rangle$ is well-defined for any $f \in Y$.

Define $Y_1 = \{g \in Y \mid \langle g, \phi_1 \rangle = 0\}$, then

$$Y = \text{span}\{\phi_1\} \oplus Y_1,$$

where $\text{span}\{\phi_1\}$ is the one-dimensional subspace. For any $u \in X,\ f \in Y$, we have

$$f = \langle f, \phi_1 \rangle \phi_1 + f_1, \tag{3.1}$$

$$u = \langle u, \phi_1 \rangle \phi_1 + u_1, \tag{3.2}$$

where $f_1 \in Y_1$ and $u_1 \in Y_1 \cap X =: X_1$. Substituting (3.1) and (3.2) into (2.4) yields

$$\langle u, \phi_1 \rangle = -\frac{\langle f, \phi_1 \rangle}{\lambda - \lambda_1}, \tag{3.3}$$

$$(A - \lambda I)u_1 = f_1. \tag{3.4}$$

And (3.4) can be rewritten as

$$(A - \lambda_1)u_1 + (\lambda_1 - \lambda)u_1 = f_1. \tag{3.5}$$

We claim that $A - \lambda_1 I : X_1 \to Y_1$ is an isomorphism. Since $N(A - \lambda_1 I) = \{u \mid (A - \lambda_1 I)u = 0\} = \text{span}\{\phi_1\}$, we obtain $\dim N(A - \lambda_1 I) = 1$. As λ_1 is simple, we have $co \dim R(A - \lambda_1 I) = \dim N(A - \lambda_1 I)^* = \dim N(A^* - \lambda_1 I) = 1$. Consequently, $A - \lambda_1 I$ is a Fredholm operator with zero index.

In addition, if $g \in R(A - \lambda_1 I)$, it means that there exists $u_1 \in X_1$ such that $(A - \lambda_1 I)u_1 = g$. Taking the inner product of both sides of (3.5) with ϕ_1, we obtain

$$\langle (A - \lambda_1 I)u_1, \phi_1 \rangle = \langle g, \phi_1 \rangle = 0,$$

and hence $g \in Y_1$, and $R(A - \lambda_1 I) \subset Y_1$. Again $co \dim\ Y_1 = 1$, we conclude that $R(A - \lambda_1 I) = Y_1$. By the definition of X_1, the map $A - \lambda_1 I : X_1 \to Y_1$ is one to one. Therefore, $A - \lambda_1 I : X_1 \to Y_1$ is an isomorphism.

Thus, there exists a constant $M_1 > 0$ such that

$$\|v\| \leq M_1 \|(A - \lambda_1 I)v\|_Y, \qquad \forall\ v \in X,$$

where $\|v\| = \max_{t\in\hat{\mathbb{T}}}|v(t)|$. Since X is finite dimensional space, the norms $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent. Substitute the above formula into (3.5), we obtain

$$\begin{aligned}\frac{1}{M_1}\|u_1\| &\le \|(A-\lambda_1 I)u_1\|_Y\\ &\le \|f_1\|_Y + |\lambda_1-\lambda|\|u_1\|_Y\\ &\le \|f_1\|_Y + |\lambda_1-\lambda|T\|u_1\|.\end{aligned}$$

Hence there exist constants $M_2>0$ and $\delta_2>0$ such that for $\lambda_1<\lambda<\lambda_1+\delta_2$,

$$\begin{aligned}\|u_1\| &\le \frac{\|f_1\|_Y}{\frac{1}{M_1}-|\lambda_1-\lambda|T}\\ &\le M_2\|f_1\|_Y\\ &= M_2\|f-\langle f,\phi_1\rangle\phi_1\|_Y\\ &\le M_2\left[\sum_{t=1}^{T} f(t) + \sum_{t=1}^{T}\langle f,\phi_1\rangle\phi_1(t)\right]\\ &= M_2\left[\frac{1}{\langle f,\phi_1\rangle}\sum_{t=1}^{T} f(t) + \sum_{t=1}^{T}\phi_1(t)\right]\langle f,\phi_1\rangle\\ &\le M_2\left[\frac{1}{\phi_*}+T\right]\langle f,\phi_1\rangle\\ &=: M_3\langle f,\phi_1\rangle,\end{aligned}$$

where $0<\delta_2\le\frac{M_2-M_1}{M_1M_2T}$ and $\phi_*=\min\left\{\phi_1(t)\mid t\in\hat{\mathbb{T}}\right\}$. Substituting this estimate into (3.2), we obtain for $\lambda_1<\lambda<\lambda_1+\delta_2$ that

$$\begin{aligned}u &\le -\frac{1}{\lambda-\lambda_1}\langle f,\phi_1\rangle\phi_1 + M_3\langle f,\phi_1\rangle\\ &= -\frac{1}{\lambda-\lambda_1}[\phi_1-(\lambda-\lambda_1)M_3]\langle f,\phi_1\rangle.\end{aligned}$$

If we choose $\lambda_1<\lambda<\lambda_1+\delta$ with

$$\delta=\min\left\{\delta_2,\frac{\phi_*}{M_3}\right\},$$

obviously,

$$u(t)<0, \qquad t\in\hat{\mathbb{T}}.$$

□

Remark 3.1. Besides the sign property of u, the proof also yields an upper bound estimate for u when λ is close to λ_1. Thus if $\lambda\in(\lambda_1,\lambda_1+\delta_0]$, where $0<\delta_0<\delta$ given in Theorem 1.1, then

$$u(t)\le-\frac{1}{\lambda-\lambda_1}[\phi_1-\delta_0M_3]\langle f,\phi_1\rangle, \qquad t\in\hat{\mathbb{T}}.$$

Remark 3.2. In Proposition 2.2 and Theorem 1.1, we deal with (2.4) for $\lambda<\lambda_1$ and $\lambda>\lambda_1$ but $\lambda-\lambda_1$ is small. Note that if $\lambda=\lambda_1$ and $f>0$, then the equatin (2.4) has no solutions. Indeed, if $f\in R(A-\lambda_1 I)$, there is $\langle f,\phi_1\rangle=0$. This is impossible and hence $f\notin R(A-\lambda_1 I)$.

4. Applications

As applications of the main result, let us consider the existence of solutions for the nonlinear problem

$$\begin{cases} -D^2u(t-1) = h(t,u(t)), \quad t \in \mathbb{T}, \\ au(0) - Du(0) = 0, \quad bu(T+1) + Du(T) = 0. \end{cases} \tag{4.1}$$

Set

$$E = \Big\{ (u(0), u(1), \cdots, u(T), u(T+1)) \;\Big|\; u : \hat{\mathbb{T}} \to \mathbb{R} \Big\}$$

with the norm

$$\|u\|_\infty = \max_{t\in\hat{\mathbb{T}}} |u(t)|,$$

then E is a Banach space. We call a pointwise partial order $\leq$ on E as

$$u \leq v \iff u(t) \leq v(t), \quad \forall\, t \in \hat{\mathbb{T}}.$$

Definition 4.1. The function $\alpha : \hat{\mathbb{T}} \to \mathbb{R}$ is said to be a lower solution of (4.1) if α satisfies

$$\begin{cases} -D^2\alpha(t-1) \leq h(t,\alpha(t)), \quad t \in \mathbb{T}, \\ a\alpha(0) - D\alpha(0) = 0, \quad b\alpha(T+1) + D\alpha(T) = 0. \end{cases} \tag{4.2}$$

Further, α is a strict lower solution if the inequality in (4.2) strictly holds. The function $\beta : \hat{\mathbb{T}} \to \mathbb{R}$ is said to be an upper solution (a strict upper solution) of (4.1) by reversing the inequalities in (4.2).

Proposition 4.1. *Let α and β be, respectively, a lower solution and an upper solution of problem* (4.1)*. Suppose that $h : \hat{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to the second variable and satisfies*
$(H1)$ $h(t,\alpha(t)) - \lambda\alpha(t) \leq h(t,s(t)) - \lambda s(t) \leq h(t,\beta(t)) - \lambda\beta(t)$, for $\beta(t) \leq s(t) \leq \alpha(t)$, $t \in \hat{\mathbb{T}}$.
If $\lambda \in (\lambda_1, \lambda_1 + \delta)$ (δ and λ_1 are introduced in Theorem 1.1 and Lemma 2.1, respectively), then there exists a solution u of (4.1) *which satisfies*

$$u \in [\beta, \alpha], \qquad t \in \hat{\mathbb{T}}.$$

Moreover if the lower (upper) solution is strict, then

$$\alpha(t) > u(t) \;\; (\beta(t) < u(t)), \qquad t \in \mathbb{T}.$$

Proof. For each $f \in E$, let $\mathcal{K}f : E \to \mathbb{R}$ be the unique solution of problem

$$\begin{cases} -D^2u(t-1) - \lambda u(t) = f(t), \quad t \in \mathbb{T}, \\ au(0) - Du(0) = 0, \quad bu(T+1) + Du(T) = 0. \end{cases}$$

Let $\mathcal{H}$ denote the Nemytskii operator associated with h,

$$(\mathcal{H}f)(t) = h(t,f(t)) - \lambda f(t), \quad \forall\, t \in \hat{\mathbb{T}},$$

and define $\mathcal{T} : E \to \mathbb{R}$ as

$$\mathcal{T} = \mathcal{K} \circ \mathcal{H},$$

that is, for each f, the function $\mathcal{T}f$ is the unique solution of problem

$$\begin{cases} -D^2u(t-1) - \lambda u(t) = h(t, f(t)) - \lambda f(t), \quad t \in \mathbb{T}, \\ au(0) - Du(0) = 0, \quad bu(T+1) + Du(T) = 0. \end{cases} \tag{4.3}$$

Clearly the operator $\mathcal{T}$ is completely continuous. For $\beta \leq \alpha$, we define the functional interval

$$[\beta, \alpha] = \{u \in E \mid \beta \leq u \leq \alpha\}.$$

Firstly, we claim that $\mathcal{K} : E \to \mathbb{R}$ is nonincreasing. Let $f_1, f_2 \in E$ with $f_1 \leq f_2$ and put $u_i = \mathcal{K}f_i,\ i = 1, 2$. Then $\chi := u_2 - u_1$ satisfies

$$\begin{cases} -D^2\chi(t-1) - \lambda\chi(t) = f_2(t) - f_1(t) \geq 0, \quad t \in \mathbb{T}, \\ a\chi(0) - D\chi(0) = 0, \quad b\chi(T+1) + D\chi(T) = 0. \end{cases}$$

Since $\lambda \in (\lambda_1, \lambda_1 + \delta)$, from Theorem 1.1, it follows that $\chi \leq 0$ and hence $u_2 \leq u_1$.

Secondly, it can be proved that $\mathcal{T}([\beta, \alpha]) \subset [\beta, \alpha]$. In fact, since α is a lower solution of problem (4.1) and $\mathcal{T}\alpha$ is the unique solution of problem (4.3), we have

$$\begin{aligned} -D^2(\mathcal{T}\alpha)(t-1) - \lambda(\mathcal{T}\alpha)(t) &= h(t, \alpha(t)) - \lambda\alpha(t) \\ &\geq -D^2\alpha(t-1) - \lambda\alpha(t), \quad \forall\ t \in \hat{\mathbb{T}}. \end{aligned}$$

Thus $\eta := \mathcal{T}\alpha - \alpha$ satisfies that

$$\begin{cases} -D^2\eta(t-1) - \lambda\eta(t) \geq 0, \quad t \in \mathbb{T}, \\ a\eta(0) - D\eta(0) = 0, \quad b\eta(T+1) + D\eta(T) = 0. \end{cases}$$

According to Theorem 1.1 again, $\eta \leq 0$, i.e., $\mathcal{T}\alpha \leq \alpha$. In an analogous way we can prove that $\beta \leq \mathcal{T}\beta$.

Based on assumption $(H1)$ and the definition of $\mathcal{T}$, we infer that

$$\mathcal{T}\beta \leq \mathcal{T}u \leq \mathcal{T}\alpha,$$

which implies $\mathcal{T}([\beta, \alpha]) \subset [\beta, \alpha]$.

Finally, it can be seen that the interval $[\beta, \alpha]$ is a nonempty closed convex and bounded subset of the Banach space E. Then by applying Schauder's fixed point theorem, we can obtain the existence of a fixed point of $\mathcal{T}$, which obviously is a solution of problem (4.1) in $[\beta, \alpha]$. □

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