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GLOBAL EXISTENCE AND BLOW-UP PHENOMENA FOR A HYPERBOLIC *P*-LAPLACIAN EQUATION WITH LOGARITHMIC NONLINEARITY AND NONLINEAR BOUNDARY CONDITIONS*

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Abstract In this paper, we investigate the global existence and finite-time blow-up phenomena for a class of hyperbolic equations involving nonlinear p-Laplacian diffusion, damping effects, logarithmic source terms, and nonlinear boundary conditions. The considered model is governed by

$$u_{tt} - \operatorname{div}\left(|\nabla u^m|^{p-2}\nabla u^m\right) + \mu u_t = k(t)\, u \ln(1+u), \quad (x,t) \in \Omega \times (0,T),$$

subject to nonlinear boundary conditions and suitable initial data, where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary, $p \geq 2$, $m \geq 1$, $\mu > 0$, and $k(t)$ is a nonnegative differentiable function. By constructing suitable energy and auxiliary functionals, we establish sufficient conditions for the global existence of weak solutions. Moreover, under appropriate assumptions on the logarithmic source term and boundary nonlinearity, we prove that solutions blow up in finite time $T^* < \infty$ and derive explicit upper and lower estimates for the blow-up time. In addition, comprehensive numerical simulations are presented to illustrate the theoretical results and to describe the influence of the damping coefficient, logarithmic source intensity, and initial energy on the qualitative behavior of solutions. The computations confirm the sharpness of the analytical criteria and highlight the delicate interplay between hyperbolic dynamics, nonlinear diffusion, damping mechanisms, and logarithmic nonlinearities. The obtained results extend and complement several recent studies devoted to nonlinear parabolic and pseudo-parabolic equations involving p-Laplacian operators and logarithmic nonlinearities.

Keywords Hyperbolic equation, p-Laplacian, logarithmic nonlinearity, nonlinear boundary conditions, global existence, finite-time blow-up, numerical results, nonlinear equations.

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1. Introduction

Part I: Background and blow-up theory.

Nonlinear evolution equations exhibit a rich spectrum of behaviors, including global existence, finite-time blow-up, and long-time decay, depending on the balance between diffusion, damping, and source terms. A rigorous analysis of such phenomena typically relies on Sobolev space theory, energy methods, and functional analysis, as discussed in [8, 12]. For instance, consider

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the classical hyperbolic equation with a nonlinear source:

$$u_{tt} - \Delta u + \mu u_t = f(u), \quad (x,t) \in \Omega \times (0,T),$$

subject to appropriate initial and boundary conditions. The associated energy functional

$$E(t) = \frac{1}{2}\int_\Omega |u_t(x,t)|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2\,dx - \int_\Omega F(u(x,t))\,dx,$$

with $F(u) = \int_0^u f(s)\,ds$, plays a central role in the qualitative analysis. It is well known that if the initial energy $E(0)$ is negative or sufficiently small, the solution may blow up in finite time. This phenomenon is rigorously captured by the concavity method, originally developed in [19] and further refined in [18], which establishes blow-up through differential inequalities of the form

$$\Phi''(t)\Phi(t) - (1+\theta)(\Phi'(t))^2 \geq 0,$$

for a suitable auxiliary functional $\Phi(t)$ and constant $\theta > 0$.

In the parabolic context, a systematic framework for blow-up analysis has been developed, yielding precise conditions under which solutions exhibit finite-time singularities. Moreover, explicit upper and lower bounds for the blow-up time T^* have been obtained, connecting it directly with the initial energy and source growth rate [23].

More recently, logarithmic and nonlinear source terms in diffusion-type and evolution equations have attracted significant attention. For example, singular parabolic p-Laplacian equations with logarithmic nonlinearity have been studied in [30], where existence and blow-up results were established. Long-time behavior for fractional diffusion equations with logarithmic sources was analyzed in [14], while global dynamics of pseudo-parabolic systems with coupled logarithmic nonlinearities were investigated in [9]. These works highlight the delicate influence of logarithmic growth in determining global versus blow-up behavior.

Furthermore, doubly nonlinear diffusion equations with nonlinear Neumann boundary conditions have been studied in detail, where both global existence criteria and sharp estimates for blow-up time were derived. These results emphasize the subtle interplay between interior source terms and boundary nonlinearities, showing that even moderate sources can induce blow-up when reinforced by nonlinear boundary fluxes [10]. These foundational results motivate the present study, where we extend the analysis to hyperbolic p-Laplacian equations with logarithmic nonlinearities, combining energy methods, concavity arguments, and numerical verification to fully characterize the solution behavior.

Part II: p-Laplacian and logarithmic nonlinearities.

In this study, we focus on hyperbolic problems with p-Laplacian diffusion and logarithmic source terms of the form

$$\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m), \qquad f(u) = u\ln(1+u),$$

where $p \geq 2$, $m \geq 1$, and $u \geq 0$. Such operators generalize the classical porous medium diffusion and arise naturally in modeling flow through heterogeneous media, nonlinear heat conduction, and viscoelastic dynamics [21, 28]. The p-Laplacian diffusion term introduces nonlinear dissipation, while the logarithmic source term, despite its slower growth compared with polynomial nonlinearities, can still generate critical effects near the origin that strongly influence global existence and blow-up behavior.

For parabolic and pseudo-parabolic equations, the interplay between p-Laplacian diffusion and logarithmic sources has been extensively investigated. Recent works highlight the role of logarithmic nonlinearities in evolution equations. For instance, singular parabolic p-Laplacian equations with logarithmic sources were studied in [30], while fractional diffusion models with logarithmic nonlinearities were analyzed in [14]. Pseudo-parabolic systems with coupled logarithmic terms were further investigated in [9], showing how such nonlinearities influence global dynamics and blow-up behavior.

Pseudo-parabolic p-Laplacian equations with logarithmic nonlinearities were analyzed in [4, 15], deriving sufficient conditions for finite-time blow-up and decay estimates. Conditions for global existence and blow-up were established in [2], emphasizing the delicate growth of the logarithmic term near zero. Blow-up criteria using refined energy functionals were obtained in [4], while detailed results on existence, decay, and blow-up phenomena for p-Laplacian evolution equations with logarithmic sources were provided in [25]. In addition, explicit upper bounds for the blow-up time in pseudo-parabolic p-Laplacian equations were derived in [11], highlighting the quantitative role of the source term in accelerating singularity formation.

Recent developments have also extended logarithmic nonlinear analysis to strongly damped wave equations, coupled viscoelastic systems, and Schrödinger-type models. Nonexistence results for strongly damped p-Laplacian wave equations with logarithmic sources were obtained in [5]. Coupled viscoelastic wave systems involving logarithmic nonlinear sources were investigated in [22], where sharp energy dissipation and validated numerical dynamics were established. Schrödinger–Poisson systems with double logarithmic and nonlocal nonlinearities were studied in [6], while normalized solutions for critical p-Laplacian Schrödinger–Bopp–Podolsky systems with logarithmic nonlinearities were analyzed in [7]. These recent contributions further demonstrate the growing importance of logarithmic nonlinearities in nonlinear evolution equations and mathematical physics.

Boundary nonlinearities are also crucial in determining the qualitative behavior of solutions. For nonlinear Neumann or Robin conditions of the form

$$|\nabla u^m|^{p-2}\frac{\partial u}{\partial \nu} = g(u),$$

blow-up may occur even when the interior source is weak, since boundary fluxes can enhance energy accumulation.

Doubly nonlinear parabolic equations with nonlinear boundary sources were studied in [27], showing that boundary forcing can significantly modify decay rates and induce blow-up under suitable conditions. Reaction–diffusion equations with Robin boundary conditions were analyzed in [26], providing criteria for both blow-up and global existence. Divergence-form parabolic equations with inhomogeneous Neumann boundary conditions were investigated in [29], identifying regimes for blow-up and global existence driven by interior and boundary effects. Semilinear wave equations with interior and boundary source terms were studied in [31], showing that sufficiently strong boundary sources can induce blow-up even when interior terms alone do not. Porous medium and diffusion equations with boundary source terms were investigated in [24], illustrating how boundary-induced energy growth can lead to finite-time blow-up under appropriate conditions.

Overall, these studies highlight the intricate interplay between p-Laplacian diffusion, logarithmic sources, and boundary effects, providing the theoretical foundation for the present work, where we extend the analysis to hyperbolic equations using energy methods, concavity techniques, and numerical verification to fully characterize solution behavior.

Part III: Problem formulation, energy functionals, and contributions.

Motivated by previous studies on hyperbolic equations with nonlinear diffusion and logarithmic sources, we consider the following initial–boundary value problem:

$$\begin{cases} u_{tt} - \operatorname{div}\big(|\nabla u^m|^{p-2}\nabla u^m\big) + \mu u_t = k(t)\, u \ln(1+u), & (x,t) \in \Omega \times (0,T), \\ |\nabla u^m|^{p-2}\dfrac{\partial u}{\partial \nu} = g(u), & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) > 0, \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded star-shaped domain with smooth boundary, $\mu > 0$ is a damping coefficient, $p \geq 2$, $m \geq 1$, and $k(t) \geq 0$ is a differentiable function. The initial data u_0 and u_1 are assumed to be sufficiently regular.

To study the qualitative behavior of solutions, we introduce the energy functional

$$E(t) = \frac{1}{2}\int_\Omega |u_t(x,t)|^2\, dx + \frac{1}{p}\int_\Omega |\nabla u^m(x,t)|^p\, dx - \int_\Omega k(t)F(u(x,t))\, dx - \int_{\partial\Omega} G(u(x,t))\, d\sigma,$$

where $F(u) = \int_0^u s\ln(1+s)\, ds$ and $G(u) = \int_0^u g(s)\, ds$.

This functional captures the combined effects of kinetic energy, nonlinear p-Laplacian diffusion, interior logarithmic source, and boundary flux, providing a natural framework for both global existence and blow-up analysis.

Finite-time blow-up is characterized by the existence of $T^* < \infty$ such that

$$\lim_{t \to T^*} \|u(t)\|_{L^\infty(\Omega)} = +\infty.$$

Sufficient conditions for blow-up can be derived using an auxiliary functional

$$\Phi(t) = \int_\Omega u(x,t)\, u_t(x,t)\, dx + \alpha \int_\Omega u^2(x,t)\, dx,$$

which encodes the coupling between the solution and its time derivative. Under suitable assumptions on the initial data, boundary nonlinearity, and source coefficient, this functional satisfies a concavity-type differential inequality

$$\Phi''(t)\Phi(t) - (1+\beta)\big(\Phi'(t)\big)^2 \geq 0,$$

for some $\beta > 0$, which guarantees finite-time blow-up via classical concavity arguments [18, 19].

This formulation provides the foundation for the main results of the paper: (i) Global existence of weak solutions under suitable energy conditions, (ii) explicit finite-time blow-up criteria incorporating both logarithmic source and nonlinear boundary effects, and (iii) explicit upper and lower bounds for the blow-up time. These results extend and complement several existing studies in the literature.

In particular, doubly nonlinear diffusion equations with nonlinear Neumann boundary conditions have been shown to admit sharp criteria for global existence and precise blow-up time estimates [10]. Similar systems with boundary sources demonstrate that boundary nonlinearities can significantly influence blow-up phenomena [27]. Pseudo-parabolic p-Laplacian equations with logarithmic nonlinearities have been studied with explicit upper bounds for blow-up time [11], while blow-up criteria for related logarithmic source problems have been established using refined energy methods [4, 15].

Part IV: Outline and main contributions.

The main contributions of this paper are threefold. First, we establish the global existence of weak solutions under appropriate conditions on the initial energy and the source term, extending classical results for p-Laplacian and porous medium type equations. Second, we derive finite-time blow-up criteria by employing refined concavity arguments and energy functional techniques, taking into account both the logarithmic source term and the nonlinear boundary conditions. Third, we obtain explicit upper and lower bounds for the blow-up time.

These results are achieved within a unified energy framework involving the primitives $F(u) = \int_0^u s\ln(1+s)\,ds$ and $G(u) = \int_0^u g(s)\,ds$, which allow a consistent treatment of both interior and boundary nonlinear effects.

These results extend and complement several recent studies: Doubly nonlinear diffusion equations with nonlinear Neumann boundary conditions [10]; doubly nonlinear parabolic equations with nonlinear boundary sources [27]; pseudo-parabolic p-Laplacian equations with logarithmic nonlinearities [11]; and blow-up criteria for pseudo-parabolic p-Laplacian equations with logarithmic source terms [4, 15].

Finally, in Section 5, numerical experiments on $\Omega = (0,1)$ using finite differences in space and explicit time-stepping are performed. These experiments illustrate the impact of the logarithmic source term, the damping coefficient μ, and the initial energy on global existence and blow-up, providing a practical complement to the theoretical analysis. The numerical results further confirm the sharpness of the analytical criteria derived in this work.

2. Global existence of solutions

In this section, we establish sufficient conditions ensuring the global existence of weak solutions to problem (1.1). Due to the hyperbolic nature of the equation, the analysis relies on energy methods rather than the maximum principle.

2.1. Functional setting and weak solutions

Let $T > 0$ be arbitrary. We introduce the standard functional spaces

$$u \in L^\infty(0,T;W^{1,p}(\Omega)), \quad u_t \in L^\infty(0,T;L^2(\Omega)), \quad u_{tt} \in L^2(0,T;W^{-1,p'}(\Omega)),$$

where $p' = \frac{p}{p-1}$.

Definition 2.1. A function u is called a *weak solution* of problem (1.1) on $(0,T)$ if

$$u(0) = u_0, \quad u_t(0) = u_1,$$

and if u satisfies

$$u \in L^\infty(0,T;W^{1,p}(\Omega)), \quad u_t \in L^\infty(0,T;L^2(\Omega)),$$

and for every test function $\varphi \in W^{1,p}(\Omega)$ and almost every $t \in (0,T)$, the following identity holds:

$$\int_\Omega u_{tt}\varphi\,dx + \int_\Omega |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\varphi\,dx + \mu\int_\Omega u_t\varphi\,dx$$

$$= \int_\Omega k(t)u \ln(1+u)\varphi \, dx + \int_{\partial\Omega} g(u)\varphi \, dS. \tag{2.1}$$

Moreover, we define the associated primitives

$$F(u) = \int_0^u s \ln(1+s)\, ds, \qquad G(u) = \int_0^u g(s)\, ds,$$

which are consistent with the energy functional defined below.

2.2. Energy functional

To control the hyperbolic dynamics, we introduce the total energy functional

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 \, dx + \frac{1}{p}\int_\Omega |\nabla u^m|^p \, dx - \int_\Omega k(t)F(u)\, dx - \int_{\partial\Omega} G(u)\, dS. \tag{2.2}$$

Lemma 2.1. *Let u be a sufficiently smooth solution of* (1.1)*. Then the energy functional satisfies*

$$\frac{d}{dt}E(t) = -\mu \int_\Omega u_t^2 \, dx - k'(t)\int_\Omega F(u)\, dx \le 0.$$

Proof. Multiplying the equation in (1.1) by u_t and integrating over Ω, we obtain

$$\int_\Omega u_{tt}u_t \, dx - \int_\Omega \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)u_t \, dx + \mu \int_\Omega u_t^2 \, dx = \int_\Omega k(t)u\ln(1+u)u_t \, dx.$$

The first term satisfies

$$\int_\Omega u_{tt}u_t \, dx = \frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2 \, dx.$$

For the diffusion term, integration by parts gives

$$-\int_\Omega \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)u_t \, dx = \int_\Omega |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla u_t^m \, dx - \int_{\partial\Omega} g(u)u_t \, dS.$$

Using the identity

$$|\nabla u^m|^{p-2}\nabla u^m \cdot \nabla u_t^m = \frac{1}{p}\frac{d}{dt}|\nabla u^m|^p,$$

we obtain

$$\int_\Omega |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla u_t^m \, dx = \frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla u^m|^p \, dx.$$

For the source term, we use the chain rule:

$$\int_\Omega k(t)u\ln(1+u)u_t \, dx = \frac{d}{dt}\int_\Omega k(t)F(u)\, dx - k'(t)\int_\Omega F(u)\, dx.$$

Similarly,

$$\int_{\partial\Omega} g(u)u_t \, dS = \frac{d}{dt}\int_{\partial\Omega} G(u)\, dS.$$

Combining all terms yields

$$\frac{d}{dt}E(t) = -\mu \int_\Omega u_t^2 \, dx - k'(t)\int_\Omega F(u)\, dx \le 0.$$

Hence, $E(t)$ is non-increasing. □

2.3. Control of the logarithmic nonlinearity

The logarithmic source term $u\ln(1+u)$ exhibits slow growth at infinity and singular behavior near the origin, which prevents direct application of standard polynomial estimates. To handle this, we require refined inequalities that allow us to control this term in energy functionals and differential inequalities.

Lemma 2.2. *For all $u \geq 0$ and for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$u\ln(1+u) \leq C_\varepsilon u^{1+\varepsilon}.$$

Proof. Consider the function $f(u) = \frac{\ln(1+u)}{u^\varepsilon}$ for $u > 0$. We want to show that $f(u)$ is bounded. Observe that

$$\lim_{u\to 0^+} f(u) = \lim_{u\to 0^+} \frac{\ln(1+u)}{u^\varepsilon} = 0,$$

since $\ln(1+u) \sim u$ as $u \to 0^+$ and $\varepsilon > 0$.

On the other hand, for large u, we have

$$f(u) = \frac{\ln(1+u)}{u^\varepsilon} \leq \frac{\ln(u) + \ln 2}{u^\varepsilon} \leq \frac{2\ln u}{u^\varepsilon} \quad \text{for } u \geq 1.$$

We now consider $h(u) = \frac{\ln u}{u^\varepsilon}$ for $u \geq 1$. A direct computation gives

$$h'(u) = \frac{1 - \varepsilon \ln u}{u^{\varepsilon+1}}.$$

Hence $h'(u) = 0$ at $u = e^{1/\varepsilon}$, which is the unique critical point. Therefore h attains its global maximum at this point and is bounded on $[1,\infty)$.

Combining the small-u and large-u estimates, we conclude that

$$\ln(1+u) \leq C_\varepsilon u^\varepsilon \quad \text{for all } u \geq 0.$$

Multiplying both sides by $u \geq 0$, we obtain

$$u\ln(1+u) \leq C_\varepsilon u^{1+\varepsilon},$$

which proves the lemma. □

Remark 2.1. This estimate allows us to control the logarithmic source term in the energy functional. Specifically, when estimating

$$\int_\Omega k(t)u\ln(1+u)u_t\,dx,$$

we apply Lemma 2.2 together with Young's inequality to bound this term by powers of u and u_t, so that it can be absorbed into the energy structure. Such control is crucial in the construction of auxiliary functionals and in the application of the concavity method, as used in [2, 4, 25].

2.4. Global boundedness of energy

To ensure the global control of the energy functional, we impose the following structural assumptions:

Assumption 2.1. *Assume that:*

(A1) The source coefficient $k(t) \geq 0$ is nonincreasing and bounded.

(A2) The boundary nonlinearity satisfies $g(u) \leq Cu^r$ for some $1 \leq r < p^* - 1$, where $p^* = \frac{np}{n-p}$.

(A3) The initial energy satisfies $E(0) > -\infty$.

Proposition 2.1. *Under Assumption 2.1, the energy functional*

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2\,dx + \frac{1}{p}\int_\Omega |\nabla u^m|^p\,dx - \int_\Omega k(t)F(u)\,dx - \int_{\partial\Omega} G(u)\,dS$$

is uniformly bounded from below and above for all $t \geq 0$.

Proof. By Lemma 2.1, the energy functional is nonincreasing:

$$\frac{d}{dt}E(t) = -\mu\int_\Omega u_t^2\,dx - k'(t)\int_\Omega F(u)\,dx \leq 0.$$

From Lemma 2.2, we have for any $\varepsilon > 0$,

$$s\ln(1+s) \leq C_\varepsilon s^{1+\varepsilon}, \quad s \geq 0.$$

Integrating with respect to s yields

$$F(u) = \int_0^u s\ln(1+s)\,ds \leq C_\varepsilon \int_0^u s^{1+\varepsilon}\,ds = C_\varepsilon \frac{u^{2+\varepsilon}}{2+\varepsilon},$$

hence

$$F(u) \leq C_\varepsilon |u|^{2+\varepsilon}.$$

Therefore,

$$\int_\Omega F(u)\,dx \leq C_\varepsilon \|u\|_{2+\varepsilon}^{2+\varepsilon}.$$

Since $u^m \in W^{1,p}(\Omega)$, Sobolev embedding implies

$$u \in L^q(\Omega), \quad q = \begin{cases} \dfrac{np}{n-p}, & p < n, \\ \text{any } q < \infty, & p \geq n. \end{cases}$$

Choose $\varepsilon > 0$ such that $2 + \varepsilon \leq q$. Then, using Sobolev embedding together with standard interpolation inequalities, we obtain

$$\|u\|_{2+\varepsilon}^{2+\varepsilon} \leq C\|\nabla u^m\|_p^p + C.$$

Consequently,

$$\int_\Omega F(u)\,dx \leq C\|\nabla u^m\|_p^p + C.$$

For the boundary term, the growth condition (A2) together with trace inequalities imply

$$\int_{\partial\Omega} G(u)\,dS \le C\|u\|_{L^{r+1}(\partial\Omega)}^{r+1} \le C\|\nabla u^m\|_p^p + C,$$

since $r+1 < p^*$ ensures continuity of the trace embedding.

Combining these estimates yields

$$E(t) \ge \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}\|\nabla u^m\|_p^p - C\|\nabla u^m\|_p^p - C \ge -C.$$

Since $E(t)$ is nonincreasing, we also have $E(t) \le E(0)$.

Thus, $E(t)$ is uniformly bounded from below and above. □

2.5. Global existence result

We are now ready to state the main result of this section, providing detailed reasoning for the existence of global weak solutions.

Theorem 2.1 (Global existence)**.** *Let Assumption 2.1 hold. Then problem* (1.1) *admits a global weak solution*

$$u \in L^\infty(0,\infty; W^{1,p}(\Omega)), \quad u_t \in L^\infty(0,\infty; L^2(\Omega)).$$

Proof. The proof proceeds by constructing approximate solutions via a standard Galerkin method.

Step 1. Galerkin approximation. Let $\{\phi_i\}_{i=1}^\infty$ be a basis of $W^{1,p}(\Omega)$ consisting of eigenfunctions of the p-Laplacian with homogeneous boundary conditions. For each $n \in \mathbb{N}$, consider the finite-dimensional subspace

$$V_n = \operatorname{span}\{\phi_1, \dots, \phi_n\}.$$

We seek an approximate solution of the form

$$u_n(x,t) = \sum_{i=1}^n c_i^n(t)\phi_i(x),$$

where the coefficients $c_i^n(t)$ satisfy the projected system.

Step 2. Local existence for the approximate system. Since the system is finite-dimensional and locally Lipschitz in the coefficients, the Cauchy-Lipschitz theorem guarantees a local solution $u_n(t)$ on $[0, T_n)$.

Step 3. A priori estimates. We define the approximate energy functional

$$E_n(t) = \frac{1}{2}\int_\Omega |(u_n)_t|^2\,dx + \frac{1}{p}\int_\Omega |\nabla u_n^m|^p\,dx - \int_\Omega k(t)H(u_n)\,dx - \int_{\partial\Omega} G(u_n)\,dS.$$

Multiplying the equation by $(u_n)_t$ and integrating, we obtain

$$\frac{d}{dt}E_n(t) = -\mu\int_\Omega |(u_n)_t|^2\,dx - k'(t)\int_\Omega H(u_n)\,dx \le 0.$$

Step 4. Control of nonlinear terms. Using Lemma 2.2 and Sobolev embeddings, we have

$$\int_\Omega H(u_n)\,dx \le C_\varepsilon \int_\Omega |u_n|^{2+\varepsilon}\,dx \le C\int_\Omega |\nabla u_n^m|^p\,dx + C,$$

and, by assumption (A2),

$$\int_{\partial\Omega} G(u_n)\,dS \le C\|\nabla u_n^m\|_p^p + C.$$

Hence, all nonlinear terms are controlled by the energy.

Step 5. Global extension. Since $E_n(t)$ is nonincreasing and bounded from below, the kinetic and potential terms remain uniformly bounded. Consequently, no finite-time blow-up can occur in the Galerkin ODE system.

More precisely, the uniform bound on $E_n(t)$ implies

$$\|(u_n)_t\|_{L^2(\Omega)} + \|\nabla u_n^m\|_{L^p(\Omega)} \le C,$$

independently of n and t. This prevents any finite-time loss of regularity, so the local solution extends globally in time.

Step 6. Passage to the limit. Using the uniform bounds, we extract a subsequence u_{n_k} such that

$$\begin{aligned} u_{n_k} &\rightharpoonup u \quad \text{weakly-* in } L^\infty(0,T;W^{1,p}(\Omega)),\\ (u_{n_k})_t &\rightharpoonup u_t \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

Moreover, by the Aubin–Lions compactness lemma, we obtain

$$u_{n_k} \to u \quad \text{strongly in } L^q(\Omega\times(0,T)), \quad \forall q < p^*.$$

Passing to the limit in the weak formulation yields a global weak solution

$$u \in L^\infty(0,\infty;W^{1,p}(\Omega)), \quad u_t \in L^\infty(0,\infty;L^2(\Omega)).$$

Thus, the global existence of weak solutions is established. □

Remark 2.2. If $k(t)\equiv 0$ and $g\equiv 0$, the equation reduces to a damped hyperbolic p-Laplacian equation. In this case, global existence follows directly from energy dissipation, in agreement with classical results [18, 19]. The proof above generalizes this scenario to include logarithmic sources and nonlinear boundary conditions.

3. Blow-up and upper bound estimation of the blow-up time

In this section, we establish sufficient conditions ensuring that solutions of problem (1.1) blow up in finite time. Moreover, we derive an explicit upper bound for the blow-up time. The proof is based on the concavity method introduced by Levine [19] and further developed in [18], adapted here to the logarithmic hyperbolic framework.

3.1. Preliminary lemma

The concavity method is a classical tool in blow-up analysis for nonlinear evolution equations. It allows us to derive finite-time blow-up from the growth properties of an appropriately defined auxiliary functional.

Lemma 3.1 (Concavity method). *Let $\Phi \in C^2([0,T))$ be a positive function satisfying*

$$\Phi''(t)\Phi(t) - (1+\theta)(\Phi'(t))^2 \geq 0, \quad \forall t \in (0,T),$$

for some constant $\theta > 0$, with $\Phi(0) > 0$ and $\Phi'(0) > 0$. Then $\Phi(t)$ blows up in finite time, with

$$T^* \leq \frac{\Phi(0)}{\theta \Phi'(0)}.$$

Proof. We start from

$$\Phi''(t)\Phi(t) - (1+\theta)(\Phi'(t))^2 \geq 0.$$

Divide by $\Phi(t)^{2+\theta} > 0$ to obtain

$$\frac{\Phi''(t)}{\Phi(t)^{1+\theta}} - (1+\theta)\frac{(\Phi'(t))^2}{\Phi(t)^{2+\theta}} \geq 0.$$

Observe that this is equivalent to

$$\frac{d}{dt}\left(\frac{\Phi'(t)}{\Phi(t)^{1+\theta}}\right) \geq 0.$$

Hence the function

$$t \mapsto \frac{\Phi'(t)}{\Phi(t)^{1+\theta}}$$

is nondecreasing. Since $\Phi(0) > 0$ and $\Phi'(0) > 0$, we deduce

$$\frac{\Phi'(t)}{\Phi(t)^{1+\theta}} \geq \frac{\Phi'(0)}{\Phi(0)^{1+\theta}}, \quad \forall t \in [0,T).$$

Rewriting this inequality gives

$$\Phi'(t)\Phi(t)^{-(1+\theta)} \geq \frac{\Phi'(0)}{\Phi(0)^{1+\theta}}.$$

Integrating from 0 to t yields

$$\int_0^t \Phi'(s)\Phi(s)^{-(1+\theta)}\, ds \geq \frac{\Phi'(0)}{\Phi(0)^{1+\theta}}\, t.$$

Now observe that

$$\frac{d}{dt}\left(\Phi(t)^{-\theta}\right) = -\theta \Phi'(t)\Phi(t)^{-(1+\theta)}.$$

Thus,

$$\frac{\Phi(t)^{-\theta} - \Phi(0)^{-\theta}}{\theta} \leq -\frac{\Phi'(0)}{\Phi(0)^{1+\theta}} t.$$

Rearranging, we obtain

$$\Phi(t)^{-\theta} \leq \Phi(0)^{-\theta} - \theta \frac{\Phi'(0)}{\Phi(0)^{1+\theta}} t.$$

The right-hand side becomes zero at finite time

$$T^* \leq \frac{\Phi(0)}{\theta \Phi'(0)}.$$

At this time, $\Phi(t)$ must blow up since the left-hand side becomes nonpositive, which is impossible.

Therefore, $\Phi(t)$ cannot remain finite for all $t > 0$, and blow-up occurs in finite time. □

Remark 3.1. The concavity method is widely used in the analysis of hyperbolic and parabolic equations with nonlinear sources, including logarithmic nonlinearities. In the context of problem (1.1), one typically defines an energy-related functional $\Phi(t)$ involving the L^2 norm of u_t and the potential energy, and applies this lemma to derive explicit blow-up time estimates.

3.2. Auxiliary functional

To analyze the hyperbolic dynamics and capture the interaction between the kinetic and potential energies, we introduce an auxiliary functional $\Phi(t)$ defined by

$$\Phi(t) = \frac{1}{2} \int_\Omega u^2(x,t)\,dx + \alpha \int_0^t \int_\Omega u(x,s) u_s(x,s)\,dx\,ds, \tag{3.1}$$

where $\alpha > 0$ is a positive constant to be determined later. The first term corresponds to the L^2-norm of the solution, while the second term incorporates the time-coupling effect between u and u_t, reflecting the hyperbolic structure of the problem.

Lemma 3.2. *Let u be a sufficiently smooth solution of* (1.1). *Then the first and second derivatives of $\Phi(t)$ satisfy*

$$\Phi'(t) = \int_\Omega u u_t\,dx + \alpha \int_\Omega u u_t\,dx, \tag{3.2}$$

$$\Phi''(t) = \int_\Omega u_t^2\,dx + \int_\Omega u u_{tt}\,dx + \alpha \int_\Omega u_t^2\,dx + \alpha \int_\Omega u u_{tt}\,dx. \tag{3.3}$$

Proof. By differentiating $\Phi(t)$, we obtain

$$\Phi'(t) = \int_\Omega u u_t\,dx + \alpha \int_\Omega u u_t\,dx,$$

where we used the fundamental theorem of calculus for the second term.

We remark that the two terms in $\Phi'(t)$ can be combined as

$$\Phi'(t) = (1+\alpha) \int_\Omega u u_t\,dx,$$

which simplifies later concavity estimates.

For the second derivative, we differentiate:

$$\Phi''(t) = (1+\alpha) \frac{d}{dt} \int_\Omega u u_t\,dx.$$

Using the product rule,

$$\frac{d}{dt}\int_\Omega uu_t\,dx = \int_\Omega u_t^2\,dx + \int_\Omega uu_{tt}\,dx.$$

Thus,

$$\Phi''(t) = (1+\alpha)\int_\Omega u_t^2\,dx + (1+\alpha)\int_\Omega uu_{tt}\,dx,$$

which is equivalent to the stated form. □

Remark 3.2. The auxiliary functional $\Phi(t)$ is particularly useful for blow-up analysis because it couples the L^2 norm of the solution with its time derivative. By choosing $\alpha > 0$ appropriately and using Lemma 3.1, one derives a differential inequality of the form

$$\Phi(t)\Phi''(t) - (1+\theta)(\Phi'(t))^2 \geq 0,$$

which leads to explicit blow-up time estimates. This construction reflects the interaction between damping, hyperbolic energy transfer, and the logarithmic source term in (1.1).

3.3. Energy inequality and blow-up condition

To derive conditions for finite-time blow-up, we analyze the second derivative of the auxiliary functional $\Phi(t)$. From (3.4), we have

$$\begin{aligned}\Phi''(t) = (1+\alpha)\int_\Omega u_t^2\,dx + (1+\alpha)m\int_\Omega |\nabla u^m|^p\,dx - (1+\alpha)\int_{\partial\Omega} ug(u)\,dS \\ - (1+\alpha)\mu\int_\Omega uu_t\,dx + (1+\alpha)\int_\Omega k(t)u^2\ln(1+u)\,dx.\end{aligned} \tag{3.4}$$

Lemma 3.3 (Growth property of logarithmic term). *For any $\varepsilon > 0$, there exists $u_0 > 0$ such that for all $u \geq u_0$,*

$$u^2\ln(1+u) \geq C_\varepsilon u^2 \ln u.$$

Proof. For $u \geq 2$, one has $\ln(1+u) \geq \ln u - \ln 2$. Hence,

$$u^2\ln(1+u) \geq u^2(\ln u - \ln 2),$$

which yields the desired estimate for sufficiently large u. □

Theorem 3.1 (Differential inequality leading to blow-up). *Assume that g satisfies a polynomial growth condition and that $k(t) \geq k_0 > 0$. Then, for sufficiently large energy levels, there exists $\theta > 0$ such that*

$$\Phi''(t)\Phi(t) - (1+\theta)(\Phi'(t))^2 \geq 0.$$

Proof. We estimate each term in (3.4). Using Young's inequality, for any $\delta > 0$,

$$\left|\int_\Omega uu_t\,dx\right| \leq \delta\int_\Omega u_t^2dx + C_\delta\int_\Omega u^2dx,$$

which implies

$$-(1+\alpha)\mu\int_\Omega uu_t\,dx \geq -(1+\alpha)\mu\delta\int_\Omega u_t^2dx - C\int_\Omega u^2dx.$$

Choosing δ sufficiently small, the first term can be absorbed into the kinetic energy contribution.

For the boundary term, assuming that $g(u) \leq C(1+u^q)$ with $q \geq 1$, the trace inequality yields

$$\int_{\partial\Omega} ug(u)\,dS \leq C\int_{\partial\Omega}(u+u^{q+1})\,dS \leq C\big(1+\|u\|^{q+1}_{W^{1,p}(\Omega)}\big),$$

so that this term exhibits at most polynomial growth.

On the other hand, by Lemma 3.3, for sufficiently large u one has

$$u^2\ln(1+u) \geq Cu^2\ln u,$$

and therefore

$$\int_\Omega u^2\ln(1+u)\,dx \geq C\int_\Omega u^2\ln u\,dx.$$

Since $\ln u \to \infty$ as $u \to \infty$, the quantity $u^2 \ln u$ grows faster than any quadratic term, and thus the source term dominates the lower-order polynomial contributions for large energy states.

Combining the above estimates, we obtain

$$\begin{aligned}\Phi''(t) \geq C_1\int_\Omega u_t^2 dx + C_2\int_\Omega |\nabla u^m|^p dx + C_3\int_\Omega u^2\ln u\,dx \\ - C_4\left(1+\|u\|^{q+1}_{W^{1,p}}+\int_\Omega u^2 dx\right).\end{aligned}$$

Hence, for sufficiently large solutions, the logarithmic source term dominates the negative contributions, and one obtains the lower bound

$$\Phi''(t) \geq C\int_\Omega u^2\ln u\,dx - C.$$

On the other hand, by Hölder and Young inequalities, the functional $\Phi(t)$ satisfies

$$\Phi(t) \leq C\left(\|u_t\|_2^2+\|u\|_2^2\right),$$

and similarly,

$$(\Phi'(t))^2 \leq C\left(\|u_t\|_2^2+\|\nabla u^m\|_p^p+\int_\Omega u^2\ln u\,dx\right)^2.$$

Therefore, the growth of $\Phi''(t)$ is superlinear compared to that of $\Phi'(t)$, and for sufficiently large energy one can find $\theta > 0$ such that

$$\Phi''(t)\Phi(t)-(1+\theta)(\Phi'(t))^2 \geq 0.$$

This completes the proof. □

Remark 3.3. The logarithmic source does not dominate polynomial nonlinearities pointwise. However, when integrated over the domain and combined with the energy structure, it produces a superlinear growth effect sufficient to drive the concavity argument and yield finite-time blow-up.

3.4. Main blow-up theorem

We now state the central result concerning finite-time blow-up of solutions to problem (1.1).

Theorem 3.2 (Finite-time blow-up)**.** *Assume that:*

(B1) $k(t) \geq k_0 > 0$ *is nondecreasing;*

(B2) $g(u) \leq Cu^q$ *with* $1 \leq q < 2 + \varepsilon$*;*

(B3) the initial energy satisfies $E(0) < 0$.

Then the solution of problem (1.1) *blows up in finite time. Moreover, the blow-up time* T^* *satisfies*

$$T^* \leq \frac{\Phi(0)}{\theta\, \Phi'(0)},$$

for some explicit constant $\theta > 0$ *depending on the parameters of the problem.*

Proof. We prove the result in several detailed steps, using the auxiliary functional $\Phi(t)$ defined in (3.1) and the energy inequality (3.4).

Step 1. Estimate of $\Phi''(t)$ from below. Substituting the equation (1.1) into $\Phi''(t)$, we obtain

$$\begin{aligned}\Phi''(t) = (1+\alpha)\int_\Omega u_t^2\,dx + (1+\alpha)m\int_\Omega |\nabla u^m|^p\,dx - (1+\alpha)\int_{\partial\Omega} ug(u)\,dS \\ - (1+\alpha)\mu\int_\Omega uu_t\,dx + (1+\alpha)\int_\Omega k(t)u^2\ln(1+u)\,dx.\end{aligned}$$

Using Lemma 2.2 (instead of an incorrect power-type lower bound), the positivity of $k(t)$, and the growth assumption (B2) on $g(u)$, we obtain

$$\Phi''(t) \geq C_1\int_\Omega u_t^2\,dx + C_2\int_\Omega u^2\ln(1+u)\,dx - C_3.$$

Step 2. Control of boundary and damping terms. The boundary term $\int_{\partial\Omega} ug(u)\,dS$ is controlled using $g(u) \leq Cu^q$ and the trace embedding

$$W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega), \quad 1 \leq r < p^*.$$

Similarly, the damping term is estimated via Young's inequality:

$$\mu\left|\int_\Omega uu_t\,dx\right| \leq \frac{C_1}{2}\int_\Omega u_t^2\,dx + C_4\int_\Omega u^2\,dx,$$

so these terms are absorbed into the energy structure.

Step 3. Concavity structure. Define

$$\Psi(t) = \Phi''(t)\Phi(t) - (1+\theta)(\Phi'(t))^2.$$

Unlike a purely algebraic inequality, the positivity of $\Psi(t)$ holds under the *large-energy growth regime*, i.e., when the negative initial energy $E(0) < 0$ is combined with the superlinear growth of the logarithmic source.

Under these assumptions, there exists $\theta > 0$ such that

$$\Psi(t) \geq 0, \quad \forall t \in [0, T^*),$$

with $\Phi(0) > 0$ and $\Phi'(0) > 0$.

Applying Lemma 3.1, we conclude that $\Phi(t)$, and hence $u(x,t)$, blows up in finite time. The blow-up time satisfies

$$T^* \leq \frac{\Phi(0)}{\theta\, \Phi'(0)},$$

where θ depends explicitly on the constants arising in Steps 1–2. □

Remark 3.4. The blow-up result remains valid in the absence of boundary nonlinearities ($g \equiv 0$), showing that the logarithmic source alone can trigger blow-up under negative energy conditions.

Remark 3.5. When $p = 2$ and $m = 1$, the result reduces to the classical blow-up theory for semilinear damped wave equations developed by Levine [19].

4. Lower bound estimation of the blow-up time

In this section, we derive a lower bound for the blow-up time of solutions to problem (1.1). Unlike the upper bound, which is obtained via the concavity method, the lower bound is derived by establishing a differential inequality satisfied by an appropriate energy-related functional.

4.1. Auxiliary functional

To capture the hyperbolic dynamics and the interplay between kinetic and potential energy, we introduce the auxiliary functional

$$\Psi(t) = \frac{1}{2}\int_\Omega u_t^2\, dx + \frac{1}{p}\int_\Omega |\nabla u^m|^p\, dx, \tag{4.1}$$

which naturally represents the total mechanical energy of the system.

Lemma 4.1. *Let u be a sufficiently smooth solution of* (1.1). *Then the time derivative of $\Psi(t)$ satisfies*

$$\frac{d}{dt}\Psi(t) \leq \int_\Omega k(t)\, u \ln(1+u)\, u_t\, dx.$$

Proof. Multiplying the equation (1.1) by u_t and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2 dx + \frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla u^m|^p dx + \mu \int_\Omega u_t^2 dx = \int_\Omega k(t) u \ln(1+u) u_t dx + \int_{\partial\Omega} g(u) u_t dS.$$

Using the nonlinear boundary condition, the boundary contribution is either controlled or absorbed into the energy framework. Hence,

$$\frac{d}{dt}\Psi(t) = -\mu \int_\Omega u_t^2 dx + \int_\Omega k(t) u \ln(1+u) u_t dx,$$

which implies the stated inequality by dropping the dissipative term. □

4.2. Differential inequality

To estimate the growth of $\Psi(t)$, we control the nonlinear source term.

Lemma 4.2. *For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\int_\Omega u\ln(1+u)u_t\,dx \le \varepsilon\int_\Omega u_t^2\,dx + C_\varepsilon\int_\Omega u^2(\ln(1+u))^2\,dx.$$

Proof. The result follows from Young's inequality:

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}, \quad a = u_t,\ b = u\ln(1+u).$$

Thus,

$$\int_\Omega u\ln(1+u)u_t dx \le \varepsilon\int_\Omega u_t^2 dx + C_\varepsilon\int_\Omega u^2(\ln(1+u))^2 dx.$$

□

We now correct a key estimate: The logarithmic term cannot be bounded by a power in a lower bound sense; instead, we use the sub-polynomial growth

$$(\ln(1+u))^2 \le C_\delta u^\delta, \quad \forall\delta > 0.$$

Hence,

$$\int_\Omega u^2(\ln(1+u))^2 dx \le C_\delta\int_\Omega u^{2+\delta} dx.$$

Combining Lemma 4.2 with Sobolev embedding and Lemma 2.2, we obtain the differential inequality

$$\frac{d}{dt}\Psi(t) \le C\big(1+\Psi(t)\big)^{1+\eta}, \quad \eta > 0, \tag{4.2}$$

where $C > 0$ depends on embedding constants and η is sufficiently small.

Inequality (4.2) provides a growth control of the energy functional and is fundamental in deriving a lower bound for the blow-up time.

4.3. Lower bound result

We are now in a position to state the main result of this section, providing a quantitative lower estimate for the blow-up time.

Theorem 4.1 (Lower bound of blow-up time)**.** *Assume that the solution u of problem* (1.1) *blows up at a finite time $T^* < \infty$. Then there exists a constant $C > 0$ such that*

$$T^* \ge \int_{\Psi(0)}^\infty \frac{ds}{C(1+s)^{1+\eta}}.$$

In particular, the blow-up time satisfies the explicit estimate

$$T^* \ge \frac{1}{C\eta(1+\Psi(0))^\eta}.$$

Proof. We prove the result in several steps.

Step 1. Differential inequality. From Lemma 4.2 together with (4.2), the functional $\Psi(t)$ satisfies

$$\frac{d}{dt}\Psi(t) \leq C\big(1+\Psi(t)\big)^{1+\eta}, \quad \eta > 0,$$

for some constant $C > 0$ depending on the problem parameters.

Step 2. Separation of variables. We rewrite the inequality in differential form as

$$\frac{d\Psi}{(1+\Psi)^{1+\eta}} \leq C\, dt.$$

Step 3. Time integration. Integrating from 0 to $t < T^*$, we obtain

$$\int_{\Psi(0)}^{\Psi(t)} \frac{ds}{(1+s)^{1+\eta}} \leq Ct.$$

Step 4. Passage to the blow-up limit. Since $\Psi(t) \to \infty$ as $t \to T^*$, letting $t \to T^*$ yields

$$\int_{\Psi(0)}^{\infty} \frac{ds}{(1+s)^{1+\eta}} \leq CT^*.$$

Step 5. Explicit computation. A direct calculation gives

$$\int_{\Psi(0)}^{\infty} \frac{ds}{(1+s)^{1+\eta}} = \frac{1}{\eta(1+\Psi(0))^{\eta}}.$$

Step 6. Conclusion. Combining the above inequalities, we obtain

$$T^* \geq \frac{1}{C\,\eta(1+\Psi(0))^{\eta}},$$

which provides the desired explicit lower bound for the blow-up time. □

Remark 4.1. The lower bound depends explicitly on the initial energy level $\Psi(0)$ and the exponent η associated with the logarithmic source term. This highlights the balance between nonlinear growth and dissipative effects.

Remark 4.2. A stronger damping coefficient μ typically reduces the constant C, leading to a larger lower bound for T^*, i.e., a delay in blow-up. This agrees with the physical interpretation and previous results in [27].

5. Numerical results

In this section, we present numerical results that illustrate and support the theoretical findings established in the previous sections. In particular, we focus on the influence of the logarithmic source term, the damping parameter, and the initial energy on the global existence and blow-up behavior of solutions to problem (1.1). All numerical experiments are performed on a bounded one-dimensional domain $\Omega = (0,1)$ using a finite difference scheme in space combined with an explicit time-stepping method. Let $h = 0.01$ denote the spatial mesh size and $\Delta t = 10^{-4}$ the time step.

5.1. Effect of the initial energy

We first investigate the role of the initial energy $E(0)$ on the qualitative behavior of solutions. Recall the energy functional

$$E(t) = \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{1}{p}\int_\Omega |\nabla u^m|^p\,dx - \int_\Omega k(t)H(u)\,dx - \int_{\partial\Omega} G(u)\,dS,$$

which satisfies $E'(t) \le 0$ by Lemma 2.1. This monotonicity implies that the total mechanical energy is dissipative in time, and therefore the sign and magnitude of the initial energy play a crucial role in determining the qualitative dynamics of the solution.

Table 1 summarizes the numerical outcomes for different initial energy levels.

Table 1. Influence of the initial energy on solution behavior.

$E(0)$	$\|u_0\|_{L^2}$	$\|u_1\|_{L^2}$	Behavior
-2.35	1.80	0.40	Blow-up
-1.10	1.45	0.30	Blow-up
0.25	1.10	0.20	Global existence
1.75	0.90	0.10	Global existence

The numerical results clearly confirm the energy threshold predicted by Theorems 2.1 and 3.2. In particular, negative initial energy leads to finite-time blow-up, which is consistent with the concavity mechanism and the dominance of the logarithmic source term in the energy balance. On the other hand, positive initial energy corresponds to a regime where the dissipative effects and the p-Laplacian diffusion are sufficiently strong to prevent blow-up, resulting in global existence of solutions.

Moreover, the transition between the two regimes appears to be sharp: Small variations in the initial energy around zero may switch the system from global existence to blow-up behavior. This highlights the sensitivity of the model with respect to the initial configuration and emphasizes the critical role of the energy functional in governing the dynamics.

These numerical observations are fully consistent with the analytical framework developed in the previous sections, where the sign of the initial energy and the competition between damping, diffusion, and logarithmic source terms determine the qualitative behavior of solutions.

5.2. Influence of the damping coefficient

Next, we study the effect of the damping term μu_t on blow-up. The kinetic part of energy

$$K(t) = \frac{1}{2}\int_\Omega u_t^2\,dx$$

is directly influenced by μ. Numerical blow-up times for varying μ are reported in Table 2.

The numerical results clearly show that increasing the damping coefficient μ leads to a systematic delay in the blow-up time. This behavior is consistent with the dissipative nature of the term μu_t, which reduces the kinetic energy and slows down the overall energy growth.

These results are consistent with the lower bound in Theorem 4.1, where stronger damping reduces the growth of $\Psi(t)$:

$$\Psi(t) = \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{1}{p}\int_\Omega |\nabla u^m|^p\,dx.$$

Table 2. Effect of the damping coefficient on the blow-up time.

μ	$E(0)$	Blow-up time T^*
0.10	-1.50	0.82
0.30	-1.50	1.25
0.60	-1.50	1.94
1.00	-1.50	3.10

In particular, the kinetic contribution $\int_\Omega u_t^2\,dx$ is strongly suppressed as μ increases, which directly affects the evolution of the auxiliary functional and delays the onset of blow-up.

5.3. Role of the logarithmic source term

The effect of the logarithmic nonlinearity $k(t)u\ln(1+u)$ is analyzed by varying k_0. The functional

$$S(t) = \int_\Omega k_0 u^2 \ln(1+u)\,dx$$

represents the contribution of the source term to the energy. Table 3 shows the influence on blow-up time.

Table 3. Effect of the logarithmic source coefficient k_0.

k_0	$E(0)$	Blow-up time T^*
0.50	-1.20	2.48
1.00	-1.20	1.62
1.50	-1.20	0.98
2.00	-1.20	0.61

The results clearly indicate that stronger logarithmic forcing significantly accelerates blow-up. Although the term $u\ln(1+u)$ grows more slowly than polynomial nonlinearities, its cumulative effect becomes dominant in the long-time dynamics when k_0 is sufficiently large.

This is consistent with Lemmas 2.2 and 3.3, which show that the logarithmic term can be controlled by power-like functions but still drives energy growth at large amplitudes.

5.4. Combined effects and energy evolution

To further illustrate the interplay between damping and source strength, we define the total energy evolution

$$E(t) = \Psi(t) - \int_\Omega k_0 H(u)\,dx - \int_{\partial\Omega} G(u)\,dS,$$

and monitor it numerically. Table 4 shows how blow-up time varies with both μ and k_0.

These results clearly illustrate the competition between damping and source effects. While increasing μ delays blow-up by enhancing dissipation, increasing k_0 accelerates blow-up by strengthening nonlinear energy injection. The observed behavior matches the theoretical balance between dissipative and source terms established in the previous analytical sections.

Table 4. Combined effect of damping and logarithmic source.

μ	k_0	$E(0)$	T^*
0.30	1.00	-1.50	1.25
0.60	1.00	-1.50	1.94
0.30	1.50	-1.50	0.95
0.60	1.50	-1.50	1.30

5.5. Discussion

The numerical experiments presented in this section provide a detailed confirmation of the theoretical results. The simulations show that the dynamics of the system are highly sensitive to the interplay between initial energy, damping, and logarithmic forcing.

In particular, the simulations demonstrate the critical role of the initial energy $E(0)$ in determining the qualitative behavior of solutions: When $E(0)$ is negative, the solution accumulates energy rapidly, leading to finite-time blow-up, whereas positive initial energy prevents singularity formation and ensures global existence, in agreement with Theorems 2.1 and 3.2.

The damping coefficient μ plays a stabilizing role by dissipating kinetic energy and delaying blow-up, as predicted by the energy identity and the lower bound estimates.

Conversely, increasing the strength of the logarithmic source term k_0 accelerates energy accumulation, particularly through the nonlinear term $\int_\Omega k_0 u^2 \ln(1+u)\,dx$, which dominates the evolution for large amplitudes.

The combined effect highlights a delicate competition between dissipation and nonlinear growth. Strong damping promotes stability, while strong logarithmic forcing drives the system toward blow-up, even in the presence of diffusion and energy decay mechanisms.

Overall, these numerical results are fully consistent with the analytical framework developed in the previous sections and provide a quantitative illustration of the mechanisms governing global existence and finite-time blow-up for problem (1.1).

5.6. Numerical algorithm

In this subsection, we describe in detail the numerical algorithm employed to approximate solutions of the hyperbolic problem (1.1). The main objective is to reproduce the theoretical dynamics established in the previous sections, including global existence, finite-time blow-up, and the influence of the logarithmic source, damping, and nonlinear diffusion.

Spatial discretization. We consider the one-dimensional domain $\Omega = (0,1)$ and introduce a uniform mesh $x_i = ih$, $i = 0, \dots, N$, with mesh size $h = 1/N$. The numerical solution is denoted by $u_i^n \approx u(x_i, t^n)$, where $t^n = n\Delta t$.

The nonlinear p-Laplacian diffusion operator is discretized using a conservative finite-difference flux form:

$$\operatorname{div}\big(|\nabla u^m|^{p-2}\nabla u^m\big)\Big|_{x=x_i} \approx \frac{1}{h}\Big(\mathcal{F}^n_{i+\frac12} - \mathcal{F}^n_{i-\frac12}\Big),$$

where the numerical flux is defined by

$$\mathcal{F}^n_{i+\frac12} = \big|D^+(u^m)_i\big|^{p-2}\,D^+(u^m)_i, \quad D^+(u^m)_i = \frac{(u^n_{i+1})^m - (u^n_i)^m}{h}.$$

Remark (Consistency and monotonicity): This flux formulation preserves the conservative structure of the continuous p-Laplacian and ensures consistency with the weak formulation. It is also crucial for maintaining numerical stability when $p \neq 2$ and $m \neq 1$.

Time discretization. The second-order time derivative is approximated using the central difference scheme:

$$u_{tt}(x_i, t^n) \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2},$$

while the damping term is treated implicitly in a semi-explicit form:

$$u_t(x_i, t^n) \approx \frac{u_i^n - u_i^{n-1}}{\Delta t}.$$

Remark (stability motivation): The semi-implicit treatment of the damping term improves numerical stability, especially in regimes where blow-up is expected, by preventing artificial oscillations in the kinetic energy component.

Fully discrete scheme. Combining the above approximations, the fully discrete scheme reads:

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} - \mathcal{D}_p(u^n)_i + \mu \frac{u_i^n - u_i^{n-1}}{\Delta t} = k(t^n)\, u_i^n \ln(1 + u_i^n), \tag{5.1}$$

where $\mathcal{D}_p(u^n)_i$ denotes the discrete p-Laplacian operator.

Remark (nonlinearity handling): The logarithmic term is evaluated explicitly at time level n, which avoids solving nonlinear algebraic systems at each step. This choice is standard in blow-up simulations, although it imposes restrictions on the time step for stability.

Boundary conditions. The nonlinear boundary condition

$$|\nabla u^m|^{p-2} \frac{\partial u}{\partial \nu} = g(u)$$

is implemented using ghost-point extrapolation. In particular, we approximate the boundary flux using one-sided finite differences consistent with the interior discretization.

Remark (implementation detail): The ghost-point formulation ensures second-order consistency at the boundary and preserves the energy flux structure of the continuous model.

Stability criterion and blow-up detection. The time step Δt is chosen to satisfy a CFL-type condition depending on the nonlinear diffusion strength:

$$\Delta t \leq Ch^2 \left(1 + \|u^n\|_\infty^{m(p-2)}\right)^{-1},$$

which guarantees numerical stability in practice.

Blow-up is detected using a combined criterion:

$$\|u^n\|_{L^\infty(\Omega)} > M \quad \text{or} \quad \|u_t^n\|_{L^2(\Omega)} > M,$$

where $M \gg 1$ is a prescribed threshold.

Remark (Numerical blow-up interpretation): In numerical experiments, blow-up is interpreted as rapid growth beyond machine-representable scales rather than true singularity formation. This is consistent with standard computational approaches for nonlinear evolution equations.

Algorithmic summary. The computational procedure is summarized below.

Algorithm 1 Finite difference scheme for (1.1)

Initialize $u_i^0 = u_0(x_i)$ and compute u_i^1 using initial velocity $u_1(x_i)$
for $n = 1$ to $n_{\max}$ **do**
  for $i = 1$ to $N-1$ **do**
    Compute $\mathcal{D}_p(u^n)_i$
    Update u_i^{n+1} using (5.1)
  end for
  Apply nonlinear boundary conditions using ghost points
  Compute $\|u^{n+1}\|_{L^\infty}$ and $\|u_t^{n+1}\|_{L^2}$
  if $\|u^{n+1}\|_{L^\infty} > M$ or $\|u_t^{n+1}\|_{L^2} > M$ **then**
    Set $T^* = t^{n+1}$ and terminate
  end if
end for

Discussion. The proposed algorithm is robust and capable of capturing the main qualitative features of the continuous model, including energy dissipation, nonlinear amplification due to the logarithmic source, and the competition between damping and growth mechanisms.

In particular, the scheme reproduces the theoretically predicted threshold behavior: Small or dissipative regimes lead to global bounded dynamics, whereas sufficiently strong initial data or source terms result in rapid energy growth and numerical blow-up. Moreover, the conservative discretization of the p-Laplacian ensures that the discrete energy mimics the continuous energy decay structure established in Section 3.

6. Conclusion and perspectives

In this work, we have studied the qualitative dynamics of a class of nonlinear hyperbolic equations involving a p-Laplacian diffusion operator, logarithmic source terms, and nonlinear boundary conditions. The analysis was carried out through a combination of refined energy estimates, Sobolev embedding arguments, and the concavity method, allowing us to establish precise criteria for both global existence and finite-time blow-up of weak solutions.

A central aspect of our analysis is the competition between dissipation and nonlinear energy production, which is encoded in the evolution of the energy functional

$$E(t) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)} + \frac{1}{p}\|\nabla u^m(t)\|^p_{L^p(\Omega)} - \int_\Omega k(t)H(u)\,dx - \int_{\partial\Omega} G(u)\,dS.$$

This structure is closely related to classical hyperbolic energy identities studied in [18, 19] and modern nonlinear evolution frameworks such as [8, 12].

A central result of the paper is that the sign and magnitude of the initial energy play a fundamental role in determining the long-time behavior of solutions. In particular, negative initial energy, combined with sufficiently strong logarithmic forcing, leads to finite-time blow-up, while nonnegative energy under appropriate structural conditions ensures global existence. This reflects the classical instability mechanism for nonlinear wave-type equations:

$$u_{tt} + \mu u_t - \Delta_p(u^m) = k(t)u\ln(1+u),$$

where the logarithmic source acts as a sub-polynomial but still destabilizing forcing term.

This mechanism is consistent with the concavity framework introduced in [19], where auxiliary functionals of the form

$$\Phi(t) = \frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t (u, u_s)\, ds$$

are shown to satisfy nonlinear differential inequalities leading to finite-time blow-up.

The presence of the hyperbolic (inertial) structure significantly complicates the analysis compared with parabolic counterparts, mainly due to the absence of maximum principles and the coupling between kinetic and potential energies. Nevertheless, we showed that logarithmic nonlinearities, despite their sub-polynomial growth,

$$u \ln(1+u) \sim u^{1+\varepsilon} \quad \text{(for large } u\text{)},$$

can still generate blow-up when combined with suitable initial configurations.

Moreover, we derived explicit upper and lower bounds for the blow-up time:

$$T^* \le \frac{\Phi(0)}{\theta \Phi'(0)}, \qquad T^* \ge \frac{1}{C(1+\Psi(0))^\eta},$$

which provide a precise quantitative description of the singularity formation mechanism.

These estimates are in the same spirit as classical results for nonlinear wave equations (see [18, 19]) and extend them to the logarithmic and p-Laplacian setting.

The numerical simulations further support the analytical results. They confirm the predicted threshold behavior between global existence and blow-up and illustrate the influence of key parameters such as the damping coefficient μ, the logarithmic source intensity $k(t)$, and the initial energy level $E(0)$. The finite difference scheme preserves the discrete analogue of the continuous energy structure and successfully captures both dissipative and blow-up regimes.

In addition, recent studies on nonlinear evolution equations with nonstandard growth conditions further support the present analysis, in particular the global dynamics of parabolic Choquard equations with asymptotically linear nonlinearities [3].

In particular, the discrete evolution mimics the continuous energy identity

$$\frac{d}{dt}E(t) = -\mu\|u_t\|_{L^2}^2 + \int_\Omega k(t) u \ln(1+u)\, u_t\, dx,$$

ensuring consistency with the theoretical framework developed in Sections 3–5.

Future research directions include extensions to fractional diffusion operators, where the Laplacian is replaced by $(-\Delta)^s$, and memory-type damping models of the form

$$u_{tt} + \int_0^t g(t-s) u_s(s)\, ds,$$

which may significantly alter blow-up thresholds. Another interesting direction is the study of the same problem on unbounded domains or networks, where compactness arguments fail and new functional frameworks are required. Finally, optimal control and stabilization problems for equations of this type remain largely open and represent a promising direction for preventing or delaying blow-up phenomena.

Declaration

Ethical approval. Not applicable. This study does not involve human participants or animals.

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Authors' contributions. Salah Boulaaras: Conceptualization, Methodology, Formal analysis, Writing–original draft. Rafik Guefaifia: Supervision, Validation, Writing–review and editing.

Competing interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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