Journal of Applied Analysis and Computation
Volume 16, Number 6, December 2026, 2945–2961
Website:http://www.jaac-online.com
DOI:10.11948/20250342

RESEARCH ON BOUNDARY VALUE PROBLEMS FOR A COUPLE OF ψ-CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH TWO DELAYS

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Abstract The purpose of this paper is to discuss boundary value problems for a couple of ψ-Caputo fractional differential equations with two delays. By applying the Banach fixed point theorem and the Leray-Schauder alternative theorem, some new results on the existence and uniqueness of solution for this type of problems have been established. Moreover, two examples are supplied to verify our main results.

Keywords ψ-Caputo fractional differential equation, delay, boundary value problem.

MSC(2010) 26A33, 34G20, 34B15.

1. Introduction

Fractional differential equations, as an extension of integer-order differential equations, have attracted the attention of many scholars in recent years and have been studied and applied in various aspects, for example, economics [20], physics [11, 17], medicine [16], mechanics [1], etc. Moreover, the research on the existence and uniqueness of solutions for fractional differential equations and fractional differential equations with delay systems has attracted increasing attention (see [7, 9, 10, 12, 15, 18, 19, 22]). In 2008, Benchorhra, Henderson, Ntouyas, Ouahab [8] used the Banach fixed point theorem and the Leray-Schauder alternative to study the existence of solutions for the following fractional order functional and neutral functional differential equations with infinite delay.

$$\begin{cases} D^{\alpha}y(t) = f(t, y_t), \quad \text{for each } t \in J = [0, b]\,, \quad 0 < \alpha \leq 1, \\ y(t) = \phi(t), \quad t \in (-\infty, 0]\,, \end{cases} \tag{1.1}$$

where D^{α} is the Riemman-Liouville fractional derivative, $f\colon\ J \times B \to \mathbb{R}$ is a given function satisfying some assumptions, $\phi \in B$, $\phi(0) = 0$, and B is a phase space. Based on the aforementioned paper, Dong, Liu, Fan [13] considered a weighted function fractional differential equation with infinite delay. By applying Schauder's fixed point theorem and others, the authors obtained the existence and continuous dependence results of the solution. Subsequently, Alghanmi and Alqurayqiri [3] disscussed a class of functional neutral differential equation as follows.

$$\begin{cases} {}^{C}D_{0^+}^{\delta}\left[u(t) - \displaystyle\int_0^t h(s, u_s)ds\right] = f(t, u_t)\ , t \in \Omega := [0, a]\,, \\ u(t) = \theta(t), \quad t \in (-\infty, 0]\,, \\ u(a) = \sum_{i=1}^{m} \lambda_i {}^{C}D_{0^+}^{\gamma} u(\mu_i) + \zeta, \ \ \mu_i \in (0, a)\,, \end{cases} \tag{1.2}$$

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where ${}^{C}D^{\delta}_{0+}$, ${}^{C}D^{\gamma}_{0+}$ are the Caputo fractional derivatives of order $1 < \delta \leq 2$, $0 < \gamma < 1$. The authors utilized the theory of infinite delay and standard fixed point theorems to investigate the existence and uniqueness of solutions for nonlinear neutral Caputo fractional differential equations involving delayed fractional derivatives and nonlocal fractional derivatives. After that, with the help of Banach fixed point theorem and Leray-Schauder alternative, Almeida [4] studied the following fractional functional differential equation:

$$\begin{cases} {}^{C}D^{\alpha,\psi}x(t) = f(t, x_t), \quad t \in [a, b]\,, \\ x_t = \phi(t), \quad t \in [a - \varepsilon, a]\,, \end{cases} \tag{1.3}$$

where ${}^{C}D^{\alpha,\psi}$ is the ψ-Caputo fractional derivative.

Since 1908, when Langevin studied the motion of particles and proposed the Langevin equation, many scholars are dedicated to the study of this equation, Ahmad, Nieto, Alsaedi and El-Shahed [2] discussed the following boundary value problem of fractional Langevin equation by using the fixed point theorems.

$$\begin{cases} {}^{C}D^{\beta}_{0+}({}^{C}D^{\alpha}_{0+} + \lambda)x(t) = f(t, x(t)), \quad t \in (0, 1)\,, \\ x(0) = 0, \quad x(\eta) = 0, \;\; x(1) = 0, \end{cases} \tag{1.4}$$

where $\alpha \in (0, 1]$, $\beta \in (1, 2]$, ${}^{C}D^{\zeta}_{0+}$ is the Caputo fractional derivative of order ζ ($\zeta = \alpha, \beta$), $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, λ is a real number, $\eta \in (0, 1)$. Subsequently, Zhang and Ni [21] studied with the nonlinear triple system of fractional Langevin equations with cycle anti-periodic boundary conditions of the form:

$$\begin{cases} {}^{C}D^{\beta}_{0+}({}^{C}D^{\alpha}_{0+} + \lambda)x_i(t) = f_i(t, x_1(t), x_2(t), x_3(t)), \quad t \in (0, 1)\,, \quad i = 1, 2, 3, \\ x_1(0) + x_2(1) = 0, \quad {}^{C}D^{\alpha}_{0+}x_1(0) + {}^{C}D^{\alpha}_{0+}x_2(1) = 0, \\ x_2(0) + x_3(1) = 0, \quad {}^{C}D^{\alpha}_{0+}x_2(0) + {}^{C}D^{\alpha}_{0+}x_3(1) = 0, \\ x_3(0) + x_1(1) = 0, \quad {}^{C}D^{\alpha}_{0+}x_3(0) + {}^{C}D^{\alpha}_{0+}x_1(1) = 0, \end{cases} \tag{1.5}$$

where $0 < \alpha$, $\beta < 1$, ${}^{C}D^{\gamma}_{0+}$ is the Caputo fractional derivative of order γ ($\gamma = \alpha, \beta$) , f_i : $[0, 1] \times \mathbb{R}^3 \to \mathbb{R}$, i= 1,2,3 are given continuous functions, $\lambda \in \mathbb{R}^+$. By using Krasnoselskii fixed point theorem and Banach contraction mapping theorem, the sufficient conditions for the existence and uniqueness of results were presented.

After reading the above-mentioned literature, it was found that the research on the delay problem in (1.3) mainly focused on discussing the existence and uniqueness of solutions for equations with a single delay; the problem (1.5) analyzed the Langevin equation with a circulatory system and obtained the ideal results. Combining the delay differential equation with the Langevin equation of a cyclic system, we obtain the following equation:

$$\begin{cases} {}^{C}D^{\beta,\psi}_{0^+}({}^{C}D^{\alpha,\psi}_{0^+} + \lambda)x_i(t) = f_i(t, x_{1t}, x_{2t}), \;\; t \in [0, 1], \\ x_1(t) = \varphi_1(t), \quad t \in [-\varepsilon_1, 0]\,, \\ x_2(t) = \varphi_2(t), \quad t \in [-\varepsilon_2, 0]\,, \\ x_1(0) + x_2(1) = 0, \\ {}^{C}D^{\alpha,\psi}_{0^+}x_2(0) + {}^{C}D^{\alpha,\psi}_{0^+}x_1(1) = 0, \end{cases} \tag{1.6}$$

where ${}^{C}D_{0+}^{\alpha,\psi}$ and ${}^{C}D_{0+}^{\beta,\psi}$ represent the ψ-Caputo fractional derivative with orders α and β respectively, $0<\alpha<1$, $0<\beta<1$, λ is a real number, $f_i:[0,1]\times C([-\varepsilon_1,0],\mathbb{R})\times C([-\varepsilon_2,0],\mathbb{R})\to\mathbb{R}$, $i=1,2$, $\psi\in C^n([0,1],\mathbb{R})$ satisfying $\psi'(t)>0$, $x_{it}:[-\varepsilon_i,0]\to\mathbb{R}$ is the translation $x_{it}(\theta)=x_i(t+\theta)$. Let us emphasize the main contributions of this paper: First, the model studied herein is more complex, and the difficulty in handling it is greater compared to the previous model of a single equation. Second, there are relatively few existing studies on cyclic boundary value problems for ψ-Caputo fractional functional differential equations, and our results enrich the existing literature. Finally, the ψ-Caputo fractional operator is a generalization of the Caputo fractional operator. Thus, our results become more general.

The paper is organized as follows. In section 2, we introduce some definitions and theorems that our work requires. In section 3, we study the existence and uniqueness of solution for the problem (1.6). In section 4, we present some examples to verify our main findings.

2. Preliminaries

Definition 2.1 ([4]). If $\alpha\in\mathbb{R}^+$, $\psi\in C^n([0,1],\mathbb{R})$, $x:[0,1]\to\mathbb{R}$ is an integrable function, then the ψ-Caputo fractional integral of the variable x and the order α is defined by

$$I_{0^+}^{\alpha,\psi}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}x(\tau)d\tau.$$

Definition 2.2 ([4]). If $\alpha>0$, $x\in C([0,1],\mathbb{R})$, the ψ-Caputo fractional derivative,

$${}^{C}D_{0^+}^{\alpha,\psi}x(t)=I_{0^+}^{n-\alpha,\psi}(\frac{1}{\psi'(t)}\frac{d}{dt})^nx(t),\quad n=[\alpha]+1.$$

Lemma 2.1 ([5]). *If $\alpha>0$, $\beta>0$, $x\in C([0,1],\mathbb{R})$, then there is*

$$I_{0^+}^{\alpha,\psi}I_{0^+}^{\beta,\psi}x(t)=I_{0^+}^{\alpha+\beta,\psi}x(t).$$

Lemma 2.2 ([6]). *Let $x:[0,1]\to\mathbb{R}$ be a continuous function. Then,*

$${}^{C}D_{0^+}^{\beta,\psi}I_{0^+}^{\beta,\psi}x(t)=x(t). \tag{2.1}$$

Moreover, if $x\in C^n([0,1],\mathbb{R})$, then

$$I_{0^+}^{\beta,\psi C}D_{0^+}^{\beta,\psi}x(t)=x(t)-\sum_{k=0}^{n-1}\frac{\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^kx(0)}{k!}[\psi(t)-\psi(0)]^k. \tag{2.2}$$

3. Main results

In this section, we consider the following problem

$${}^{C}D_{0^+}^{\beta,\psi}({}^{C}D_{0^+}^{\alpha,\psi}+\lambda)x_i(t)=f_i(t,x_{1t},x_{2t}),t\in[0,1],\ 0<\alpha+\beta<2,\ i=1,2, \tag{3.1}$$

supplemented with the boundary conditions

$$\begin{cases}x_1(t)=\varphi_1(t),\ t\in[-\varepsilon_1,0],\\ x_2(t)=\varphi_2(t),\ t\in[-\varepsilon_2,0],\\ x_1(0)+x_2(1)=0,\\ {}^{C}D_{0^+}^{\alpha,\psi}x_2(0)+{}^{C}D_{0^+}^{\alpha,\psi}x_1(1)=0.\end{cases} \tag{3.2}$$

Let $C([0,1],\mathbb{R})$ and $C([-\varepsilon_i,1],\mathbb{R})$ be equipped with the norms

$$\|u\|_{[0,1]}=\sup_{t\in[0,1]}|u(t)| \text{ and } \|u_t\|_{[-\varepsilon_i,0]}=\sup_{\theta\in[-\varepsilon_i,0]}|u_t(\theta)|,$$

respectively. Moreover, define $C([-\varepsilon_1,1],\mathbb{R})\times C([-\varepsilon_2,1],\mathbb{R})$ possessing the norm

$$\|(u_1,u_2)\|=\max\left\{\|u_1\|_{[-\varepsilon_1,1]},\|u_2\|_{[-\varepsilon_2,1]}\right\}.$$

Theorem 3.1. *Let $(x_1,x_2)\in C([-\varepsilon_1,1],\mathbb{R})\times C([-\varepsilon_2,1],\mathbb{R})$ is a solution of problem (3.1) and (3.2) if and only if x_i satisfy the following equations*

$$x_1(t)=\begin{cases}\varphi_1(t), \quad t\in[-\varepsilon_1,0],\\ I_{0^+}^{\alpha+\beta,\psi}f_1(t,x_{1t},x_{2t})-\lambda I_{0^+}^{\alpha,\psi}x_1(t)+\varphi_1(0)\\ +\dfrac{(\psi(t)-\psi(0))^\alpha}{\Gamma(\alpha+1)}(\lambda x_2(t)|_{t=0}+\lambda x_1(t)|_{t=1}\\ -I_{0^+}^{\alpha+\beta,\psi}f_1(t,x_{1t},x_{2t})\Big|_{t=1})-\dfrac{(\psi(t)-\psi(0))^\alpha}{(\psi(1)-\psi(0))^\alpha}(\lambda I_{0^+}^{\alpha,\psi}x_2(t)\Big|_{t=1}\\ -I_{0^+}^{\alpha+\beta,\psi}f_1(t,x_{1t},x_{2t})\Big|_{t=1}-\varphi_1(0)-\varphi_2(0)), \ t\in[0,1],\end{cases} \tag{3.3}$$

$$x_2(t)=\begin{cases}\varphi_2(t), \quad t\in[-\varepsilon_2,0],\\ I_{0^+}^{\alpha+\beta,\psi}f_2(t,x_{1t},x_{2t})-\lambda I_{0^+}^{\alpha,\psi}x_2(t)+\varphi_2(0)\\ +\dfrac{(\psi(t)-\psi(0))^\alpha}{(\psi(1)-\psi(0))^\alpha}\left(\lambda I_{0^+}^{\alpha,\psi}x_2(t)\Big|_{t=1}-I_{0^+}^{\alpha+\beta,\psi}f_2(t,x_{1t},x_{2t})\Big|_{t=1}\right.\\ \left.-\varphi_1(0)-\varphi_2(0)\right), \quad t\in[0,1].\end{cases} \tag{3.4}$$

Proof. Applying the operator $I_{0^+}^{\beta,\psi}$ on both sides of (3.1), we get

$$({}^CD_{0^+}^{\alpha,\psi}+\lambda)x_i(t)=I_{0^+}^{\beta,\psi}f_i(t,x_{1t},x_{2t})+C_{i0}. \tag{3.5}$$

Acting the operator $I_{0^+}^{\alpha,\psi}$ on both sides of (3.5), it follows

$$x_i(t)=I_{0^+}^{\alpha+\beta,\psi}f_i(t,x_{1t},x_{2t})+I_{0^+}^{\alpha,\psi}C_{i0}-\lambda I_{0^+}^{\alpha,\psi}x_i(t)+C_{i1}, \tag{3.6}$$

which together with the boundary value conditions (3.2) yield that

$$\begin{aligned}
C_{11}&=\varphi_1(0),\\
C_{21}&=\varphi_2(0),\\
C_{10}&=-I_{0^+}^{\beta,\psi}f_1(t,x_{1t},x_{2t})\Big|_{t=1}+\lambda(x_2(0)+x_1(1))\\
&\quad-\frac{\Gamma(\alpha+1)}{(\psi(1)-\psi(0))^\alpha}(\lambda I_{0^+}^{\alpha,\psi}x_2(t)\Big|_{t=1}-\varphi_1(0)-\varphi_2(0)\\
&\quad-I_{0^+}^{\alpha+\beta,\psi}f_2(t,x_{1t},x_{2t})\Big|_{t=1}),\\
C_{20}&=\frac{\Gamma(\alpha+1)}{(\psi(1)-\psi(0))^\alpha}(\lambda I_{0^+}^{\alpha,\psi}x_2(t)\Big|_{t=1}-\varphi_1(0)-\varphi_2(0)
\end{aligned}$$

$$-I_{0^+}^{\alpha+\beta,\psi} f_2(t,x_{1t},x_{2t})\Big|_{t=1}).$$

Thus, substituting the values of C_{i0}, C_{i1}, $i=1,2$ in (3.6), the (3.3) and (3.4) hold. On the other hand, if x_i satisfy (3.3) and (3.4), by Lemma 2.4, it follows that (x_1,x_2) is a solution of (3.1) and (3.2). □

The following fixed point lemma will be used in this paper.

Lemma 3.1 ([14]). *Let E be a Banach space, D a closed convex subset of E, and $U \subseteq D$ an open subset with $0 \in U$. If $T\colon \overline{U} \to D$ is a continuous function and if $T(\overline{U})$ is contained in a compact set, then either*

(i) T has a fixed point in $\overline{U}$, or

(ii) there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda T(u)$.

Theorem 3.2. *Suppose there exist the functions $L_i \in C([0,1],\mathbb{R}^+)$ such that $\forall u_1,u_2 \in C([-\varepsilon_1,0],\mathbb{R}), \forall v_1,v_2 \in C([-\varepsilon_2,0],\mathbb{R})$,*

$$|f_i(t,u_1,v_1)-f_i(t,u_2,v_2)| \le L_i(t)(\|u_1-u_2\|_{[-\varepsilon_1,0]}+\|v_1-v_2\|_{[-\varepsilon_2,0]}), \quad \forall t\in[0,1],$$

where $\mathbb{R}^+=[0,+\infty)$. Then, there exists a unique solution for the problem (3.1) and (3.2), provided that $k=\max\{k_1,k_2\}<1$, where

$$k_1 = \frac{\|L_1\|_{[0,1]}(\psi(1)-\psi(0))^{\alpha+\beta}+\|L_2\|_{[0,1]}(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{2\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\frac{\|L_1\|_{[0,1]}(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)},$$
$$k_2 = \frac{2\|L_2\|_{[0,1]}(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}.$$

Proof. Define the set

$$\begin{aligned} U := \{&(x_1,x_2)\in C([-\varepsilon_1,1],R)\times C([-\varepsilon_2,1],R): \\ &{}^C D_{0^+}^{\beta,\psi}({}^C D_{0^+}^{\alpha,\psi}+\lambda)x_i(t) \text{ is continuous in } [0,1]\Big\}, \end{aligned} \tag{3.7}$$

and the operator $T=(T_1,T_2)$, where

$$T_1(x_1,x_2)(t) = \begin{cases} \varphi_1(t), \quad t\in[-\varepsilon_1,0], \\ I_{0^+}^{\alpha+\beta,\psi} f_1(t,x_{1t},x_{2t}) - \lambda I_{0^+}^{\alpha,\psi} x_1(t) + \varphi_1(0) \\ +\dfrac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}(\lambda x_2(t)|_{t=0}+\lambda x_1(t)|_{t=1} \\ -I_{0^+}^{\alpha+\beta,\psi} f_1(t,x_{1t},x_{2t})\Big|_{t=1}) - \dfrac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}(\lambda I_{0^+}^{\alpha,\psi} x_2(t)\Big|_{t=1} \\ -I_{0^+}^{\alpha+\beta,\psi} f_1(t,x_{1t},x_{2t})\Big|_{t=1} - \varphi_1(0)-\varphi_2(0)), \ \ t\in[0,1], \end{cases} \tag{3.8}$$

$$T_2(x_1,x_2)(t) = \begin{cases} \varphi_2(t), \quad t\in[-\varepsilon_2,0], \\ I_{0^+}^{\alpha+\beta,\psi} f_2(t,x_{1t},x_{2t}) - \lambda I_{0^+}^{\alpha,\psi} x_2(t) + \varphi_2(0) \\ +\dfrac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\Big(\lambda I_{0^+}^{\alpha,\psi} x_2(t)\Big|_{t=1} - I_{0^+}^{\alpha+\beta,\psi} f_2(t,x_{1t},x_{2t})\Big|_{t=1} \\ -\varphi_1(0)-\varphi_2(0)\Big), \ \ t\in[0,1]. \end{cases} \tag{3.9}$$

By the standard process, it follows $T : U \to U$. Next, we plan to prove that T is a contraction map. Letting (x_1, x_2), $(y_1, y_2) \in U$ and $t \in [-\varepsilon_1, 0]$, one has $|T_1(x_1, x_2)(t) - T_1(y_1, y_2)(t)| = 0$. Moreover, for $t \in [-\varepsilon_2, 0]$, we can also get $|T_2(x_1, x_2)(t) - T_2(y_1, y_2)(t)| = 0$. On the other hand, letting $z_i = x_i - y_i$, for $t \in [0, 1]$, we can deduce

$$\begin{aligned}
&|T_1(x_1, x_2)(t) - T_1(y_1, y_2)(t)| \\
\leq{}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t \psi'(\tau)(\psi(1) - \psi(\tau))^{\alpha+\beta-1} |f_1(\tau, x_{1\tau}, x_{2\tau}) - f_1(\tau, y_{1\tau}, y_{2\tau})| \, d\tau \\
&+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(1) - \psi(\tau))^{\alpha-1} |y_1(\tau) - x_1(\tau)| \, d\tau \\
&+ \frac{(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 \psi'(\tau)(\psi(1) - \psi(\tau))^{\beta-1} |f_1(\tau, y_{1\tau}, y_{2\tau}) - f_1(\tau, x_{1\tau}, x_{2\tau})| \, d\tau \\
&+ \frac{\lambda(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} |x_2(0) - y_2(0)| + \frac{\lambda(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} |x_1(1) - y_1(1)| \\
&+ \frac{\lambda(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha)(\psi(1) - \psi(0))^\alpha} \int_0^1 \psi'(\tau)(\psi(1) - \psi(\tau))^{\alpha-1} |y_2(\tau) - x_2(\tau)| \, d\tau \\
&+ \frac{(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+\beta)(\psi(1) - \psi(0))^\alpha} \\
&\times \int_0^1 \psi'(\tau)(\psi(1) - \psi(\tau))^{\alpha+\beta-1} |f_2(\tau, x_{1\tau}, x_{2\tau}) - f_2(\tau, y_{1\tau}, y_{2\tau})| \, d\tau \\
\leq{}& \frac{\|L_1\|_{[0,1]}(\psi(t) - \psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} (|z_1| + |z_2|) + \frac{\lambda(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} (|z_1| + |z_2|) \\
&+ \frac{\|L_1\|_{[0,1]}(\psi(t) - \psi(0))^\alpha(\psi(1) - \psi(0))^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} (|z_1| + |z_2|) + \frac{\lambda(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} |z_1| \\
&- \frac{\lambda(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} |z_2| + \frac{\|L_2\|_{[0,1]}(\psi(t) - \psi(0))^\alpha(\psi(1) - \psi(0))^\beta}{\Gamma(\alpha+\beta+1)} (|z_1| + |z_2|) \\
\leq{}& \left(\frac{\|L_1\|_{[0,1]}(\psi(1) - \psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{2\lambda(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha+1)} + \frac{\|L_1\|_{[0,1]}(\psi(1) - \psi(0))^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \right. \\
&\left. + \frac{\|L_2\|_{[0,1]}(\psi(1) - \psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right) (\|z_1\|_{[-\varepsilon_1,1]} + \|z_2\|_{[-\varepsilon_2,1]}),
\end{aligned}$$

and

$$\begin{aligned}
&|T_2(x_1, x_2)(t) - T_2(y_1, y_2)(t)| \\
\leq{}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1} |f_2(\tau, x_{1\tau}, x_{2\tau}) - f_2(\tau, y_{1\tau}, y_{2\tau})| \, d\tau \\
&+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} |y_2(\tau) - x_2(\tau)| \, d\tau \\
&+ \frac{(\psi(t) - \psi(0))^\alpha}{(\psi(1) - \psi(0))^\alpha} \frac{\lambda}{\Gamma(\alpha)} \int_0^1 \psi'(\tau)(\psi(1) - \psi(\tau))^{\alpha-1} |x_2(\tau) - y_2(\tau)| \, d\tau \\
&+ \frac{(\psi(t) - \psi(0))^\alpha}{(\psi(1) - \psi(0))^\alpha} \frac{1}{\Gamma(\alpha+\beta)}
\end{aligned}$$

$$\begin{aligned}
&\times \int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha+\beta-1}\left|f_2(\tau, y_{1\tau}, y_{2\tau})-f_2(\tau, x_{1\tau}, x_{2\tau})\right| d\tau \\
\leq\ & \frac{\|L_2\|_{[0,1]}(\psi(t)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}(|z_1|+|z_2|)+\frac{\lambda(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}|z_2| \\
&+\frac{\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}|z_1|+\frac{\|L_2\|_{[0,1]}(\psi(t)-\psi(0))^{\alpha}(\psi(1)-\psi(0))^{\beta}}{\Gamma(\alpha+\beta+1)}(|z_1|+|z_2|) \\
\leq\ & \left(\frac{2\|L_2\|_{[0,1]}(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)(\|z_1\|_{[-\varepsilon_1,1]}+\|z_2\|_{[-\varepsilon_2,1]}).
\end{aligned}$$

Therefore, we proved that

$$\|T_i(x_1,x_2)-T_i(y_1,y_2)\| \leq k_i(\|z_1\|+\|z_2\|),$$

that is, T is a contraction map. By the Banach fixed point theorem, T admits a unique fixed point. □

(H_1) There exist some functions $p_i \in C([0,1],\mathbb{R}^+)$, $\gamma_i \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $q_i \in C\left(\mathbb{R}_0^+,\mathbb{R}_0^+\right)$ such that γ_i, q_i are nondecreasing and

$$|f_i(t,u,v)| \leq p_i(t)\left[q_i(\|u\|_{[-\varepsilon_1,0]})+\gamma_i(\|v\|_{[-\varepsilon_2,0]})\right],\ \forall t\in[0,1].$$

(H_2) There exist a positive constant M such that

$$\begin{aligned}
&\|p_1\|_{[0,1]}(q_1(M)+\gamma_1(M))\left(\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right) \\
&+\frac{4\lambda M(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}+\|p_2\|_{[0,1]}(q_2(M)+\gamma_2(M))\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&+2\varphi_1(0)+\varphi_2(0)<M, \\
&\|p_2\|_{[0,1]}(q_2(M)+\gamma_2(M))\frac{2(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2M\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&+\varphi_1(0)+2\varphi_2(0)<M,
\end{aligned}$$

and

$$M>\sup_{t\in[-\varepsilon_i,0]}|\varphi_i(t)|.$$

Theorem 3.3. *(H_1) and (H_2) are satisfied, then, the problem (3.1) and (3.2) possesses at least one solution.*

Proof. The proof will be divided in several steps.

Step 1. T is continuous.

Let $\{x_{in}\}$ be a sequence in $C([-\varepsilon_i,1],\mathbb{R})$ whose limit is $x_i \in C([-\varepsilon_i,1],\mathbb{R})$. Since f is a continuous function, for all $t\in[0,1]$, we have

$$\begin{aligned}
&|T_1(x_{1n},x_{2n})(t)-T_1(x_1,x_2)(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}|f_1(\tau,x_{1n\tau},x_{2n\tau})-f_1(\tau,x_{1\tau},x_{2\tau})|d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\left|x_{1n}(\tau)-x_1(\tau)\right|d\tau\\
&+\frac{\lambda(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\left|x_{2n}(0)-x_2(0)\right|+\frac{\lambda(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\left|x_{1n}(1)-x_1(1)\right|\\
&+\frac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta)}\\
&\times\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\beta-1}\left|f_1(\tau,x_{1n\tau},x_{2n\tau})-f_1(\tau,x_{1\tau},x_{2\tau})\right|d\tau\\
&+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{\lambda}{\Gamma(\alpha)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}\left|x_{2n}(\tau)-x_2(\tau)\right|d\tau\\
&+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{1}{\Gamma(\alpha+\beta)}\\
&\times\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha+\beta-1}\left|f_2(\tau,x_{1n\tau},x_{2n\tau})-f_2(\tau,x_{1\tau},x_{2\tau})\right|d\tau\\
\le{}&\sup_{t\in[0,1]}\left|f_1(t,x_{1nt},x_{2nt})-f_1(t,x_{1t},x_{2t})\right|\left[\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right]\\
&+\sup_{t\in[0,1]}\left|f_2(t,x_{1nt},x_{2nt})-f_2(t,x_{1t},x_{2t})\right|\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}\left|x_{1n}(\tau)-x_1(\tau)\right|d\tau\\
&+\frac{\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\left|x_{2n}(0)-x_2(0)\right|+\frac{\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\left|x_{1n}(1)-x_1(1)\right|\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}\left|x_{2n}(\tau)-x_2(\tau)\right|d\tau\\
\to{}&0 \text{ as } n\to+\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|T_2(x_{1n},x_{2n})(t)-T_2(x_1,x_2)(t)\right|\\
\le{}&\frac{1}{\Gamma(\alpha+\beta)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}\left|f_2(\tau,x_{1n\tau},x_{2n\tau})-f_2(\tau,x_{1\tau},x_{2\tau})\right|d\tau\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\left|x_2(\tau)-x_{2n}(\tau)\right|d\tau\\
&+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{\lambda}{\Gamma(\alpha)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}\left|x_{2n}(\tau)-x_2(\tau)\right|d\tau\\
&+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{1}{\Gamma(\alpha+\beta)}\\
&\times\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha+\beta-1}\left|f_2(\tau,x_{1\tau},x_{2\tau})-f_2(\tau,x_{1n\tau},x_{2n\tau})\right|d\tau\\
\le{}&\sup_{t\in[0,1]}\left|f_2(t,x_{1nt},x_{2nt})-f_2(t,x_{1t},x_{2t})\right|\frac{2(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}\left|x_2(\tau)-x_{2n}(\tau)\right|d\tau \\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}\left|x_{2n}(\tau)-x_2(\tau)\right|d\tau \\
&\to 0 \text{ as } n\to+\infty.
\end{aligned}
$$

Step 2. T is uniformly bounded.

Let

$$
\overline{B_R}:=\left\{(x_1,x_2)\in U:\|(x_1,x_2)\|\le R\right\},
$$

where $R>0$ is a real number. It is noted that the case of $t\in[-\varepsilon_i,0]$ is clear. For given $(x_1,x_2)\in\overline{B_R}$ and $t\in[0,1]$, we can get

$$
\begin{aligned}
&\left|T_1(x_1,x_2)(t)\right| \\
\le&\frac{1}{\Gamma(\alpha+\beta)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}\left|f_1(\tau,x_{1\tau},x_{2\tau})\right|d\tau+|\varphi_1(0)| \\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\left|x_1(\tau)\right|d\tau+\frac{\lambda(\psi(t)-\psi(0))^{\alpha}\left|x_2(t)|_{t=0}\right|}{\Gamma(\alpha+1)} \\
&+\frac{\lambda(\psi(t)-\psi(0))^{\alpha}\left|x_1(t)|_{t=1}\right|}{\Gamma(\alpha+1)}+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}|\varphi_1(0)+\varphi_2(0)| \\
&+\frac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\beta-1}\left|f_1(\tau,x_{1\tau},x_{2\tau})\right|d\tau \\
&+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{\lambda}{\Gamma(\alpha)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}\left|x_2(\tau)\right|d\tau \\
&+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha+\beta-1}\left|f_2(\tau,x_{1\tau},x_{2\tau})\right|d\tau \\
\le&\frac{\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\|x_1\|+\|p_1\|_{[0,1]}\left[q_1(\|x_1\|_{[-\varepsilon_1,1]})+\gamma_1(\|x_2\|_{[-\varepsilon_2,1]})\right] \\
&\times\left(\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right) \\
&+\|p_2\|_{[0,1]}\left[q_2(\|x_1\|_{[-\varepsilon_1,1]})+\gamma_2(\|x_2\|_{[-\varepsilon_2,1]})\right]\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&+\frac{\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}(\|x_1\|+\|x_2\|)+\frac{\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\|x_2\|+2\varphi_1(0)+\varphi_2(0) \\
\le&\|p_1\|_{[0,1]}(q_1(R)+\gamma_1(R))\left(\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right) \\
&+\|p_2\|_{[0,1]}(q_2(R)+\gamma_2(R))\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{4R\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&+2\varphi_1(0)+\varphi_2(0),
\end{aligned}
$$

which does not depend on (x_1,x_2). Thus, T_1 maps bounded sets into bounded sets of $C([-\varepsilon_1,1],\mathbb{R})$. Moreover, we have

$$
\left|T_2(x_1,x_2)(t)\right|
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}\,|f_2(\tau,x_{1\tau},x_{2\tau})|d\tau \\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_0^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\,|x_2(\tau)|d\tau+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\,|\varphi_1(0)+\varphi_2(0)| \\
&\quad+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{\lambda}{\Gamma(\alpha)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}\,|x_2(\tau)|d\tau+|\varphi_2(0)| \\
&\quad+\frac{(\psi(t)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 \psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha+\beta-1}\,|f_2(\tau,x_{1\tau},x_{2\tau})|d\tau \\
&\leq \|p_2\|_{[0,1]}\left[q_2(\|x_1\|_{[-\varepsilon_1,1]})+\gamma_2(\|x_2\|_{[-\varepsilon_2,1]})\right]\frac{2(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+|\varphi_1(0)+\varphi_2(0)| \\
&\quad+|\varphi_2(0)|+\frac{2\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\,\|x_2\| \\
&\leq \|p_2\|_{[0,1]}(q_2(R)+\gamma_2(R))\frac{2(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2R\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad+\varphi_1(0)+2\varphi_2(0),
\end{aligned}
$$

which does not depend on (x_1,x_2). Thus, T_2 maps bounded sets into bounded sets of $C([-\varepsilon_2,1],\mathbb{R})$.

Step 3. T is equicontinuous.

For any $(x_1,x_2)\in\overline{B_R}$, $t_1,t_2\in[0,1]$, assuming $t_1>t_2$, we have

$$
\begin{aligned}
&|T_1(x_1,x_2)(t_1)-T_1(x_1,x_2)(t_2)| \\
&\leq \frac{1}{\Gamma(\alpha+\beta)} \\
&\quad\times\left|\int_0^{t_2}\psi'(\tau)\left[(\psi(t_1)-\psi(\tau))^{\alpha+\beta-1}-(\psi(t_2)-\psi(\tau))^{\alpha+\beta-1}\right]f_1(\tau,x_{1\tau},x_{2\tau})d\tau\right| \\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\left|\int_0^{t_2}\psi'(\tau)\left[(\psi(t_2)-\psi(\tau))^{\alpha-1}-(\psi(t_1)-\psi(\tau))^{\alpha-1}\right]x_1(\tau)d\tau\right| \\
&\quad+\frac{1}{\Gamma(\beta)}\int_0^1\psi'(\tau)(\psi(1)-\psi(\tau))^{\beta-1}f_1(\tau,x_{1\tau},x_{2\tau})d\tau \\
&\quad\times\left|\frac{(\psi(t_2)-\psi(0))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right| \\
&\quad+(\lambda x_2(0)+\lambda x_1(1))\left|\frac{(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right| \\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_0^1\psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}x_2(\tau)d\tau\left|\frac{(\psi(t_2)-\psi(0))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right| \\
&\quad+\frac{1}{\Gamma(\alpha+\beta)}\int_0^1\psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha+\beta-1}f_2(\tau,x_{1\tau},x_{2\tau})d\tau \\
&\quad\times\left|\frac{(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right| \\
&\quad+(\varphi_1(0)+\varphi_2(0))\left|\frac{(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha+\beta)}\int_{t_2}^{t_1}\psi'(\tau)(\psi(t_1)-\psi(\tau))^{\alpha+\beta-1}f_1(\tau,x_{1\tau},x_{2\tau})d\tau\\
&-\frac{\lambda}{\Gamma(\alpha)}\int_{t_2}^{t_1}\psi'(\tau)(\psi(t_1)-\psi(\tau))^{\alpha-1}x_1(\tau)d\tau\\
\le{}& \|p_1\|_{[0,1]}(q_1(R)+\gamma_1(R))\frac{(\psi(t_1)-\psi(0))^{\alpha+\beta}-(\psi(t_2)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\\
&+\frac{\lambda R\left[(\psi(t_2)-\psi(0))^{\alpha}+(\psi(t_1)-\psi(t_2))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}\right]}{\Gamma(\alpha+1)}\\
&+\|p_1\|_{[0,1]}(q_1(R)+\gamma_1(R))\frac{(\psi(1)-\psi(0))^{\beta}}{\Gamma(\beta+1)}\left|\frac{(\psi(t_2)-\psi(0))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right|\\
&+2R\lambda\left|\frac{(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right|\\
&+\frac{\lambda R(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\left|\frac{(\psi(t_2)-\psi(0))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right|\\
&+\|p_2\|_{[0,1]}(q_2(R)+\gamma_2(R))\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\\
&\times\left|\frac{(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right|\\
&+(\varphi_1(0)+\varphi_2(0))\left|\frac{(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right|\\
&-\frac{\lambda R}{\Gamma(\alpha+1)}(\psi(t_1)-\psi(t_2))^{\alpha}\\
\to{}& 0 \text{ as } t_2\to t_1.
\end{aligned}
$$

Let us observe that for $t_2, t_1 \in [-\varepsilon_1, 0]$,

$$|T_1(x_1,x_2)(t_1)-T_1(x_1,x_2)(t_2)| = |\varphi_1(t_1)-\varphi_1(t_2)| \to 0 \text{ as } t_2\to t_1.$$

Moreover, for $t_1 \in [0,1]$ and $t_2 \in [-\varepsilon_1, 0]$,

$$
\begin{aligned}
&|T_1(x_1,x_2)(t_1)-T_1(x_1,x_2)(t_2)|\\
={}&\left|\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha+\beta-1}f_1(\tau,x_{1\tau},x_{2\tau})d\tau\right.\\
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^{t_1}\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}x_1(\tau)+\frac{\lambda(\psi(t_1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}(x_2(0)+x_1(1))\\
&+\varphi_1(0)-\frac{(\psi(t_1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta)}\int_0^{1}\psi'(\tau)(\psi(1)-\psi(\tau))^{\beta-1}f_1(\tau,x_{1\tau},x_{2\tau})d\tau\\
&-\frac{(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{\lambda}{\Gamma(\alpha)}\int_0^{1}\psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}x_2(\tau)d\tau\\
&+\frac{(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{1}{\Gamma(\alpha+\beta)}\int_0^{1}\psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha+\beta-1}f_2(\tau,x_{1\tau},x_{2\tau})d\tau\\
&\left.+\frac{(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\,(\varphi_1(0)+\varphi_2(0))-\varphi_1(t_2)\right|\\
\to{}& 0 \text{ as } t_2\to t_1.
\end{aligned}
$$

By the Ascoli-Arzela theorem, $T_1(\overline{B_R})$ is contained in a compact set. On the other hand, by the same way, it follows

$$\begin{aligned}
&|T_2(x_1,x_2)(t_1)-T_2(x_1,x_2)(t_2)| \\
\le\ & \frac{1}{\Gamma(\alpha+\beta)} \\
&\times\left|\int_0^{t_2}\psi'(\tau)\left[(\psi(t_1)-\psi(\tau))^{\alpha+\beta-1}-(\psi(t_2)-\psi(\tau))^{\alpha+\beta-1}\right]f_2(\tau,x_{1\tau},x_{2\tau})d\tau\right| \\
&+\frac{\lambda}{\Gamma(\alpha)}\left|\int_0^{t_2}\psi'(\tau)\left[(\psi(t_2)-\psi(\tau))^{\alpha-1}-(\psi(t_1)-\psi(\tau))^{\alpha-1}\right]x_2(\tau)d\tau\right|+\frac{\lambda}{\Gamma(\alpha)} \\
&\times\int_0^1\psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}x_2(\tau)d\tau\left|\frac{(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right| \\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_0^1\psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha+\beta-1}f_2(\tau,x_{1\tau},x_{2\tau})d\tau \\
&\times\left|\frac{(\psi(t_2)-\psi(0))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right| \\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_{t_2}^{t_1}\psi'(\tau)(\psi(t_1)-\psi(\tau))^{\alpha+\beta-1}|f_2(\tau,x_{1\tau},x_{2\tau})|d\tau \\
&-\frac{\lambda}{\Gamma(\alpha)}\int_{t_2}^{t_1}\psi'(\tau)(\psi(t_1)-\psi(\tau))^{\alpha-1}|x_2(\tau)|d\tau \\
\le\ & \|p_2\|_{[0,1]}(q_2(R)+\gamma_2(R)) \\
&\times\frac{\left[(\psi(t_1)-\psi(0))^{\alpha+\beta}+(\psi(t_1)-\psi(t_2))^{\alpha+\beta}-(\psi(t_2)-\psi(0))^{\alpha+\beta}\right]}{\Gamma(\alpha+\beta+1)} \\
&+\frac{R\lambda}{\Gamma(\alpha+1)}\left[(\psi(t_2)-\psi(0))^{\alpha}+(\psi(t_1)-\psi(t_2))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}\right] \\
&+\frac{R\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\left|\frac{(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right| \\
&+\|p_2\|_{[0,1]}(q_2(R)+\gamma_2(R))\frac{(\psi(1)-\psi(0))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta+1)} \\
&\times\left|\frac{(\psi(t_2)-\psi(0))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\right|-\frac{R\lambda(\psi(t_1)-\psi(t_2))^{\alpha}}{\Gamma(\alpha+1)} \\
&+\|p_2\|_{[0,1]}(q_2(R)+\gamma_2(R))\frac{(\psi(t_1)-\psi(t_2))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
\to\ & 0 \text{ as } t_2\to t_1.
\end{aligned}$$

Let us observe that for $t_2, t_1 \in [-\varepsilon_2, 0]$,

$$|T_2(x_1,x_2)(t_1)-T_2(x_1,x_2)(t_2)| = |\varphi_2(t_1)-\varphi_2(t_2)| \to 0, \quad \text{when} \quad t_2 \to t_1.$$

Moreover, for $t_1 \in [0,1]$ and $t_2 \in [-\varepsilon_2, 0]$,

$$\begin{aligned}
&|T_2(x_1,x_2)(t_1)-T_2(x_1,x_2)(t_2)| \\
=\ & \left|\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}\psi'(\tau)(\psi(t_1)-\psi(\tau))^{\alpha+\beta-1}f_2(\tau,x_{1\tau},x_{2\tau})d\tau\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^{t_1}\psi'(\tau)(\psi(t_1)-\psi(\tau))^{\alpha-1}x_2(\tau)d\tau-\frac{(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}(\varphi_1(0)+\varphi_2(0))\\
&+\varphi_2(0)+\frac{(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{\lambda}{\Gamma(\alpha)}\int_0^1\psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha-1}x_2(\tau)d\tau\\
&-\frac{(\psi(t_1)-\psi(0))^{\alpha}}{(\psi(1)-\psi(0))^{\alpha}}\frac{1}{\Gamma(\alpha+\beta)}\\
&\times\int_0^1\psi'(\tau)(\psi(1)-\psi(\tau))^{\alpha+\beta-1}f_2(\tau,x_{1\tau},x_{2\tau})d\tau-\varphi_2(t_2)|\\
\to{}&0 \text{ as } t_2\to t_1.
\end{aligned}
$$

By the Ascoli-Arzela theorem, $T_2(\overline{B_R})$ is contained in a compact set. Thus, $T(\overline{B_R})$ is contained in a compact set.

Step 4. Let

$$\overline{B_M}:=\{(x_1,x_2)\in U:\|(x_1,x_2)\|<M\}.$$

For $(x_1,x_2)\in\overline{B_M}$, $t\in[-\varepsilon_1,0]$, one has

$$|T_1(x_1,x_2)(t)|=|\varphi_1(t)|<M.$$

Moreover, for $t\in[0,1]$, we have

$$
\begin{aligned}
&|T_1(x_1,x_2)(t)|\\
\le{}&\|p_1\|_{[0,1]}(q_1(M)+\gamma_1(M))\left(\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right)\\
&+\|p_2\|_{[0,1]}(q_2(M)+\gamma_2(M))\frac{(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+2\varphi_1(0)+\varphi_2(0)\\
&+\frac{4\lambda M(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\\
<{}&M.
\end{aligned}
$$

And, for $t\in[-\varepsilon_2,0]$, we get

$$|T_2(x_1,x_2)(t)|=|\varphi_2(t)|<M.$$

Furthermore, for $t\in[0,1]$, it follows

$$
\begin{aligned}
&|T_2(x_1,x_2)(t)|\\
\le{}&\|p_2\|_{[0,1]}(q_2(M)+\gamma_2(M))\frac{2(\psi(1)-\psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2M\lambda(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\\
&+\varphi_1(0)+2\varphi_2(0)\\
<{}&M.
\end{aligned}
$$

Next, we plan to conclude that (H_2) of Theorem 3.3 does not hold. Given $(x_1,x_2)\in B_M$, we can get $\|T(x_1,x_2)\|<M$. If there exists a pair of functions $(x_1,x_2)\in\partial B_M$ and a real number $\lambda\in(0,1)$ such that $(x_1,x_2)=\lambda T(x_1,x_2)$, then we have

$$M=\|(x_1,x_2)\|=\lambda\|T(x_1,x_2)\|<\lambda M<M,$$

which implies that it is a contradiction. □

4. Example

Example 4.1. Letting $t \in [0,1]$, $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$, $\lambda = \frac{1}{10}$, $\psi(t) = t$, $\varepsilon_1 = \varepsilon_2 = 1$, consider the following problem:

$$\begin{cases} {}^C D_{0^+}^{2/3,t}({}^C D_{0^+}^{1/2,t} + (1/10))x_i(t) = f_i(t, x_{1t}, x_{2t}),\ i = 1,2, \\ x_1(t) = \dfrac{1}{3}t + \dfrac{1}{6},\ \ t \in [-1,0], \\ x_2(t) = -\dfrac{1}{4},\ \ t \in [-1,0], \\ x_1(0) + x_2(1) = 0, \\ {}^C D_{0^+}^{1/2,t} x_2(0) + {}^C D_{0^+}^{1/2,t} x_1(1) = 0, \end{cases} \tag{4.1}$$

where

$$f_1(t, x_{1t}, x_{2t}) = \frac{1}{5(1+t)} + \frac{|x_{1t}|}{15(1+|x_{1t}|)} + \frac{|x_{2t}|}{20(1+|x_{2t}|)},$$
$$f_2(t, x_{1t}, x_{2t}) = \frac{\cos(\pi t)}{8} + \frac{|x_{1t}|}{18(1+|x_{1t}|)} + \frac{|x_{2t}|}{25(1+|x_{2t}|)}.$$

Choosing

$$L_1(t) = \frac{1}{15}t,\ L_2(t) = \frac{1}{18}t,$$

then, we have

$$\begin{aligned} k_1 &= \frac{\frac{1}{15} + \frac{1}{18}}{\Gamma(13/6)} + \frac{\frac{1}{5}}{\Gamma(3/2)} + \frac{\frac{1}{15}}{\Gamma(3/2)\Gamma(5/3)} \approx 0.46 < 1, \\ k_2 &= \frac{\frac{1}{9}}{\Gamma(13/6)} + \frac{\frac{1}{10}}{\Gamma(3/2)} \approx 0.24 < 1, \\ k &= \max\{k_1, k_2\} < 1. \end{aligned}$$

So, the conditions of Theorem 3.2 are satisfied. Therefore, the problem (4.1) has a unique solution.

Example 4.2. Letting $t \in [0,1]$, $\alpha = \frac{1}{10}$, $\beta = \frac{3}{10}$ $\lambda = \frac{1}{150}$, $\psi(t) = 50t$, $\varepsilon_1 = \varepsilon_2 = 0.2$, consider the following problem:

$$\begin{cases} {}^C D_{0^+}^{3/10,50t}({}^C D_{0^+}^{1/10,50t} + (1/150))x_i(t) = f_i(t, x_{1t}, x_{2t}),\ i = 1,2, \\ x_1(t) = \dfrac{1}{100}(t + 0.5),\ \ t \in [-0.2, 0], \\ x_2(t) = \dfrac{1}{80}(t + 0.3),\ \ t \in [-0.2, 0], \\ x_1(0) + x_2(1) = 0, \\ {}^C D_{0^+}^{1/10,50t} x_2(0) + {}^C D_{0^+}^{1/10,50t} x_1(1) = 0, \end{cases} \tag{4.2}$$

where

$$f_1(t, x_{1t}, x_{2t}) = \frac{t}{30}(|x_{1t}|^{1/4} + |x_{2t}|^{1/4} + 2),$$
$$f_2(t, x_{1t}, x_{2t}) = \frac{t}{40}(|x_{1t}|^{1/5} + |x_{2t}|^{1/5} + 2).$$

Choosing

$$p_1(t) = \frac{t}{30},\ q_1(u) = u^{1/4} + 1,\ \gamma_1(v) = v^{1/4} + 1,$$
$$p_2(t) = \frac{t}{40},\ q_2(u) = u^{1/5} + 1,\ \gamma_2(v) = v^{1/5} + 1,$$

then in Theorem 3.3, (H_1) is satisfied. Moreover, if $M = 5$, we can get

$$\begin{aligned}
&\|p_1\|_{[0,1]}(q_1(M) + \gamma_1(M)) \left(\frac{(\psi(1) - \psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(\psi(1) - \psi(0))^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \right) \\
&+\|p_2\|_{[0,1]}(q_2(M) + \gamma_2(M)) \frac{(\psi(1) - \psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + 2\varphi_1(0) + \varphi_2(0) \\
&+\frac{4\lambda M(\psi(1) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
\approx\ & 3.92 < 5, \\
&\|p_2\|_{[0,1]}(q_2(M) + \gamma_2(M)) \frac{2(\psi(1) - \psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{2M\lambda(\psi(1) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\
&+\varphi_1(0) + 2\varphi_2(0) \\
\approx\ & 2.04 < 5,
\end{aligned}$$

and

$$5 > \max\left\{\sup|\varphi_1(t)|, \sup|\varphi_2(t)|\right\} = \max\{0.005, 0.00375\} = 0.005.$$

Therefore, (H_2) is satisfied and there exists at least one solution for the problem (4.2).

5. Conclusions

In this paper, the cyclic anti-periodic boundary value problems for ψ-Caputo fractional differential equations with two delays were studied. By several kinds of fixed point theorems, some new results on existence and uniqueness of solution for the problem (1.6) were obtained. The ψ-Caputo fractional operator is a generalization of the Caputo fractional operator, which makes our results more general. Moreover, there are relatively few existing studies on cyclic boundary value problems for ψ-Caputo fractional functional differential equations. So, our main results enrich some existing papers regarding to the field of fractional functional differential equations. Cyclic boundary value problems are more complex than coupling ones. In this paper, we do not consider the Ulam-Hyers-Rassias stability of solutions or discuss the p-Laplacian three-cycle problem. We will study these problems in future work.

Acknowledgements

Thank you for the reviewers' careful review and valuable comments. This research is supported by Anhui Provincial Excellent Young Teachers Cultivation Program for Higher Education Institutions (No. YQYB2024023).

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Received November 2025; Accepted May 2026; Available online June 2026.