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GRAPHICAL VIEWS OF ORTHOGONAL $\mathfrak{M}$-METRIC SPACES WITH AN APPLICATION

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Abstract In this paper, we present the concept of an orthogonal double controlled orthogonal $\mathfrak{m}$-metric space by employing two control maps α and ξ on right-hand side of the triangular inequality of $\mathfrak{m}$-metric space. We also present an example to illustrate our results. Using the resulting conclusions, analytical solutions to integral equations have been discovered. A numerical solution is used to validate the analytical solutions.

Keywords Fixed point, orthogonal double controlled orthogonal $\mathfrak{m}$-metric space, integral equation.

MSC(2010) 54H25, 46H20, 45B05, 45D05.

1. Introduction

Following the axiomatic interpretation of a metric space by the French mathematician Frechet in 1906, discoveries in mathematics in general and functional analysis in particular went in new directions. We can see how vast is the subject of fixed point theory and its applications in various areas. Fixed point results deal with the existence and properties of fixed points. These observations are critical in proving the existence and uniqueness of mathematical model solutions. Fixed point outcomes are well known tools in the mathematical analysis and have

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strong applications. Fixed point techniques are also used to solve both integral and differential equations, [12].

The notion of a partial metric space was initiated by Matthews [15] in 1994. Asadi et al. [5] introduced the $\mathfrak{m}_\mathfrak{b}$-metric space in 2014, which is the expanded form of a partial metric space. Mlaiki et al. [18] improved the definition of an "$\mathfrak{m}_\mathfrak{b}$"-metric space in 2016. In 2018, Mlaiki at al. [17] expanded the concept of extended $\mathfrak{m}_\mathfrak{b}$-metric spaces. Also, in 2018, Abdeljawad et al. [1] improved the notion of a controlled metric space [16,19] by using two control mappings. Recently, Eshaghi Gordji et al. [10] introduced the concept of orthogonal sets and gave a new expansion for the Banach fixed point result. Baghani et al. [6] proved the existence and uniqueness of fixed point and as an application they found the solution for Volterra integral equations. Gordji and Habibi [9] proved the existence and uniqueness of fixed point in generalized orthogonal metric space in 2017. An orthogonal L- contraction mapping is introduced by Mukheimer et al. [2] and solved integral equations. Gnanaprakasam et al. [8] investigated existence of fixed point using an orthogonal triangular $\alpha-$ orbital admissible mapping under orthogonal complete Branciari metric spaces. Prakasam et al. [21] introduces the concept of orthogonally modified $\mathcal{F}-$ contraction of type - I and type - II and proved fixed point theorems under orthogonal complete metric like space. Again Prakasam et al. [20] demonstrate the concepts of orthogonal generalized $\mathcal{F}$- contraction of type - (1) and type - (2) and prove several fixed point theorems. Mani et al. [11] proved common fixed point theorems via simulation functions in orthogonal Rectangular metric spaces. Uddin et al. [28] established the concept of orthogonal m- metric space and proved fixed point theorems under the same metric. For more related details, kindly refer to ([3,4,7,13,14,22–27]).

In this article, we present a new orthogonal double controlled orthogonal $\mathfrak{m}_\mathfrak{b}$-metric space (Shortly ODCOMMS), by applying two control functions $\alpha, \xi : \mathtt{G}\times\mathtt{G} \to [1,\infty)$ with the following new triangle inequality as a characterization of the ODCOMM.

$$(\Delta_\perp(\varsigma,\varpi) - \Delta_{\perp_{\varsigma,\varpi}}) \leq \alpha(\varsigma,\xi)\big(\Delta_\perp(\varsigma,\xi) - \Delta_{\perp_{\varsigma,\xi}}\big) + \xi(\xi,\varpi)\big(\Delta_\perp(\xi,\varpi) - \Delta_{\perp_{\xi,\varpi}}\big).$$

In this regard, we enhance and expand certain well-known theorems from the literature as well as established some fixed point results for the suggested contraction. Additionally, we present some significant examples to support our claims. In the final section, we provide an application to guarantee the existence of a solution of a Fredholm integral equation.

2. Preliminaries

In this section, we define many terms that are essential to understand this work.

Definition 2.1. [17] Let $\mathtt{G} \neq \emptyset$. A map $\mathfrak{m} : \mathtt{G}\times\mathtt{G} \to \mathbb{R}_+$ is called a partial metric if the below axioms hold for all $\varsigma, \varpi, \mathfrak{z} \in \mathtt{G}$,

1. $\varsigma = \varpi$ iff $\mathfrak{m}(\varsigma,\varsigma) = \mathfrak{m}(\varsigma,\varpi) = \mathfrak{m}(\varpi,\varpi)$;
2. $\mathfrak{m}(\varsigma,\varpi) = \mathfrak{m}(\varpi,\varsigma)$;
3. $\mathfrak{m}(\varsigma,\varsigma) \leq \mathfrak{m}(\varsigma,\varpi)$;
4. $\mathfrak{m}(\varsigma,\varpi) \leq \mathfrak{m}(\varsigma,\mathfrak{z}) + \mathfrak{m}(\mathfrak{z},\varpi) - \mathfrak{m}(\mathfrak{z},\mathfrak{z})$.

Definition 2.2. [19] Let $\mathtt{G} \neq \emptyset$. A map $\mathfrak{m}_\mathfrak{b} : \mathtt{G}\times\mathtt{G} \to \mathbb{R}_+$ is called an $\mathfrak{m}_\mathfrak{b}$-metric with $\mathfrak{s} > 1$ if the following conditions hold for all $\varsigma, \varpi, \mathfrak{z} \in \mathtt{G}$,

1. $\varsigma = \varpi$ iff $\mathfrak{m}_{\mathfrak{b}}(\varsigma,\varsigma) = \mathfrak{m}_{\mathfrak{b}}(\varsigma,\varpi) = \mathfrak{m}_{\mathfrak{b}}(\varpi,\varpi)$;
2. $\mathfrak{m}_{\mathfrak{b}\ \varsigma,\varpi} \leq \mathfrak{m}_{\mathfrak{b}}(\varsigma,\varpi)$;
3. $\mathfrak{m}_{\mathfrak{b}}(\varsigma,\varpi) = \mathfrak{m}_{\mathfrak{b}}(\varpi,\varsigma)$;
4. $\mathfrak{m}_{\mathfrak{b}}(\varsigma,\varpi) - \mathfrak{m}_{\mathfrak{b}\varsigma,\varpi} \leq \mathfrak{s}[(\mathfrak{m}_{\mathfrak{b}}(\varsigma,\mathfrak{z}) - \mathfrak{m}_{\mathfrak{b}\varsigma,\mathfrak{z}}) + (\mathfrak{m}_{\mathfrak{b}}(\mathfrak{z},\varpi) - \mathfrak{m}_{\mathfrak{b}\mathfrak{z},\varpi})] - \mathfrak{m}_{\mathfrak{b}}(\mathfrak{z},\mathfrak{z})$.

Definition 2.3. [17] Let $\mathtt{G} \neq \emptyset$ and $\alpha\colon \mathtt{G}\times\mathtt{G} \to \mathbb{R}_+$. A map $\mathfrak{m}_{\mathfrak{b}}\colon \mathtt{G}\times\mathtt{G}\to\mathbb{R}_+$ is termed as an extended $\mathfrak{m}_{\mathfrak{b}}$-metric if the following axioms hold for all $\varsigma,\varpi,\mathfrak{z}\in\mathtt{G}$,

1. $\varsigma = \varpi$ iff $\mathfrak{m}_{\mathfrak{b}}(\varsigma,\varsigma) = \mathfrak{m}_{\mathfrak{b}}(\varsigma,\varpi) = \mathfrak{m}_{\mathfrak{b}}(\varpi,\varpi)$;
2. $\mathfrak{m}_{\mathfrak{b}\ \varsigma,\varpi} \leq \mathfrak{m}_{\mathfrak{b}}(\varsigma,\varpi)$;
3. $\mathfrak{m}_{\mathfrak{b}}(\varsigma,\varpi) = \mathfrak{m}_{\mathfrak{b}(\varpi,\varsigma)}$;
4. $\mathfrak{m}_{\mathfrak{b}}(\varsigma,\varpi - \mathfrak{m}_{\mathfrak{b}\varsigma,\varpi}) \leq \alpha(\varsigma,\varpi)[(\mathfrak{m}_{\mathfrak{b}}(\varsigma,\mathfrak{z}) - \mathfrak{m}_{\mathfrak{b}\varsigma,\mathfrak{z}}) + (\mathfrak{m}_{\mathfrak{b}}(\mathfrak{z},\varpi) - \mathfrak{m}_{\mathfrak{b}\mathfrak{z},\varpi})]$.

The pair of $(\mathtt{G},\mathfrak{m}_{\mathfrak{b}})$ is termed as an extended $\mathfrak{m}_{\mathfrak{b}}$-metric space.

Eshaghi Gordji et al. [10] proposed a generalized Banach fixed point theorem via the notion of orthogonality.

Definition 2.4. [10] Let $\mathtt{G}\neq\emptyset$ and $\perp\subseteq \mathtt{G}\times\mathtt{G}$ be a binary relation. If

$$\exists\ \varsigma_0\in\mathtt{G} : (\forall\ \varsigma\in\mathtt{G}, \varsigma\perp\varsigma_0) \quad \text{or} \quad (\forall\ \varsigma\in\mathtt{G}, \varsigma_0\perp\varsigma),$$

then $(\mathtt{G},\perp)$ is said to be an orthogonal set.

Definition 2.5. [10] Let $(\mathtt{G},\perp)$ be an orthogonal set. A sequence $\{\varsigma_\varrho\}$ is called an orthogonal sequence (O_s) if

$$(\forall\ \varrho\in\mathbb{N}, \varsigma_\varrho\perp\varsigma_{\varrho+1}) \quad or \quad (\forall\ \varrho\in\mathbb{N}, \varsigma_{\varrho+1}\perp\varsigma_\varrho).$$

Definition 2.6. [10] The triplet $(\mathtt{G},\mathfrak{m},\perp)$ is known as an orthogonal metric space (OMS) if $(\mathtt{G},\perp)$ is an O_s and $(\mathtt{G},\mathfrak{m})$ is a metric space.

Definition 2.7. [10] Let $(\mathtt{G},\perp)$ be an O_s. A map $\Omega:\mathtt{G}\to\mathtt{G}$ is known as orthogonal-preserving (shortly, O_p) if $\Omega\varsigma\perp\Omega\varpi$, whenever $\varsigma\perp\varpi$.

3. Main results

This section will begin with the definition of an $\mathfrak{m}_{\mathfrak{b}}$-metric space. But, first we will present certain notions that will be useful during the proof.

Definition 3.1. Let $\mathtt{G}$ be a non-void orthogonal set and $\alpha,\xi\colon\mathtt{G}\times\mathtt{G}\to[1,+\infty)$ be two functions. A map $\Delta_\perp\colon\mathtt{G}\times\mathtt{G}\to[0,+\infty)$ is called an ODCOMM if the following axioms hold:

(m1) $\varsigma=\varpi \iff \Delta_\perp(\varsigma,\varsigma)=\Delta_\perp(\varsigma,\varpi)=\Delta_\perp(\varpi,\varpi), \forall\ \varsigma\perp\varpi,\ \varsigma,\varpi\in\mathtt{G}$;

(m2) $\Delta_{\perp_{\varsigma,\varpi}}\leq\Delta_\perp(\varsigma,\varpi),\ \forall\ \varsigma\perp\varpi,\ \varsigma,\varpi\in\mathtt{G}$;

(m3) $\Delta_\perp(\varsigma,\varpi)=\Delta_\perp(\varpi,\varsigma),\ \forall\ \varsigma\perp\varpi$;

(m4) $(\Delta_\perp(\varsigma,\varpi)-\Delta_{\perp_{\varsigma,\varpi}})\leq\alpha(\varsigma,\mathfrak{z})(\Delta_\perp(\varsigma,\mathfrak{z})-\Delta_{\perp_{\varsigma,\mathfrak{z}}})+\xi(\mathfrak{z},\varpi)(\Delta_\perp(\mathfrak{z},\varpi)-\Delta_{\perp_{\mathfrak{z},\varpi}}), \forall\ \varsigma\perp\varpi,\varsigma\perp\mathfrak{z}$.

Example 3.1. Take $\mathtt{G}=[0,3]$. For distinct ς,ϖ and $\mathfrak{z}\in\mathtt{G}$, we take

1. $\Delta_\perp(\varsigma,\varpi)=18=\Delta_\perp(\varpi,\varsigma),\ \Delta_\perp(1,1)=1$,
2. $\Delta_\perp(\varsigma,\mathfrak{z})=6=\Delta_\perp(\mathfrak{z},\varsigma),\ \Delta_\perp(2,2)=8$,
3. $\Delta_\perp(\mathfrak{z},\varpi)=7=\Delta_\perp(\varpi,\mathfrak{z}),\ \Delta_\perp(3,3)=3$.

Define the binary relation $\perp$ on $\mathtt{G}$: If there exists $\varpi\in\mathtt{G}\ \forall\ \varsigma\in\mathtt{G}$ such that $\varsigma\perp\varpi$ then, $(\mathtt{G},\perp)$ is orthogonal set. Also define $\alpha,\xi\colon\mathtt{G}\times\mathtt{G}\to[0,\infty)$ by

$$\alpha(\varsigma,\varpi)=\begin{cases}2\varsigma+3, & \text{if } \varsigma>\varpi,\\ 3\varpi^2+1, & \text{otherwise,}\end{cases}$$

and

$$\xi(\varsigma,\varpi)=\begin{cases}8\varsigma+4, & \text{if } \varsigma>\varpi,\\ 2\varpi^2+1, & \text{otherwise.}\end{cases}$$

Clearly, $(m1)-(m3)$ are satisfied. Now, to prove $(m4)$ first, we verify the triangle inequality:

$$\begin{aligned}&\Delta_\perp(\varsigma,\varpi)-\Delta_{\perp_{\varsigma,\varpi}}\le\alpha(\varsigma,\mathfrak{z})(\Delta_\perp(\varsigma,\mathfrak{z})-\Delta_{\perp_{\varsigma,\mathfrak{z}}})+\xi(\mathfrak{z},\varpi)(\Delta_\perp(\mathfrak{z},\varpi)-\Delta_{\perp_{\mathfrak{z},\varpi}}),\\ &18-8\le 7(3)+8(3),\\ &10\le 45.\end{aligned}$$

Clearly, $\Delta_\perp$ is an $\mathfrak{m}_\mathfrak{b}$-metric. Hence, $\Delta_\perp$ is an $\mathfrak{m}_\mathfrak{b}$-metric, but it is an ODCOMM.

Example 3.2. Consider the set of fourth roots of unity with squared modulus distance. i. e., $\mathtt{G}=\{\varsigma\in\mathbb{C}:\varsigma^4=1\}$, the fourth roots of unity,

$$\mathtt{G}=\{1,-1,\mathtt{i},-\mathtt{i}\}.$$

Let us define the orthogonality as two roots are orthogonal if their product is primitive fourth root of unity

$$\varsigma\perp\varpi \quad \text{iff} \quad \varsigma\cdot\varpi\in\{\mathtt{i},-\mathtt{i}\}.$$

Let, $\alpha,\xi:\mathtt{G}\times\mathtt{G}\to[1,+\infty)$. Now, define the control function α by

$$\alpha(\varsigma,\varpi)=2+\frac{\max(\arg\varsigma,\arg\varpi)}{\pi},$$

and ξ for an orthogonal pairs $\varsigma\perp\varpi$ by,

$$\xi(\varsigma,\varpi)=2+\frac{|\arg(\varsigma)-\arg(\varpi)|}{\pi}.$$

Now, define $\Delta_{\perp_{\varsigma,\varpi}}=\max\{\Delta_\perp(\varsigma,\varsigma),\Delta_\perp(\varpi,\varpi)\}$ and define the distance function $\Delta_\perp$ by,

$$\Delta_\perp(\varsigma,\varpi)=|\varsigma-\varpi|^2+\frac{\arg(\varsigma)+\arg(\varpi)}{2\pi}, \qquad \varsigma\perp\varpi.$$

Let us check for the orthogonal pairs:

- $1\perp\mathtt{i}:\ |1-\mathtt{i}|^2=2,\ \ \arg(1)=0,\ \ \arg(\mathtt{i})=\frac{\pi}{2}$, then, $\Delta_\perp(1,\mathtt{i})=2.25$.

- $1 \perp -\mathtt{i}: \ |1+\mathtt{i}|^2 = 2, \ \ \arg(1) = 0, \ \ \arg(-\mathtt{i}) = \frac{3\pi}{2}$, then, $\Delta_\perp(1, -\mathtt{i}) = 2.75$.
- $\mathtt{i} \perp -1: \ |1+\mathtt{i}|^2 = 2, \ \ \arg(\mathtt{i}) = \frac{\pi}{2}, \ \ \arg(-1) = \pi$, then, $\Delta_\perp(\mathtt{i}, -1) = 2.75$.
- $-1 \perp -\mathtt{i}: \ |-1+\mathtt{i}|^2 = 2, \ \ \arg(-1) = \pi, \ \ \arg(-\mathtt{i}) = \frac{3\pi}{2}$, then, $\Delta_\perp(-\mathtt{i}, -1) = 3.25$.

Now, to verify:

(m1) $\varsigma = \varpi \iff \Delta_\perp(\varsigma, \varsigma) = \Delta_\perp(\varsigma, \varpi) = \Delta_\perp(\varpi, \varpi)$; holds true.

(m2) For orthogonal pairs:

$$\Delta_{\perp_{\varsigma,\varpi}} = \max\{\Delta_\perp(\varsigma, \varsigma), \Delta_\perp(\varpi, \varpi)\}.$$

1. For $1 \perp \mathtt{i}$:

$$\Delta_{\perp_{1,\mathtt{i}}} = \max\Big(\Delta_\perp(1,1), \Delta_\perp(\mathtt{i},\mathtt{i})\Big) = \max\Big(1, 1.25\Big) = 1.25.$$

   Also, $1.25 \leq 2.25$ satisfied.

2. For $1 \perp -\mathtt{i}$:

$$\Delta_{\perp_{1,-\mathtt{i}}} = \max\Big(\Delta_\perp(1,1), \Delta_\perp(-\mathtt{i},-\mathtt{i})\Big) = \max\Big(1, 1.75\Big) = 1.75.$$

   Also, $1.75 \leq 2.75$.

3. For $\mathtt{i} \perp -1$:

$$\Delta_{\perp_{\mathtt{i},-1}} = \max\Big(\Delta_\perp(-1,-1), \Delta_\perp(\mathtt{i},\mathtt{i})\Big) = \max\Big(1.5, 1.75\Big) = 1.75.$$

   Also, $1.75 \leq 2.75$.

4. For $-\mathtt{i} \perp -1$:

$$\Delta_{\perp_{-\mathtt{i},-1}} = \max\Big(\Delta_\perp(-1,-1), \Delta_\perp(-\mathtt{i},-\mathtt{i})\Big) = \max\Big(1.5, 1.75\Big) = 1.75.$$

   Also, $1.75 \leq 3.25$.

Hence, (m2) holds for all the orthogonal pairs.

(m3) To verify (m3), Since, $|\varsigma - \varpi|^2 = |\varpi - \varsigma|^2$ holds for all pair $\varsigma, \ \varpi$, (m3) holds good.

(m4) To verify, $(\Delta_\perp(\varsigma, \varpi) - \Delta_{\perp_{\varsigma,\varpi}}) \leq \alpha(\varsigma, \mathfrak{z})(\Delta_\perp(\varsigma, \mathfrak{z}) - \Delta_{\perp_{\varsigma,\mathfrak{z}}}) + \xi(\mathfrak{z}, \varpi)(\Delta_\perp(\mathfrak{z}, \varpi) - \Delta_{\perp_{\mathfrak{z},\varpi}})$, $\forall \ \ \varsigma \perp \varpi, \varsigma \perp \mathfrak{z}$.

- For $1 \perp \mathtt{i}$: $\mathfrak{z} = -\mathtt{i}$,

$$\begin{aligned}
&\alpha(1, -\mathtt{i}) = 2 + \frac{\max(\arg(1), \arg(-\mathtt{i}))}{\pi} = 2 + \frac{\max(0, \frac{3\pi}{2})}{\pi} = 2 + \frac{3}{2} = 3.5, \\
&\xi(-\mathtt{i}, \mathtt{i}) = 2 + \frac{|\arg(-\mathtt{i}) - \arg(\mathtt{i})|}{\pi} = 2 + \frac{|\frac{3\pi}{2} - \frac{\pi}{2}|}{\pi} = 3, \\
&1 \leq 3.5(2.75 - 1.75) + 3(5 - 1.75) = 13.25.
\end{aligned}$$

Similarly, for other pairs of elements of $\mathtt{G}$ we have, $1 \leq$ L. H. S $\leq 4 \leq$ R. H. S.

Definition 3.2. Let $(\mathtt{G}, \Delta_\perp, \perp)$ be an ODCOMMS.

1. An O_s $\{\varsigma_\varrho\}$ in $\mathtt{G}$ converges to a point ς if
$$\lim_{\varrho\to\infty}(\Delta_\perp(\varsigma_\varrho,\varsigma)-\Delta_{\perp_{\varsigma_\varrho,\varsigma}})=0.$$
2. An O_s $\{\varsigma_\varrho\}$ in $\mathtt{G}$ is said to be a orthogonal $\Delta_\perp$-Cauchy sequence if
$$\lim_{\varrho,\mathfrak{m}_\mathfrak{b}\to\infty}(\Delta_\perp(\varsigma_\varrho,\varsigma_m)-\Delta_{\perp_{\varsigma_\varrho,\varsigma_\mathfrak{m}}})\text{ and }\lim_{\varrho\to\infty}(\Delta_{\perp_{\varsigma_\varrho,\varsigma_\mathfrak{m}}}-\mathfrak{m}_{\mathfrak{b}_{\varsigma_\varrho,\varsigma_\mathfrak{m}}})<\infty.$$
3. An orthogonal double controlled orthogonal $\Delta_\perp$- metric space is said to be an orthogonal $\Delta_\perp$-complete if every orthogonal $\Delta_\perp$-Cauchy sequence $\{\varsigma_\varrho\}$ is convergent to ς, that is,
$$\lim_{\varrho,\mathfrak{m}\to\infty}(\Delta_\perp(\varsigma_\varrho,\varsigma)-\Delta_{\perp_{\varsigma_\varrho,\varsigma}})=0\text{ and }\lim_{\varrho\to\infty}(\Delta_{\perp_{\varsigma_\varrho,\varsigma}}-\mathfrak{m}_{\varsigma_\varrho,\varsigma})=0.$$

Theorem 3.1. *Let $(\mathtt{G},\Delta_\perp,\perp)$ be an orthogonal complete DCOMMS with the functions $\alpha,\xi\colon\mathtt{G}\times\mathtt{G}\to[0,\infty)$. Suppose that an orthogonal continuous mapping $\Omega\colon\mathtt{G}\to\mathtt{G}$ satisfies*
$$\Delta_\perp(\Omega\varsigma,\Omega\varpi)\le(\Delta_\perp(\varsigma,\varpi)+\rho\Delta_{\perp_{\varsigma,\varpi}}),\tag{3.1}$$
for all $\varsigma,\varpi\in\mathtt{G}$ where $\rho\in(0,1)$. For $\varsigma_0\in\mathtt{G}$, let $\varsigma_\varrho=\Omega^\varrho\varsigma_0$. Assume that
$$\sup_{\mathfrak{m}\ge1}\lim_{\mathfrak{a}\to\infty}\frac{\alpha(\varsigma_{\mathfrak{i}+1},\varsigma_{\mathfrak{i}+2})}{\alpha(\varsigma_\mathfrak{a},\varsigma_{\mathfrak{i}+1})}\xi(\alpha(\varsigma_{\mathfrak{i}+1},\mathfrak{m}_\mathfrak{b}))<\frac{1}{\rho}.\tag{3.2}$$
In addition, for each $\varsigma\in\mathtt{G}$, suppose that
$$\lim_{\varrho\to\infty}\alpha(\varsigma,\varsigma_\varrho)\text{ and }\lim_{\varrho\to\infty}\xi(\varsigma_\varrho,\varsigma)<\infty.\tag{3.3}$$
Then the mapping Ω has a unique fixed point (UFP).

Proof. By the orthogonality, there is $\varsigma_0\in\mathtt{G}$ such that
$$\forall\ \varpi\in\mathtt{G},\ \varsigma_0\perp\varpi\ \text{ or }\ \forall\ \varpi\in\mathtt{G},\ \varpi\perp\varsigma_0.$$
It follows that $\varsigma_0\perp\Omega\varsigma_0$ or $\Omega\varsigma_0\perp\varsigma_0$. Let $\varsigma_1=\Omega\varsigma_0,\varsigma_2=\Omega\varsigma_1,\varsigma_3=\Omega\varsigma_2,\ldots\ldots\ \varsigma_{\varrho+1}=\Omega\varsigma_\varrho,\ \forall\ \varrho\in\mathbb{N}$. Since Ω is O_p, $\{\varsigma_\varrho\}=\Omega^\varrho\varsigma_0$ is an O_s in $\mathtt{G}$ fulfilling the conditions of theorem. By (3.1), we get
$$\Delta_\perp(\varsigma_\varrho,\varsigma_{\varrho+1})\le\rho^\varrho(\Delta_\perp(\varsigma_0,\varsigma_1)+\Delta_{\perp_{\varsigma_0,\varsigma_1}}),\ \forall\ \varrho\ge0.$$
For $\varrho,\mathfrak{m}\in\mathbb{N}$ with $\mathfrak{m}\ge\varrho$, we get
$$\begin{aligned}\Delta_\perp(\varsigma_\varrho,\varsigma_\mathfrak{m})+\Delta_{\perp_{\varsigma_\varrho,\varsigma_\mathfrak{m}}}&\le\alpha(\varsigma_\varrho,\varsigma_{\varrho+1})(\Delta_\perp(\varsigma_\varrho,\varsigma_{\varrho+1})-\Delta_{\perp_{\varsigma_\varrho,\varsigma_{\varrho+1}}})\\&\quad+\xi(\varsigma_{\varrho+1},\varsigma_\mathfrak{m})(\Delta_\perp(\varsigma_{\varrho+1},\varsigma_\mathfrak{m})-\Delta_{\perp_{\varsigma_{\varrho+1},\varsigma_\mathfrak{m}}})\\&\le\alpha(\varsigma_\varrho,\varsigma_{\varrho+1})(\Delta_\perp(\varsigma_\varrho,\varsigma_{\varrho+1})-\Delta_{\perp_{\varsigma_\varrho,\varsigma_{\varrho+1}}})\\&\quad+\xi(\varsigma_{\varrho+1},\varsigma_\mathfrak{m})\big[\alpha(\varsigma_{\varrho+1},\varsigma_{\varrho+2})(\Delta_\perp(\varsigma_{\varrho+1},\varsigma_{\varrho+2})-\Delta_{\perp_{\varsigma_{\varrho+1},\varsigma_{\varrho+2}}})\\&\quad+\xi(\varsigma_{\varrho+2},\varsigma_\mathfrak{m})(\Delta_\perp(\varsigma_{\varrho+2},\varsigma_\mathfrak{m})-\Delta_{\perp_{\varsigma_{\varrho+2},\varsigma_\mathfrak{m}}})\big]\\&\le\alpha(\varsigma_\varrho,\varsigma_{\varrho+1})(\Delta_\perp(\varsigma_\varrho,\varsigma_{\varrho+1})-\Delta_{\perp_{\varsigma_\varrho,\varsigma_{\varrho+1}}})\\&\quad+\xi(\varsigma_{\varrho+1},\varsigma_\mathfrak{m})\alpha(\varsigma_{\varrho+1},\varsigma_{\varrho+2})[(\Delta_\perp(\varsigma_{\varrho+1},\varsigma_{\varrho+2})-\Delta_{\perp_{\varsigma_{\varrho+1},\varsigma_{\varrho+2}}})]\end{aligned}$$

$$
\begin{aligned}
&+\xi(\varsigma_{\varrho+1},\varsigma_{\mathfrak{m}})\xi(\varsigma_{\varrho+2},\varsigma_{\mathfrak{m}})[(\Delta_{\perp}(\varsigma_{\varrho+2},\varsigma_{\mathfrak{m}})-\Delta_{\perp_{\varsigma_{\varrho+2},\varsigma_{\mathfrak{m}}}})]\\
&\leq \alpha(\varsigma_{\varrho+2},\varsigma_{\varrho+3})(\Delta_{\perp}(\varsigma_{\varrho+2},\varsigma_{\varrho+3})-\Delta_{\perp_{\varsigma_{\varrho+2},\varsigma_{\varrho+3}}})\\
&\quad+\xi(\varsigma_{\varrho+3},\varsigma_{\mathfrak{m}})(\Delta_{\perp}(\varsigma_{\varrho+3},\varsigma_{\mathfrak{m}})-\Delta_{\perp_{\varsigma_{\varrho+3},\varsigma_{\mathfrak{m}}}}).
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\Delta_{\perp}(\varsigma_{\varrho},\varsigma_{\mathfrak{m}})-\Delta_{\perp_{\varsigma_{\varrho},\varsigma_{\mathfrak{m}}}}\\
&\leq \alpha(\varsigma_{\varrho},\varsigma_{\varrho+1})(\Delta_{\perp}(\varsigma_{\varrho},\varsigma_{\varrho+1})-\Delta_{\perp_{\varsigma_{\varrho},\varsigma_{\varrho+1}}})\\
&\quad+\xi(\varsigma_{\varrho+1},\varsigma_{\mathfrak{m}})\alpha(\varsigma_{\varrho+1},\varsigma_{\varrho+2})[\Delta_{\perp}(\varsigma_{\varrho+1},\varsigma_{\varrho+2})-\Delta_{\perp_{\varsigma_{\varrho+1},\varsigma_{\varrho+2}}}]\\
&\quad+\xi(\varsigma_{\varrho+1},\varsigma_{\mathfrak{m}})\xi(\varsigma_{\varrho+2},\varsigma_{\mathfrak{m}})\big[\alpha(\varsigma_{\varrho+2},\varsigma_{\varrho+3})(\Delta_{\perp}(\varsigma_{\varrho+2},\varsigma_{\varrho+3})-\Delta_{\perp_{\varsigma_{\varrho+2},\varsigma_{\varrho+3}}})\\
&\quad+\xi(\varsigma_{\varrho+3},\mathfrak{m}_{\mathfrak{b}})(\Delta_{\perp}(\varsigma_{\varrho+3},\mathfrak{m}_{\mathfrak{b}})-\Delta_{\perp_{\varsigma_{\varrho+3},\mathfrak{m}_{\mathfrak{b}}}})\big]\\
&\leq \alpha(\varsigma_{\varrho},\varsigma_{\varrho+1})(\Delta_{\perp}(\varsigma_{\varrho},\varsigma_{\varrho+1})-\Delta_{\perp_{\varsigma_{\varrho},\varsigma_{\varrho+1}}})\\
&\quad+\xi(\varsigma_{\varrho+1},\varsigma_{\mathfrak{m}})\alpha(\varsigma_{\varrho+1},\varsigma_{\varrho+2})[\Delta_{\perp}(\varsigma_{\varrho+1},\varsigma_{\varrho+2})-\Delta_{\perp_{\varsigma_{\varrho+1},\varsigma_{\varrho+2}}}]\\
&\quad+\xi(\varsigma_{\varrho+1},\varsigma_{\mathfrak{m}})\xi(\varsigma_{\varrho+2},\varsigma_{\mathfrak{m}})\alpha(\varsigma_{\varrho+2},\varsigma_{\varrho+3})(\Delta_{\perp}(\varsigma_{\varrho+2},\varsigma_{\varrho+3})-\Delta_{\perp_{\varsigma_{\varrho+2},\varsigma_{\varrho+3}}})\\
&\quad+\xi(\varsigma_{\varrho+1},\varsigma_{\mathfrak{m}})\xi(\varsigma_{\varrho+2},\varsigma_{\mathfrak{m}})\xi(\varsigma_{\varrho+3},\mathfrak{m}_{\mathfrak{b}})(\Delta_{\perp}(\varsigma_{\varrho+3},\mathfrak{m}_{\mathfrak{b}})-\Delta_{\perp_{\varsigma_{\varrho+3},\mathfrak{m}_{\mathfrak{b}}}})\\
&\leq\\
&\quad\vdots\\
&\leq \alpha(\varsigma_{\varrho},\varsigma_{\varrho+1})(\Delta_{\perp}(\varsigma_{\varrho},\varsigma_{\varrho+1})-\Delta_{\perp_{\varsigma_{\varrho},\varsigma_{\varrho+1}}})\\
&\quad+\sum_{\mathfrak{a}=\varrho+1}^{\mathfrak{m}_{\mathfrak{b}}-2}\left(\prod_{\mathrm{j}=\varrho+1}^{\mathfrak{a}}\xi(\varsigma_{\mathrm{j}},\varsigma_{\mathfrak{m}})\right)\alpha(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1})\big[\Delta_{\perp}(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1})-\Delta_{\perp_{\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1}}}\big]\\
&\quad+\prod_{\rho=\varrho+1}^{\mathfrak{m}_{\mathfrak{b}}-1}\xi(\varsigma_{\rho},\varsigma_{\mathfrak{m}})\big[\Delta_{\perp}(\varsigma_{\mathfrak{m}_{\mathfrak{b}}-1},\varsigma_{\mathfrak{m}})-\Delta_{\perp_{\varsigma_{\mathfrak{m}_{\mathfrak{b}}-1},\varsigma_{\mathfrak{m}}}}\big]\\
&\leq \alpha(\varsigma_{\varrho},\varsigma_{\varrho+1})\rho^{\varrho}(\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}})\\
&\quad+\sum_{\mathfrak{a}=\varrho+1}^{\mathfrak{m}_{\mathfrak{b}}-2}\left(\prod_{\mathrm{j}=\varrho+1}^{\mathfrak{a}}\xi(\varsigma_{\mathrm{j}},\varsigma_{\mathfrak{m}})\right)\alpha(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1})\rho^{\mathfrak{a}}\big[\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}}\big]\\
&\quad+\left(\prod_{\mathfrak{a}=\varrho+1}^{\mathfrak{m}_{\mathfrak{b}}-1}\xi(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{m}})\right)\rho^{\mathfrak{m}_{\mathfrak{b}}-1}(\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}})\\
&\leq \alpha(\varsigma_{\varrho},\varsigma_{\varrho+1})\rho^{\varrho}(\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}})\\
&\quad+\sum_{\mathfrak{a}=\varrho+1}^{\mathfrak{m}_{\mathfrak{b}}-2}\left(\prod_{\mathrm{j}=\varrho+1}^{\mathfrak{a}}\xi(\varsigma_{\mathrm{j}},\varsigma_{\mathfrak{m}})\right)\alpha(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1})\rho^{\mathfrak{a}}\big[\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}}\big]\\
&\quad+\left(\prod_{\mathrm{j}=\varrho+1}^{\mathfrak{m}_{\mathfrak{b}}-1}\xi(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{m}})\right)\rho^{\mathfrak{m}_{\mathfrak{b}}-1}\alpha(\varsigma_{\mathfrak{m}_{\mathfrak{b}}-1},\varsigma_{\mathfrak{m}})(\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}})\\
&\leq \alpha(\varsigma_{\varrho},\varsigma_{\varrho+1})\rho^{\varrho}(\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}})\\
&\quad+\sum_{\mathfrak{a}=\varrho+1}^{\mathfrak{m}_{\mathfrak{b}}-1}\left(\prod_{\mathrm{j}=\varrho+1}^{\mathfrak{a}}\xi(\varsigma_{\mathrm{j}},\varsigma_{\mathfrak{m}})\right)\alpha(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1})\rho^{\mathfrak{a}}\big[\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}}\big]
\end{aligned}
$$

$$\begin{aligned}&\leq \alpha(\varsigma_{\varrho},\varsigma_{\varrho+1})\rho^{\varrho}(\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}})\\&\quad+\sum_{\mathfrak{a}=\varrho+1}^{\mathfrak{m}_{\mathfrak{b}}-1}\left(\prod_{\mathrm{j}=0}^{\mathfrak{a}}\xi(\varsigma_{\mathrm{j}},\varsigma_{\mathfrak{m}})\right)\alpha(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1})\rho^{\mathfrak{a}}\big[\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}}\big].\end{aligned}$$

Taking

$$\mathfrak{s}_{\mathfrak{e}}=\sum_{\mathfrak{a}=0}^{\mathfrak{e}}\left(\prod_{\mathrm{j}=0}^{\mathfrak{a}}\xi(\varsigma_{\mathrm{j}},\varsigma_{\mathfrak{m}})\right)\alpha(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1})\rho^{\mathfrak{a}},$$

we get

$$(\Delta_{\perp}(\varsigma_{\varrho},\varsigma_{\mathfrak{m}})-\Delta_{\perp_{\varsigma_{\varrho},\varsigma_{\mathfrak{m}}}})\leq(\Delta_{\perp}(\varsigma_0,\varsigma_1)-\Delta_{\perp_{\varsigma_0,\varsigma_1}})+[\rho^{\varrho}\alpha(\varsigma_{\varrho},\varsigma_{\varrho+1})+\mathfrak{s}_{\mathfrak{m}_{\mathfrak{b}}-1}-\mathfrak{s}_{\varrho}].\tag{3.4}$$

The equation (3.2) implies that an O_s exists, and so $\{\mathfrak{s}_{\varrho}\}$ is an orthogonal Cauchy sequence. Indeed, we obtain

$$\bar{x}_{\mathfrak{a}}=\left(\prod_{\mathrm{j}=0}^{\mathfrak{a}}\xi(\varsigma_{\mathrm{j}},\varsigma_{\mathfrak{m}})\right)\alpha(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1}).$$

Taking $\varrho,\mathfrak{m}_{\mathfrak{b}}\to\infty$ in (3.4), we have

$$\lim_{\varrho,\mathfrak{m}_{\mathfrak{b}}\to\infty}(\Delta_{\perp}(\varsigma_{\varrho},\varsigma_{\mathfrak{m}})-\Delta_{\perp_{\varsigma_{\varrho},\varsigma_{\mathfrak{m}}}}=0,$$

which implies that $\{\varsigma_{\varrho}\}$ is an orthogonal Cauchy sequence, and by the help of orthogonal completeness of $\Delta_{\perp}$, there is $\varsigma\in\mathtt{G}$ such that

$$\Delta_{\perp_{\varsigma,\Omega\varsigma}}\leq\Delta_{\perp}(\varsigma,\Omega\varsigma),$$

and

$$\Delta_{\perp}(\varsigma,\Omega\varsigma)-\Delta_{\perp_{\varsigma,\Omega\varsigma}}\leq\alpha(\varsigma,\psi_{\varrho})\big[\Delta_{\perp}(\varsigma,\psi_{\varrho})-\Delta_{\perp_{\varsigma,\psi_{\varrho}}}\big]+\xi(\psi_{\varrho},\Omega\varsigma)\big[\Delta_{\perp}(\psi_{\varrho},\Omega\varsigma)-\Delta_{\perp_{\psi_{\varrho},\Omega\varsigma}}\big].$$

By the orthogonality continuity of Ω and taking the limit, we have

$$\begin{aligned}&\Delta_{\perp}(\varsigma,\Omega\varsigma)-\Delta_{\perp_{\varsigma,\Omega\varsigma}}\leq 0,\\&\Delta_{\perp}(\varsigma,\Omega\varsigma)=\Delta_{\perp_{\varsigma,\Omega\varsigma}}.\end{aligned}$$

Now, let

$$\begin{aligned}&\mathcal{L}^{\star}_{\varsigma,\Omega\varsigma}=\Delta_{\perp}(\varsigma,\Omega\varsigma)\text{ where }\mathcal{L}^{\star}_{\varsigma,\Omega\varsigma}=\max(\Delta_{\perp}(\varsigma,\varsigma),\Delta_{\perp}(\Omega\varsigma,\Omega\varsigma)),\\&\mathcal{L}^{\star}_{\psi_{\varrho},\Omega\psi_{\varrho}}=\Delta_{\perp}(\psi_{\varrho},\psi_{\varrho})\leq\mathcal{L}^{\varrho}\big(\Delta_{\perp}(\psi_0,\psi_1)+\Delta_{\perp_{\psi_0,\psi_1}}\big).\end{aligned}$$

Since $\mathcal{L}\in(0,1)$, by applying limit, we have

$$\mathcal{L}^{\star}_{\psi_{\varrho},\psi_{\varrho}}=0.$$

Finally, since

$$\Delta_{\perp}(\varsigma,\Omega\varsigma)=\Delta_{\perp_{\varsigma,\Omega\varsigma}}\leq\mathcal{L}^{\star}_{\varsigma,\Omega\varsigma},$$

one writes

$$\begin{aligned}&\Delta_\perp(\Omega\varsigma,\Omega\varsigma)=\Delta_{\perp_{\varsigma,\Omega\varsigma}},\\&\varsigma=\Omega\varsigma.\end{aligned}$$

Hence, ς is a fixed point of Ω.

Now, we show the uniqueness. Let $\bar{x}\in\mathtt{G}$ be another fixed point of Ω. Then we have $\Omega^\varrho\varsigma=\varsigma,\Omega^\varrho\bar{x}=\bar{x}\ \forall\ \varrho\in\mathbb{N}$. By the choice of ς_0, we have

$$[\varsigma_0\perp\varsigma \text{ and } \varsigma_0\perp\bar{x}] \text{ or } [\varsigma\perp\varsigma_0 \text{ and } \bar{x}\perp\varsigma_0].$$

Since Ω is O_p, we get

$$[\Omega^\varrho\varsigma_0\perp\Omega^\varrho\varsigma \text{ and } \Omega^\varrho\varsigma_0\perp\Omega^\varrho\bar{x}]$$

or

$$[\Omega^\varrho\varsigma\perp\Omega^\varrho\varsigma_0 \text{ and } \Omega^\varrho\bar{x}\perp\Omega^\varrho\varsigma_0],\quad\forall\ \varrho\in\mathbb{N}.$$

Thus,

$$\Delta_\perp(\varsigma,\bar{x})=\Delta_\perp(\Omega^\varrho\varsigma,\Omega^\varrho\bar{x})\leq\rho\Delta_\perp(\varsigma,\bar{x})<\Delta_\perp(\varsigma,\bar{x}),$$

which implies

$$\Delta_\perp(\varsigma,\bar{x})=0,$$

and so

$$\varsigma=\bar{x}.$$

□

Example 3.3. Let $\mathtt{G}=[0,\infty)$. Consider the orthogonal complete DCOMM defined by $\Delta_\perp(\varsigma,\varpi)$ $=\varsigma+\varpi$, for all $\varsigma,\varpi\in\mathtt{G}$. Given the functions $\alpha,\xi\colon\mathtt{G}^2\to[0,\infty)$ as

$$\alpha(\varsigma,\varpi)=\begin{cases}8, & \text{if }\ \varsigma,\varpi\geq1,\\8(\varsigma+2), & \text{otherwise,}\end{cases}$$

and

$$\xi(\varsigma,\varpi)=\begin{cases}7, & \text{if }\ \varsigma,\varpi\geq1,\\7(\varpi+2), & \text{otherwise.}\end{cases}$$

Define the binary relation $\perp$ on $\mathtt{G}$ by $\varsigma\perp\varpi$ if $\varpi\leq8\varsigma$. Taking $\Omega\varsigma=1,\ \forall\ \varsigma\in\mathtt{G}$ and $\varsigma_0=1$ and $\rho=\frac{1}{4}$, we obtain

$$\sup_{\mathfrak{m}_\mathfrak{b}\geq1}\lim_{\mathfrak{a}\to\infty}\frac{\alpha(\varsigma_{\mathfrak{a}+1},\varsigma_{\mathfrak{a}+2})}{\alpha(\varsigma_\mathfrak{a}),\varsigma_{\mathfrak{a}+1}}\xi(\varsigma_{\mathfrak{a}+1},\varsigma_\mathfrak{m})=1<4=\frac{1}{\rho},$$

that is, (3.2) holds. Moreover, for every $\varsigma\in[0,\infty)$ we get

$$\begin{aligned}&\lim_{\varrho\to\infty}\alpha(\varsigma,\varsigma_\varrho)=\max(1,\varsigma)<\infty\ \text{ and}\\&\lim_{\varrho\to\infty}\xi(\varsigma_\varrho,\varsigma)=\max(\varsigma,1)<\infty.\end{aligned}$$

Thus, (3.3) holds. All the axioms of Theorem 3.1 hold, and $\varsigma=1$ is the UFP.

Theorem 3.2. *Let $(\mathtt{G}, \Delta_{\perp}, \perp)$ be an orthogonal complete DCOMMS and $\Omega\colon \mathtt{G} \to \mathtt{G}$ be an orthogonal continuous mapping. Suppose that there are $\bar{k}, \breve{i} \in [0, \infty)$ with*

$$\lim_{\varrho\to\infty} \frac{\bar{k}\alpha(\varsigma_{\varrho}, \varsigma_{\varrho-1})}{1-\breve{i}\xi(\varsigma_{\varrho}, \varsigma_{\varrho-1})} < 1, \tag{3.5}$$

where $\varsigma_{\varrho} = \Omega^{\varrho}\varsigma_0$ and for any $\varsigma \perp \varsigma_0, \quad \varsigma, \varsigma_0 \in \mathtt{G}$,

$$\bar{k}\alpha(\varsigma, \Omega\varsigma) + \breve{i}\xi(\varsigma, \Omega\varsigma) < 1.$$

If

$$\Delta_{\perp}(\Omega\varsigma, \Omega\varpi) \leq \bar{k}\alpha(\varsigma, \Omega\varsigma)\Delta_{\perp}(\varsigma, \Omega\varsigma) + \breve{i}\xi(\varpi, \Omega\varpi)\Delta_{\perp}(\varpi, \Omega\varpi),$$

then Ω has a UFP.

Proof. By orthogonality, there is $\varsigma_0 \in \mathtt{G}$ such that

$$\forall \ \ \varpi \in \mathtt{G}, \ \varsigma_0 \perp \varpi \ \text{ or } \ \forall \ \ \varpi \in \mathtt{G}, \ \varpi \perp \varsigma_0.$$

It follows that $\varsigma_0 \perp \Omega\varsigma_0$ or $\Omega\varsigma_0 \perp \varsigma_0$. Let $\varsigma_1 = \Omega\varsigma_0, \varsigma_2 = \Omega\varsigma_1, \varsigma_3 = \Omega\varsigma_2, \cdots \varsigma_{\varrho+1} = \Omega\varsigma_{\varrho}$, for every $\varrho \in \mathbb{N}$. Since Ω is O_p, we first prove that

$$\Delta_{\perp}(\varsigma_{\varrho}, \varsigma_{\varrho+1}) \leq \bar{k}^{\varrho} \prod_{\mathfrak{a}=0}^{\varrho} \left[\frac{\alpha(\varsigma_{\mathfrak{a}}, \varsigma_{\mathfrak{a}-1})}{1-\breve{i}\xi(\varsigma_{\mathfrak{a}}, \varsigma_{\mathfrak{a}+1})}\right] \Delta_{\perp}(\varsigma_0, \varsigma_1).$$

Let $\varrho \in \mathbb{N}$. Then

$$\begin{aligned}
\Delta_{\perp}(\varsigma_{\varrho}, \varsigma_{\varrho+1}) &= \Delta_{\perp}(\Omega\varsigma_{\varrho-1}, \varsigma_{\varrho}) \\
&\leq \bar{k}\alpha(\varsigma_{\varrho-1}, \Omega\varsigma_{\varrho-1})\Delta_{\perp}(\varsigma_{\varrho-1}, \Omega\varsigma_{\varrho-1}) + \breve{i}\xi(\varsigma_{\varrho}, \Omega\varsigma_{\varrho})\Delta_{\perp}(\varsigma_{\varrho}, \Omega\varsigma_{\varrho}) \\
&\leq \bar{k}\alpha(\varsigma_{\varrho-1}, \varsigma_{\varrho})\Delta_{\perp}(\varsigma_{\varrho-1}, \varsigma_{\varrho}) + \breve{i}\xi(\varsigma_{\varrho}, \varsigma_{\varrho+1})\Delta_{\perp}(\varsigma_{\varrho}, \varsigma_{\varrho+1}),
\end{aligned}$$

and so

$$\begin{aligned}
&\Delta_{\perp}(\varsigma_{\varrho}, \varsigma_{\varrho+1}) - \breve{i}\xi(\varsigma_{\varrho}, \varsigma_{\varrho+1})\Delta_{\perp}(\varsigma_{\varrho}, \varsigma_{\varrho+1}) \leq \bar{k}\alpha(\varsigma_{\varrho-1}, \varsigma_{\varrho})\Delta_{\perp}(\varsigma_{\varrho-1}, \varsigma_{\varrho}), \\
&\Delta_{\perp}(\varsigma_{\varrho}, \varsigma_{\varrho+1})(1-\breve{i}\xi(\varsigma_{\varrho}, \varsigma_{\varrho+1})) \leq \bar{k}\alpha(\varsigma_{\varrho-1}, \varsigma_{\varrho})\Delta_{\perp}(\varsigma_{\varrho-1}, \varsigma_{\varrho}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Delta_{\perp} &\leq \frac{\bar{k}\alpha(\varsigma_{\varrho-1}, \varsigma_{\varrho})\Delta_{\perp}(\varsigma_{\varrho-1}, \varsigma_{\varrho})}{1-\breve{i}\xi(\varsigma_{\varrho}, \varsigma_{\varrho+1})} \\
&= \frac{\bar{k}\alpha(\varsigma_{\varrho-1}, \varsigma_{\varrho})}{1-\breve{i}\xi(\varsigma_{\varrho}, \varsigma_{\varrho+1})}\Delta_{\perp}(\Omega\varsigma_{\varrho-1}, \Omega\varsigma_{\varrho-2}) \\
&\leq \frac{\bar{k}^2\alpha(\varsigma_{\varrho}, \varsigma_{\varrho-1})\alpha(\varsigma_{\varrho-1}, \varsigma_{\varrho-2})}{[(1-\breve{i}\xi(\varsigma_{\varrho}, \varsigma_{\varrho+1}))(1-\breve{i}\xi(\varsigma_{\varrho-1}, \varsigma_{\varrho}))]}\Delta_{\perp}(\varsigma_{\varrho-1}, \varsigma_{\varrho}) \\
&\ \ \vdots \\
&\leq \bar{k}^{\varrho} \prod_{\mathfrak{a}=1}^{\varrho} \left[\frac{\alpha(\varsigma_{\mathfrak{a}}, \varsigma_{\mathfrak{a}-1})}{1-\breve{i}\xi(\varsigma_{\mathfrak{a}}, \varsigma_{\mathfrak{a}+1})}\right] \Delta_{\perp}(\varsigma_0, \varsigma_1).
\end{aligned}$$

Since

$$\lim_{\varrho\to\infty}\frac{\alpha(\varsigma_\varrho,\varsigma_{\varrho-1})}{1-\breve{i}\xi(\varsigma_\varrho,\varsigma_{\varrho+1})}<1,$$

by the ratio test,

$$\sum_{\mathfrak{a}=1}^{\infty}\bar{k}^{\varrho}\prod_{\mathfrak{a}=0}^{\varrho}\left[\frac{\alpha(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}-1})}{1-\breve{i}\xi(\varsigma_{\mathfrak{a}},\varsigma_{\mathfrak{a}+1})}\right]\to 0 \quad\Longrightarrow\quad \Delta_\perp(\varsigma_\varrho,\varsigma_{\varrho+1})\to 0.$$

Further, for $\varrho,\mathfrak{m}\in\mathbb{N}$,

$$\begin{aligned}\Delta_\perp(\varsigma_\varrho,\varsigma_{\mathfrak{m}})&=\Delta_\perp(\Omega\varsigma_{\varrho-1},\Omega\varsigma_{\mathfrak{m}-1})\\&\le\bar{k}\alpha(\varsigma_{\varrho-1},\Omega\varsigma_{\varrho-1})\Delta_\perp(\varsigma_{\varrho-1},\Omega\varsigma_{\varrho-1})+\breve{i}\xi(\varsigma_{\mathfrak{m}_\mathfrak{b}-1},\Omega\varsigma_{\mathfrak{m}-1})\Delta_\perp(\varsigma_{\mathfrak{m}_\mathfrak{b}-1},\Omega\varsigma_{\mathfrak{m}-1})\\&\le\bar{k}\alpha(\varsigma_{\varrho-1},\varsigma_\varrho)\Delta_\perp(\varsigma_{\varrho-1},\varsigma_\varrho)+\breve{i}\xi(\varsigma_{\mathfrak{m}-1},\varsigma_{\mathfrak{m}})\Delta_\perp(\varsigma_{\mathfrak{m}-1},\varsigma_{\mathfrak{m}}).\end{aligned}$$

It implies that $\Delta_\perp(\varsigma_\varrho,\varsigma_{\mathfrak{m}})$ converges to 0. Since

$$\Delta_{\perp\varsigma_\varrho,\varsigma_{\mathfrak{m}}}=\min(\Delta_\perp(\varsigma_\varrho,\varsigma_\varrho),\Delta_\perp(\varsigma_{\mathfrak{m}},\varsigma_{\mathfrak{m}}))\le\Delta_\perp(\varsigma_{\mathfrak{m}},\varsigma_{\mathfrak{m}}),$$

we conclude that

$$\lim_{\varrho,\mathfrak{m}_\mathfrak{b}\to\infty}(\Delta_\perp(\varsigma_\varrho,\varsigma_{\mathfrak{m}})-\Delta_{\perp_{\varsigma_\varrho,\varsigma_{\mathfrak{m}}}})=0.$$

Also, suppose that

$$\mathcal{L}^\star_{\varsigma_\varrho,\varsigma_{\mathfrak{m}}}=\max(\Delta_\perp(\varsigma_\varrho,\varsigma_\varrho),\Delta_\perp(\varsigma_{\mathfrak{m}},\varsigma_{\mathfrak{m}}))=\Delta_\perp(\varsigma_\varrho,\varsigma_\varrho).$$

Then

$$\begin{aligned}\mathcal{L}^\star_{\varsigma_\varrho,\varsigma_{\mathfrak{m}}}-\Delta_{\perp_{\varsigma_\varrho},\varsigma_{\mathfrak{m}}}&\le\mathcal{L}^\star_{\varsigma_\varrho,\varsigma_{\mathfrak{m}}}\\&=\Delta_\perp(\varsigma_\varrho,\varsigma_\varrho)\\&=\Delta_\perp(\Omega\varsigma_{\varrho-1},\Omega\varsigma_{\varrho-1})\\&\le\bar{k}\alpha(\varsigma_{\varrho-1},\varsigma_\varrho)\Delta_\perp(\varsigma_{\varrho-1},\varsigma_\varrho)+\breve{i}\xi(\varsigma_{\mathfrak{m}_\mathfrak{b}-1},\varsigma_{\mathfrak{m}})\Delta_\perp(\varsigma_{\mathfrak{m}_\mathfrak{b}-1},\varsigma_{\mathfrak{m}}).\end{aligned}$$

Let us take $\varrho\to\infty$, we get

$$\lim_{\varrho,\mathfrak{m}_\mathfrak{b}\to\infty}(\mathcal{L}^\star_{\varsigma_\varrho,\varsigma_{\mathfrak{m}}}-\Delta_{\perp_{\varsigma_\varrho},\varsigma_{\mathfrak{m}}})=0.$$

Thus, $\{\varsigma_\varrho\}$ is an orthogonal Cauchy sequence, and by using the orthogonal completeness of $\Delta_\perp$ which implies that $\{\varsigma_\varrho\}\to\varsigma\in\mathtt{G}$ and so $\{\Omega\varsigma_\varrho\}=\varsigma_{\varrho+1}\to\varsigma,\varsigma\in\mathtt{G}$. Furthermore, by the hypothesis of Theorem 3.2, i.e., $\Omega\colon(\mathtt{G},\Delta_\perp)\to(\mathtt{G},\Delta_\perp)$ is orthogonal continuous, we show that $\{\Omega\varsigma_\varrho\}\to\Omega\iota$. By 3.1, we get

$$\Delta_\perp(\iota,\Omega\iota)-\Delta_{\perp_{\iota,\Omega\iota}}=0,$$

and

$$\Delta_\perp(\iota,\Omega\iota)=\Delta_\perp(\Omega\iota,\Omega\iota)$$

$$\begin{aligned}&\leq \bar{k}\alpha(\iota,\Omega\iota)\Delta_{\perp}(\iota,\Omega\iota)+\breve{i}\xi(\iota,\Omega\iota)\Delta_{\perp}(\iota,\Omega\iota)\\ &\leq (\bar{k}\alpha(\iota,\Omega\iota)+\breve{i}\xi(\iota,\Omega\iota))\Delta_{\perp}(\iota,\Omega\iota)\\ &\leq \Delta_{\perp}(\iota,\Omega\iota).\end{aligned}$$

Hence,

$$\Delta_{\perp}(\iota,\Omega\iota)=\Delta_{\perp}(\Omega\iota,\Omega\iota)=0.$$

By the same observation given in the above equality, we obtain

$$\Delta_{\perp}(\Omega\iota,\Omega\iota)=\Delta_{\perp}(\Omega^2\iota,\Omega^2\iota)=0. \tag{3.6}$$

Since $(\mathtt{G},\Delta_{\perp},\perp)$ is orthogonal complete, one writes

$$\Omega^2\iota=\Omega\iota.$$

Thus, we get

$$\iota'=\Omega\iota,$$

is a fixed point of Ω.

Now, let $\bar{x}\in\mathtt{G}$ be one more fixed point of Ω. Then we obtain $\Omega^{\varrho}\varsigma=\varsigma,\Omega^{\varrho}\bar{x}=\bar{x}$ for each $\varrho\in\mathbb{N}$. By the choice of ς_0, we have

$$[\varsigma_0\perp\varsigma \text{ and } \varsigma_0\perp\bar{x}] \text{ or } [\varsigma\perp\varsigma_0 \text{ and } \bar{x}\perp\varsigma_0].$$

Since Ω is O_p, we get

$$[\Omega^{\varrho}\varsigma_0\perp\Omega^{\varrho}\varsigma \text{ and } \Omega^{\varrho}\varsigma_0\perp\Omega^{\varrho}\bar{x}],$$

or

$$[\Omega^{\varrho}\varsigma\perp\Omega^{\varrho}\varsigma_0 \text{ and } \Omega^{\varrho}\bar{x}\perp\Omega^{\varrho}\varsigma_0],\quad \forall\ \varrho\in\mathbb{N}.$$

Thus,

$$\Delta_{\perp}(\varsigma,\bar{x})=\Delta_{\perp}(\Omega^{\varrho}\varsigma,\Omega^{\varrho}\bar{x})\leq\mathfrak{s}\Delta_{\perp}(\varsigma,\bar{x})<\Delta_{\perp}(\varsigma,\bar{x}),$$

which implies

$$\Delta_{\perp}(\varsigma,\bar{x})=0,$$

and so

$$\varsigma=\bar{x}.$$

That is, Ω has a UFP. □

Example 3.4. Let $\mathtt{G}=\{1,2,3\}$ and $\alpha,\xi\colon\mathtt{G}\times\mathtt{G}\to[0,\infty)$ be defined by

$$\alpha(\varsigma,\varpi)=\begin{cases}\varsigma+1, & \text{if } \varsigma>\varpi,\\ \varpi^2, & \text{otherwise,}\end{cases}$$

and

$$\xi(\varsigma,\varpi)=\begin{cases}3\varsigma, & \text{if } \varsigma>\varpi,\\ 2\varpi, & \text{otherwise.}\end{cases}$$

Given

$$\Delta_\perp(\varsigma,\varpi)=\varsigma+\varpi,$$

and define the binary relation $\perp$ on $\mathtt{G}$ by $\varsigma\perp\varpi$ if $\Omega\colon \mathtt{G}\to\mathtt{G}$ is given as

$$\Omega(\varsigma)=\frac{\varsigma}{2}.$$

Clearly,

$$\Delta_\perp(\varpi\varsigma,\varpi\varpi)\le \bar{k}\alpha(\varsigma,\varpi\varsigma)\Delta_\perp(\varsigma,\varpi\varsigma)+\breve{i}\xi(\varpi,\varpi\varpi)\Delta_\perp(\varpi,\varpi\varpi).$$

Taking $\varpi\varsigma=1$ for $\varsigma_0=1$, we get

$$\lim_{\varrho\to\infty}\frac{\bar{k}\alpha(\varsigma_\varrho,\varsigma_{\varrho-1})}{1-\breve{i}\xi(\varsigma_\varrho,\varsigma_{\varrho+1})}<1.$$

That is, the equation (3.5) holds.

$$\lim_{\varrho\to\infty}\alpha(\varsigma,\varsigma_\varrho)=\lim_{\varrho\to\infty}\xi(\varsigma_\varrho,\varsigma)<\infty,$$

hence all the axioms in Theorem 3.2 are fulfilled, and therefore a fixed point is $\varsigma=1$.

4. Application

Let $\mathtt{G}=\mathcal{C}([0,1],\mathbb{R})$, and

$$\varsigma'(\mathfrak{e})=\int_0^1\Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e}))d\eta,\ \text{ for }\ \mathfrak{e},\eta\in[0,1], \tag{4.1}$$

be an integral equation, where $\Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e}))$ is a continuous function on $\mathtt{G}$. Now, we define

$$\begin{aligned}&\Delta_\perp\colon \mathtt{G}\times\mathtt{G},\\ &\Delta_\perp(\varsigma,\varpi)\to\sup_{\mathfrak{e}\in[0,1]}\frac{|\varsigma'(\mathfrak{e})+\varpi(\eta)|}{2}.\end{aligned}$$

Note that the double controlled $\mathfrak{m}_\mathfrak{b}$-metric space $(\mathtt{G},\Delta_\perp,\perp)$ is already as shown in Example 3.1.

Theorem 4.1. *Suppose that for all $\varsigma,\varpi\in\mathtt{G}$,*

1.

$$\begin{aligned}|\Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e}))+\Lambda(\mathfrak{e},\eta,\varpi(\mathfrak{e}))| &\le \sup_{\mathfrak{e}\in[0,1]}(\{|\varsigma'(\mathfrak{e})(\varpi(\mathfrak{e})|)\})(|\varsigma'(\mathfrak{e})+\varpi(\mathfrak{e})|)\\ &\le \Big(\sup_{\mathfrak{e}\in[0,1]}(\{|\varsigma'(\mathfrak{e})(\varpi(\mathfrak{e})|)\})(|\varsigma'(\mathfrak{e})+\varpi(\mathfrak{e})|)+\Omega_{|\varsigma(\mathfrak{e})\varpi(\mathfrak{e})|}\Big),\end{aligned}$$

where $\rho\in\Big[0,\dfrac{1}{(1+\sup_{\mathfrak{e},\eta}|\Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e}))+\Lambda(\mathfrak{e},\eta,\varpi(\mathfrak{e}))|)^2}\Big]$, *and for some* $\mathfrak{e}\in[0,1]$

2.

$$\Lambda\Big(\mathfrak{e},\eta,\int_0^1 \Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e}))\eth\eta\Big) \leq \Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e})),\ \forall \ \ \mathfrak{e},\eta\in[0,1]. \tag{4.2}$$

Then the equation (4.2) *has a unique solution.*

Proof. Let $\Omega\colon \mathtt{G}\to\mathtt{G}$ be defined by $\Omega\varsigma'(\mathfrak{e}) = \int_0^1 \Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e}))$. Then

$$\Delta_\perp(\Omega(\varsigma),\Omega(\varpi)) = \sup_{\mathfrak{e}\in[0,1]}\left(\frac{|\Omega(\varsigma')+\Omega(\varpi)|}{2}\right).$$

So we get

$$\begin{aligned}
\frac{|\Omega(\varsigma')+\Omega(\varpi)|}{2} &= \frac{|\int_0^1 \Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e}))\eth\eta| + |\int_0^1 \Lambda(\mathfrak{e},\eta,\varpi(\mathfrak{e}))\eth\eta|}{2}\\
&\leq \frac{\int_0^1 |\Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e}))|\eth\eta + \int_0^1 |\Lambda(\mathfrak{e},\eta,\varpi(\mathfrak{e}))|\eth\eta}{2}\\
&= \frac{\int_0^1 \big(|\Lambda(\mathfrak{e},\eta,\varsigma'(\mathfrak{e}))| + |\Lambda(\mathfrak{e},\eta,\varpi(\mathfrak{e}))|\big)\eth\eta}{2}\\
&\leq \frac{\int_0^1 \sup_{\mathfrak{e}\in[0,1]}(\{\varsigma'(\mathfrak{e}|(\varpi(\mathfrak{e})))\})(|\varsigma'(\mathfrak{e})+\varpi(\mathfrak{e})|)\eth\eta}{2}\\
&\leq \frac{\int_0^1 \rho(\sup_{\mathfrak{e}\in[0,1]}(\{\varsigma'(\mathfrak{e}|(\varpi(\mathfrak{e})))\})(|\varsigma'(\mathfrak{e})+\varpi(\mathfrak{e})|))\eth\eta}{2}\\
&\quad + \rho(\Omega_{|\varsigma(\mathfrak{e})\varpi(\mathfrak{e})|}) \text{ by (1)},\\
\frac{|\Omega(\varsigma)+\Omega(\varpi)|}{2} &\leq \rho(\Delta_\perp(\varsigma,\varpi)+\Delta_{\perp_{\varsigma,\varpi}}),\\
\Delta_\perp(\Omega\varsigma,\Omega\varpi) &\leq \rho(\Delta_\perp(\varsigma,\varpi)+\Delta_{\perp_{\varsigma,\varpi}}).
\end{aligned}$$

Furthermore, let $\varrho\in\mathbb{N}$, and $\varsigma\in\mathtt{G}$. Then

$$\begin{aligned}
\Omega^\varrho\varsigma(\mathfrak{e}) &= \Omega(\Omega^{\varrho-1}\varsigma'(\mathfrak{e}))\\
&= \int_0^1 \Lambda(\mathfrak{e},\eta,\Omega^{\varrho-1}\varsigma'(\mathfrak{e}))d\eta\\
&= \int_0^1 \Lambda(\mathfrak{e},\eta,\Omega(\Omega^{\varrho-2}\varsigma'(\mathfrak{e})))d\eta\\
&\leq \int_0^1 \Lambda(\mathfrak{e},\eta,\int_0^1 \Lambda(\mathfrak{e},\eta,\Omega^{\varrho-2}\varsigma'(\mathfrak{e})))\eth\eta\\
&\leq \int_0^1 \Lambda(\mathfrak{e},\eta,\Omega^{\varrho-2}(\varsigma'(\mathfrak{e})))\eth\eta\\
&= \Omega^{\varrho-1}\varsigma'(\mathfrak{e}).
\end{aligned}$$

Thus, for all $\mathfrak{e}\in[0,1]$, we find that the sequence $\{\Omega^\varrho\varsigma'(\mathfrak{e})\}$ converges to ℓ. Since $\Omega^\varrho\to\ell$ is monotonic, by using Denis theorem, we obtain that $\Delta_\perp(\Omega(\varsigma),\Omega(\varpi)) = \sup_{\mathfrak{e}\in[0,1]}\left(\frac{|\Omega_{\varsigma(\mathfrak{e})}+\Omega_{\varpi(\mathfrak{e})}|}{2}\right)\to\ell$. Thus, all the axioms of Theorem 3.1 are fulfilled and the equation (4.1) has a unique solution. $\square$

Example 4.1. Given an integral equation as follows:

$$f(t) = \int_0^t \cos(t-s)\kappa(s)ds = t\sin(t). \tag{4.3}$$

Now, we are going to find the approximate solution of this integral equation under this metric. Then, we will find the relative error between our approximate solution and the analytical solution as follows:

Proof. From (4.3), the exact solution is $\kappa(t) = 2\sin(t)$, for $0 \leq t < 1$. Table 1 shows the numerical values.

Table 1. Comparison of $\kappa(t)$ and approximation solutions.

t	$\kappa(t)$	approximation solution (m=64)	approximation solution (m=128)
0.0	0	0.010417	0.005208
0.1	0.199667	0.197570	0.192399
0.2	0.397339	0.382942	0.398412
0.3	0.591040	0.605205	0.589930
0.4	0.778837	0.781174	0.785758
0.5	0.958851	0.967335	0.963098
0.6	1.129285	1.126666	1.122812
0.7	1.288435	1.276056	1.289847
0.8	1.434712	1.446451	1.433200
0.9	1.566654	1.569934	1.572171

The graph below combines analytical and numerical results. Figure 1 and Figure 2 show

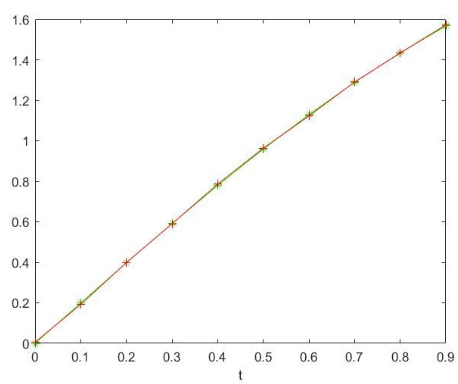

Figure 1. Graph of $f(t)$ (m = 64) compared to $\kappa(t)$ for (h=0.1).

that the error between $f(t)$ and $\kappa(t)$ is also relatively very small. □

Example 4.2. Consider the integral equation

$$f(s) = \int_0^1 K(s,t)x(t)\mathfrak{d}\mathfrak{t}, \quad 0 \leq s \leq 1,$$

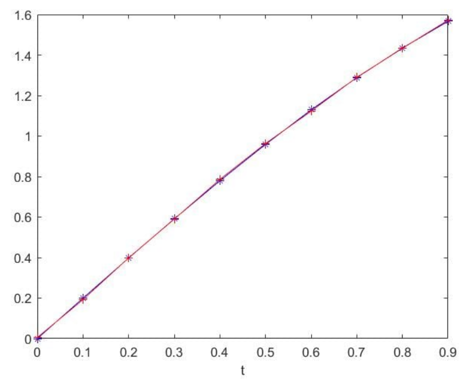

Figure 2. Graph of $f(t)$ (m = 128) compared to $\kappa(t)$ for h=0.1.

where $K(s,t) = e^{s^2 t}$ and $f(s) = \frac{e^{(s^2+1)}-1}{(s^2+1)}$, with the exact solution $x(s) = e^s$. Consider Table 2

Table 2. Numerical results for a test problem, using the proposed method.

s	$\mathfrak{f}(\mathfrak{s})$	$\mathfrak{x}(\mathfrak{s})$	Error
0.100	1.727	1.105	0.622
0.200	1.758	1.221	0.537
0.300	1.811	1.349	0.462
0.400	1.887	1.491	0.396
0.500	1.992	1.648	0.344
0.600	2.129	1.822	0.307
0.700	2.306	2.013	0.293
0.800	2.533	2.225	0.308
0.900	2.823	2.459	0.364
1.000	3.194	2.718	0.476

shows that the error of $f(s)$ compared to the solution $x(s)$ is relatively different.

5. Conclusion

In this article, we proved some fixed point theorems for an orthogonal complete DCOMMS. The derived results have been supplemented with suitable trivial and non-trivial examples. We have also presented an application to find the solution of an integral equation. The solved analytical results have been stimulated with related numerical results. It is an open problem to generalize and extend to the derived results by using other contractive conditions.

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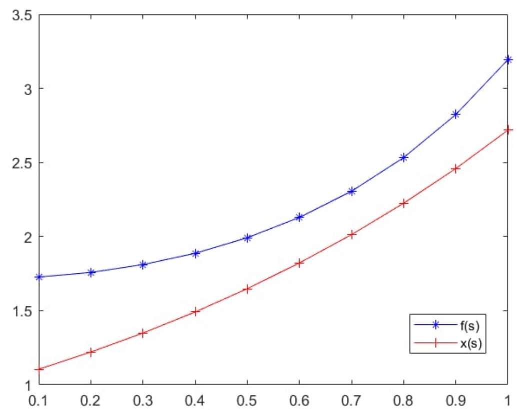

Figure 3. Graph of $f(s)$ compared to $x(s)$ for h=0.1.

Competing interests. The authors declare that they do not have any competing interests. All authors read and approved the final manuscript.

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