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LOCAL WELL-POSEDNESS IN CRITICAL BESOV SPACES, BLOW-UP ANALYSIS, AND GLOBAL EXISTENCE OF SOLUTIONS FOR THE B-FAMILY FOKAS-OLVER-ROSENAU-QIAO EQUATIONS

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Abstract This work establishes the local well-posedness of the b-family Fokas-Olver-Rosenau-Qiao (bFORQ) equations in the critical Besov space $B^{5/2}_{2,1}(\mathbb{R})$. We then analyze two special cases: For $b = 1$, we prove finite-time blow-up of solutions with the explicit blow-up time $T^* \leq -1/(\epsilon u_{0x}(x_1)m_0(x_1))$; for $b = 0$, we show the existence of global solutions in time.

Keywords Local well-posedness, Besov space, blow-up analysis, global existence, b-family of FORQ equations.

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1. Introduction

The b-family of the Camassa-Holm equations

$$\begin{cases} m_t + um_x + bu_x m = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ m = (1-\partial_x^2)u, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \end{cases} \tag{1.1}$$

where $b \in \mathbb{R}$ denotes a dimensionless parameter and $u(t,x)$ denotes the fluid velocity field, constitutes a family of nonlinear dispersive partial differential equations. This system, introduced by Holm and Staley [13], was established to investigate stability exchange in solitary wave interactions within shallow water regimes. The governing equations unify multiple integrable models in hydrodynamics through the parameter b: When $b = 2$, the system reduces to the celebrated Camassa-Holm (CH) equation, renowned for its peakon solutions and complete integrability. Similarly, the case $b = 3$ corresponds to the Degasperis-Procesi (DP) equation, another integrable model exhibiting nonlinear wave phenomena. These specializations highlight the critical role of the parameter b in modulating nonlinearity and dispersion characteristics across shallow water wave theory.

The Camassa-Holm (CH) equations

$$\begin{cases} m_t + um_x + 2u_x m = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ m = (1-\partial_x^2)u, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \end{cases} \tag{1.2}$$

originated in the mathematical framework of hereditary symmetries developed by Fuchssteiner and Fokas [12]. This nonlinear dispersive PDE gained prominence when Camassa and Holm [2]

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rigorously derived it in 1993 as a model for unidirectional shallow water wave propagation over flat topography. As a completely integrable system, the CH equation exhibits an infinite hierarchy of conservation laws [19, 20]. Its solution behavior in Besov spaces has attracted particular analytical attention. The well-posedness theory for (1.2) was systematically established by Danchin [7], who proved existence and uniqueness of solutions for initial data $u_0 \in B^s_{p,r}$ with

$$s > \max\left\{1 + \frac{1}{p}, \frac{3}{2}\right\} \quad \text{and} \quad (p, r) \in [1, \infty] \times [1, \infty).$$

Danchin [8] extended these results to the critical Besov space $B^{\frac{3}{2}}_{2,1}$, demonstrating well-posedness in the sense of Hadamard through continuous dependence on initial data. In [16], McKean established necessary and sufficient conditions for the blow-up scenario. These conditions depend solely on the geometric configuration of initial data, without requiring any constraints on their smoothness. Specifically, let $y_0 = (1 - \partial_x^2)u_0$, the CH equation (1.2) breaks if and only if some portion of the positive part of y_0 lies to the left of some portion of its negative part. McKean provided the proof via contradiction in [17], showing that global existence of solutions satisfying these conditions would lead to a contradiction. Nevertheless, this approach offered a limited understanding of the equation's intrinsic analytical structure. Subsequent work by Zhou [26] and Jiang et al. [14] offered analytical proofs that further indicated the structural properties underlying the blow-up mechanism.

The Degasperis-Procesi (DP) equations

$$\begin{cases} m_t + um_x + 3u_x m = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ m = (1 - \partial_x^2)u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \end{cases} \tag{1.3}$$

originated in the mathematical physics framework through the seminal work of Degasperis and Procesi [10]. This third-order nonlinear dispersive partial differential equation shares fundamental integrability characteristics with the Camassa-Holm equation, notably possessing a bi-Hamiltonian structure [9] that underpins its complete integrability. The analytical theory for (1.3) has been substantially developed through rigorous analysis: Yin [23, 24] established its well-posedness in Sobolev and Besov spaces via refined energy estimates. In particular, Yin characterized in [24] the blow-up mechanism precisely: For initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, the corresponding strong solution u blows up in finite time $T^* < +\infty$ if and only if

$$\liminf_{t \uparrow T^*} \left(\inf_{x \in \mathbb{R}} u_x(t, x) \right) = -\infty. \tag{1.4}$$

Furthermore, Zhao and Yan [25] established a quantitative estimate for the maximal existence time by proving that the strong solution u exhibits finite-time blow-up at precisely T^*, where the blow-up time satisfies the upper bound

$$T^* \leq T_0 := \frac{2}{3\|u_0\|_{L^\infty(\mathbb{R})}}. \tag{1.5}$$

For other properties of the solutions to the DP equation, we refer the reader to [4, 11].

For general values of the parameter $b \neq 0$, the b-family equation (1.1) does not represent a completely integrable system. The foundational well-posedness theory was established by Zhou [27], who proved that for initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, there exists a maximal time $T = T(\|u_0\|_{H^s}) > 0$ yielding a unique solution $u \in \mathcal{C}\big([0, T); H^s(\mathbb{R})\big) \cap C^1\big([0, T); H^{s-1}(\mathbb{R})\big)$.

Moreover, equation (1.1) exhibits an identical blow-up scenario to that of (1.2). For the case where $b \in [2,3]$ and the initial data $u_0 \in H^2(\mathbb{R})$ satisfies the following conditions at some point $x_0 \in \mathbb{R}$ with $m_0 := (1-\partial_x^2)u_0$:

(1) $m_0(x_0) = 0$,

(2) $m_0(x) \geq 0$ ($\neq 0$) for all $x < x_0$,

(3) $m_0(x) \leq 0$ ($\neq 0$) for all $x > x_0$,

the solution experiences finite-time blow-up with an upper bound on its lifespan given by

$$T^* \leq \max\left\{-\frac{4}{u_{0x}(x_0)}, -\frac{u_{0x}(x_0)}{(u_{0x}^2 - u_0^2)(x_0)}\right\}. \tag{1.6}$$

For the well-posedness theory, we refer to the foundational work of [3, 21], which provides comprehensive analyses and additional references on this topic.

This paper investigates the following bFORQ equations

$$\begin{cases} m_t + (u^2 - u_x^2)m_x + bu_x m^2 = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ m = (1-\partial_x^2)u, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \end{cases} \tag{1.7}$$

The local well-posedness theory for this system was developed by Yang et al. [22]. Their main result states that for initial data $m_0 \in B^s_{p,r}(\mathbb{R})$ with parameters satisfying

$$1 \leq p, r \leq \infty, \quad s > \max\left\{\frac{1}{2}, \frac{1}{p}\right\},$$

there exists a maximal time $T > 0$ such that the Cauchy problem (1.7) admits a unique solution m in the following spaces

$$E^s_{p,r}(T) := C([0,T], B^s_{p,r}(\mathbb{R})) \cap C^1([0,T], B^{s-1}_{p,r}(\mathbb{R})), \quad r < \infty,$$
$$E^s_{p,\infty}(T) := L^\infty([0,T], B^s_{p,\infty}(\mathbb{R})) \cap \mathrm{Lip}([0,T], B^{s-1}_{p,\infty}(\mathbb{R})).$$

Notably, the critical case $p = r = 2$ recovers the classical Sobolev space framework: $B^s_{2,2}(\mathbb{R}) \cong H^s(\mathbb{R})$. This yields local well-posedness for initial data $m_0 \in H^s(\mathbb{R})$ with $s > \frac{1}{2}$, corresponding to solutions $u_0 \in H^s(\mathbb{R})$ when $s > \frac{5}{2}$. Moreover, they established the following blow-up scenario: Let $b \in \mathbb{R}$, $m_0 \in H^s(\mathbb{R})$ with $s > \frac{1}{2}$ and $T > 0$ be the maximal existence time of strong solution m to the Cauchy problem (1.7). Then the solution m blows up in finite time if and only if

$$\liminf_{t \to T} \inf_{x \in \mathbb{R}} m u_x(t,x) = -\infty.$$

Furthermore, they require the solution to satisfy the estimate $\|u(t)\|_{L^\infty(\mathbb{R})} \leq Ce^{Ct}$ along with additional conditions, which determine the precise blow-up time. We refer the reader to the complete statement of the theorem for a detailed argument. In [28], Zhu et al. conducted a study of the solutions to (1.7). Specifically, they discussed the persistence property in weighted Sobolev spaces and established infinite propagation speed. To the best of our knowledge, the well-posedness in critical Besov spaces remains unresolved. In addition, since equation (1.7) lacks conservation laws, the question of global existence for its solutions remains open.

The structure of this paper is organized as follows. In Section 2, the foundational tools from Littlewood-Paley theory are presented. In Section 3, the local well-posedness result in the critical Besov space $B^{5/2}_{2,1}(\mathbb{R})$ is established. Section 4 is devoted to proving the finite-time blow-up phenomenon for solutions of equation (1.7) with parameter $b = 1$. In Section 5, the existence of global-in-time solutions is shown for the case $b = 0$.

2. Preliminaries

For readers' convenience, we briefly review some fundamental results on the Littlewood-Paley theory and Besov spaces in [1,7].

For every $u \in \mathcal{S}'(\mathbb{R})$, the nonhomogeneous dyadic blocks Δ_j are defined as follows

$$\Delta_j u = \begin{cases} 0, & \text{if } j \leq 2, \\ \chi(D)u = \mathcal{F}^{-1}(\chi \mathcal{F}u), & \text{if } j = -1, \\ \phi(2^{-j}D)u = \mathcal{F}^{-1}(\phi(2^{-j\cdot})\mathcal{F}u), & \text{if } j \geq 0. \end{cases}$$

The nonhomogeneous low-frequency cut-off operators S_j is defined by

$$S_j u = \sum_{j' \leq j-1} \Delta_{j'} u.$$

Definition 2.1 (Besov space [1,7])**.** Given $1 \leq p,\ r \leq \infty$ and $s \in \mathbb{R}$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R})$ is defined for $s < \infty$ as follows

$$B^s_{p,r}(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}}(\mathbb{R}) := \begin{cases} \Big(\sum\limits_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j f\|^r_{L^p} \Big)^{\frac{1}{r}}, & r < \infty, \\ \sup\limits_{j \in \mathbb{Z}} 2^{js} \|\Delta_j f\|_{L^p}, & r = \infty. \end{cases}$$

If $s = \infty$, $B^\infty_{p,r}(\mathbb{R}) := \cap_{s \in \mathbb{R}} B^s_{p,r}(\mathbb{R})$.

Lemma 2.1 ([1,7])**.** *Let $s \in \mathbb{R}, 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$.*

(1) If $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B^s_{p_1,r_1}(\mathbb{R}) \hookrightarrow B^{s-(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R})$. If $s_1 < s_2$ or $s_1 = s_2, r_1 \leq r_2$, then the embedding $B^{s_1}_{p,r_1}(\mathbb{R}) \hookrightarrow B^{s_2}_{p,r_2}(\mathbb{R})$ is locally compact.

(2) For $s > 0, B^s_{p,r}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is an algebra. Moreover, $B^s_{p,r}(\mathbb{R})$ is a Banach algebra $\Leftrightarrow$ $B^s_{p,r}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \Leftrightarrow s > \frac{1}{p}$ (or $s \geq \frac{1}{p}$ and $r = 1$).

(3) If s_1 and s_2 are real numbers such that $s_1 < s_2, \theta \in (0,1)$ and for any $(p,r) \in [1,+\infty]^2$, then we have

$$\|u\|_{B^{\theta s_1 + (1-\theta)s_2}_{p,r}(\mathbb{R})} \leq \|u\|^\theta_{B^{s_1}_{p,r}(\mathbb{R})} \|u\|^{1-\theta}_{B^{s_2}_{p,r}(\mathbb{R})},$$

and

$$\|u\|_{B^{\theta s_1 + (1-\theta)s_2}_{p,1}(\mathbb{R})} \leq \frac{C}{s_2 - s_1}\Big(\frac{1}{\theta} + \frac{1}{1-\theta}\Big) \|u\|^\theta_{B^{s_1}_{p,\infty}(\mathbb{R})} \|u\|^{1-\theta}_{B^{s_2}_{p,\infty}(\mathbb{R})},$$

where $C > 0$ is a constant independent of u.

(4) If $s \in \mathbb{R}$, $p \in [1,+\infty]$, $\epsilon > 0$, there exists a constant $C = C(\epsilon) > 0$ such that

$$\|f\|_{B^s_{p,1}(\mathbb{R})} \leq C \|f\|_{B^s_{p,\infty}(\mathbb{R})} \ln\Big(e + \frac{\|f\|_{B^{s+\epsilon}_{p,\infty}(\mathbb{R})}}{\|f\|_{B^s_{p,\infty}(\mathbb{R})}}\Big), \quad \forall f \in B^{s+\epsilon}_{p,\infty}(\mathbb{R}).$$

Now we state some useful results in the theory of transport equations, which are crucial to the proofs of our main theorems.

Lemma 2.2 ([1,7]). *Let $1 \leq p, r \leq \infty$ and $s > -\min\{\frac{1}{p}, 1-\frac{1}{p}\}$. Assume that $f_0 \in B^s_{p,r}, F \in L^1([0,T]; B^s_{p,r})$ and $\partial_x v \in L^1([0,T]; B^{s-1}_{p,r})$ if $s > 1+\frac{1}{p}$ or $\partial_x v \in L^1([0,T]; B^{1/p}_{p,r} \cap L^\infty)$ otherwise. If $f \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ solves the linear transport equation*

$$\begin{cases} \partial_t f + v f_x = F, & t > 0, x \in \mathbb{R}, \\ f|_{t=0} = f_0, & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Then there exists a constant C depending only on s, p, r, such that the following inequalities hold:
(1) *If $r = 1$ or $s \neq 1 + \frac{1}{p}$. Then, the solution f satisfies the following inequalities:*

$$\|f\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}}\, d\tau + C \int_0^t V'(\tau)\|f(\tau)\|_{B^s_{p,r}}\, d\tau, \tag{2.2}$$

and hence,

$$\|f\|_{B^s_{p,r}} \leq e^{CV(t)} \left(\|f_0\|_{B^s_{p,r}} + C \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B^s_{p,r}}\, d\tau \right), \tag{2.3}$$

where

$$V(t) = \begin{cases} \int_0^t \|\nabla v(\tau)\|_{B^{1/p}_{p,r} \cap L^\infty}\, d\tau & \text{if } s < 1 + \frac{1}{p}, \\ \int_0^t \|\nabla v(\tau)\|_{B^{s-1}_{p,r}}\, d\tau & \text{otherwise.} \end{cases}$$

(2) *Let $s \leq 1 + \frac{1}{p}$. Assume the following regularity conditions hold:*

$$\nabla f_0 \in L^\infty(\mathbb{R}), \quad \nabla f \in L^\infty([0,T] \times \mathbb{R}), \quad \nabla F \in L^1([0,T]; L^\infty(\mathbb{R})).$$

Then, the solution f satisfies the following inequality:

$$\begin{aligned} \|f\|_{B^s_{p,r}} + \|\nabla f(t)\|_{L^\infty} \leq e^{CV(t)} \Big(& \|f_0\|_{B^s_{p,r}} + \|\nabla f_0\|_{L^\infty} \\ & + \int_0^t e^{-CV(\tau)} \left(\|F(\tau)\|_{B^s_{p,r}} + \|\nabla F(\tau)\|_{L^\infty} \right) d\tau \Big), \end{aligned} \tag{2.4}$$

where

$$V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{1/p}_{p,r} \cap L^\infty}\, d\tau.$$

(3) *If $r < +\infty$, then $f \in C([0,T]; B^s_{p,r})$. If $r = \infty$, then $f \in C([0,T]; B^{s'}_{p,r})$ for all $s' < s$.*

Lemma 2.3 ([1]). *Let p, r, s, f_0, F be as in the statement of Lemma 2.2. Assume that $v \in L^\rho([0,T]; B^{-M}_{\infty,\infty})$ for some $\rho > 1, M > 0$. Suppose that one of the following holds:*
(1) *$v_x \in L^1([0,T]; B^{1/p}_{p,\infty} \cap L^\infty)$, if $s < 1 + 1/p$,*
(2) *$v_x \in L^1([0,T]; B^{s-1}_{p,r})$, if $s > 1 + 1/p$ or ($s = 1 + 1/p$ and $r = 1$).*
Then the transport equations (2.1) has a unique solution $f \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; B^{s'}_{p,1})$ for any $s' < s$ and the inequalities of Lemma 2.2 hold true. Moreover if $r < \infty$, then $f \in C([0,T]; B^s_{p,r})$.

Lemma 2.4 (1-D Moser-type estimates [6,7]). *Assume that* $(p,r)\in[1,+\infty]^2$, *then*
(1) *For* $s>0$, *there exists* $C=C(s)$ *suth that* $\|fg\|_{B^s_{p,r}}\le C\big(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty}+\|f\|_{L^\infty}\|g\|_{B^s_{p,r}}\big)$.
(2) *For all* $s_1\le\frac{1}{p}$, $s_2>\frac{1}{p}$ $(s_2\ge\frac{1}{p}$ *if* $r=1)$ *and* $s_1+s_2>0$, *there exists* $C=C(s_1,s_2,p,r)$

$$\|fg\|_{B^{s_1}_{p,r}}\le C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}.$$

Lemma 2.5 (Osgood Lemma [1,5]). *Let* ρ *be a measurable function from* $[t_0,T]$ *to* $[0,c]$, γ *be a locally integrable function from* $[t_0,T]$ *to* $\mathbb{R}^+$, *and* μ *be a continuous and nondecreasing function from* $[0,c]$ *to* $\mathbb{R}^+$. *Assume that, for some nonnegative real number* a, *the function* ρ *satisfies*

$$\rho(t)\le a+\int_{t_0}^t\gamma(t')\mu(\rho(t'))dt',\quad t\in[t_0,T].$$

If $a>0$, *then for* $t\in[t_0,T]$, *we have*

$$-\mathcal{M}(\rho(t))+\mathcal{M}(a)\le\int_{t_0}^t\gamma(t')dt',\quad \textit{with } \mathcal{M}(x)=\int_x^c\frac{dr}{\mu(r)}.$$

If $a=0$ *and* μ *satisfies the condition* $\int_0^1\frac{dr}{\mu(r)}=+\infty$, *then* $\rho=0$.

Lemma 2.6 ([1]). *Let* $v_0\in B^{\frac12}_{2,1}(\mathbb{R})$, *and* $f\in L^1([0,T];B^{\frac12}_{2,1}(\mathbb{R}))$. *Suppose there exists* $\beta\in L^1(0,T)$ *such that*

$$\sup_{n\in\mathbb{N}}\|\partial_x a^{(n)}(t)\|_{B^{\frac12}_{2,1}}\le\beta(t),$$

and $a^{(n)}\to a^{(\infty)}$ *in* $L^1([0,T];B^{\frac12}_{2,1}(\mathbb{R}))$ *as* $n\to\infty$. *Let* $\{v^{(n)}\}_{n\in\mathbb{N}\cup\{\infty\}}\in C([0,T];B^{\frac12}_{2,1})$ *be the solution to*

$$\begin{cases}\partial_t v^{(n)}+a^{(n)}\partial_x v^{(n)}=f, & x\in\mathbb{R},\, t\in[0,T],\\ v^{(n)}|_{t=0}=v_0,\end{cases}\tag{2.5}$$

then the corresponding solutions satisfy

$$v^{(n)}\to v^{(\infty)}\quad \textit{in}\ \ C([0,T];B^{\frac12}_{2,1}(\mathbb{R})).$$

Lemma 2.7 (Fatou-type lemma [8]). *If* $\{u^k\}_{k\in\mathbb{N}}$ *is a bounded sequence in* $B^s_{p,r}$, *which tends to* u *in* $\mathcal{S}'$, *then* $u\in B^s_{p,r}$, *and*

$$\|u\|_{B^s_{p,r}}\le\liminf_{k\to\infty}\|u^{(k)}\|_{B^s_{p,r}}.$$

3. Local well-posedness

This section is devoted to establishing the local well-posedness of the Cauchy problem (1.7) in the critical Besov spaces.

We begin by reformulating the original system (1.7) as follows:

$$\partial_t u+\big(u^2-\frac13u_x^2\big)u_x+p*F(u)+\partial_x p*G(u)=0,\tag{3.1}$$

where

$$F(u)=\frac{3b-5}{3}u_x^3+(b-2)u_xu_{xx}^2,\ G(u)=\frac{b}{3}u^3+(3-b)uu_x^2,$$

and $p(x) = \frac{1}{2}e^{-|x|}$, denote the fundamental solution of $1-\partial_x^2$ on $\mathbb{R}$. Furthermore, we define the convolution operators p_+ and p_- as

$$p_+ * f(x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{y} f(y)\,dy,$$
$$p_- * f(x) = \frac{e^{x}}{2}\int_{x}^{\infty} e^{-y} f(y)\,dy,$$

which satisfy the fundamental relations:

$$p = p_+ + p_-, \quad \partial_x p = p_- - p_+.$$

We introduce the following spaces:

Definition 3.1. Given $T > 0, s \in \mathbb{R}$ and $p \in [1, +\infty]$, we define

$$E^s_{p,r}(T) := C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}), \quad r < \infty,$$
$$E^s_{p,\infty}(T) := L^\infty([0,T]; B^s_{p,\infty}) \cap Lip([0,T]; B^{s-1}_{p,\infty}),$$

and

$$E^s_{p,r} := \cap_{T>0} E^s_{p,r}(T).$$

Theorem 3.1. *Suppose the initial data $u_0(x) \in B^{5/2}_{2,1}$. Then there exist a $T > 0$, which depends only on $\|u_0\|_{B^{5/2}_{2,1}}$, and a unique solution $u(t,x)$ to the Cauchy problem (1.7) such that*

$$u = u(\cdot, u_0) \in E^{5/2}_{2,1}(T).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u:\ B^{5/2}_{2,1} \longrightarrow E^{5/2}_{2,1}(T),$$
$$u_0 \longmapsto u(\cdot, u_0)$$

is continuous.

Remark 3.1. The momentum equation in (1.7) admits the transport formulation

$$m_t + \left(u^2 - u_x^2\right) m_x = -b u_x m^2.$$

Roughly speaking, to propagate the regularity of the solution m to the Cauchy problem (1.7) in terms of its initial data m_0, the coefficient $u^2 - u_x^2$ of m_x needs to satisfy the Lipschitz condition. This necessitates $u \in W^{2,\infty}$, achieved through the embedding chain

$$B^{\frac{5}{2}}_{2,1} \hookrightarrow B^2_{\infty,1} \hookrightarrow W^{2,\infty} \hookrightarrow B^2_{\infty,\infty}.$$

Consequently, the critical regularity threshold emerges at $s = \frac{5}{2}$, as evidenced by the space hierarchy:

$$H^s \hookrightarrow B^{\frac{5}{2}}_{2,1} \hookrightarrow H^{\frac{5}{2}} \hookrightarrow B^{\frac{5}{2}}_{2,\infty} \hookrightarrow H^{s'} \quad \forall s' < \frac{5}{2} < s.$$

The proof of Theorem 3.1 relies on the following three key lemmas.

Lemma 3.1. *Let $u_0 \in B^{\frac{5}{2}}_{2,1}(\mathbb{R})$. There exists a time $T > 0$ depending only on $\|u_0\|_{B^{\frac{5}{2}}_{2,1}}$, such that the Cauchy problem* (1.7) *admits a solution $u \in E^{\frac{5}{2}}_{2,1}(T)$.*

Proof. We construct a solution of (1.7) via an iterative scheme. Initialize with $u^{(0)} = 0$ and consider the sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ generated by solving the linearized transport equations:

$$\begin{cases} \partial_t m^{(n+1)} + \big((u^{(n)})^2 - (u_x^{(n)})^2\big) \cdot \partial_x m^{(n+1)} = -bu_x^{(n)}(m^{(n)})^2, & t > 0\ x \in \mathbb{R}, \\ u^{(n+1)}|_{t=0} = u_0^{(n+1)}(x) = S_{n+1}u_0, & x \in \mathbb{R}. \end{cases} \tag{3.2}$$

Now we prove that there exists a $T > 0$ such that
(1) $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $E^{5/2}_{2,1}(T)$,
(2) $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; B^{3/2}_{2,1})$.

Step 1. We first prove $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $B^{5/2}_{2,1}$.

Since

$$\begin{aligned}
&\|m_0^{(n+1)}\|_{B^\infty_{2,1}} \le C\|u_0^{(n+1)}\|_{B^\infty_{2,1}} \le C\|S_{n+1}u_0\|_{B^\infty_{2,1}} \le C\|u_0\|_{B^\infty_{2,1}}, \\
&\|(u^{(n)})^2 - (u_x^{(n)})^2\|_{B^{-1/4}_{\infty,\infty}} \le \|(u^{(n)})^2 - (u_x^{(n)})^2\|_{B^{1/4}_{2,1}} \le \|(u^{(n)})\|^2_{B^\infty_{2,1}}, \\
&\|bu_x^{(n)}(m^{(n)})^2\|_{B^\infty_{2,1}} \le C\|u^{(n)}\|^3_{B^\infty_{2,1}}, \\
&\|\nabla\big((u^{(n)})^2 - (u_x^{(n)})^2\big)\|_{B^\infty_{2,1}} \le C\|u^{(n)}\|^2_{B^\infty_{2,1}},
\end{aligned}$$

from Lemma 2.3 with $p = 2$, $r = 1$, $s = \infty$, $M = 1/4$, we obtain by induction that for all $n \ge 1$, (3.2) has a global solution $u^{(n+1)} \in C([0,T]; B^\infty_{2,1})$, which implies $\|u^{(n+1)}\|_{B^{5/2}_{2,1}}$ is uniformly bounded.

Applying Lemma 2.1, we obtain the following estimates:

$$\|m_0^{(n+1)}\|_{B^{1/2}_{2,1}} \le C\|u_0^{(n+1)}\|_{B^{5/2}_{2,1}} \le C\|S_{n+1}u_0\|_{B^{5/2}_{2,1}} \le C\|u_0\|_{B^{5/2}_{2,1}}, \tag{3.3}$$

$$\|bu_x^{(n)}(m^{(n)})^2\|_{B^{1/2}_{2,1}} \le C\|u_x^{(n)}\|_{L^\infty}\|(m^{(n)})^2\|_{B^{1/2}_{2,1}} + C\|u_x^{(n)}\|_{B^{1/2}_{2,1}}\|m^{(n)}\|^2_{L^\infty} \le C\|u^{(n)}\|^3_{B^{5/2}_{2,1}}, \tag{3.4}$$

$$\|\nabla\big((u^{(n)})^2 - (u_x^{(n)})^2\big)\|_{B^{1/2}_{2,1}\cap L^\infty} \le \|\nabla\big((u^{(n)})^2 - (u_x^{(n)})^2\big)\|_{B^{1/2}_{2,1}} \le C\|u^{(n)}\|^2_{B^{5/2}_{2,1}}. \tag{3.5}$$

Applying the estimates in Lemma 2.2 to transport equation (3.2), we obtain:

$$\begin{aligned}
\|u^{(n+1)}\|_{B^{5/2}_{2,1}} \le \|m^{(n+1)}\|_{B^{1/2}_{2,1}} \\
&\le e^{C\int_0^t \|\nabla\big((u^{(n)})^2-(u_x^{(n)})^2\big)(\tau)\|_{B^{1/2}_{2,1}\cap L^\infty} d\tau}\|m_0^{(n+1)}\|_{B^{1/2}_{2,1}} \\
&\quad + C\int_0^t e^{C\int_s^t \|\nabla\big((u^{(n)})^2-(u_x^{(n)})^2\big)(\tau)\|_{B^{1/2}_{2,1}\cap L^\infty} d\tau}\|bu_x^{(n)}(m^{(n)})^2(s)\|_{B^{1/2}_{2,1}}ds\Big).
\end{aligned} \tag{3.6}$$

Substituting (3.3)-(3.5) into (3.6), we have

$$\|u^{(n+1)}\|_{B^{5/2}_{2,1}} \le e^{C\int_0^t \|u^{(n)}(\tau)\|^2_{B^{5/2}_{2,1}}d\tau}\|u_0\|_{B^{5/2}_{2,1}} + C\int_0^t e^{C\int_s^t \|u^{(n)}(\tau)\|^2_{B^{5/2}_{2,1}}d\tau}\|u^{(n)}(s)\|^3_{B^{5/2}_{2,1}}ds. \tag{3.7}$$

Let

$$T := \frac{1}{8C\|u_0\|^2_{B^{5/2}_{2,1}}}.$$

Suppose by induction that for all $t \in [0, T]$, it holds that

$$\|u^{(n)}\|_{B^{5/2}_{2,1}} \le \frac{\|u_0\|_{B^{5/2}_{2,1}}}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} t)^{1/2}}. \tag{3.8}$$

It is obvious that (3.8) holds for $t = 0$. Plugging (3.8) into (3.7) yields

$$\begin{aligned}
\|u^{(n+1)}\|_{B^{5/2}_{2,1}} &\le \frac{\|u_0\|_{B^{5/2}_{2,1}}}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} t)^{1/4}} \\
&\quad + \frac{C}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} t)^{1/4}} \int_0^t (1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} s)^{1/4} \frac{\|u_0\|^3_{B^{5/2}_{2,1}}}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} s)^{3/2}} ds \\
&\le \frac{\|u_0\|_{B^{5/2}_{2,1}}}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} t)^{1/4}} \left(1 + C \int_0^t \frac{\|u_0\|^2_{B^{5/2}_{2,1}}}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} s)^{5/4}} ds \right) \\
&\le \frac{\|u_0\|_{B^{5/2}_{2,1}}}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} t)^{1/4}} \times \frac{1}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} t)^{1/4}} \\
&= \frac{\|u_0\|_{B^{5/2}_{2,1}}}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} t)^{1/2}},
\end{aligned}$$

which implies for $0 \le t \le T$ that

$$\|u^{(n+1)}\|_{B^{5/2}_{2,1}} \le \frac{\|u_0\|_{B^{5/2}_{2,1}}}{(1 - 4C\|u_0\|^2_{B^{5/2}_{2,1}} t)^{1/2}}.$$

Therefore, $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $C([0, T]; B^{5/2}_{2,1})$.

Step 2. We now establish that $\partial_t u^{(n+1)} \in C([0, T]; B^{3/2}_{2,1})$.

By (3.2), it suffices to bound the terms

$$\Big((u^{(n)})^2 - (u^{(n)}_x)^2\Big)\partial_x m^{(n+1)}, \ -b u^{(n)}_x (m^{(n)})^2 \tag{3.9}$$

in $B^{-1/2}_{2,1}$, we estimate the first term in (3.9). Using the Moser-type estimates (see Lemma 2.4 (2) with $s_1 = -1/2, \ s_2 = 1$), we get

$$\begin{aligned}
\|\Big((u^{(n)})^2 - (u^{(n)}_x)^2\Big)\partial_x m^{(n+1)}\|_{B^{-1/2}_{2,1}} &\le C\|\partial_x m^{(n+1)}\|_{B^{-1/2}_{2,1}} \|(u^{(n)})^2 - (u^{(n)}_x)^2\|_{B^1_{2,1}} \\
&\le C\|u^{(n+1)}\|_{B^{5/2}_{2,1}} \|u^{(n)}\|^2_{B^{5/2}_{2,1}}.
\end{aligned}$$

Similarly, using Lemma 2.4-(2) with the same indices, we obtain

$$\| -bu_x^{(n)}(m^{(n)})^2\|_{B^{-1/2}_{2,1}} \leq \|bu_x^{(n)}\|_{B^1_{2,1}}\|(m^{(n)})^2\|_{B^{-1/2}_{2,1}} \leq C\|u^{(n)}\|^3_{B^{5/2}_{2,1}}.$$

Combining these estimates with (3.2), we conclude that $\partial_t u^{(n+1)}$ is uniformly bounded in $C([0,T]; B^{3/2}_{2,1})$, which yields that the sequence $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $E^{5/2}_{2,1}(T)$.

Step 3. We now demonstrate that $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; B^{3/2}_{2,1})$.

From (3.1), for any $m, n \in \mathbb{N}$, we derive the following equation:

$$\partial_t(u^{(n+m+1)} - u^{(n+1)}) + \left((u^{(n)})^2 - \frac{1}{3}(u_x^{(n)})^2\right)\partial_x(u^{(n+m+1)} - u^{(n+1)}) = Q, \tag{3.10}$$

where the nonlinear term

$$\begin{aligned} Q = \partial_x u^{(n+m+1)}&\left[(u^{(n)})^2 - (u^{(n+m)})^2 + \frac{1}{3}\left((u_x^{(n+m)})^2 - (u_x^{(n)})^2\right)\right] \\ &+ p * \left(F(u^{(n)}) - F(u^{(n+m)})\right) + \partial_x p * \left(G(u^{(n)}) - G(u^{(n+m)})\right). \end{aligned}$$

The convolution terms admit the following explicit representations:

$$\begin{aligned} p * \left(F(u^{(n)}) - F(u^{(n+m)})\right) = p * &\left(\frac{3b-5}{3}\left((u_x^{(n)})^3 - (u_x^{(n+m)})^3\right)\right) \\ &+ p * \left((b-2)\left(u_x^{(n)}(u_{xx}^{(n)})^2 - u_x^{(n+m)}(u_{xx}^{(n+m)})^2\right)\right), \end{aligned}$$

and

$$\begin{aligned} \partial_x p * \left(G(u^{(n)}) - G(u^{(n+m)})\right) = \partial_x p * &\left[\frac{b}{3}\left((u^{(n)})^3 - (u^{(n+m)})^3\right)\right. \\ &\left. +(3-b)\left(u^{(n)}(u_x^{(n)})^2 - u^{(n+m)}(u_x^{(n+m)})^2\right)\right]. \end{aligned}$$

Applying Lemma 2.1 yields

$$\begin{aligned} \|\nabla((u^{(n)})^2 - \frac{1}{3}(u_x^{(n)})^2)\|_{B^{1/2}_{2,1}\cap L^\infty} &\leq \|\nabla((u^{(n)})^2 - \frac{1}{3}(u_x^{(n)})^2)\|_{B^{1/2}_{2,1}} \\ &\leq \|((u^{(n)})^2 - \frac{1}{3}(u_x^{(n)})^2)\|_{B^{3/2}_{2,1}} \\ &\leq \|((u^{(n)})^2\|_{B^{3/2}_{2,1}} + \|\frac{1}{3}(u_x^{(n)})^2)\|_{B^{3/2}_{2,1}} \\ &\leq C\|u^{(n)}\|_{L^\infty}\|u^{(n)}\|_{B^{3/2}_{2,1}} + C\|u_x^{(n)}\|_{L^\infty}\|u_x^{(n)}\|_{B^{3/2}_{2,1}} \\ &\leq \|u^{(n)}\|^2_{B^{5/2}_{2,1}}, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \|\nabla(u^{(n+m+1)} - u^{(n+1)})\|_{L^\infty} \leq &\|\nabla(u^{(n+m+1)} - u^{(n+1)})\|_{B^{1/2}_{2,1}} \\ \leq &\|(u^{(n+m+1)} - u^{(n+1)})\|_{B^{3/2}_{2,1}} \end{aligned} \tag{3.12}$$

$$\leq \|u^{(n+m+1)}\|_{B^{5/2}_{2,1}} + \|u^{(n+1)}\|_{B^{5/2}_{2,1}},$$

and

$$\|u_0^{(n+m+1)} - u_0^{(n+1)}\|_{B^{3/2}_{2,1}} + \|\nabla(u_0^{(n+m+1)} - u_0^{(n+1)})\|_{L^\infty} \leq C\|u_0^{(n+m+1)} - u_0^{(n+1)}\|_{B^{3/2}_{2,1}}.$$

By the definition of the cut-off operator S_{n+1}, the following inequality chain holds:

$$\begin{aligned}
\|u_0^{(n+m+1)} - u_0^{(n+1)}\|_{B^{3/2}_{2,1}} &= \|S_{n+m+1}u_0^{(n+m+1)} - S_{n+1}u_0^{(n+1)}\|_{B^{3/2}_{2,1}} \\
&= \sum_{k\geq -1} 2^{\frac{3}{2}k}\|\Delta_k \sum_{q=n+1}^{n+m} \Delta_q u_0\|_{L^2} \\
&\leq \sum_{n\leq k\leq m+n+1} 2^{\frac{3}{2}k}\Big(\|\Delta_k\Delta_{k-1}u_0\|_{L^2} + \|\Delta_k\Delta_k u_0\|_{L^2} + \|\Delta_k\Delta_{k+1}u_0\|_{L^2}\Big) \\
&\leq C\Big(2^{-n}\sum_{n\leq k\leq n+m+1} 2^{\frac{5}{2}k}\|\Delta_k u_0\|_{L^2}\Big) \\
&\leq C2^{-n}\|u_0\|_{B^{\frac{5}{2}}_{2,1}}, \qquad (3.13)
\end{aligned}$$

and

$$\|\nabla Q\|_{L^\infty} + \|Q\|_{B^{3/2}_{2,1}} \leq \|\nabla Q\|_{B^{1/2}_{2,1}} + \|Q\|_{B^{3/2}_{2,1}} \leq C\|Q\|_{B^{3/2}_{2,1}}.$$

Since the estimation process for each term in Q is similar, we only estimate one of them. Application of Lemma 2.1 produces

$$\begin{aligned}
&\|\partial_x p * \big(G(u^{(n)}) - G(u^{(n+m)})\big)\|_{B^{3/2}_{2,1}} \\
=&\left\|\partial_x p * \Big(\frac{b}{3}\big((u^{(n)})^3 - (u^{(n+m)})^3\big) + (3-b)\big(u^{(n)}(u_x^{(n)})^2 - u^{(n+m)}(u_x^{(n+m)})^2\big)\Big)\right\|_{B^{3/2}_{2,1}} \\
\leq&\left\|\frac{b}{3}\big((u^{(n)})^3 - (u^{(n+m)})^3\big) + (3-b)\big(u^{(n)}(u_x^{(n)})^2 - u^{(n+m)}(u_x^{(n+m)})^2\big)\right\|_{B^{1/2}_{2,1}} \\
\leq& C\|u^{(n+m)} - u^{(n)}\|_{B^{3/2}_{2,1}}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|\nabla Q\|_{L^\infty} + \|Q\|_{B^{3/2}_{2,1}} &\leq C\|u^{(n+m)} - u^{(n)}\|_{B^{3/2}_{2,1}} \\
&\leq C\|u^{(n+m)} - u^{(n)}\|_{B^{3/2}_{2,\infty}} \ln\Big(e + \frac{\|u^{(n+m)} - u^{(n)}\|_{B^{5/2}_{2,\infty}}}{\|u^{(n+m)} - u^{(n)}\|_{B^{3/2}_{2,\infty}}}\Big). \qquad (3.14)
\end{aligned}$$

Applying the estimates in Lemma 2.2 to transport equation (3.10) yields:

$$\begin{aligned}
&\|u^{(n+m+1)} - u^{(n+1)}\|_{B^{3/2}_{2,\infty}} + \|\nabla(u^{(n+m+1)} - u^{(n+1)})\|_{L^\infty} \\
\leq& e^{C\int_0^t \|\nabla((u^{(n)})^2 - \frac{1}{3}(u_x^{(n)})^2)(s)\|_{B^{1/2}_{2,1}\cap L^\infty} ds}\Big(\|u_0^{(n+m+1)} - u_0^{(n+1)}\|_{B^{3/2}_{2,1}} + \|\nabla(u_0^{(n+m+1)} - u_0^{(n+1)})\|_{L^\infty}\Big)
\end{aligned}$$

$$+\int_0^t e^{C\int_s^t \|\nabla((u^{(n)})^2-\frac{1}{3}(u_x^{(n)})^2)(\tau)\|_{B^{1/2}_{2,1}\cap L^\infty}d\tau}\big(\|Q\|_{B^{3/2}_{2,1}}+\|\nabla Q\|_{L^\infty}\big)(s)ds. \tag{3.15}$$

Substituting (3.11)-(3.14) into (3.15), we can obtain

$$\|u^{(n+m+1)}-u^{(n+1)}\|_{B^{3/2}_{2,\infty}}\le C2^{-n}+C\int_0^t\|u^{(n+m)}-u^{(n)}\|_{B^{3/2}_{2,\infty}}\ln\Big(e+\frac{\|u^{(n+m)}-u^{(n)}\|_{B^{5/2}_{2,\infty}}}{\|u^{(n+m)}-u^{(n)}\|_{B^{3/2}_{2,\infty}}}\Big).$$

We define the difference between successive approximations as

$$\omega_{n,m}(t):=\|u^{(n+m)}-u^{(n)}\|_{B^{3/2}_{2,\infty}}.$$

Hence, we obtain the estimate

$$\omega_{n+1,m}(t)\le C2^{-n}+C\int_0^t\omega_{n,m}(\tau)\ln\left(e+\frac{2C}{\omega_{n,m}(\tau)}\right)d\tau,$$

where $C>0$ is a constant depending on $\|u^{(n)}\|_{B^{5/2}_{2,1}}$.

Let

$$\omega_n(t):=\sup_{m\in\mathbb{N}}\omega_{n,m}(t)\quad\text{and}\quad\tilde\omega(t):=\limsup_{n\to\infty}\omega_n(t).$$

Then for any $\epsilon>0$, there exists $N\in\mathbb{N}$, such that for any $n>N$, we have

$$\omega_n(t)\le\tilde\omega(t)+\epsilon.$$

Taking the supremum over $m\in\mathbb{N}$, we refine the estimate for $\omega_{n+1}(t)$ as

$$\omega_{n+1}(t)\le C2^{-n}+C\int_0^t(\tilde\omega(\tau)+\epsilon)\ln\left(e+\frac{2C}{\tilde\omega(\tau)+\epsilon}\right)d\tau.$$

Passing to the limit $n\to\infty$, we deduce

$$\tilde\omega(t)\le C\int_0^t(\tilde\omega(\tau)+\epsilon)\ln\left(e+\frac{2C}{\tilde\omega(\tau)+\epsilon}\right)d\tau.$$

Since $\epsilon>0$ is arbitrary, sending $\epsilon\to0^+$ yields

$$\tilde\omega(t)\le C\int_0^t\tilde\omega(\tau)\ln\left(e+\frac{2C}{\tilde\omega(\tau)}\right)d\tau.$$

Applying Lemma 2.5 with $a=0$ and $\mu(r)=r\ln\big(e+\frac{2M}{r}\big)$ is a continuous and increasing function, we conclude that $\tilde\omega(t)=0$ for all $t\in[0,T]$. This establishes that $(u^{(n)})$ is a Cauchy sequence in $C([0,T];B^{3/2}_{2,\infty})$.

Finally, by Lemma 2.1, we derive the interpolation bound

$$\begin{aligned}\|u^{(n+m)}-u^{(n)}\|_{B^{3/2}_{2,1}}&\le\|u^{(n+m)}-u^{(n)}\|_{B^{2}_{2,1}}\\&\le\|u^{(n+m)}-u^{(n)}\|^{1/2}_{B^{3/2}_{2,\infty}}\|u^{(n+m)}-u^{(n)}\|^{1/2}_{B^{5/2}_{2,\infty}}\end{aligned}$$

$$
\begin{aligned}
&\leq \|u^{(n+m)} - u^{(n)}\|^{1/2}_{B^{3/2}_{2,\infty}} \|u^{(n+m)} - u^{(n)}\|^{1/2}_{B^{5/2}_{2,1}} \\
&\leq (2C)^{1/2} \|u^{(n+m)} - u^{(n)}\|^{1/2}_{B^{3/2}_{2,\infty}}.
\end{aligned}
$$

Hence, $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{3/2}_{2,1})$.

Thanks to above calculation, we have that $(u^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence $C([0,T];B^{3/2}_{2,1})$, so it converges to some function $u \in C([0,T];B^{3/2}_{2,1})$. Next, we need to verify that u belongs to $E^{5/2}_{2,1}$ and solves the Cauchy problem (1.7). Since $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,T];B^{5/2}_{2,1})$, it follows from the Fatou property (Lemma 2.7) for Besov spaces that $u \in L^\infty([0,T];B^{5/2}_{2,1})$.

On the other hand, since $(u^{(n)})_{n\in\mathbb{N}}$ converges to u in $C([0,T];B^{3/2}_{2,1})$, an interpolation argument ensures that the convergence holds in $C([0,T];B^{s'}_{2,1})$, for any $s' < \frac{5}{2}$. It is then easy to pass to the limit in (3.2) and conclude that u is indeed a solution to the Cauchy problem (1.7). It is easy to check that $u_t \in L^\infty([0,T];B^{3/2}_{2,1})$. Lemma 2.3 guarantees that $u \in C([0,T];B^{5/2}_{2,1})$. Moreover, from equation (1.7), we can also verify that $u_t \in C([0,T];B^{3/2}_{2,1})$. Thus, we conclude that $u \in E^{5/2}_{2,1}$. □

The above discussions demonstrate that solutions to equation (1.7) indeed exist. We now present the following lemma to establish the uniqueness of these solutions.

Lemma 3.2. *Let $u, v \in L^\infty([0,T];B^{5/2}_{2,1}) \cap C([0,T];B^{3/2}_{2,1})$ be solutions to the Cauchy problem* (1.7) *with the initial data $u_0 \in B^{\frac{5}{2}}_{2,1}$. If the solutions satisfy the differential inequality*

$$
\|u - v\|_{B^{3/2}_{2,1}} \leq \|u_0 - v_0\|_{B^{3/2}_{2,1}} \exp\left(C\int_0^t \left(\|u\|^2_{B^{5/2}_{2,1}} + \|v\|^2_{B^{5/2}_{2,1}}\right)d\tau\right), \tag{3.16}
$$

then uniqueness holds for the Cauchy problem (1.7)*.*

Proof. By virtue of (3.1), the difference $w := u - v$ satisfies the transport equation

$$
\partial_t w + \left(u^2 - \frac{1}{3}u_x^2\right)\partial_x w = M, \tag{3.17}
$$

where

$$
M := \left(v^2 - u^2 + \frac{1}{3}(u_x^2 - v_x^2)\right)v_x + p * (F(v) - F(u)) + \partial_x p * (G(v) - G(u)).
$$

Here, the nonlinear terms F and G are given explicitly by

$$
p * (F(v) - F(u)) = p * \left(\frac{3b-5}{3}(v_x^3 - u_x^3) + (b-2)(v_x v_{xx}^2 - u_x u_{xx}^2)\right),
$$

and

$$
\partial_x p * (G(v) - G(u)) = \partial_x p * \left(\frac{b}{3}(v^3 - u^3) + (3-b)(v v_x^2 - u u_x^2)\right).
$$

Lemma 2.1 first provides

$$
\|\partial_x(u^2 - \frac{1}{3}u_x^2)\|_{B^{1/2}_{2,1}} \leq \|u^2 - \frac{1}{3}u_x^2\|_{B^{3/2}_{2,1}} \leq \|u\|^2_{B^{5/2}_{2,1}}. \tag{3.18}
$$

We estimate M as follows

$$\begin{aligned}&\|M\|_{B^{3/2}_{2,1}}\\&=\|(v^2-u^2+\frac{1}{3}(u_x^2-v_x^2))v_x+p*(F(v)-F(u))+\partial_x p*(G(v)-G(u))\|_{B^{3/2}_{2,1}}.\end{aligned}\tag{3.19}$$

Since the estimates for each term in (3.19) follow analogous arguments, we only estimate one of them,

$$\begin{aligned}\|\partial_x p*(G(v)-G(u))\|_{B^{3/2}_{2,1}}&\leq C(\|v^3-u^3\|_{B^{1/2}_{2,1}}+\|vv_x^2-uu_x^2\|_{B^{1/2}_{2,1}})\\&\leq C(\|u\|^2_{B^{5/2}_{2,1}}+\|v\|^2_{B^{5/2}_{2,1}})\|w\|_{B^{3/2}_{2,1}}.\end{aligned}$$

Applying the estimates in Lemma 2.2 to transport equation (3.17), we have

$$\|w\|_{B^{3/2}_{2,1}}\leq\|w_0\|_{B^{3/2}_{2,1}}+\int_0^t\|M\|_{B^{3/2}_{2,1}}ds\quad+C\int_0^t\|\partial_x(u^2-\frac{1}{3}u_x^2)\|_{B^{1/2}_{2,1}}\|w\|_{B^{3/2}_{2,1}}ds.\tag{3.20}$$

Substituting (3.18) and (3.19) into (3.20) yields

$$\|w\|_{B^{3/2}_{2,1}}\leq\|w_0\|_{B^{3/2}_{2,1}}+C\int_0^t(\|u\|^2_{B^{5/2}_{2,1}}+\|v\|^2_{B^{5/2}_{2,1}})\|w\|_{B^{3/2}_{2,1}}ds.$$

An application of Grönwall's inequality in integral form consequently gives

$$\|w\|_{B^{3/2}_{2,1}}\leq\|w_0\|_{B^{3/2}_{2,1}}\exp(C\int_0^t(\|u\|^2_{B^{5/2}_{2,1}}+\|v\|^2_{B^{5/2}_{2,1}})d\tau).\tag{3.21}$$

For solutions sharing identical initial data ($w_0=u_0-v_0=0$), the exponential term remains finite while the prefactor vanishes. This immediately yields $w(t)\equiv 0$ for all $t\in[0,T]$, establishing the uniqueness property. □

We now verify the continuity of solutions with respect to the initial data in $B^{5/2}_{2,1}$.

Lemma 3.3. *For any $u_0\in B^{5/2}_{2,1}$, there exist a neighborhood V of u_0 in $B^{5/2}_{2,1}$ and a time $T>0$ such that for any $v_0\in V$ with $v\in E^{5/2}_{2,1}(T)$ being the corresponding solution to Cauchy problem* (1.7)*, the map*

$$\begin{aligned}\Phi: V\subset B^{5/2}_{2,1}&\longrightarrow E^{5/2}_{2,1}(T),\\ v_0&\longmapsto v(\cdot,v_0)\end{aligned}$$

is continuous.

Proof. Step 1. We establish the Lipschitz continuity of Φ in the space $C([0,T];B^{3/2}_{2,1})$.

Fix an initial datum $u_0\in B^{5/2}_{2,1}$, there exist a neighborhood V and a constant $T>0$ such that for any $v_0\in V$, the corresponding solution $v=\Phi(v_0)$ to the Cauchy problem (1.7) belongs to $C([0,T];B^{5/2}_{2,1})$ and satisfies $\|v\|_{L^\infty([0,T];B^{5/2}_{2,1})}\leq\sqrt{2}\|v_0\|_{B^{5/2}_{2,1}}$. We have from (3.16) that

$$\|\Phi(v_0)-\Phi(u_0)\|_{L^\infty([0,T];B^{3/2}_{2,1})}\leq\|v_0-u_0\|_{B^{5/2}_{2,1}}e^{CM^2T},$$

where $M = \sqrt{2}\max\{\|v_0\|_{B^{5/2}_{2,1}}, \|u_0\|_{B^{5/2}_{2,1}}\}$. Hence, Φ is Lipschitz continuous from $B^{5/2}_{2,1}$ into $C([0,T];B^{3/2}_{2,1})$.

Step 2. We present the continuity of the map $\Phi\in C([0,T];B^{5/2}_{2,1})$.

Let $u_0^{(n)}\in B^{5/2}_{2,1}$, $u_0^{(n)}$ tends to $u_0^{(\infty)}$ in $B^{5/2}_{2,1}$. According to Lemma 3.1, we know that $u^{(n)}$, $u^{(\infty)}\in L^\infty([0,T];B^{5/2}_{2,1})$. In order to prove $u^{(n)}$ tends to $u^{(\infty)}$ in $C([0,T];B^{5/2}_{2,1})$, it is sufficient to prove $m^{(n)}=u^{(n)}-\partial_{xx}u^{(n)}$ tends to $m^{(\infty)}=u^{(\infty)}-\partial_{xx}u^{(\infty)}$ in $C([0,T];B^{1/2}_{2,1})$.

Note that $m^{(n)}$ solves the following transport equation:

$$\begin{cases}\partial_t m^{(n)}+\big((u^{(n)})^2-(u_x^{(n)})^2\big)\partial_x m^{(n)}=-bu_x^{(n)}(m^{(n)})^2, & t>0, x\in\mathbb{R},\\ m^{(n)}|_{t=0}=m_0^{(n)}=u_0^{(n)}-u_{0xx}^{(n)}, & x\in\mathbb{R}.\end{cases}$$

Following the Kato theory [15,18], we decompose $m^{(n)}$ into $m^{(n)}=z^{(n)}+\omega^{(n)}$ with

$$\begin{cases}\partial_t z^{(n)}+\big((u^{(n)})^2-(u_x^{(n)})^2\big)\partial_x z^{(n)}=-bu_x^{(n)}(m^{(n)})^2+bu_x^{(\infty)}(m^{(\infty)})^2, & t>0, x\in\mathbb{R},\\ z^{(n)}|_{t=0}=m_0^{(n)}(x)-m_0^{(\infty)}(x), & x\in\mathbb{R},\end{cases}\tag{3.22}$$

and

$$\begin{cases}\partial_t \omega^{(n)}+\big((u^{(n)})^2-(u_x^{(n)})^2\big)\partial_x \omega^{(n)}=-bu_x^{(\infty)}(m^{(\infty)})^2, & t>0, x\in\mathbb{R},\\ \omega^{(n)}|_{t=0}=m_0^{(\infty)}(x), & x\in\mathbb{R}.\end{cases}\tag{3.23}$$

Denote $z^{(\infty)}=0$, $\omega^{(\infty)}=m^{(\infty)}$. Making use of the fact that $u^{(n)}$, $u^{(\infty)}$ are uniformly bounded in $B^{5/2}_{2,1}$ and applying Lemma 2.1, we have

$$\begin{aligned}\|\nabla((u^{(n)})^2-(u_x^{(n)})^2)\|_{B^{1/2}_{2,1}\cap L^\infty}&\le\|\nabla((u^{(n)})^2-(u_x^{(n)})^2)\|_{B^{1/2}_{2,1}}\\&\le\|(u^{(n)})^2-(u_x^{(n)})^2\|_{B^{3/2}_{2,1}}\\&\le\|(u^{(n)})^2\|_{B^{3/2}_{2,1}}+\|(u_x^{(n)})^2\|_{B^{3/2}_{2,1}}\\&\le C\|u^{(n)}\|_{L^\infty}\|u^{(n)}\|_{B^{3/2}_{2,1}}+C\|u_x^{(n)}\|_{L^\infty}\|u_x^{(n)}\|_{B^{3/2}_{2,1}}\\&\le C\|u^{(n)}\|_{B^{1/2}_{2,1}}\|u^{(n)}\|_{B^{3/2}_{2,1}}+C\|u_x^{(n)}\|_{B^{1/2}_{2,1}}\|u_x^{(n)}\|_{B^{3/2}_{2,1}}\\&\le C\|u^{(n)}\|^2_{B^{5/2}_{2,1}},\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}&\|-bu_x^{(n)}(m^{(n)})^2+bu_x^{(\infty)}(m^{(\infty)})^2\|_{B^{1/2}_{2,1}}\\&\le C\|u_x^{(n)}(m^{(n)})^2-u_x^{(n)}(m^{(\infty)})^2\|_{B^{1/2}_{2,1}}+\|u_x^{(n)}(m^{(\infty)})^2-u_x^{(\infty)}(m^{(\infty)})^2\|_{B^{1/2}_{2,1}}\\&\le\|u_x^{(n)}\|_{L^\infty}\|(m^{(n)}+m^{(\infty)})(m^{(n)}-m^{(\infty)})\|_{B^{1/2}_{2,1}}+\|u_x^{(n)}\|_{B^{1/2}_{2,1}}\|(m^{(n)}+m^{(\infty)})(m^{(n)}-m^{(\infty)})\|_{L^\infty}\\&\quad+\|m^{(\infty)}\|^2_{L^\infty}\|u_x^{(n)}-u_x^{(\infty)}\|_{B^{1/2}_{2,1}}+\|(m^{(\infty)})^2\|_{B^{1/2}_{2,1}}\|u_x^{(n)}-u_x^{(\infty)}\|_{L^\infty}\end{aligned}$$

$$=: I_1 + I_2 + I_3 + I_4. \tag{3.25}$$

The following provides an estimation of each component of (3.25), respectively,

$$\begin{aligned}
I_1 &= \|u_x^{(n)}\|_{L^\infty} \|(m^{(n)} + m^{(\infty)})(m^{(n)} - m^{(\infty)})\|_{B^{1/2}_{2,1}} \\
&\le \|u_x^{(n)}\|_{L^\infty} \|m^{(n)} + m^{(\infty)}\|_{L^\infty} \|m^{(n)} - m^{(\infty)}\|_{B^{1/2}_{2,1}} \\
&\quad + \|u_x^{(n)}\|_{L^\infty} \|m^{(n)} + m^{(\infty)}\|_{B^{1/2}_{2,1}} \|m^{(n)} - m^{(\infty)}\|_{L^\infty} \\
&\le \|u_x^{(n)}\|_{B^{1/2}_{2,1}} \|m^{(n)} + m^{(\infty)}\|_{B^{1/2}_{2,1}} \|m^{(n)} - m^{(\infty)}\|_{B^{1/2}_{2,1}} \\
&\quad + \|u_x^{(n)}\|_{B^{1/2}_{2,1}} \|m^{(n)} + m^{(\infty)}\|_{B^{1/2}_{2,1}} \|m^{(n)} - m^{(\infty)}\|_{B^{1/2}_{2,1}} \\
&\le C\|m^{(n)} - m^{(\infty)}\|_{B^{1/2}_{2,1}}, \\
I_2 &= \|u_x^{(n)}\|_{B^{1/2}_{2,1}} \|(m^{(n)} + m^{(\infty)})(m^{(n)} - m^{(\infty)})\|_{L^\infty} \\
&\le \|u^{(n)}\|_{B^{3/2}_{2,1}} \|m^{(n)} + m^{(\infty)}\|_{L^\infty} \|m^{(n)} - m^{(\infty)}\|_{L^\infty} \\
&\le \|u^{(n)}\|_{B^{3/2}_{2,1}} \|m^{(n)} + m^{(\infty)}\|_{B^{1/2}_{2,1}} \|m^{(n)} - m^{(\infty)}\|_{L^\infty} \\
&\le C\|m^{(n)} - m^{(\infty)}\|_{B^{1/2}_{2,1}}, \\
I_3 &= \|m^{(\infty)}\|^2_{L^\infty} \|u_x^{(n)} - u_x^{(\infty)}\|_{B^{1/2}_{2,1}} \\
&\le C\|m^{(\infty)}\|^2_{B^{1/2}_{2,1}} \|u^{(n)} - u^{(\infty)}\|_{B^{3/2}_{2,1}} \\
&\le C\|u^{(n)} - u^{(\infty)}\|_{B^{3/2}_{2,1}}, \\
I_4 &= \|(m^{(\infty)})^2\|_{B^{1/2}_{2,1}} \|u_x^{(n)} - u_x^{(\infty)}\|_{L^\infty} \\
&\le C\|m^{(\infty)}\|_{L^\infty} \|m^{(\infty)}\|_{B^{1/2}_{2,1}} \|u_x^{(n)} - u_x^{(\infty)}\|_{B^{1/2}_{2,1}} \\
&\le \|m^{(\infty)}\|^2_{B^{1/2}_{2,1}} \|u_x^{(n)} - u_x^{(\infty)}\|_{B^{1/2}_{2,1}} \\
&\le C\|u^{(n)} - u^{(\infty)}\|_{B^{3/2}_{2,1}}.
\end{aligned}$$

Hence, we obtain

$$\|-bu_x^{(n)}(m^{(n)})^2 + bu_x^{(\infty)}(m^{(\infty)})^2\|_{B^{1/2}_{2,1}} \le C\|m^{(n)} - m^{(\infty)}\|_{B^{1/2}_{2,1}} + C\|u^{(n)} - u^{(\infty)}\|_{B^{3/2}_{2,1}}. \tag{3.26}$$

Applying the estimates in Lemma 2.2 to transport equation (3.22), we have

$$\begin{aligned}
&\|z^{(n)}(t)\|_{B^{1/2}_{2,1}} \\
\le& e^{C\int_0^t \|\nabla((u^{(n)})^2 - (u_x^{(n)})^2)(s)\|_{B^{1/2}_{2,1}\cap L^\infty} ds} \cdot \Big(\|m_0^{(n)}(x) - m_0^{(\infty)}(x)\|_{B^{1/2}_{2,1}} \\
&+ \int_0^t e^{-C\int_0^{t'} \|\nabla((u^{(n)})^2 - (u_x^{(n)})^2)(s)\|_{B^{1/2}_{2,1}\cap L^\infty} ds} \|-bu_x^{(n)}(m^{(n)})^2 + bu_x^{(\infty)}(m^{(\infty)})^2\|_{B^{1/2}_{2,1}}(t')dt' \Big).
\end{aligned} \tag{3.27}$$

Substituting (3.24) and (3.26) into (3.27), we have

$$\begin{aligned}\|z^{(n)}(t)\|_{B^{1/2}_{2,1}} \le & e^{Ct}\Big(\|m_0^{(n)}(x)-m_0^{(\infty)}(x)\|_{B^{1/2}_{2,1}} \\ & + C\int_0^t \Big(\|z^{(n)}\|_{B^{1/2}_{2,1}}+\|\omega^{(n)}-\omega^{(\infty)}\|_{B^{1/2}_{2,1}}+\|u^{(n)}-u^{(\infty)}\|_{B^{3/2}_{2,1}}\Big)(t')dt'\Big). \end{aligned} \tag{3.28}$$

From Lemma 2.1 and Step 1, it yields that

$$\begin{aligned}\|bu_x^{(\infty)}(m^{(\infty)})^2\|_{B^{1/2}_{2,1}} &\le C\|u_x^{(\infty)}\|_{L^\infty}\|(m^{(\infty)})^2\|_{B^{1/2}_{2,1}}+C\|u_x^{(\infty)}\|_{B^{1/2}_{2,1}}\|(m^{(\infty)})^2\|_{L^\infty} \\ &\le C\|u_x^{(\infty)}\|_{B^{1/2}_{2,1}}\|m^{(\infty)}\|^2_{B^{1/2}_{2,1}} \\ &\le C\|u^{(\infty)}\|^3_{B^{5/2}_{2,1}}, \end{aligned} \tag{3.29}$$

$$\begin{aligned}\|\nabla((u^{(n)})^2-(u_x^{(n)})^2)\|_{B^{1/2}_{2,1}} &\le \|(u^{(n)})^2-(u_x^{(n)})^2\|_{B^{3/2}_{2,1}} \\ &\le \|(u^{(n)})^2\|_{B^{3/2}_{2,1}}+\|(u_x^{(n)})^2\|_{B^{3/2}_{2,1}} \\ &\le C\|u^{(n)}\|_{L^\infty}\|u^{(n)}\|_{B^{3/2}_{2,1}}+C\|u_x^{(n)}\|_{L^\infty}\|u_x^{(n)}\|_{B^{3/2}_{2,1}} \\ &\le C\|u^{(n)}\|_{B^{1/2}_{2,1}}\|u^{(n)}\|_{B^{3/2}_{2,1}}+C\|u_x^{(n)}\|_{B^{1/2}_{2,1}}\|u_x^{(n)}\|_{B^{3/2}_{2,1}} \\ &\le C\|u^{(n)}\|^2_{B^{5/2}_{2,1}}, \end{aligned} \tag{3.30}$$

and

$$0 \le \lim_{n\to\infty}\|((u^{(n)})^2-(u_x^{(n)})^2)-((u^{(\infty)})^2-(u_x^{(\infty)})^2)\|_{B^{1/2}_{2,1}} \le C\lim_{n\to\infty}\|u^{(n)}-u^{(\infty)}\|_{B^{3/2}_{2,1}}=0. \tag{3.31}$$

By combining estimates (3.29)-(3.31) and applying Lemma 2.6 to (3.23), we obtain the convergence result:

$$\omega^{(n)} \to \omega^{(\infty)} \quad \text{in} \quad C([0,T];B^{1/2}_{2,1}). \tag{3.32}$$

Recall that the initial data satisfies

$$u_0^{(n)} \to u_0^{(\infty)} \quad \text{in} \quad B^{5/2}_{2,1}, \tag{3.33}$$

and the solution sequence converges as

$$u^{(n)} \to u^{(\infty)} \quad \text{in} \quad C([0,T];B^{3/2}_{2,1}).$$

Applying the integral form of Gronwall's inequality to (3.28), we deduce that

$$z^{(n)} \to 0 \quad \text{in} \quad C([0,T];B^{1/2}_{2,1}).$$

Using the definition of $m^{(n)}$, we conclude that

$$\|m^{(n)}-m^{(\infty)}\|_{B^{1/2}_{2,1}} \le \|z^{(n)}-z^{(\infty)}\|_{B^{1/2}_{2,1}}+\|\omega^{(n)}-\omega^{(\infty)}\|_{B^{1/2}_{2,1}},$$

which yields that

$$u^{(n)} \to u^{(\infty)} \quad \text{in} \quad C([0,T];B^{5/2}_{2,1}).$$

Therefore we complete the continuity of the map Φ in $C([0,T];B^{5/2}_{2,1})$.

Step 3. Continuity in $C^1([0,T];B^{3/2}_{2,1})$.

Finally, applying ∂_t to equation in (1.7) and using the same argument to the resulting equation in terms of $\partial_t u$, we may verify the continuity of the map Φ in $C^1([0,T];B^{3/2}_{2,1})$. This completes the proof of Lemma 3.3. $\square$

Combining Lemma 3.1-3.3, we complete the proof of Theorem 3.1.

4. Blow-up criteria

Theorem 4.1. *Let $b=1$, $u_0\in H^s(\mathbb{R})$ with $s>\frac{5}{2}$. Suppose that there exists a point $x_1\in\mathbb{R}$, such that $u_{0x}(x_1)<0$ and $m_0(x_1)>\frac{13\sqrt{2}\|m_0\|^3_{L^2}}{12(1-\epsilon)\epsilon u^2_{0x}(x_1)}$, where $0<\epsilon<1$. Then the solution $u(t,x)$ of (1.7) blows up in finite time with an estimate of the blow up time T^* as*

$$T^*\leq -\frac{1}{\epsilon u_{0x}(x_1)m_0(x_1)}.$$

Remark 4.1. Compared with Theorem 3.3 in [22], our theorem does not require any additional assumptions on $\|u(t,\cdot)\|_{L^\infty}$.

Lemma 4.1 ([22]). *Let $b\in\mathbb{R}$, $m_0\in H^s(\mathbb{R})$ with $s>\frac{1}{2}$ and $T>0$ be the maximal existence time of strong solution m to the Cauchy problem (1.7). Then the solution m blows up in finite time if and only if*

$$\liminf_{t\to T}\inf_{x\in\mathbb{R}} mu_x(t,x)=-\infty.$$

Before deriving the exact blow-up time, we introduce the following conservation law:

Lemma 4.2. *Suppose that $m_0\in H^s(\mathbb{R})$ with $s>\frac{1}{2}$. Then any solution of equation (1.7) with $b=1$ satisfies*

$$\|m\|_{L^2}=\|m_0\|_{L^2}.$$

Proof. When $b=1$, equation (1.7) reduces to

$$m_t+(u^2-u_x^2)m_x+u_xm^2=0.$$

Multiplying both sides by m and integrating over $\mathbb{R}$, we obtain through integration by parts:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}m^2\,\mathrm{d}x&=-\int_{\mathbb{R}}(u^2-u_x^2)mm_x\,\mathrm{d}x-\int_{\mathbb{R}}u_xm^3\,\mathrm{d}x\\&=\int_{\mathbb{R}}u_xm^3\,\mathrm{d}x-\int_{\mathbb{R}}u_xm^3\,\mathrm{d}x\\&=0.\end{aligned}$$

This completes the proof of Lemma 4.2. $\square$

Let q be the particle trajectory satisfying

$$\begin{cases}q_t(t,x)=(u^2-u_x^2)(t,q(t,x)), & x\in\mathbb{R},t\in[0,T),\\ q(0,x)=x, & x\in\mathbb{R},t\in[0,T),\end{cases}\tag{4.1}$$

where T is the lifespan of solution u. Taking derivative (4.1) with respect to x, we obtain

$$q_{tx}(t,x) = 2u_x(t,q)m(t,q)q_x, \quad t \in [0,T).$$

Therefore,

$$\begin{cases} q_x(t,x) = e^{\int_0^t 2u_x m(t,q(t,x))ds}, & x \in \mathbb{R}, t \in [0,T), \\ q_x(0,x) = 1, & x \in \mathbb{R}. \end{cases} \tag{4.2}$$

Since q_x is always positive before blow-up, $q(t,x)$ is increasing with respect to x and trajectories never coincide before blow-up. According to definition of (3.1) and (4.1), direct calculation yields

$$\begin{aligned} &\frac{d}{dt}u(t,q(t,x)) = u_t + u_x q_t(t,x) = -\frac{2}{3}u_x^3 + p*(\frac{2}{3}u_x^3 + u_x u_{xx}^2) - \partial_x p*(\frac{1}{3}u^3 + 2uu_x^2), \\ &\frac{d}{dt}u_x(t,q(t,x)) = u_{xt} + u_{xx}q_t(t,x) = \frac{1}{3}u^3 + \partial_x p*(\frac{2}{3}u_x^3 + u_x u_{xx}^2) - p*(\frac{1}{3}u^3 + 2uu_x^2), \\ &\frac{d}{dt}m(t,q(t,x)) = m_t + m_x q_t(t,x) = -u_x m^2. \end{aligned} \tag{4.3}$$

Proof of Theorem 4.1. By the Sobolev embedding and Lemma 4.2, we obtain

$$\begin{aligned} &\|u\|_{L^\infty} \le \frac{1}{\sqrt{2}}\|u\|_{H^1} \le \frac{1}{\sqrt{2}}\|m_0\|_{L^2}, \\ &\|u_x\|_{L^\infty} \le \frac{1}{\sqrt{2}}\|u\|_{H^2} \le \frac{1}{\sqrt{2}}\|m_0\|_{L^2}, \\ &\|u_{xx}\|_{L^2} \le \|m\|_{L^2} = \|m_0\|_{L^2}, \\ &\|u_x\|_{L^3}^3 \le \|u_x\|_{L^\infty}\|u_x\|_{L^2}^2 \le \frac{1}{\sqrt{2}}\|m_0\|_{L^2}^3. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{3}\|u\|_{L^\infty}^3 \le \frac{\sqrt{2}}{12}\|m_0\|_{L^2}^3, \\ &|\partial_x p*(\frac{2}{3}u_x^3 + u_x u_{xx}^2)| \le \frac{1}{2}\big(\frac{2}{3}\|u_x\|_{L^\infty}\|u_x\|_{L^2}^2 + \|u_x\|_{L^\infty}\|u_{xx}\|_{L^2}^2\big) \le \frac{5\sqrt{2}}{12}\|m_0\|_{L^2}^3, \end{aligned}$$

and

$$|p*(\frac{1}{3}u^3 + 2uu_x^2)| \le \frac{1}{2}\big(\frac{1}{3}\|u\|_{L^\infty}\|u\|_{L^2}^2 + 2\|u\|_{L^\infty}\|u_x\|_{L^2}^2\big) \le \frac{7\sqrt{2}}{12}\|m_0\|_{L^2}^3.$$

Substituting the above estimates into equation (4.3) and taking x to be x_1 given in Theorem 4.1, we obtain

$$|\frac{d}{dt}u_x(t,q(t,x_1))| \le \frac{13\sqrt{2}}{12}\|m_0\|_{L^2}^3,$$

which is equivalent to

$$-\frac{13\sqrt{2}}{12}\|m_0\|_{L^2}^3 t + u_{0x}(x_1) \le u_x(t,q(t,x_1)) \le \frac{13\sqrt{2}}{12}\|m_0\|_{L^2}^3 t + u_{0x}(x_1).$$

From the condition $u_{0x}(x_1) < 0$ in Theorem 4.1, it holds for any $t \in [0, T')$, $T' = -\frac{6\sqrt{2}u_{0x}(x_1)}{13\|m_0\|_{L^2}^3}$ that

$$u_x(t, q(t, x_1)) \leq 0.$$

Take $T'' = (1-\epsilon)T' = -\frac{6\sqrt{2}(1-\epsilon)u_{0x}(x_1)}{13\|m_0\|_{L^2}^3}$ with $0 < \epsilon < 1$ given in Theorem 4.1. Then for any $t \in [0, T'']$, we have

$$u_x(t, q(t, x_1)) \leq \frac{13\sqrt{2}}{12}\|m_0\|_{L^2}^3 T'' + u_{0x}(x_1) \leq \epsilon u_{0x}(x_1) < 0. \tag{4.4}$$

From the preceding discussion, we arrive at

$$\frac{d}{dt}m(t, q(t, x_1)) = -u_x m^2 \geq -\epsilon u_{0x}(x_1) m^2(t, q(t, x_1)) > 0.$$

Integrating from 0 to t yields

$$m(t, q(t, x_1)) \geq \frac{m_0(x_1)}{1 + \epsilon u_{0x}(x_1) m_0(x_1) t}.$$

From the condition $m_0(x_1) > -\frac{1}{\epsilon u_{0x}(x_1) T''}$ in Theorem 4.1, we have $T'' > -\frac{1}{\epsilon u_{0x}(x_1) m_0(x_1)}$. There exits $T^* \leq -\frac{1}{\epsilon u_{0x}(x_1) m_0(x_1)}$, such that

$$\lim_{t \to T^*} m(t, (q(t, x_1))) = +\infty. \tag{4.5}$$

Combining (4.4), (4.5), and Lemma 4.1, we complete the proof. □

5. Global existence

Theorem 5.1. *Let $b = 0$, $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Then the solution to (1.7) exists globally in time.*

Lemma 5.1. *Let $b = 0$, $m_0 \in H^s$ with $s > \frac{1}{2}$ and $T > 0$ be the maximal existence time of strong solution m to the Cauchy problem (1.7). Then we have*

$$\|m(t, \cdot)\|_{L^\infty} = \|m_0\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|m_0\|_{H^s},$$

for all $t \in [0, T)$.

Proof. By direct calculation, we have

$$\frac{d}{dt}m(t, q(t, x)) = m_t + m_x q_t(t, x) = m_t + m_x(u^2 - u_x^2) = 0,$$

which completes the proof of Lemma 5.1. □

Proof of Theorem 5.1. According to Lemma 4.1, the solution exists globally if and only if the condition $\|mu_x\| \leq C$ is satisfied. Furthermore, applying the Sobolev embedding theorem, convolution form of Young's inequality, Lemma 5.1, and the identities $\|\partial_x p\|_{L^1} = \|p\|_{L^1} = 1$, we derive

$$\|u_x\|_{L^\infty} = \|\partial_x p * m\|_{L^\infty} \leq \|\partial_x p\|_{L^1}\|m\|_{L^\infty} \leq \|m_0\|_{L^\infty},$$

$$\|m\|_{L^\infty} = \|m_0\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|m_0\|_{H^s},$$

$$\|mu_x\|_{L^\infty} \leq \|u_x\|_{L^\infty}\|m\|_{L^\infty} \leq \|m_0\|_{L^\infty}^2 \leq \frac{1}{2}\|m_0\|_{H^s}^2,$$

which complete the proof of Thorem 5.1. □

6. Conclusions

In this paper, we established three results that characterized the solution behavior of the b-FORQ equation. First, we obtained local well-posedness in the critical Besov spaces. We then analyzed the solutions in two special cases: For $b = 1$, we proved that the solution blows up in finite time, whereas for $b = 0$, we established the global existence of the solution. These results extend the existing well-posedness theory of the bFORQ equation by establishing critical Besov space results and developing, for a specific parameter, both the blow-up analysis and the global existence of solutions.

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