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ON WELL-POSEDNESS OF A GENERAL FRACTIONAL INTEGRAL EQUATION IN ORLICZ SPACES

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Abstract This article has two objectives. The first objective is to utilize Darbo's fixed-point theorem ($\mathcal{FPT}$) and the measure of noncompactness ($\mathcal{MNC}$) technique to study the well-posedness (i.e., the existence and the uniqueness in addition to continuous dependence on the data) of a generalized fractional quadratic integral equation in Orlicz spaces L_ψ. Our second objective is to prove and explain the novel properties of g-fractional operators, such as boundedness, continuity, acting, and monotonicity within L_ψ. As a result of our work, several fractional operators are generalized, unified, and extended, including Riemann-Liouville, Hadamard, and Erdélyi-Kober, as well as related classical and quadratic fractional problems. Our theories are supported by several examples.

Keywords Orlicz spaces L_ψ, ($\mathcal{MNC}$) measure of noncompactness, g-fractional operator, ($\mathcal{FPT}$) fixed point theorem.

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1. Introduction

Fractional calculus has emerged as a powerful and flexible framework for modeling phenomena in which memory, nonlocality, and hereditary effects play a fundamental role. Unlike classical integer-order models, fractional operators can capture long-range temporal and spatial interactions that naturally arise in complex systems. This distinctive feature has made fractional calculus an indispensable tool in a wide range of applications, including economics, biology, physics, viscoelasticity, engineering, fluid dynamics, electrical circuits, earthquake modeling, electrochemistry, and traffic flow (cf. [21, 23, 32, 36, 38]).

Despite this rapid development of fractional modeling, the majority of existing studies focus on specific fractional kernels such as the Riemann–Liouville, Hadamard, or Erdélyi–Kober operators (cf. [2, 9, 18, 40–42]). While these kernels have proved useful, they restrict the admissible memory structures and limit the modeling flexibility of the associated integral equations. In

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contrast, the g-fractional operator provides a unifying and more general framework, allowing the memory kernel to be adapted to the underlying physical or analytical context through an appropriate choice of the function g. This generality enables the treatment of a broader class of fractional models, including many classical cases as particular examples.

Quadratic fractional integral equations arise naturally in several areas of applied science, such as the kinetic theory of gases [19], radiative transfer [12], and neutron transport theory [20], where nonlinear interactions between states play a crucial role. When such equations are combined with generalized fractional kernels, as in the g-fractional operators, they provide an effective mathematical description of nonlinear systems with complex and nonlocal memory effects. However, the presence of both quadratic nonlinearities and weakly singular fractional kernels introduces substantial analytical challenges, particularly with respect to the regularity and integrability of solutions.

In many realistic models, solutions are not expected to be continuous or bounded in the classical sense, and they may exhibit nonstandard or fast growth behavior. This makes the traditional functional spaces of continuous functions or Lebesgue spaces L^p with fixed $p \geq 1$ inadequate or overly restrictive. In particular, the quadratic structure of the integral equation, together with the generalized g-fractional kernel, may lead to solutions that fall outside any single L^p space. This limitation motivates the use of Orlicz spaces L_ψ, which extend the Lebesgue space and are specifically designed to handle nonlinear growth conditions, irregular data, and non-polynomial integrability behaviors frequently encountered in statistical physics and nonlinear analysis (cf. [33, 34]).

Orlicz spaces provide a natural and flexible framework for the analysis of nonlinear integral equations, as they allow the growth of solutions to be controlled by a suitable Young function rather than by a fixed power. Moreover, these spaces exhibit favorable structural properties with respect to nonlinear (Nemytskii-type) operators and are well-suited for the application of compactness methods based on the $(\mathcal{MNC})$. Consequently, establishing fundamental properties of fractional-type operators such as boundedness, continuity, and monotonicity in Orlicz spaces is essential for proving well-posedness results for nonlinear fractional integral equations under weak and physically meaningful assumptions.

The main objective of this paper is to investigate essential analytical properties of the g-fractional operator in Orlicz spaces L_ψ, including boundedness, continuity, and monotonicity. These results are then applied to study the well-posedness of a class of quadratic fractional integral equations of the form

$$y(v) = f(v) + \frac{h_1(v, y(v))}{\Gamma(\alpha)} \int_0^v \frac{h_2(s, y(s))}{\big(g(v) - g(s)\big)^{1-\alpha}} g'(s)\, ds, \qquad v \in [0, d], \quad 0 < \alpha < 1, \tag{1.1}$$

within the Orlicz space L_ψ. Equation (1.1) encompasses, as particular cases, several well-known fractional integral equations corresponding to specific choices of the function g, including the Riemann–Liouville, Hadamard, and Erdélyi–Kober operators.

By employing the $(\mathcal{MNC})$ and suitable $(\mathcal{FPT})$ techniques, including Darbo-type $(\mathcal{FPT})$, we derive sufficient conditions ensuring the existence, uniqueness, and continuous dependence of solutions of (1.1) on the given data in L_ψ (cf. [4, 30]). These results demonstrate that equation (1.1) is stable and mathematically well-behaved within the proposed functional framework.

Our approach generalizes and unifies several existing results in the literature. For instance, the author in [26] presented some basic properties of the $(\mathcal{RL})$ Riemann-Liouville type fractional integral operator and explored its solutions

$$y(v) \;=\; f(v)+G(y)(v)\int_0^v \frac{(v-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))\,ds,\quad 0<\alpha<1,\; v\in[0,d]$$

in Orlicz spaces L_ψ, see also [27].

The authors in [3,29] demonstrated and studied fundamental features of the Hadamard-type fractional operator within L_ψ and utilized them to solve the following equation

$$y(v) \;=\; G_2(y)(v)+\frac{G_1(y)(v)}{\Gamma(\alpha)}\int_1^v \left(\log\frac{v}{s}\right)^{\alpha-1}G_2(y)(s)\,ds,\quad v\in[1,e],\; 0<\alpha<1.$$

The authors in [31] showed the basic characteristics of the Erdélyi–Kober fractional operators in Lebesgue and Orlicz spaces and used them to analyze the problem

$$y(v) \;=\; f(v)+\; f_1(v,y(v))+f_2\left(v,\;\frac{\beta h_1(v,y(v))}{\Gamma(\alpha)}\cdot\int_0^v \frac{t^{\beta-1}h_2(s,y(s))}{(v^\beta-s^\beta)^{1-\alpha}}\,ds\right),\quad v\in[0,d],$$

where $0<\alpha<1$ and $\beta>0$ in the indicated spaces (see also [11]).

In addition, the $(\mathcal{MNC})$ and $(\mathcal{FPT})$ have been successfully applied in solving various classes of quadratic integral equations in Orlicz spaces under different sets of assumptions (cf. [5,7,14, 15,28]).

Inspired by these developments, as well as by models arising in statistical physics and applied analysis (cf. [22,24]), the present work extends the existing theory to the general g-fractional operator. This provides a robust and versatile analytical setting for the study of nonlinear fractional models with generalized memory kernels. Several examples are included to illustrate and substantiate the theoretical results.

2. Preliminaries

Let $\mathbb{R}=(-\infty,\infty)$ and $[0,d]=\mathbb{I}\subset\mathbb{R}^+=[0,\infty)$. Denote the Young function (YF) by $\psi:\mathbb{R}^+\to\mathbb{R}^+$, where

$$\psi(\theta)=\int_0^\theta f(s)ds,\quad \text{for}\quad \theta\geq 0$$

and $f:\mathbb{R}^+\to(0,\infty)$ is an increasing and left continuous function on $\mathbb{R}^+$. The pair (ψ,ϕ) is said to be a complementary pair of YF if $\phi(y)=\sup_{z\geq 0}(yz-\psi(y))$.

The function ψ is known as N-function when it is a finite-valued and verifies:

- $\lim_{\theta\to\infty}\frac{\psi(\theta)}{\theta}=\infty,\;\; \lim_{\theta\to 0}\frac{\psi(\theta)}{\theta}=0,$
- $\psi(\theta)>0$ if $\theta>0$ ($\psi(\theta)=0 \iff \theta=0$).

Denote by $L_\psi=L_\psi(\mathbb{I})$ the *Orlicz space* of all measurable functions $y:\mathbb{I}\to\mathbb{R}$ with the norm

$$\|y\|_\psi=\inf_{\lambda>0}\left\{\int_{\mathbb{I}}\psi\left(\frac{y(s)}{\lambda}\right)\,ds\leq 1\right\}<\infty.$$

It is important to recall that for any YF ψ, we have $\psi(\theta+s)\leq\psi(\theta)+\psi(s)$ and $\psi(\rho\theta)\leq\rho\psi(\theta)$, where $\theta,s\in\mathbb{R}$, and $\rho\in[0,1]$.

Assume that $E_\psi(\mathbb{I})$ is the set that contains all bounded functions in $L_\psi(\mathbb{I})$ with absolutely continuous norms.

Moreover, we get $E_\psi = L_\psi$ if ψ satisfies the condition:

$$\exists\ \omega,\ \theta_0 \geq 0 \ \text{ such that } \psi(2\theta) \leq \omega \cdot \psi(\theta),\ \theta \geq \theta_0, \qquad (\Delta_2\text{−condition}).$$

It is noted that, the classical Lebesgue spaces $L_p(\mathbb{I})$ shall be considered as a particular case of Orlicz spaces $L_{\psi_p}(\mathbb{I})$ with corresponding N-function $\psi_p = s^p$, satisfying the above Δ_2- condition.

Lemma 2.1. [22] *Assume that the function $h(v, y) : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ verifies Carathéodory conditions (i.e., it is continuous in the variable y for almost all $v \in \mathbb{I}$ and measurable in v for any $y \in \mathbb{R}$). The superposition operator $F_h = h(v, y) : E_{\psi_1} \to L_\psi = E_\psi$ is bounded and continuous if*

$$|h(v, y)| \leq a(v) + b \cdot \psi^{-1}\big(\psi_1(y)\big), \quad y \in \mathbb{R}, \quad v \in \mathbb{I},$$

where $a \in L_\psi$, $b \geq 0$, and the function $\psi(y)$ satisfies the Δ_2-condition.

Lemma 2.2. [25] *Assume that ψ, ψ_1, and ψ_2 are arbitrary different N-functions. The following given conditions are equivalent:*

1. *For every functions $y_1 \in L_{\psi_1}$ and $y_2 \in L_{\psi_2}$, $y_1 \cdot y_2 \in L_\psi$.*
2. *There exists $k > 0$ such that for all measurable y_1, y_2 on $\mathbb{I}$, we get $\|y_1 y_2\|_\psi \leq k \cdot \|y_1\|_{\psi_1} \|y_1\|_{\psi_2}$.*
3. *There exists $C > 0$, $s_0 \geq 0$ such that for all $v, s \geq s_0$, $\psi\left(\frac{sv}{C}\right) \leq \psi_1(s) + \psi_2(v)$.*
4. $\limsup_{v\to\infty} \frac{\psi_1^{-1}(v)\psi_2^{-1}(v)}{\psi(v)} < \infty$.

The set $S = S(\mathbb{I})$ is the set of Lebesgue measurable functions "means" on $(\mathbb{I})$ connected with the metric

$$d(y, x) = \inf_{\rho>0}[\epsilon + meas\{v : |y(v) - x(v)| \geq \rho\}]$$

is a complete space. Additionally, the convergence in measure on $\mathbb{I}$ is equivalent to the convergence regarding d (cf. [39]). The compactness in S is called "compactness in measure".

Lemma 2.3. [14] *Assume that $Y \subset L_\psi(\mathbb{I})$ is a bounded set, and there exists a family $(\Omega_r)_{0\leq r\leq d}$ $\subset \mathbb{I}$ such that meas $\Omega_r = r$ for every $r \in [0, d]$, and for every $y \in Y$,*

$$y(s_1) \geq y(s_2), \quad (s_1 \in \Omega_r,\ s_2 \notin \Omega_r).$$

Then, Y is compact in measure in $L_\psi(\mathbb{I})$.

Definition 2.1. [8] The Hausdorff measure of noncompactness $\beta_H(X)$-$(\mathcal{MNC})$ for a bounded set $\emptyset \neq Y \subset L_\psi$ is assigned by

$$\beta_H(Y) = \inf\{r > 0 :\ \exists\ X \subset L_\psi \text{ such that } Y \subset\ X\ +\ B_r\ \},$$

where the ball $B_r = \{y \in L_\psi(\mathbb{I}) : \|y\|_\psi \leq r\}$ is centered at the origin with radius r.

Denote a measure of equi-integrability c of $Y \in L_\psi(\mathbb{I})$ by :

$$c(Y)\ =\ \lim_{\epsilon\to 0}\ \sup_{measD\leq\epsilon}\ \sup_{y\in Y} \|y \cdot \chi_D\|_\psi,$$

where $\epsilon > 0$ and χ_A points to the characteristic function of $A \subset \mathbb{I}$ (see [39] or [17]).

Lemma 2.4. [14,17] *Let Y be a nonempty subset of L_ψ that is bounded and compact in measure. Then*

$$c(Y) = \beta_H(Y).$$

Theorem 2.1. [6] *Let Ω be a nonempty, bounded, closed, and convex subset of L_ψ, and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant b with $0 \le b < 1$ such that*

$$\beta_H(T(Y)) \le b\, \beta_H(Y)$$

for every nonempty subset $Y \subset \Omega$. Then T has at least one fixed point in Ω.

3. Generalized fractional operators

We provide and prove some properties and concepts of the generalized fractional (or g-fractional) operator in L_ψ.

Definition 3.1. [21,36] The g-fractional, or generalized fractional, integral of a function y of order α concerning a different function g is given by

$$J_g^\alpha y(v) = \frac{1}{\Gamma(\alpha)} \int_0^v \frac{y(s)g'(s)}{\Big(g(v) - g(s)\Big)^{1-\alpha}} ds, \;\; \alpha > 0, \tag{3.1}$$

where g is a positive and increasing function on $(0, \infty]$ and has a continuous derivative on $(0, \infty)$.

Table 1. Some particular cases of Definition 3.1 corresponding to different forms of $g(t)$.

$g(t)$	**Common name**	**Fractional integral**	**Applications**
t	Riemann–Liouville	$\frac{1}{\Gamma(\alpha)} \int_0^v \frac{y(s)\, ds}{(v-s)^{1-\alpha}}$	Viscoelasticity, control theory [21].
$\ln t$	Hadamard	$\int_1^v \left(\ln \frac{v}{s}\right)^{\alpha-1} \frac{y(s)\, ds}{\Gamma(\alpha)s}$	Finance, biological growths [36].
$t^\beta, \beta > 0$	Erdélyi–Kober	$\frac{1}{\Gamma(\alpha)} \int_0^v \frac{\beta s^{\beta-1} y(s)\, ds}{(v^\beta - s^\beta)^{1-\alpha}}$	Diffusion equation, geophysics [21].
t^2	Sneddon–type	$\frac{2}{\Gamma(\alpha)} \int_0^v \frac{y(s)\, s ds}{(v^2 - s^2)^{1-\alpha}}$	Potential theory, elasticity [37].
$t, \; \alpha = 1$	Hardy–Littlewood	$\frac{1}{t} \int_0^t y(s)\, ds.$	Harmonic analysis, PDE theory [13].

Now, we examine the monotonicity of the operator J_g^α (3.1).

Lemma 3.1. *The operator J_g^α, $\alpha > 0$ with $g(0) = 0$ maps nonnegative and a.e. nondecreasing functions into functions having similar properties.*

Proof. Let $v_1, v_2 \in \mathbb{I}$, $v_1 \le v_2$, and y be a.e. nondecreasing and nonnegative function, then by putting $g(s) = g(v_1) \cdot u$, we have

$$J_g^\alpha y(v_1) = \frac{1}{\Gamma(\alpha)} \int_0^{v_1} \frac{y(s)}{\Big(g(v_1) - g(s)\Big)^{1-\alpha}} g'(s) ds$$

$$= \frac{g^{\alpha}(v_1)}{\Gamma(\alpha)} \int_0^1 \frac{y\Big(g^{-1}\big(g(v_1)\cdot u\big)\Big)}{\big(1-u\big)^{1-\alpha}} du.$$

Since the function g is an increasing and positive function on $(0,\infty]$, then its inverse g^{-1} exists and has also the same properties (cf. [10]), then we get

$$J_g^{\alpha} y(v_1) \leq \frac{g^{\alpha}(v_2)}{\Gamma(\alpha)} \int_0^1 \frac{y\Big(g^{-1}\big(g(v_2)\cdot u\big)\Big)}{\big(1-u\big)^{1-\alpha}} du = J_g^{\alpha} y(v_2).$$

Therefore, $0 \leq J_g^{\alpha} y(v_1) \leq J_g^{\alpha} y(v_2)$ for $v_1 \leq v_2$. □

Proposition 3.1. [35] *Let P be a (YF) Young function, then for $v \in \mathbb{R}^+$ and any $0 < \alpha < 1$, the set*

$$\mathbb{P}(s) = \left\{ \epsilon > 0 : \quad \frac{1}{\|g'\|_P} \int_0^{g(v)\sigma^{\frac{1}{1-\alpha}}} P(u^{\alpha-1})\, du \leq \sigma^{\frac{1}{1-\alpha}} \right\} \neq \emptyset, \quad \sigma = \frac{\epsilon}{\|g'\|_P},$$

is nonempty with $\mathbb{P}(0) = 0$, where the function g is defined in Definition 3.1.

Lemma 3.2. *Let (P, Q) be a complementary pair of N-functions and ψ be an N-function, where P verifies $\int_0^{g(v)} P(u^{\alpha-1})\, du < \infty$, $\alpha \in (0,1)$. Then the operator $J_g^{\alpha} : L_Q(\mathbb{I}) \to L_{\psi}(\mathbb{I})$ is continuous, where*

$$k(v) = \frac{\sigma^{\frac{1}{\alpha-1}}}{\|g'\|_P} \int_0^{g(v)\sigma^{\frac{1}{1-\alpha}}} P(u^{\alpha-1})\, du \in E_{\psi}(\mathbb{I}), \quad \epsilon > 0, \quad \sigma = \frac{\epsilon}{\|g'\|_P}$$

for a.e. $v \in \mathbb{I}$.

Proof. Suppose that

$$K(v,s) = \begin{cases} \dfrac{\Big(g(v) - g(s)\Big)^{\alpha-1} g'(s)}{\Gamma(\alpha)}, & \text{if } \; s \in [0,v],\; v > 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $y \in L_Q(\mathbb{I})$ and by utilizing Hölder inequality, we get:

$$\begin{aligned} |J_g^{\alpha} y(v)| &= \left| \int_0^v K(v,s) y(s)\, ds \right| \\ &\leq 2\|K(v,\cdot)\|_P \|y\|_Q \\ &\leq \frac{2}{\Gamma(\alpha)} \inf_{\epsilon>0} \left\{ \int_{\mathbb{I}} P\left(\frac{\Big(g(v)-g(s)\Big)^{\alpha-1} g'(s)}{\epsilon} \right) ds \leq 1 \right\} \|y\|_Q \\ &= \frac{2}{\Gamma(\alpha)} \inf_{\epsilon>0} \left\{ \int_{\mathbb{I}} P\left(\frac{\Big(g(v)-g(s)\Big)^{\alpha-1} g'(s)\big\|g'\big\|_P}{\epsilon \cdot \big\|g'\big\|_P} \right) ds \leq 1 \right\} \|y\|_Q \end{aligned}$$

$$\leq \frac{2}{\Gamma(\alpha)} \inf_{\epsilon>0} \left\{ \int_{\mathbb{I}} P\left(\frac{\Big(g(v)-g(s)\Big)^{\alpha-1} \|g'\|_P}{\epsilon} \right) \frac{g'(s)}{\|g'\|_P}\, ds \leq 1 \right\} \|y\|_Q.$$

Put $u = \big(g(v)-g(s)\big)\sigma^{\frac{1}{1-\alpha}}$, where $\sigma = \frac{\epsilon}{\|g'\|_P}$, we have

$$\begin{aligned} \|J_g^\alpha y\|_\psi &\leq \frac{2}{\Gamma(\alpha)} \left\| \inf_{\epsilon>0} \left\{ \frac{1}{\|g'\|_P} \int_0^{g(v)\sigma^{\frac{1}{1-\alpha}}} P(u^{\alpha-1})\, du \leq \sigma^{\frac{1}{1-\alpha}} \right\} \right\|_\psi \|y\|_Q \\ &\leq \frac{2}{\Gamma(\alpha)} \|k\|_\psi \|y\|_Q, \end{aligned}$$

where $k \in E_\psi(\mathbb{I})$. Then, by recalling Proposition 3.1 and [22, Lemma 16.3], we get $J_g^\alpha : L_Q \to L_\psi$ and is continuous. □

4. Main results

First, we define the operator B associated with the right-hand side of equation (1.1) as follows:

$$y = B(y) = f + U(y),$$

where

$$U(y) = F_{h_1}(y) \cdot A(y), \quad A(y)(v) = J_g^\alpha F_{h_2}(y)(v),$$

such that J_g^α is defined in Definition 3.1 and $F_{h_i}, i = 1,2$ are known as the superposition operators.

Next, we shall discuss our existence theorem in L_ψ-spaces in the most interesting case, where the generating N-functions verifying the Δ_2-condition (cf. [1,15,28,31]). These allow us to utilize some general conditions for the studied functions.

Theorem 4.1. *Assume that $\psi_1, \psi_2,$ and ψ are N-functions and (P,Q) is a complementary pair of N-functions, such that Q, ψ, ψ_1 satisfy the Δ_2-condition and $\int_0^{g(v)} P(u^{\alpha-1})\, du < \infty$, $\alpha \in (0,1)$ and that:*

- *(G1) There exists $k_1 > 0$ such that for $y_1 \in L_{\psi_1}(\mathbb{I})$ and $y_2 \in L_{\psi_2}(\mathbb{I})$ we get $\|y_1 y_2\|_\psi \leq k_1 \|y_1\|_{\psi_1} \|y_2\|_{\psi_2}$.*
- *(C1) $f \in E_\psi(\mathbb{I})$ is a.e. nondecreasing on the interval $\mathbb{I}$.*
- *(C2) $h_i : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ satisfy Carathéodory conditions and $(s,y) \to h_i(s,y)$ are nondecreasing with respect to each variable separately.*
- *(C3) There exist $d_i \geq 0$, $i = 1,2$ and functions $b_1 \in E_{\psi_1}(\mathbb{I})$, and $b_2 \in E_Q(\mathbb{I})$, such that*

$$|h_1(s,y)| \leq b_1(s) + d_1 \psi_1^{-1}\Big(\psi(y)\Big), \quad |h_2(s,y)| \leq b_2(s) + d_2 Q^{-1}\Big(\psi(y)\Big). \tag{4.1}$$

- *(C4) Assume that for a.e. $v \in \mathbb{I}$, there exists $\epsilon > 0$, such that*

$$k(v) = \frac{\sigma^{\frac{1}{\alpha-1}}}{\|g'\|_P} \int_0^{\sigma^{\frac{1}{1-\alpha}} g(v)} P(u^{\alpha-1})\, du \in E_{\psi_2}(\mathbb{I}), \quad \sigma = \frac{\epsilon}{\|g'\|_P}.$$

(C5) Assume that there exists a constant $r > 0$ such that the following inequalities holds on $I_0 = [0, d_0] \subset \mathbb{I}$

$$\int_{I_0} \psi\left(|f(v)| + \frac{2k_1}{\Gamma(\alpha)}\|k\|_{\psi_2}\left(\|b_1\|_{\psi_1} + d_1 r\right)\left(\|b_2\|_Q + d_2 r\right)\right) dv \le r$$

and

$$\left[\frac{2d_1k_1}{\Gamma(\alpha)}\|k\|_{\psi_2}\left(\|b_2\|_Q + d_2 \cdot r\right)\right] < 1.$$

Then, there is a.e. nondecreasing solution $y \in E_\psi(I_0)$ of equation (1.1) on $I_0 \subset \mathbb{I}$.

Proof. **Step I.** Lemma 2.1, together with assumptions *(C2)* and *(C3)*, implies that

$$F_{h_1} : E_\psi(\mathbb{I}) \to L_{\psi_1}(\mathbb{I}) \quad \text{and} \quad F_{h_2} : E_\psi(\mathbb{I}) \to L_Q(\mathbb{I})$$

are continuous operators. Moreover, by assumption *(C4)* and Lemma 3.2, the operator

$$A = J_g^\alpha F_{h_2} : E_\psi(\mathbb{I}) \to E_{\psi_2}(\mathbb{I})$$

is continuous. Assumption *(G1)* then ensures that

$$U = F_{h_1} \cdot A : E_\psi(\mathbb{I}) \to E_\psi(\mathbb{I})$$

is well defined and continuous. Finally, by assumption *(C1)*, the operator

$$B : E_\psi(\mathbb{I}) \to E_\psi(\mathbb{I})$$

is continuous.

Step II. Next, we verify that the operator B is bounded in $E_\psi(\mathbb{I})$.

Let Ω denote the closure of the set

$$\left\{y \in E_\psi(I_0) : \int_0^{d_0} \psi(|y(s)|)\, ds \le r - 1\right\}.$$

Clearly, Ω is not a ball in $E_\psi(I_0)$; however, $\Omega \subset B_r\big(E_\psi(I_0)\big)$ (see [22], p. 222). Moreover, the set $\overline{\Omega} \subset E_\psi(I_0)$ is bounded, convex, and closed.

By applying [22, Theorem 10.5] with constant $k = 1$ (see [22]), for arbitrary $y \in \Omega$ and $v \in I_0$, we obtain

$$\begin{aligned} &\left\|\psi_1^{-1}\big(\psi(|y|)\big)\right\|_{\psi_1} \le \|y\|_\psi = 1 + \int_{I_0} \psi\big(|y(v)|\big)\, dv, \quad \text{and} \\ &\left\|Q^{-1}\big(\psi(|y|)\big)\right\|_Q \le \|y\|_\psi = 1 + \int_{I_0} \psi\big(|y(v)|\big)\, dv. \end{aligned} \tag{4.2}$$

Therefore, in view of Lemma 3.2 and the imposed assumptions, we conclude that

$$\begin{aligned} &|B(y)(v)| \\ &\le |f(v)| + |U(y)(v)| \\ &\le |f(v)| + k_1\|F_{h_1}(y)\|_{\psi_1}\|A(y)\|_{\psi_2} \end{aligned}$$

$$
\begin{aligned}
&\leq |f(v)| + k_1 \Big\| b_1 + d_1 \psi_1^{-1}\Big(\psi(|y|)\Big)\Big\|_{\psi_1} \cdot \|J_g^{\alpha} F_{h_2}(y)\|_{\psi_2} \\
&\leq |f(v)| + k_1 \bigg(\|b_1\|_{\psi_1} + d_1 \Big\| \psi_1^{-1}\Big(\psi(|y|)\Big)\Big\|_{\psi_1} \bigg) \frac{2}{\Gamma(\alpha)} \|k\|_{\psi_2} \bigg(\|b_2\|_Q + d_2 \Big\| Q^{-1}\Big(\psi(|y|)\Big)\Big\|_Q \bigg) \\
&\leq |f(v)| + \frac{2k_1}{\Gamma(\alpha)} \|k\|_{\psi_2} \bigg(\|b_1\|_{\psi_1} + d_1 + d_1 \int_{I_0} \psi(y(v))dv \bigg) \bigg(\|b_2\|_Q + d_2 + d_2 \int_{I_0} \psi(y(v))dv \bigg) \\
&\leq |f(v)| + \frac{2k_1}{\Gamma(\alpha)} \|k\|_{\psi_2} \Big(\|b_1\|_{\psi_1} + d_1 + d_1(r-1) \Big) \Big(\|b_2\|_Q + d_2 + d_2(r-1) \Big).
\end{aligned}
$$

Recalling assumption *(C5)*, we obtain

$$
\int_{I_0} \psi\big(B(y)(v)\big)\, dv \leq \int_{I_0} \psi\bigg(|f(v)| + \frac{2k_1}{\Gamma(\alpha)} \|k\|_{\psi_2} \big(\|b_1\|_{\psi_1} + d_1 r\big)\big(\|b_2\|_Q + d_2 r\big) \bigg)\, dv \leq r.
$$

Hence, $B(\Omega) \subset \Omega$, and consequently

$$
B(\overline{\Omega}) \subset \overline{B(\Omega)} \subset \overline{\Omega} = \Omega.
$$

Therefore, the operator $B : \Omega \to \Omega$ is well defined and continuous on $\Omega \subset B_r\big(E_{\psi}(I_0)\big)$.

Step III. Assume that $\Omega_r \subset \Omega$ contains all a.e. monotonic (nondecreasing) functions on the interval I_0. The set $\emptyset \neq \Omega_r$ is bounded, closed, compact in measure, and convex in $L_{\psi}(I_0)$ (cf. [15]).

Step IV. The monotonicity of the functions is preserved via the operator B.

Take $y \in \Omega_r$, then y is a.e. nondecreasing on I_0 and, consequently, the operators $F_{h_i}, i = 1, 2$ are also a.e. nondecreasing on I_0. By Lemma 3.1, A is a.e. nondecreasing on I_0, then $U = F_{h_1} \cdot A$ is also a.e. nondecreasing on I_0. Using (C1), we get $B : \Omega_r \to \Omega_r$ is continuous.

Step V. Now, we show that B satisfies the contraction condition with respect to β_H.

Suppose there is a set $D \subset I_0$, with measure $D \leq \varepsilon$, $\varepsilon > 0$. Therefore, for $y \in Y$ and $\emptyset \neq Y \subset \Omega_r$, we have:

$$
\begin{aligned}
\|B(y) \cdot \chi_D\|_{\psi} &\leq \|f \cdot \chi_D\|_{\psi} + \|F_{h_1}(y) A(y) \cdot \chi_D\|_{\psi} \\
&\leq \|f \cdot \chi_D\|_{\psi} + k_1 \|F_{h_1}(y) \cdot \chi_D\|_{\psi_1} \|A(y) \cdot \chi_D\|_{\psi_2} \\
&\leq \|f \cdot \chi_D\|_{\psi} + k_1 \bigg\| \Big(b_1 + d_1 \psi_1^{-1}\Big(\psi(|y|)\Big)\Big) \cdot \chi_D \bigg\|_{\psi_1} \cdot \Big\| J_g^{\alpha} F_{h_2} \Big\|_{\psi_2} \\
&\leq \|f \cdot \chi_D\|_{\psi} + k_1 \bigg(\|b_1 \cdot \chi_D\|_{\psi_1} + d_1 \Big\| \psi_1^{-1}\Big(\psi(|y|)\Big) \cdot \chi_D \Big\|_{\psi_1} \bigg) \\
&\quad \times \frac{2}{\Gamma(\alpha)} \|k\|_{\psi_2} \bigg(\|b_2\|_Q + d_2 \Big\| Q^{-1}\Big(\psi(|y|)\Big)\Big\|_Q \bigg) \\
&\leq \|f \cdot \chi_D\|_{\psi} + \frac{2k_1}{\Gamma(\alpha)} \|k\|_{\psi_2} \Big(\|b_1 \cdot \chi_D\|_{\psi_1} + d_1 \|y \cdot \chi_D\|_{\psi} \Big) \Big(\|b_2\|_Q + d_2 \cdot r \Big).
\end{aligned}
$$

Since $f \in E_{\psi}$, $b_1 \in E_{\psi_1}$, then we get

$$
\lim_{\varepsilon \to 0} \{ \sup_{meas\ D \leq \varepsilon} [\sup_{y \in Y} \{\|f \cdot \chi_D\|_{\psi}\}]\} = 0
$$

and

$$
\lim_{\varepsilon \to 0} \{ \sup_{meas\ D \leq \varepsilon} [\sup_{y \in Y} \{\|b_1 \cdot \chi_D\|_{\psi_1}\}]\} = 0.
$$

By using the formula of $c(Y)$, we get

$$c(B(Y)) \leq \left(\frac{2d_1k_1}{\Gamma(\alpha)}\|k\|_{\psi_2}\Big(\|b_2\|_Q + d_2 \cdot r\Big)\right)c(Y).$$

Based on the previously established properties, we may apply Lemma 2.4 to get

$$\beta_H(B(Y)) \leq \left(\frac{2d_1k_1}{\Gamma(\alpha)}\|k\|_{\psi_2}\Big(\|b_2\|_Q + d_2 \cdot r\Big)\right)\beta_H(Y).$$

The above inequality with $\left(\frac{2d_1k_1}{\Gamma(\alpha)}\|k\|_{\psi_2}\Big[\|b_2\|_Q + d_2 \cdot r\Big]\right) < 1$ allows us to apply Theorem 2.1. That ends the proof. □

4.1. Uniqueness of the solution

In this section, we may prove and discuss the uniqueness of the solutions of (1.1).

Theorem 4.2. *Assume that assumptions of Theorem 4.1 are verified, but replace the inequalities (4.1) with:*

(C6) $|h_i(v,0)| \leq b_i(v),\ b_1 \in E_{\psi_1}(\mathbb{I}),\quad b_2 \in E_Q(\mathbb{I})$ *and*

$$|h_1(v,y) - h_1(v,z)| \leq d_1\psi_1^{-1}\Big(\psi\big(|y-z|\big)\Big),$$
$$|h_2(v,y) - h_2(v,z)| \leq d_2Q^{-1}\Big(\psi\big(|y-z|\big)\Big),\quad y,z \in \Omega_r,$$

where $d_i \geq 0$, *and* Ω_r *is as in Theorem 4.1 for* $i = 1,2$.

(C7) Assume that

$$C = \frac{2k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\left(d_2\Big(\|b_1\|_{\psi_1} + d_1 \cdot r\Big) + d_1\Big(\|b_2\|_Q + d_2 \cdot r\Big)\right) < 1,$$

where r is given in assumption (C5).

Then (1.1) has a unique solution $y \in E_\psi$ in Ω_r.

Proof. Using assumption (C6), we get

$$\Big||h_1(v,y)| - |h_1(v,0)|\Big| \leq |h_1(v,y) - h_1(v,0)| \leq d_1\psi_1^{-1}\Big(\psi(y)\Big)$$
$$\Rightarrow |h_1(v,y)| \leq |h_1(v,0)| + d_1\psi_1^{-1}\Big(\psi(y)\Big) \leq b_1(v) + d_1\psi_1^{-1}\Big(\psi(y)\Big).$$

Similarly, $|h_2(v,y)| \leq b_2(v) + d_2Q^{-1}\Big(\psi(y)\Big)$. Thus, Theorem 4.1 implies that, there exists a.e. nondecreasing solution $y \in E_\psi$ of (1.1) in Ω_r.

Next, Let $y,\ z \in \Omega_r$ be two distinct solutions of equation (1.1), then by using the inequalities (4.2) and assumption (C6), we obtain

$$\|y - z\|_\psi$$

$$
\begin{aligned}
&\leq \big\|F_{h_1}(y)A(y) - F_{h_1}(z)A(z)\|_{\psi} \\
&\leq \big\|F_{h_1}(y)A(y) - F_{h_1}(y)A(z)\|_{\psi} + \|F_{h_1}(y)A(z) - F_{h_1}(z)A(z)\|_{\psi} \\
&\leq k_1\big\|F_{h_1}(y)\|_{\psi_1}\|A(y) - A(z)\|_{\psi_2} + k_1\|F_{h_1}(y) - F_{h_1}(z)\|_{\psi_1}\|A(z)\|_{\psi_2} \\
&\leq k_1\left\|b_1 + d_1\psi_1^{-1}\Big(\psi(y)\Big)\right\|_{\psi_1}\left\|J_g^{\alpha}\Big(F_{h_2}(y) - F_{h_2}(z)\Big)\right\|_{\psi_2} \\
&\quad +k_1\left\|d_1\psi_1^{-1}\Big(\psi\big(|y-z|\big)\Big)\right\|_{\psi_1}\left\|J_g^{\alpha}F_{h_2}(z)\right\|_{\psi_2} \\
&\leq k_1\Big(\|b_1\|_{\psi_1} + d_1\cdot r\Big)\frac{2\|k\|_{\psi_2}}{\Gamma(\alpha)}\left\|F_{h_2}(y) - F_{h_2}(z)\right\|_{Q} + d_1k_1\|y-z\|_{\psi}\frac{2\|k\|_{\psi_2}}{\Gamma(\alpha)}\left\|F_{h_2}(z)\right\|_{Q} \\
&\leq \frac{2k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\Big(\|b_1\|_{\psi_1} + d_1\cdot r\Big)\left\|d_2Q^{-1}\Big(\psi(|y-z|)\Big)\right\|_{Q} \\
&\quad +\frac{2d_1k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\|y-z\|_{\psi}\left\|b_2 + d_2Q^{-1}\Big(\psi(|z|)\Big)\right\|_{Q} \\
&\leq \frac{2k_1d_2\|k\|_{\psi_2}}{\Gamma(\alpha)}\Big(\|b_1\|_{\psi_1} + d_1\cdot r\Big)\Big\|y-z\Big\|_{\psi} + \frac{2d_1k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\Big\|y-z\Big\|_{\psi}\Big(\|b_2\|_Q + d_2\cdot r\Big) \\
&= \frac{2k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\bigg(d_2\Big(\|b_1\|_{\psi_1} + d_1\cdot r\Big) + d_1\Big(\|b_2\|_Q + d_2\cdot r\Big)\bigg)\big\|y-z\big\|_{\psi} \\
&= C\cdot\big\|y-z\big\|_{\psi}.
\end{aligned}
$$

The above estimate, based on assumption (C7), concludes the proof. □

4.2. Continuous dependence on the function

In this section, we investigate the third condition of well-posedness for the studied equation, i.e., we examine how solutions of equation (1.1) depend continuously on the function f.

Remark 4.1. The concepts of stability and continuous dependence on the data in the context of integral equations are closely related, essentially equivalent in both practical and theoretical terms.

Definition 4.1. [4,30] A solution $y \in L_{\psi}$ of (1.1) is said to be continuously dependent on the function f if $\forall\ \epsilon > 0$, there exists $\delta > o$ such that $\|f - \bar{f}\|_{\psi} \leq \delta$ implies that $\|y - \bar{y}\|_{\psi} \leq \epsilon$, where

$$
\bar{y}(v) \ = \bar{f}(v) + \frac{h_1(v,\bar{y}(v))}{\Gamma(\alpha)}\cdot\int_0^v \frac{h_2(s,\bar{y}(s))}{\Big(g(v)-g(s)\Big)^{1-\alpha}}g'(s)\,ds, \quad t \in \mathbb{I}.
$$

Theorem 4.3. *Let all the assumptions of Theorem 4.2 be held. Then the solutions $y \in L_{\psi}$ of the equation (1.1) depend continuously on the function f.*

Proof. Let $y, \bar{y}$ be two distinct solutions of (1.1), then similar to what is indicated in Theorem 4.2, we obtain

$$
\begin{aligned}
&\|y-\bar{y}\|_{\psi} \\
\leq &\left\|f + \frac{h_1(v,y)}{\Gamma(\alpha)}\int_0^v \frac{h_2(s,y(s))}{\Big(g(v)-g(s)\Big)^{1-\alpha}}g'(s)\,ds - \bar{f} - \frac{h_1(v,\bar{y})}{\Gamma(\alpha)}\int_0^v \frac{h_2(s,\bar{y}(s))}{\Big(g(v)-g(s)\Big)^{1-\alpha}}g'(s)\,ds\right\|_{\psi}
\end{aligned}
$$

$\leq \|f-\bar{f}\|_\psi + C\|y-\bar{y}\|_\psi,$

where C is given by assumption (C7). Then, we get

$$\|y-\bar{y}\|_\psi \leq \Big(1-C\Big)^{-1}\|f-\bar{f}\|_\psi.$$

Therefore, if $\|f-\bar{f}\|_\psi \leq \delta(\epsilon)$, then $\|y-\bar{y}\|_\psi \leq \epsilon$, and $\delta(\epsilon)=\epsilon\cdot(1-C)$. □

5. Examples

Finally, we provide examples that demonstrate and clarify our outcomes.

Example 5.1. Here are a few additional interesting forms of the functions $g(v)$, that can be considered as examples of Definition 3.1:

- Let $g(v)=t^\beta+c, \beta>0$. One can call it the shifting scaling kernel, and it is similar to the Erdélyi-Kober operator [21], but shifted by $c>0$, which can be applied in Memory effects with delay.
- Let $g(v)=\sin v,\ g'(v)=\cos v$. One can refer to it as a wave-modulated kernel, and the operator J_g^α takes the following form

$$J_{\sin}^\alpha\ y(v)=\frac{1}{\Gamma(\alpha)}\int_0^v \frac{\cos s\ y(s)\ ds}{(\sin v-\sin s)^{1-\alpha}},$$

  which can be applied in Wave propagation with memory.
- Let $g(v)=e^v,\ g'(v)=e^v$. One could call it the exponential kernel, and the operator J_g^α takes the form

$$J_e^\alpha y(v)=\frac{1}{\Gamma(\alpha)}\int_0^v \frac{e^s\ y(s)}{(e^v-e^s)^{1-\alpha}}\ ds,$$

  which can be applied to population growth and chemical kinetics.

Example 5.2. Considering the N-functions $P(\theta)=Q(\theta)=\theta^2$ and $\psi_2(\theta)=\exp|\theta|-|\theta|-1$. We can examine that Lemma 3.2 is verified and the fractional operator $J_g^\alpha : L_Q(\mathbb{I})\to L_{\psi_2}(\mathbb{I})$ is continuous, where $\mathbb{I}=[0,d]$.

Therefore: For any $\alpha,\beta>0$ and $v\in\mathbb{I}$, we get

$$k(v)=\int_0^{g(v)} P\big(\theta^{\alpha-1}\big)\,d\theta=\int_0^{g(v)}\theta^{2\alpha-2}\ d\theta=\frac{g(v)^{2\alpha-1}}{2\alpha-1}.$$

Moreover,

$$\int_0^d \psi_2\big(k(v)\big)\,dv=\int_0^d\left(\exp\left|\frac{g(v)^{2\alpha-1}}{2\alpha-1}\right|-\left|\frac{g(v)^{2\alpha-1}}{2\alpha-1}\right|-1\right)dv\ <\infty,$$

where g is a positive and increasing function on $(0,\infty]$. Then, Lemma 3.2 is verified. Therefore, for $y\in L_Q(\mathbb{I})$, we obtain $J_g^\alpha : L_{Q=\theta^2}(\mathbb{I})\to L_{\psi_2}(\mathbb{I})$ is continuous.

For further particular cases verifying Example 5.2, see [26,29,31], and for more information and numerous examples of the functions ψ and (P,Q) verifying Lemma 3.2, see [22].

Example 5.3. Let $\alpha = \frac{2}{3}$ and consider the N-functions $P(\theta) = Q(\theta) = \theta^2,\ \psi(\theta) = |\theta^\gamma| \ln\big(|\theta| + 1\big), \gamma \geq \frac{3+\sqrt{5}}{2},\ \psi_1(\theta) = \big(|\theta|+1\big)\ln\big(|\theta|+1\big) - |\theta|$, and $\psi_2(t) = \exp|\theta| - |\theta| - 1$, with the equation

$$y(v) = e^{2v} + \Big(e^v + d_1\psi_1^{-1}\big(\psi\big(y(v)\big)\big)\Big)\int_0^v \frac{e^s + d_2 Q^{-1}\Big(\psi\big(y(s)\big)\Big)}{\Gamma(\frac{2}{3})\Big(\sin v - \sin s\Big)^{\frac{1}{3}}}\cos s\, ds,\ \ v \in I_0 = [0,1].$$

- Note the N functions Q, ψ and ψ_1 verify Δ_2-condition.
- The N functions ψ, ψ_1 and ψ_2 verify the inequality

$$\psi(\frac{s\cdot t}{1}) = \psi(st) \leq |s|^\gamma |t|^\gamma \ln\big(|st|+1\big) \leq |s|^\gamma|t|^\gamma\big[\ln(|s|+1) + \ln(|t|+1)\big].$$

  Since $\gamma > 2$,Young-type inequalities imply that there exist $c_1, c_2 > 0$ such that

$$|s|^\gamma|t|^\gamma \leq c_1(|s|+1)\ln(|s|+1) + c_2(\exp|t| - 1), \qquad s,t \in \mathbb{R}.$$

  Combining the above estimates, we obtain

$$\psi(st) \leq \big(|s|+1\big)\ln\big(|s|+1\big) - |s| + \exp|t| - |t| - 1 \leq \psi_1(s) + \psi_2(t),$$

  then by recalling Lemma $2.2_{3.}$, with constant $C = 1$, we have for $y_1 \in L_{\psi_1}, y_2 \in L_{\psi_2}$, and $y_1 \cdot y_2 \in L_\psi$ that

$$\|y_1y_2\|_\psi \leq k_1\|y_1\|_{\psi_1}\|y_1\|_{\psi_2},$$

  which verifies assumption (G1).
- Assumptions (C2) and (C3) are verified for the functions

$$|h_1(v,y)| \leq e^v + d_1\psi_1^{-1}\big(\psi\big(y\big)\big),\quad |h_2(v,y)| \leq e^v + d_2Q^{-1}\Big(\psi\big(y\big)\Big),$$

  where $b_1(v) = e^v \in L_{\psi_1},\ b_2(v) = e^v \in L_Q$ and $Q^{-1}(t) = \sqrt{t},\ \psi_1^{-1}(t) = \frac{t-1}{W(\frac{t-1}{e})} - 1$, where W denotes the Lambert W function $W(a)e^{W(a)} = a$ see [16].
- Assumption (C4) is verified (see Example 5.2), where

$$g(t) = \sin t \ \Rightarrow\ g'(t) = \cos t \ \Rightarrow\ \|g\|_P \leq 1.$$

- With proper constants $d_1 > 0, d_2 > 0, k_1 > 0$ and the function $f(v) = e^{2v} \in L_\psi$, we can find $r > 0$ on $I_0 = [0, d_0]$ with $0 < d_0 \leq 1$, where

$$k(v) = \int_0^{\sin v} \theta^{2(\frac{2}{3}-1)}\, d\theta = \int_0^{\sin v}\theta^{\frac{2}{3}}\, d\theta = 3\,(\sin v)^{1/3},$$

so that $|k(v) \leq 3(\sin d_0)^{1/3} =: C_{\sin,2/3}$, Then

$$\|k\|_{\psi_2} \leq C_{\sin,2/3}, \qquad v \in I_0.$$

Moreover, for all $v \in I_0$,

$$|b_1(v)| = |b_2(v)| = e^v \leq e, \quad |f(v)| = e^{2v} \leq e^2$$
$$\Longrightarrow \|b_1\|_{\psi_1} \leq (e+1)\ln(e+1) - e, \quad \|b_2\|_Q \leq e,$$

$$\begin{aligned}
&|f(v)| + \frac{2k_1}{\Gamma(2/3)}\|k\|_{\psi_2}\big(\|b_1\|_{\psi_1} + d_1 r\big)\big(\|b_2\|_Q + d_2 r\big)\\
&\le e^2 + \frac{2k_1}{\Gamma(2/3)} C_{\sin,2/3}\big((e+1)\ln(e+1) - e + d_1\big)(e + d_2)\\
&=: e^2 + A_0,
\end{aligned}$$

where $A_0 = \frac{2k_1}{\Gamma(2/3)} C_{\sin,2/3}\big((e+1)\ln(e+1) - e + d_1\big)(e + d_2)$.

Since ψ is increasing on $[0,\infty)$, it follows that

$$\psi\Big(|f(v)| + \frac{2k_1}{\Gamma(2/3)}\|k\|_{\psi_2}\big(\|b_1\|_{\psi_1} + d_1 r\big)\big(\|b_2\|_Q + d_2 r\big)\Big) \le \psi(e^2 + A_0), \qquad v \in I_0.$$

Therefore,

$$\int_{I_0} \psi\Big(|f(v)| + \frac{2k_1}{\Gamma(2/3)}\|k\|_{\psi_2}\big(\|b_1\|_{\psi_1} + d_1 r\big)\big(\|b_2\|_Q + d_2 r\big)\Big)\, dv \le d_0\, \psi(e^2 + A_0) \le \psi(e^2 + A_0).$$

Defining $r = \min\{1, \psi(e^2 + A_0)\}$ gives

$$\int_{I_0} \psi\Big(|f(v)| + \frac{2k_1}{\Gamma(2/3)}\|k\|_{\psi_2}\big(\|b_1\|_{\psi_1} + d_1 r\big)\big(\|b_2\|_Q + d_2 r\big)\Big)\, dv\Big)\, dv \le r,$$

that verifies assumption (C5).

Therefore, all assumptions of Theorem 4.1 are verified, and equation (1.1) has a solution $y \in E_\psi(I_0)$.

6. Conclusion

In this article, we prove and illustrate several novel features of the g-fractional type operator, encompassing boundedness, monotonicity, acting, and continuity within the Orlicz spaces L_ψ. The g-fractional operator combines and unifies many forms of fractional operators, such as Hadamard, Riemann-Liouville, and Erdélyi–Kober fractional operators, into one form. We examine the well-posedness (existence, uniqueness, continuous dependence on the data) for the quadratic integral equation related to the g-fractional type operator in the L_ψ, utilizing the aforementioned properties and proper $(\mathcal{MNC})$ concerning $(\mathcal{FPT})$. We present several examples that substantiate our outcomes. Upcoming research in this domain will investigate the fundamental properties of numerous fractional operators across different function spaces and utilize these properties to address diverse fractional problems.

Conflicts of interest

The authors proclaim that there is no conflict of interest.

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