Journal of Applied Analysis and Computation
Volume 16, Number 6, December 2026, 3098–3144
Website:http://www.jaac-online.com
DOI:10.11948/20250238

DYNAMIC SYSTEMS INVOLVING HUMAN UNCERTAINTY AND PROBABILITY DISTRIBUTION WITH AMBIGUITY: PART I. WELL-POSEDNESS, STABILITY AND NUMERICAL METHOD*

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Abstract The operation of a system in the real world is inevitably influenced by various indeterminate factors. In many cases, a hybrid system always exists both human uncertainty and probability distribution with ambiguity. To model this complex phenomenon, uncertain-sublinear (U-S) chance theory offers an effective mathematical method by introducing the concept of uncertain random variable under U-S chance spaces, and it has proved significant effectiveness in handling issues related to static systems. To model the evolution of this type of uncertain stochastic dynamic system, this paper introduces an uncertain stochastic differential equation driven by canonical Liu process and G-Brownian motion (LG-USDE). Firstly, we construct the method of uncertain stochastic integral (named as G-Itô-Liu integral) and present a G-Itô-Liu formula under U-S chance framework. Next, the well-posedness (existence and uniqueness of solution) of the uncertain stochastic dynamic system is proved under the linear growth condition and standard Lipschitz condition. Additionally, we propose three different concepts of stability related to this dynamic system. Finally, we apply the Euler-Maruyama numerical method to solve the dynamic system and prove that the numerical approximate solutions produced by this method converge to the exact solution.

Keywords Uncertain stochastic dynamic system, uncertain stochastic differential equation, well-posedness, Euler-Maruyama method, U-S chance theory.

MSC(2010) 34F05, 60H10, 28E10.

1. Introduction

The complexities of real-world systems give rise to diverse forms of indeterminacy in the events we encounter, and randomness is regarded as the objective indeterminacy. The axiom system of probability theory was constructed by Kolmogorov [20] and provides a framework for studying random events and phenomena. However, in most stochastic systems, it is difficult to find an ideal situation where the probability distributions of random variables are precisely determined. The inherent indeterminacy of probability itself is fundamental and irremovable for some stochastic systems, and is difficult to solve through classical probability theory and linear expectation. Sublinear expectation (also known as G-expectation) theory proposed by Peng [32] provides a robust framework for addressing this randomness with ambiguous probability distributions. Under G-expectation theory, the noise of a stochastic dynamic system is modeled by

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*The authors were supported by the National Natural Science Foundation of China (11801307), the Natural Science Foundation of Shandong Province of China (ZR2021MA009), and the Special Funds for Taishan Scholars Program of Shandong Province (tsqn202507172).
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a G-Brownian motion, which is a nonlinear Wiener process introduced by Peng [32], and the evolution of such dynamic system is described by a stochastic differential equation driven by G-Brownian motion (G-SDE). Peng [32] studied the well-posedness to G-SDEs by the construction mapping theorem, and further theoretical research on G-SDEs can be found in Gao [12], Yuan and Zhu [43] and others. It is worth mentioning that the applications of G-Brownian motion and related G-SDEs to stochastic dynamic systems are much wider than the classical case. For example, in financial markets with indeterminate volatility, G-SDEs can more accurately model the dynamic evolution of asset prices, leading to more robust pricing and risk assessment (see, e.g., Epstein and Ji [9], Vorbrink [36]). In addition, sublinear expectation theory has been employed in stochastic control (see, e.g., Hu and Ji [16], Ren et al. [35]), mathematical economics (see, e.g., Chen et al. [5]), systems science (see, e.g., Xu et al. [38]) and so on.

In addition to randomness, another kind of indeterminacy in systems is human uncertainty. When a system has no or only limited historical data, making it impossible to obtain the distribution function, decision-makers often invite experts to assess the belief degree associated with the occurrence of an event. Liu [22] referred to this indeterminacy of human subjective judgment about the belief degree as human uncertainty, and pointed out that both probability theory and fuzzy set theory (Zadeh [44]) exist limitations in comprehensively capturing the complexities inherent in human uncertainty. In 2007, Liu [21] founded uncertainty theory to handle human uncertainty more reasonably from a mathematical perspective. With the development and deepening of this research, uncertain programming (see, e.g., Liu [22]), uncertain statistics (see, e.g., Liu [24]), uncertain risk analysis (see, e.g., Liu [23]), and other sub-fields have been developed. To model the evolution of uncertain dynamic systems, Liu [24] introduced a class of uncertain differential equations (UDEs) driven by canonical Liu processes. Later, Chen and Liu [1] proved the solvability to UDEs. Liu [24], Yao et al. [42] studied the stability of UDEs. Up to now, UDEs have found effective applications in a wide range of fields, including uncertain finance and uncertain optimal control (see, e.g., Gao et al. [13], Najafi and Taleghani [30], Peng and Yao [31], Wang et al. [37], Zhu [45]).

In numerous situations, a complex system is often influenced by both human uncertainty and randomness. To model this uncertain random phenomenon, Liu [26] first introduced the concept of uncertain random variable and developed chance measure, which integrates uncertain measure and probability measure to offer a more comprehensive and reliable method for quantifying events containing both uncertainty and randomness. Furthermore, chance theory has successfully been applied to uncertain random programming (see, e.g., Liu [25]), risk analysis (see, e.g., Liu and Ralescu [27]), and other fields. In order to model dynamic uncertain random events, Gao and Yao [14] introduced the definition of uncertain random process. Chen et al. [3], Chen and Zhu [2] studied discrete-time uncertain random systems and their optimal control problems. Chen et al. [4], Jia and Liu [19] established continuous-time uncertain stochastic dynamic systems by employing SDEs and UDEs to characterize different dynamic features, and investigated the corresponding optimal control problems.

In real life, there is always a class of complex systems that exist both human uncertainty and probability distribution with ambiguity simultaneously. Fu et al. [11] first introduced a pair of U-S chance spaces to model these complex systems, which are product spaces composed of uncertainty spaces and sublinear expectation spaces. Meanwhile, the concepts of U-S chance measures, uncertain random variables, and chance distributions within U-S chance spaces were also defined by Fu et al. [11]. Based on sublinear expectation theory and uncertainty theory, a new framework called U-S chance theory has been proposed. Furthermore, Fu et al. [11] studied the law of large numbers of uncertain random variables. Following the developments of U-S chance theory, substantial advancements have been made in its theoretical study and application. Hu et al. [15] proposed four types of programming models that involve both human uncertainty and probability distribution with ambiguity, which enhances its utility in real-world decision-making. Xu and Hu [39] applied U-S chance theory to a government-regulated incomplete financial market and studied the problems related to risk aversion. Yang et al. [41] investigated various types of convergences for uncertain random variables and continuities of U-S chance measures. Uncertainty theory specifically deals with belief degree based cognitive uncertainty (i.e., human subjective uncertainty), while sublinear expectation theory primarily applies to randomness with ambiguous probability distributions (i.e., objective random phenomena with unclear distribution characteristics). The fundamental difference of U-S chance theory from these two approaches lies in its construction of product spaces comprising uncertainty space and sublinear expectation spaces, which achieves unified modeling of two types of indeterminacy. Furthermore, compared to the classic chance theory, U-S chance theory takes into account the random characteristic of distributional ambiguity.

However, the evolution of a dynamic system that simultaneously exhibits human uncertainty and probability distribution with ambiguity has not yet been studied. Our initial goal was to study the optimal control problems of these uncertain stochastic dynamic systems. However, due to space constraints, this paper mainly focuses on establishing a unified modeling framework and theoretical foundation for such systems. First, based on U-S chance theory, the G-Itô-Liu integral is rigorously defined, and the chain rule for uncertain random processes (referred to as the G-Itô-Liu formula) is proved. Subsequently, an uncertain stochastic differential equation driven by canonical Liu process and G-Brownian motion (LG-USDE) is introduced to model a dynamic system that involves both human uncertainty and probability distribution with ambiguity. Following this, a series of investigations are carried out regarding this uncertain stochastic dynamic system, including its well-posedness, stability, and numerical approximate solutions.

From the perspective of system modeling, differences between theoretical frameworks lie in their characterization of system indeterminacy. Table 1 provides a comparison between the framework presented herein and major existing dynamic system models, and focuses on the origins of indeterminacy and driving mechanisms, thereby clarifying the novel contributions of this study.

Table 1. Comparison with existing literatures.

Reference	**Indeterminacy**	**Driven by**
[18]	randomness	Wiener process $\{W_t\}_{t\geq 0}$
[32]	randomness with ambiguity	G-Brownian motion $\{B_t\}_{t\geq 0}$
[24]	human uncertainty	canonical Liu process $\{C_t\}_{t\geq 0}$
[4]	human uncertainty and randomness	canonical Liu process $\{C_t\}_{t\geq 0}$ and Wiener process $\{W_t\}_{t\geq 0}$
Proposed method	human uncertainty and randomness with ambiguity	canonical Liu process $\{C_t\}_{t\geq 0}$ and G-Brownian motion $\{B_t\}_{t\geq 0}$

Table 1 provides a systematic comparison of modeling frameworks based on their capacity to handle different types of indeterminacy. Compared with existing models, [18, 32] and [24] only address a single type of indeterminacy, although [4] considers both types of indeterminacy, it employs separate modeling approaches using UDEs and SDEs and fails to address the distributional ambiguity of randomness. The primary innovation lies in the newly established G-Itô-Liu integral and its corresponding differential rules, which provide a rigorous mathematical framework for addressing the issue of mathematical consistency when simultaneously handling human uncertainty and randomness with ambiguity. Based on this calculus system, we introduce the LG-USDE, a novel differential equation driven by a mixture of the canonical Liu process and G-Brownian motion, designed to model dynamic systems with hybrid indeterminacy.

The organization of this paper is as follows. Some definitions and properties of U-S chance theory that will be used are reviewed in Subsection 2.1, and Subsection 2.2 introduces the model of uncertain stochastic dynamic system. Section 3 first proposes the definition of G-Itô-Liu integral and proves the chain rule for uncertain random processes. In Section 4, we prove the well-posedness to uncertain stochastic dynamic system under the standard Lipschitz condition and linear growth condition. Section 5 carries out a stability analysis of the dynamic system and presents three different concepts of stabilities. In Section 6, we extend the Euler-Maruyama numerical method to this system, and the convergence property of numerical solutions are proved. Furthermore, a special numerical example demonstrates the effectiveness of the proposed numerical method. Section 7 offers a conclusion and prospect for the research. In Appendix A, some fundamental concepts for uncertainty theory are given. Appendix B describes the theory of sublinear expectation and related stochastic calculus.

2. Preliminaries and model description

2.1. U-S chance theory

In this subsection, some basic concepts of U-S chance theory are introduced. For more details, please refer to Fu et al. [11], and Yang et al. [41].

Definition 2.1. (Fu et al. [11]) (U-S chance spaces) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, $(\Omega, \mathcal{F})$ be a measurable space, and $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. Let $\mathbb{V}$, v be non-additive probabilities generated by sublinear expectation $\hat{\mathbb{E}}$. Then $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, \mathbb{V})$ and $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, v)$ are called a pair of U-S chance spaces, denoted by

$$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, \mathbb{V}) = (\Gamma \times \Omega, \mathcal{L} \times \mathcal{F}, \mathcal{M} \times \mathbb{V}), \tag{2.1}$$

and

$$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, v) = (\Gamma \times \Omega, \mathcal{L} \times \mathcal{F}, \mathcal{M} \times v), \tag{2.2}$$

where $\Gamma \times \Omega$ is the universal set, $\mathcal{L} \times \mathcal{F}$ is the product σ-algebra, and $\mathcal{M} \times \mathbb{V}$ and $\mathcal{M} \times v$ are two product measures.

Here, the notion of $(\Gamma, \mathcal{L}, \mathcal{M})$ is defined within uncertainty theory, which is presented by Liu [21] (see Appendix A for some necessary concepts and conclusions). The notions of $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, v and $\mathbb{V}$ are defined within the theory of sublinear expectation introduced by Chen et al. [7] and Peng [34] (see Appendix B for more details).

Definition 2.2. (Fu et al. [11]) (U-S chance measures) Let $\Lambda \in \mathcal{L} \times \mathcal{F}$ be an uncertain random event. Then a pair of chance measures of Λ is defined by

$$\tilde{ch}\{\Lambda\} := \int_0^1 v\big\{\omega \in \Omega | \mathcal{M}\left\{\gamma \in \Gamma | (\gamma, \omega) \in \Lambda\right\} \geq y\big\} \mathrm{d}y$$

and

$$CH\{\Lambda\} := \int_0^1 \mathbb{V}\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Lambda\} \geq y\}\,\mathrm{d}y.$$

Proposition 2.1. (Fu et al. [11]) *A pair of* U-S *chance spaces are defined by* (2.1) *and* (2.2). *Then we have*
(i) $\tilde{ch}\{\mathcal{A} \times \mathcal{B}\} = \mathcal{M}\{\mathcal{A}\} \times v\{\mathcal{B}\}$, $CH\{\mathcal{A} \times \mathcal{B}\} = \mathcal{M}\{\mathcal{A}\} \times \mathbb{V}\{\mathcal{B}\}$, *for each* $\mathcal{A} \in \mathcal{L}$, $\mathcal{B} \in \mathcal{F}$. *In particular,* $\tilde{ch}\{\emptyset\} = 0$, $\tilde{ch}\{\Gamma \times \Omega\} = 1$, $CH\{\emptyset\} = 0$, $CH\{\Gamma \times \Omega\} = 1$;
(ii) $CH\{\Lambda\} + \tilde{ch}\{\Lambda^c\} = 1$, *for* $\Lambda \in \mathcal{L} \times \mathcal{F}$;
(iii) $\tilde{ch}\{\Lambda_1\} \leq \tilde{ch}\{\Lambda_2\}$, $CH\{\Lambda_1\} \leq CH\{\Lambda_2\}$, *for* $\Lambda_1, \Lambda_2 \in \mathcal{L} \times \mathcal{F}$, $\Lambda_1 \subseteq \Lambda_2$;
(iv) $\tilde{ch}\{\Lambda\} \leq CH\{\Lambda\}$, *for* $\Lambda \in \mathcal{L} \times \mathcal{F}$.

Definition 2.3. (Fu et al. [11]) An uncertain random variable on U-S chance spaces is a function $\xi : \Gamma \times \Omega \mapsto \mathbb{R}$ such that

$$\{\xi \in B\} = \{(\gamma, \omega) \in \Gamma \times \Omega | \xi(\gamma, \omega) \in B\} \in \mathcal{L} \times \mathcal{F}$$

for each $B \in \mathcal{B}(\mathbb{R})$. Two U-S chance distributions for ξ are defined by
(i) lower distribution: $\phi(x) = \tilde{ch}\{\xi \leq x\}$, $\forall x \in \mathbb{R}$;
(ii) upper distribution: $\Phi(x) = CH\{\xi \leq x\}$, $\forall x \in \mathbb{R}$.

Remark 2.1. To clarify the specific meanings of upper and lower distributions, we take financial risk measurement as an example. The potential portfolio loss is modeled as an uncertain random variable ξ. $\Phi(x) = CH\{\xi \leq x\}$ denotes the maximum confidence level for the loss ξ not exceeding threshold x, while $\phi(x) = \tilde{ch}\{\xi \leq x\}$ denotes the minimum confidence level for the loss ξ not exceeding threshold x. Risk-averse decision-makers focus more on upper distribution $\Phi(x)$ as they tend to maintain losses at lower levels, while risk-seeking decision-makers pay greater attention to lower distribution $\phi(x)$ as they are willing to accept higher risks in pursuit of potential returns. Together, these dual distributions form a complete risk assessment interval $[\phi(x), \Phi(x)]$, providing a more comprehensive reference for financial decision-making under hybrid indeterminacy.

Definition 2.4. (Yang et al. [41]) **Upper continuous:** A chance measure CH ($\tilde{ch}$, respectively) on U-S chance spaces is called upper continuous if $\lim_{n\to\infty} CH\{\Lambda_n\} = CH\{\Lambda\}$ ($\lim_{n\to\infty} \tilde{ch}\{\Lambda_n\} = \tilde{ch}\{\Lambda\}$, respectively), as $\Lambda_n \downarrow \Lambda$.
Lower continuous: A chance measure CH ($\tilde{ch}$, respectively) on U-S chance spaces is called lower continuous if $\lim_{n\to\infty} CH\{\Lambda_n\} = CH\{\Lambda\}$ ($\lim_{n\to\infty} \tilde{ch}\{\Lambda_n\} = \tilde{ch}\{\Lambda\}$, respectively), as $\Lambda_n \uparrow \Lambda$.
Continuous: U-S chance measure CH or $\tilde{ch}$ is called continuous if it both satisfies upper continuous and lower continuous.

Lemma 2.1. (Yang et al. [41]) *Assume that uncertain measure* $\mathcal{M}$ *is continuous. Let* $\{P_\theta\}_{\theta\in\Theta}$ *be a family of probability measures on measurable space* $(\Omega, \mathcal{F})$. *If* $\{P_\theta\}_{\theta\in\Theta}$ *is weakly compact, then U-S chance measures* CH *and* $\tilde{ch}$ *are continuous.*

2.2. Modeling of uncertain stochastic dynamic systems

To describe a dynamic uncertain random phenomenon, we introduce the concept of uncertain random process.

Definition 2.5. $\{X_t(\gamma, \omega)\}_{t\geq 0}$ is called an uncertain random process if for each $t \geq 0$, X_t is an uncertain random variable on U-S chance spaces.

Remark 2.2. $\{X_t(\gamma,\cdot)\}_{t\geq 0}$ is a stochastic process for each fixed $\gamma\in\Gamma$, and $\{X_t(\cdot,\omega)\}_{t\geq 0}$ is an uncertain process for each fixed $\omega\in\Omega$.

Example 2.1. The canonical Liu process $\{C_t\}_{t\geq 0}$ and G-Brownian motion $\{B_t\}_{t\geq 0}$ are all special uncertain random processes (The notions of canonical Liu process and G-Brownian motion please refer to Appendix A and Appendix B).

Example 2.2. Let $\{C_t\}_{t\geq 0}$ be a canonical Liu process, and $\{B_t\}_{t\geq 0}$ be a G-Brownian motion. Then for each measurable function $f:\mathbb{R}^2\mapsto\mathbb{R}$, $\{X_t=f(C_t,B_t)\}_{t\geq 0}$ is an uncertain random process.

Under the framework of sublinear expectation, a stochastic dynamic system with ambiguous probability distributions is described by a G-SDE (see, e.g., Hu and Ji [16]), and can be written in the following form:

$$\begin{cases}\mathrm{d}Y_s=\hat{b}(s,Y_s,\mu_s)\mathrm{d}s+\hat{h}(s,Y_s,\mu_s)\mathrm{d}\left\langle B\right\rangle_s+\hat{g}(s,Y_s,\mu_s)\mathrm{d}B_s,\\ Y_0=y_0,\end{cases}\tag{2.3}$$

where stochastic process $\{Y_s\}_{s\geq 0}$ is a state variable with the initial condition $Y_0=y_0$, stochastic process $\{\mu_s\}_{s\geq 0}$ is the control variable, the functions $\hat{b},\hat{h},\hat{g}:[0,T]\times\mathbb{R}\times\mathbb{R}\times\Omega\mapsto\mathbb{R}$, $\{B_s\}_{s\geq 0}$ is a G-Brownian motion, and $\{\left\langle B\right\rangle_s\}_{s\geq 0}$ is the quadratic variation process of G-Brownian motion.

Example 2.3. In Epstein and Ji [9], volatility cannot be observed directly and must be inferred approximately from high-frequency data. Due to statistical noise and indeterminacy over model specification, agents can neither identify the true dynamic process of volatility precisely nor rely on any single parametric specification. Consequently, volatility can only be confined to a range $\underline{\sigma}\leq\sigma_t\leq\overline{\sigma}$. This volatility ambiguity directly makes quadratic variation $\{\langle B\rangle_t\}_{t\geq 0}$ of driving process $\{B_t\}_{t\geq 0}$ non deterministic but instead satisfying $\underline{\sigma}^2t\leq\langle B\rangle_t\leq\overline{\sigma}^2t$ for each $t\geq 0$. The probability measures governing asset price evolution lose uniqueness and constitute a family of mutually singular measures $\mathcal{P}=\{P^{\sigma_t}:\underline{\sigma}\leq\sigma_t\leq\overline{\sigma}\}$, and under this framework the driving process $\{B_t\}_{t\geq 0}$ is called G-Brownian motion. Consider asset price satisfying $\mathrm{d}S_t/S_t=b_t\mathrm{d}t+\mathrm{d}B_t$ with constant risk-free rate r and suppose $b_t-r=b\sigma_t^2$ for a constant $b>0$, then the state price process is given by $\pi_t=\exp\left\{-rt-bB_t-\frac{1}{2}b^2\langle B\rangle_t\right\}$. For a European call option with strike price Z and terminal payoff $\Phi(S_T)=\max(S_T-Z,0)$, the super hedging price $\overline{C}(S_t,t)=\hat{\mathbb{E}}\left[\frac{\pi_T}{\pi_t}\Phi(S_T)|\mathcal{F}_t\right]$ is Black-Scholes price under maximum volatility $\overline{\sigma}$ representing worst case, while the sub-hedging price $\underline{C}(S_t,t)=-\hat{\mathbb{E}}\left[-\frac{\pi_T}{\pi_t}\Phi(S_T)|\mathcal{F}_t\right]$ is the Black-Scholes price under minimum volatility $\underline{\sigma}$. Options no longer have a unique arbitrage-free price, instead forming a price interval $[\underline{C},\overline{C}]$. This interval exists not because of parameter indeterminacy but because of model ambiguity itself, i.e., different volatility assumptions correspond to mutually incompatible probability measures.

A UDE driven by canonical Liu process is used to describe dynamic system with human uncertainty. Its state equation is a controlled UDE (see, e.g., Zhu [45]) as follows:

$$\begin{cases}\mathrm{d}Z_t=\tilde{b}(s,Z_s,\nu_s)\mathrm{d}s+\tilde{f}(s,Z_s,\nu_s)\mathrm{d}C_s,\\ Z_0=z_0,\end{cases}\tag{2.4}$$

where uncertain process $\{Z_s\}_{s\geq 0}$ is a state variable with the initial condition $Z_0=z_0$, uncertain process $\{\nu_s\}_{s\geq 0}$ is the control variable, the functions $\tilde{b},\tilde{f}:[0,T]\times\mathbb{R}\times\mathbb{R}\times\Gamma\mapsto\mathbb{R}$, and $\{C_s\}_{s\geq 0}$ is a canonical Liu process.

Now we consider a dynamic system that simultaneously involves human uncertainty and probability distribution with ambiguity. The dynamic property of this system displays unique coupling traits: It incorporates human uncertainty stemming from subjective cognition, such as psychological expectations and decision preferences, and also involves probability distribution with ambiguity caused by information asymmetry and environmental disturbances. Assume that the system state $\{X_s\}_{s\geq 0}$ is an uncertain random process as defined in Definition 2.5, the noise is affected by the canonical Liu process and G-Brownian, and an uncertain random process $\{u_s\}_{s\geq 0}$ serves as a control variable. Then an uncertain stochastic dynamic system driven by canonical Liu process and G-Brownian can be expressed as follows:

$$\begin{cases} \mathrm{d}X_t = b(s, X_s, u_s)\mathrm{d}s + f(s, X_s, u_s)\mathrm{d}C_s + h(s, X_s, u_s)\mathrm{d}\left\langle B\right\rangle_s + g(s, X_s, u_s)\mathrm{d}B_s, \\ X_0 = x_0, \end{cases} \tag{2.5}$$

where $X_0 = x_0$ is the initial condition of the state variable $\{X_s\}_{s\geq 0}$, and $b, f, g, h : [0, T] \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \Omega \mapsto \mathbb{R}$ are given functions.

Remark 2.3. Stochastic dynamic system (2.3), uncertain dynamic system (2.4), and uncertain stochastic dynamic system (2.5) are fundamentally distinguished by their modeling objectives. They respectively aim to describe randomness with ambiguous probability distributions, human uncertainty, and the hybrid of both in dynamic systems, consequently employing distinct modeling purposes and research methods. The uncertain stochastic dynamic system (2.5) mathematically encompasses both systems (2.3) and (2.4). It degenerates to the stochastic dynamic system (2.3) when the uncertain noise $\{C_t\}_{t\geq 0}$ is absent, and to the uncertain dynamic system (2.4) when the stochastic noise $\{B_t\}_{t\geq 0}$ is disregarded.

Example 2.4. Consider the scheduling problem for an intelligent shipping system within global trade as an example, which must handle two distinct types of indeterminacy. On one hand, the system needs to handle randomness with ambiguous probability distributions presented by the marine meteorological environment. The complexity of meteorological system results in indeterminacy inherent in its dynamic processes being impossible to accurately describe by a single probability distribution. The evolution of this indeterminacy requires that it could be driven by G-Brownian motion $\{B_t\}_{t\geq 0}$ and its quadratic variation process $\{\langle B\rangle_t\}_{t\geq 0}$.

On the other hand, the system must also address human uncertainty arising from sudden events such as waterway blockades, changes in trade agreements, regional conflicts, and pirate attacks. Unlike weather-related risks, the occurrence mechanisms of these risks do not follow statistical laws, and thus represent a typical form of human uncertainty, which can be driven by canonical Liu process $\{C_t\}_{t\geq 0}$.

The state evolution of an intelligent ship system can be described by a USDE. Let X_t represent the state vector (e.g., position, fuel) of a ship at time t, and u_t be the control vector (e.g., heading, speed). Its dynamic equation can be expressed as:

$$dX_t = \mu(X_t, u_t, t)\mathrm{d}t + \sigma(X_t, u_t, t)\mathrm{d}B_t + h(X_t, u_t, t)\mathrm{d}\langle B\rangle_t + g(X_t, u_t, t)\mathrm{d}C_t.$$

Remark 2.4. The uncertain stochastic dynamic system (2.5) proposed in this paper is described by USDE and driven simultaneously by canonical Liu process and G-Brownian motion. Referring to discrete-time uncertain random system models in [2,3], their representation within U-S chance framework can be expressed by the following difference equation:

$$\begin{cases} x(j+1) = \mu(x(j), u(j), j) + \Delta(x(j), u(j), j)\xi_j, \\ j = 0, 1, 2, \cdots, M-1, x(0) = x_0, \end{cases}$$

where $x(j)$ is the state variable of system at stage j with the initial condition $x(0) = x_0$, $u(j)$ is the control vector, μ, $\Delta : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$, and ξ_j, $j = 0, 1, \cdots, M-1$ are independent uncertain random variables U-S chance framework. Obviously, within U-S chance framework, uncertain random dynamic system and uncertain stochastic dynamic system are essentially different.

If the control variable in the system (2.5) is predetermined function or fixed constant, then the uncertain stochastic system (2.5) related to the control problem can be simplified to a USDE. As the dynamic behavior of this system demonstrates a coupling effect between these two forms of indeterminacy, the traditional single framework of G-stochastic calculus or uncertainty calculus is no longer directly applicable. The form of "differential" in the system (2.5) essentially serves as a symbolic shorthand for integration. Without integral rules specifically aimed at uncertain random processes, the differential expressions in the system (2.5) are simply symbols that lack strict mathematical significance. Consequently, a rigorous mathematical model and well-posedness analysis of this uncertain stochastic dynamic system require a well-completed theoretical framework for support. Next, this paper will construct a foundational theoretical framework by rigorously defining the uncertain stochastic integral and deriving the chain rule for uncertain random processes. Subsequently, we will progressively study the mathematical properties of this uncertain stochastic dynamic system.

3. Uncertain stochastic calculus

This section initially introduces a completed square-integrable space for uncertain random processes based on U-S chance theory framework. With this foundation, the uncertain stochastic integral (i.e., G-itô-Liu integral) is defined, and the chain rule for uncertain random processes (i.e., G-itô-Liu formula) is established. Throughout this paper, we adopt the following convention: Constants that depend on $\gamma \in \Gamma$ are denoted with a subscript or superscript γ (e.g., $K_\gamma, P_\gamma, \bar{P}_\gamma, M_\gamma, c_\gamma, C_1^\gamma, C_2^\gamma, D_1^\gamma, D_2^\gamma$); constants that are independent of γ (e.g., $L, K, c_1, c_2, k_1, k_2, \overline{\sigma}, \underline{\sigma}, M$) are written without a subscript or superscript γ.

Suppose that $\Omega = C_0(\mathbb{R}^+)$, and $\{B_t\}_{t\geq 0}$ is a canonical G-Brownian motion. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, $\mathcal{F} = \sigma(B_s, s \geq 0)$, and $\mathbb{V} = \sup_{P\in\mathcal{P}} P$ and $v = \inf_{P\in\mathcal{P}} P$ be a pair of non-additive probabilities generated by G-expectation (The related notions please refer to Appendix B). From Definition 2.1 and Remark B.1, the triplets

$$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, \mathbb{V}) = (\Gamma \times \Omega, \mathcal{L} \times \sigma(B_s), \mathcal{M} \times \sup_{P\in\mathcal{P}} P)$$

and

$$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, \mathbb{V}) = (\Gamma \times \Omega, \mathcal{L} \times \sigma(B_s), \mathcal{M} \times \inf_{P\in\mathcal{P}} P)$$

are a pair of U-S chance spaces.

In the rest of this paper, for $T \in [0, \infty)$, we consider an uncertain random process $\{\varphi_t(\gamma, \omega)\}_{0\leq t\leq T}$ satisfying:
(i) $\varphi_t(\cdot, \omega)$ is an uncertain variable, for each $t \in [0, T]$, $\omega \in \Omega$;
(ii) $\varphi_t(\gamma, \cdot)$ is a random variable, and $\varphi_t(\gamma, \cdot) \in L_G^2(\Omega_t)$, for each $t \in [0, T]$, $\gamma \in \Gamma$ (The notion of $L_G^2(\Omega_t)$ please refer to Appendix B).
The collection of such uncertain random processes are denoted by $\mathrm{H}^2(0, T)$.

For each fixed $\gamma \in \Gamma$, let $\hat{\mathbb{E}}[\varphi(\gamma, \cdot)]$ represent the sublinear expectation of $\varphi(\gamma, \cdot)$. To simplify the notation, this paper uniformly uses $\hat{\mathbb{E}}[\cdot]$ to denote the sublinear expectation of random variable. Denote

$$\mathbb{L}^2(\Gamma \times \Omega_t) := \big\{\zeta(\gamma,\omega) \big| \zeta(\gamma,\omega) \text{ is an uncertain random variable} \\ \text{satisfing } \zeta(\gamma,\cdot) \in L_G^2(\Omega_t), \forall \gamma \in \Gamma\big\}$$

under the norm $\|\zeta(\gamma,\cdot)\|_{\mathbb{L}^2(\Gamma\times\Omega_t)} = (\hat{\mathbb{E}}[|\zeta(\gamma,\cdot)|^2])^{1/2}$, for each fixed $\gamma \in \Gamma$, and

$$\mathbb{L}^2(\Gamma \times \Omega) := \bigcup_{t>0} \mathbb{L}^2(\Gamma \times \Omega_t).$$

Let $\pi_T = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ be a given partition of $[0,T]$. Set a simple process with the form

$$\varphi_t(\gamma,\omega) = \sum_{k=0}^{N-1} \zeta_k(\gamma,\omega) \mathbf{1}_{[t_k,t_{k+1})}(t), \quad t \in [0,T], \tag{3.1}$$

where $\zeta_k(\gamma,\omega) \in \mathbb{L}^2(\Gamma \times \Omega_{t_k})$, $k = 0,1,\cdots,N-1$. We denote

$$\mathbb{M}^{2,0}(0,T) := \Big\{\varphi_t(\gamma,\omega), 0 \le t \le T \Big|\ \varphi_t(\gamma,\omega), 0 \le t \le T \text{ is a simple process,} \\ \text{and } \varphi_t(\gamma,\omega) \in \mathbb{L}^2(\Gamma \times \Omega_t) \text{ for each } t \in [0,T]\Big\}.$$

Furthermore, let $\mathbb{M}^2(0,T)$ be the complete space of $\mathbb{M}^{2,0}(0,T)$ under the norm

$$\|\varphi_t(\gamma,\cdot)\|_{\mathbb{M}^2(0,T)} = \left\{\hat{\mathbb{E}}\left[\int_0^T |\varphi_t(\gamma,\cdot)|^2 \mathrm{d}t\right]\right\}^{1/2}, \text{ for each fixed } \gamma \in \Gamma.$$

Obviously, $\mathbb{M}^2(0,T) \subseteq \mathrm{H}^2(0,T)$.

Let $\{\pi_T^N = \{0 = t_0^N, t_1^N, \cdots, t_N^N = T\}, N \ge 1\}$ be a sequence of partitions of $[0,T]$ with $\lim_{N\to\infty} \mu(\pi_T^N) = 0$, where

$$\mu(\pi_T^N) = \max\left\{\left|t_{i+1}^N - t_i^N\right| : i = 0,1,\cdots,N-1\right\}.$$

The definition of G-Itô-Liu integral about uncertain random processes is as follows:

Definition 3.1. Let $\{C_t\}_{t\ge0}$ be a canonical Liu process, $\{B_t\}_{t\ge0}$ be a G-Brownian motion, and $\{\langle B\rangle_t\}_{t\ge0}$ be the quadratic variation process of G-Brownian motion. The G-Itô-Liu integral of uncertain random process $\{\varphi_t(\gamma,\omega)\}_{0\le t\le T} \in \mathbb{M}^2(0,T)$ is defined by

$$\begin{aligned} &\int_0^T \varphi_t(\gamma,\omega)\mathrm{d}C_t(\gamma) + \int_0^T \varphi_t(\gamma,\omega)\mathrm{d}B_t(\omega) + \int_0^T \varphi_t(\gamma,\omega)\mathrm{d}\langle B\rangle_t(\omega) \\ :=& \lim_{\mu(\pi_T^N)\to0} \sum_{i=0}^{N-1} \left[\varphi_{t_i^N}(\gamma,\omega)\cdot\left(C_{t_{i+1}^N} - C_{t_i^N}\right)(\gamma)\right] \\ &+ \lim_{\mu(\pi_T^N)\to0} \sum_{i=0}^{N-1} \left[\varphi_{t_i^N}(\gamma,\omega)\left(B_{t_{i+1}^N} - B_{t_i^N}\right)(\omega)\right] \\ &+ \lim_{\mu(\pi_T^N)\to0} \sum_{i=0}^{N-1} \left[\varphi_{t_i^N}(\gamma,\omega)\left(\langle B\rangle_{t_{i+1}^N} - \langle B\rangle_{t_i^N}\right)(\omega)\right]. \end{aligned}$$

Remark 3.1. (i) For each simple process $\{\varphi_t(\gamma,\omega)\}_{0\le t\le T}\in\mathbb{M}^{2,0}(0,T)$ defined by (3.1), let

$$\mathcal{I}_1\left(\varphi_t\right)=\int_0^T\varphi_t(\gamma,\omega)\mathrm{d}C_t(\gamma):=\sum_{k=0}^{N-1}\Big[\zeta_k(\gamma,\omega)\cdot\left(C_{t_{k+1}}-C_{t_k}\right)(\gamma)\Big].$$

From the Hölder inequality and Remark A.1, we observe that

$$\hat{\mathbb{E}}\left[\left|\int_0^T\varphi_t(\gamma,\cdot)\mathrm{d}C_t(\gamma)\right|^2\right]\le K_\gamma^2T\hat{\mathbb{E}}\left[\int_0^T\varphi_t^2(\gamma,\cdot))\mathrm{d}t\right],$$

where K_γ is a Lipschitz constant of $\{C_t\left(\gamma\right)\}_{t\ge 0}$ defined in Remark A.1. Hence the mapping $\mathcal{I}_1:\mathbb{M}^{2,0}(0,T)\mapsto\mathbb{L}^2(\Gamma\times\Omega_T)$ is a linear continuous mapping, and can be continuously extend to $\mathcal{I}_1:\mathbb{M}^2(0,T)\mapsto\mathbb{L}^2(\Gamma\times\Omega_T)$. For each $\{\varphi_t(\gamma,\omega)\}_{0\le t\le T}\in\mathbb{M}^2(0,T)$, the limit of

$$\sum_{i=0}^{N-1}\left[\varphi_{t_i^N}(\gamma,\omega)\cdot\left(C_{t_{i+1}^N}-C_{t_i^N}\right)(\gamma)\right]$$

exists in $\mathbb{L}^2(\Gamma\times\Omega_T)$. Thus the integral

$$\int_0^T\varphi_t(\gamma,\omega)\mathrm{d}C_t(\gamma):=\lim_{\mu(\pi_T^N)\to 0}\sum_{i=0}^{N-1}\left[\varphi_{t_i^N}(\gamma,\omega)\cdot\left(C_{t_{i+1}^N}-C_{t_i^N}\right)(\gamma)\right]$$

can be well-defined.

(ii) For each simple process $\{\varphi_t(\gamma,\omega)\}_{0\le t\le T}\in\mathbb{M}^{2,0}(0,T)$ defined by (3.1), let

$$\mathcal{I}_2\left(\varphi_t\right)=\int_0^T\varphi_t(\gamma,\omega)\mathrm{d}B_t(\omega):=\sum_{k=0}^{N-1}\Big[\zeta_k(\gamma,\omega)\left(B_{t_{k+1}}-B_{t_k}\right)(\omega)\Big],$$

and

$$\mathcal{I}_3\left(\varphi_t\right)=\int_0^T\varphi_t(\gamma,\omega)\mathrm{d}\langle B\rangle_t(\omega):=\sum_{k=0}^{N-1}\Big[\zeta_k(\gamma,\omega)\left(\langle B\rangle_{t_{k+1}}-\langle B\rangle_{t_k}\right)(\omega)\Big].$$

From Peng [34], it is obvious that the linear continuous mappings $\mathcal{I}_2:\mathbb{M}^{2,0}(0,T)\mapsto\mathbb{L}^2(\Gamma\times\Omega_T)$, and $\mathcal{I}_3:\mathbb{M}^{2,0}(0,T)\mapsto\mathbb{L}^2(\Gamma\times\Omega_T)$ can be continuously extended to $\mathcal{I}_2:\mathbb{M}^2(0,T)\mapsto\mathbb{L}^2(\Gamma\times\Omega_T)$, and $\mathcal{I}_3:\mathbb{M}^2(0,T)\mapsto\mathbb{L}^2(\Gamma\times\Omega_T)$. Thus for each $\{\varphi_t(\gamma,\omega)\}_{0\le t\le T}\in\mathbb{M}^2(0,T)$, the integrals

$$\int_0^T\varphi_t(\gamma,\omega)\mathrm{d}B_t(\omega):=\lim_{\mu(\pi_T^N)\to 0}\sum_{i=0}^{N-1}\Big[\varphi_{t_i^N}(\gamma,\omega)\left(B_{t_{i+1}^N}-B_{t_i^N}\right)(\omega)\Big],$$

and

$$\int_0^T\varphi_t(\gamma,\omega)\mathrm{d}\langle B\rangle_t(\omega):=\lim_{\mu(\pi_T^N)\to 0}\sum_{i=0}^{N-1}\left[\varphi_{t_i^N}(\gamma,\omega)\left(\langle B\rangle_{t_{i+1}^N}-\langle B\rangle_{t_i^N}\right)(\omega)\right]$$

can be well-defined.

In summary, our construction ensures that for each fixed γ, the aforementioned $\mathcal{I}_1(\varphi_t)$, $\mathcal{I}_2(\varphi_t)$, and $\mathcal{I}_3(\varphi_t)$ all converge within the constructed space $\mathbb{L}^2(\Gamma\times\Omega_T)$, thereby guaranteeing the well-definedness of G-Itô-Liu integral for any process in $\mathbb{M}^2(0,T)$.

Remark 3.2. When $\overline{\sigma} = \underline{\sigma} > 0$, G-Brownian motion $\{B_t\}_{t\geq 0}$ degenerates to classical Brownian motion, the quadratic variation process $\{\langle B\rangle_t\}_{t\geq 0}$ has no mean-uncertainty, and $\langle B\rangle_t = \overline{\sigma}^2 t$ for $t \geq 0$, hence

$$\int_0^T \varphi_t(\gamma,\omega)\mathrm{d}\langle B\rangle_t(\omega) = \overline{\sigma}^2 \int_0^T \varphi_t(\gamma,\omega)\mathrm{d}t.$$

Proposition 3.1. *For each $\{\varphi_t(\gamma,\omega)\}_{0\leq t\leq T} \in \mathbb{M}^2(0,T)$, we have*

(i)

$$\hat{\mathbb{E}}\left[\sup_{0\leq s\leq t}\left|\int_0^s \varphi_u(\gamma,\cdot)\mathrm{d}C_u\right|^2\right] \leq K_\gamma^2 t\hat{\mathbb{E}}\left[\int_0^t |\varphi_u(\gamma,\cdot)|^2\,\mathrm{d}u\right];$$

(ii)

$$\hat{\mathbb{E}}\left[\sup_{0\leq s\leq t}\left|\int_0^s \varphi_u(\gamma,\cdot)\mathrm{d}B_u\right|^2\right] \leq 4\hat{\mathbb{E}}\left[\left|\int_0^t \varphi_s(\gamma,\cdot)\mathrm{d}B_s\right|^2\right] \leq 4\overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^t |\varphi_u(\gamma,\cdot)|^2\,\mathrm{d}u\right];$$

(iii)

$$\hat{\mathbb{E}}\left[\sup_{0\leq s\leq t}\left|\int_0^s \varphi_u(\gamma,\cdot)\mathrm{d}\langle B\rangle_u\right|^2\right] \leq \overline{\sigma}^4 t\hat{\mathbb{E}}\left[\int_0^t |\varphi_u(\gamma,\cdot)|^2\,\mathrm{d}u\right].$$

Proof. For (i), let $\pi_s^N = \{0 = t_0^N, t_1^N, \cdots, t_N^N = s\}$ be any partition of $[0,s]$ with $\lim_{N\to\infty}\mu(\pi_s^N) = 0$, where

$$\mu(\pi_s^N) = \max\left\{\left|t_{i+1}^N - t_i^N\right| : i = 0,1,\cdots,N-1\right\}.$$

It follows from Definition 3.1 and Remark A.1 that

$$\begin{aligned}\left|\int_0^s \varphi_s(\gamma,\omega)\mathrm{d}C_u\right| &= \left|\lim_{\mu(\pi_s^N)\to 0}\sum_{i=0}^{N-1}\left[\varphi_{t_i^N}(\gamma,\omega)\cdot\left(C_{t_{i+1}^N} - C_{t_i^N}\right)(\gamma)\right]\right| \\ &\leq \lim_{\mu(\pi_s^N)\to 0}\sum_{i=0}^{N-1}\left|\varphi_{t_i^N}(\gamma,\omega)\right|\cdot\left|C_{t_{i+1}^N} - C_{t_i^N}\right| \\ &\leq K_\gamma \lim_{\mu(\pi_s^N)\to 0}\sum_{i=0}^{N-1}\left|\varphi_{t_i^N}(\gamma,\omega)\right|\cdot\left|t_{i+1}^N - t_i^N\right| \\ &= K_\gamma\int_0^s |\varphi_u(\gamma,\omega)|\,\mathrm{d}u.\end{aligned}$$

By the Hölder inequality,

$$\hat{\mathbb{E}}\left[\sup_{0\leq s\leq t}\left|\int_0^s \varphi_u(\gamma,\cdot)\mathrm{d}C_u\right|^2\right] \leq \hat{\mathbb{E}}\left[K_\gamma^2\left(\int_0^t |\varphi_u(\gamma,\cdot)|\,\mathrm{d}u\right)^2\right] \leq K_\gamma^2 t\hat{\mathbb{E}}\left[\int_0^t |\varphi_u(\gamma,\cdot)|^2\,\mathrm{d}u\right].$$

The proofs of (ii) and (iii) are basically the same as those of Lemma 3.3.4 and Lemma 3.4.3 in Peng [34], and Theorem 2.2 in Gao [12]. □

Lemma 3.1. *Let $\{\varphi_t(\gamma,\omega)\}_{0\leq t\leq T} \in \mathbb{M}^2(0,T)$, and π_T^N be a given partition of $[0,T]$ with $\mu(\pi_T^N) \to 0$ as $N\to\infty$. Then in $\mathbb{L}^2(\Gamma\times\Omega_T)$, for each $\gamma\in\Gamma$, we have*

$$\sum_{k=0}^{N-1}\varphi_{t_k^N}(\gamma,\omega)\left(B_{t_{k+1}^N} - B_{t_k^N}\right)^2 \to \int_0^T \varphi_t(\gamma,\omega)\mathrm{d}\langle B\rangle_t$$

as $N\to\infty$.

Proof. The proof is basically the same as that of Lemma 3.4.6 in Peng [34]. $\square$

Denote $C_{b,\mathrm{Lip}}(\mathbb{R})$ be the space of functions $\varphi(\cdot)$ satisfying the following bounded and Lipschitz continuous conditions:

$$|\varphi(x)| \leq M, \text{ and } |\varphi(x) - \varphi(y)| \leq M|x-y|, \quad \forall x, y \in \mathbb{R},$$

where $M > 0$ is a positive constant. Next, based on Theorem A.2 and Theorem B.3 (G-Itô formula), we introduce a G-Itô-Liu formula about uncertain random process $\{X_t\}_{0\leq t\leq T}$.

Lemma 3.2. *Consider an uncertain random process $\{X_t\}_{0\leq t\leq T} \in \mathbb{M}^2(0,T)$ such that*

$$X_t = X_s + \alpha(t-s) + \zeta(C_t - C_s) + \rho\left(\langle B\rangle_t - \langle B\rangle_s\right) + \beta(B_t - B_s), \quad 0 \leq s \leq t \leq T, \tag{3.2}$$

where α, ζ, ρ, β belong to $\mathbb{L}^2(\Gamma\times\Omega_s)$ and are bounded with respect to $\omega \in \Omega$, i.e., $\max_{\omega\in\Omega}\{|\alpha|, |\zeta|, |\rho|, |\beta|\} \leq P_\gamma$, where P_γ is a positive constant for each fixed $\gamma \in \Gamma$. Let $\ell \in C^2(\mathbb{R})$ with $\partial_x\ell, \partial^2_{xx}\ell \in C_{b,Lip}(\mathbb{R})$, then in $\mathbb{L}^2(\Gamma\times\Omega_t)$, for $0 \leq s \leq t \leq T$, we have

$$\begin{aligned} \ell(X_t) - \ell(X_s) &= \int_s^t \partial_x\ell(X_u)\alpha \mathrm{d}u + \int_s^t \partial_x\ell(X_u)\zeta \mathrm{d}C_u \\ &\quad + \int_s^t \partial_x\ell(X_u)\beta \mathrm{d}B_u + \int_s^t \left[\partial_x\ell(X_u)\rho + \frac{1}{2}\partial^2_{xx}\ell(X_u)\beta^2\right]\mathrm{d}\langle B\rangle_u. \end{aligned} \tag{3.3}$$

Proof. Let $\delta = (t-s)/N$ for any positive integer N, and set a partition

$$\pi^N_{[s,t]} = \{t^N_0, t^N_1, \cdots, t^N_N\} = \{s, s+\delta, \cdots, s+N\delta = t\}.$$

For $0 \leq s \leq t \leq T$, it follows from Taylor expansion formula that

$$\begin{aligned} \ell(X_t) - \ell(X_s) &= \sum_{k=0}^{N-1}\left[\ell\big(X_{t^N_{k+1}}\big) - \ell\left(X_{t^N_k}\right)\right] \\ &= \sum_{k=0}^{N-1}\partial_x\ell\left(X_{t^N_k}\right)\big(X_{t^N_{k+1}} - X_{t^N_k}\big) \\ &\quad + \frac{1}{2}\sum_{k=0}^{N-1}\left[\partial^2_{xx}\ell\left(X_{t^N_k}\right)\left(X_{t^N_{k+1}} - X_{t^N_k}\right)^2\right] + \frac{1}{2}\sum_{k=0}^{N-1}\Upsilon^N_k, \end{aligned} \tag{3.4}$$

where

$$\Upsilon^N_k = \partial^2_{xx}\ell\left(X_{t^N_k} + \theta_k\left(X_{t^N_{k+1}} - X_{t^N_k}\right)\right)\left(X_{t^N_{k+1}} - X_{t^N_k}\right)^2 - \partial^2_{xx}\ell\left(X_{t^N_k}\right)\left(X_{t^N_{k+1}} - X_{t^N_k}\right)^2,$$

and $\theta_k \in [0,1]$. From $\partial^2_{xx}\ell \in C_{b,\mathrm{Lip}}(\mathbb{R})$, Proposition B.1 and the boundedness of α, ζ, ρ, β with respect to $\omega \in \Omega$, one can show that

$$\begin{aligned} \hat{\mathbb{E}}\left[\left|\Upsilon^N_k\right|^2\right] &= \hat{\mathbb{E}}\left[\left|\left(\partial^2_{xx}\ell\left(X_{t^N_k} + \theta_k(X_{t^N_{k+1}} - X_{t^N_k})\right) - \partial^2_{xx}\ell\left(X_{t^N_k}\right)\right)\left(X_{t^N_{k+1}} - X_{t^N_k}\right)^2\right|^2\right] \\ &\leq \hat{\mathbb{E}}\left[M^2\left|\theta_k\left(X_{t^N_{k+1}} - X_{t^N_k}\right)\right|^2\left|\left(X_{t^N_{k+1}} - X_{t^N_k}\right)\right|^4\right] \end{aligned}$$

$$
\begin{aligned}
&\leq M^2\hat{\mathbb{E}}\left[\left|X_{t_{k+1}^N}-X_{t_k^N}\right|^6\right]\\
&\leq M^2\hat{\mathbb{E}}\left[\left|\alpha\left(t_{k+1}^N-t_k^N\right)+\zeta\big(C_{t_{k+1}^N}-C_{t_k^N}\big)+\rho\big(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N}\big)+\beta\big(B_{t_{k+1}^N}-B_{t_k^N}\big)\right|^6\right]\\
&\leq M^2P_\gamma^6\hat{\mathbb{E}}\left[\left(\left|t_{k+1}^N-t_k^N\right|+\left|C_{t_{k+1}^N}-C_{t_k^N}\right|+\left|\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N}\right|+\left|B_{t_{k+1}^N}-B_{t_k^N}\right|\right)^6\right]\\
&\leq \bar{P}_\gamma\hat{\mathbb{E}}\left[\left|t_{k+1}^N-t_k^N\right|^6+\left|C_{t_{k+1}^N}-C_{t_k^N}\right|^6+\left|\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N}\right|^6+\left|B_{t_{k+1}^N}-B_{t_k^N}\right|^6\right],
\end{aligned}
$$

where $\bar{P}_\gamma$ is a positive constant for each fixed $\gamma\in\Gamma$. Hence for $0\leq s\leq t\leq T$, from Remark A.1 and Proposition B.2, it yields that

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\left|\Upsilon_k^N\right|^2\right]&\leq \bar{P}_\gamma\left|t_{k+1}^N-t_k^N\right|^6+\bar{P}_\gamma K_\gamma^6\left|t_{k+1}^N-t_k^N\right|^6+\bar{P}_\gamma\overline{\sigma}^{12}\left|t_{k+1}^N-t_k^N\right|^6+15\bar{P}_\gamma\overline{\sigma}^6\left|t_{k+1}^N-t_k^N\right|^3\\
&=\bar{P}_\gamma\left(1+K_\gamma^6+\overline{\sigma}^{12}\right)\delta^6+15\bar{P}_\gamma\overline{\sigma}^6\delta^3\\
&=\bar{P}_\gamma\left(1+K_\gamma^6+\overline{\sigma}^{12}\right)\frac{(t-s)^6}{N^6}+15\bar{P}_\gamma\overline{\sigma}^6\frac{(t-s)^3}{N^3}.
\end{aligned}
$$

Thus

$$
\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1}\Upsilon_k^N\right|^2\right]\leq N\sum_{k=0}^{N-1}\hat{\mathbb{E}}\left[\left|\Upsilon_k^N\right|^2\right]\to 0
$$

as $N\to\infty$. The remaining terms of (3.4) are denoted as $\mu_k^N+\nu_k^N$ with

$$
\begin{aligned}
\mu_k^N=\sum_{k=0}^{N-1}\Bigg\{&\partial_x\ell\left(X_{t_k^N}\right)\Big[\alpha\left(t_{k+1}^N-t_k^N\right)+\zeta\left(C_{t_{k+1}^N}-C_{t_k^N}\right)+\rho\left(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N}\right)\\
&+\beta\left(B_{t_{k+1}^N}-B_{t_k^N}\right)\Big]+\frac{1}{2}\partial_{xx}^2\ell\left(X_{t_k^N}\right)\beta^2\left(B_{t_{k+1}^N}-B_{t_k^N}\right)^2\Bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_k^N=\frac{1}{2}\sum_{k=0}^{N-1}\partial_{xx}^2\ell\left(X_{t_k^N}\right)\Bigg\{&\left[\alpha\left(t_{k+1}^N-t_k^N\right)+\zeta\left(C_{t_{k+1}^N}-C_{t_k^N}\right)+\rho\left(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N}\right)\right]^2\\
&+2\left[\alpha\left(t_{k+1}^N-t_k^N\right)+\zeta\left(C_{t_{k+1}^N}-C_{t_k^N}\right)+\rho\left(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N}\right)\right]\beta\left(B_{t_{k+1}^N}-B_{t_k^N}\right)\Bigg\}.
\end{aligned}
$$

For μ_k^N, from Definition 3.1 and Lemma 3.1, for each $0\leq s\leq t\leq T$,

$$
\begin{aligned}
\mu_k^N\to&\int_s^t\partial_x\ell(X_u)\alpha\mathrm{d}u+\int_s^t\partial_x\ell(X_u)\zeta\mathrm{d}C_u+\int_s^t\partial_x\ell(X_u)\beta\mathrm{d}B_u\\
&+\int_s^t\left[\partial_x\ell(X_u)\rho+\frac{1}{2}\partial_{xx}^2\ell(X_u)\beta^2\right]\mathrm{d}\langle B\rangle_u
\end{aligned}
$$

as $N\to\infty$ in $\mathbb{L}^2(\Gamma\times\Omega_t)$. For ν_k^N, it follows that

$$
\nu_k^N\leq\frac{1}{2}\sum_{k=0}^{N-1}\partial_{xx}^2\ell\left(X_{t_k^N}\right)\left[3\alpha^2\left(t_{k+1}^N-t_k^N\right)^2+3\zeta^2\left(C_{t_{k+1}^N}-C_{t_k^N}\right)^2+3\rho^2\left(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N}\right)^2\right.
$$

$$+ 2\alpha\beta \left(t_{k+1}^{N} - t_{k}^{N}\right) \left(B_{t_{k+1}^{N}} - B_{t_{k}^{N}}\right) + 2\zeta\beta \left(C_{t_{k+1}^{N}} - C_{t_{k}^{N}}\right) \left(B_{t_{k+1}^{N}} - B_{t_{k}^{N}}\right)$$
$$+ 2\rho\beta \left(\langle B\rangle_{t_{k+1}^{N}} - \langle B\rangle_{t_{k}^{N}}\right) \left(B_{t_{k+1}^{N}} - B_{t_{k}^{N}}\right) \Bigg].$$

We need to show that $\nu_k^N \to 0$ as $N \to \infty$ in $\mathbb{L}^2(\Gamma \times \Omega_t)$. Hence from $\partial_x \ell, \partial_{xx}^2 \ell \in C_{b,\mathrm{Lip}}(\mathbb{R})$, Remark A.1, Proposition B.2 and the boundedness of α, ζ, ρ, β with respect to $\omega \in \Omega$, for $0 \le s \le t \le T$, when $N \to \infty$, we obtain

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1} \partial_{xx}^2 \ell\left(X_{t_k^N}\right) \alpha^2 \left(t_{k+1}^N - t_k^N\right)^2\right|^2\right] \\
&\le N\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} \left|\partial_{xx}^2 \ell\left(X_{t_k^N}\right)\right|^2 \alpha^4 \left(t_{k+1}^N - t_k^N\right)^4\right] \\
&\le M^2 P_\gamma^4 N \sum_{k=0}^{N-1} \left(t_{k+1}^N - t_k^N\right)^4 \\
&= M^2 P_\gamma^4 \frac{(t-s)^4}{N^2} \\
&\to 0, \\
&\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1} \partial_{xx}^2 \ell\left(X_{t_k^N}\right) \zeta^2 \left(C_{t_{k+1}^N} - C_{t_k^N}\right)^2\right|^2\right] \\
&\le N\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} \left|\partial_{xx}^2 \ell\left(X_{t_k^N}\right)\right|^2 \zeta^4 \left(C_{t_{k+1}^N} - C_{t_k^N}\right)^4\right] \\
&\le M^2 P_\gamma^4 N \sum_{k=0}^{N-1} \left(C_{t_{k+1}^N} - C_{t_k^N}\right)^4 \\
&\le M^2 P_\gamma^4 K_\gamma^4 N \sum_{k=0}^{N-1} \left(t_{k+1}^N - t_k^N\right)^4 \\
&= M^2 P_\gamma^4 K_\gamma^4 \frac{(t-s)^4}{N^2} \\
&\to 0, \\
&\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1} \partial_{xx}^2 \ell\left(X_{t_k^N}\right) \rho^2 \left(\langle B\rangle_{t_{k+1}^N} - \langle B\rangle_{t_k^N}\right)^2\right|^2\right] \\
&\le N\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} \left|\partial_{xx}^2 \ell\left(X_{t_k^N}\right)\right|^2 \rho^4 \left(\langle B\rangle_{t_{k+1}^N} - \langle B\rangle_{t_k^N}\right)^4\right] \\
&\le M^2 P_\gamma^4 N \sum_{k=0}^{N-1} \hat{\mathbb{E}}\left[\left(\langle B\rangle_{t_{k+1}^N} - \langle B\rangle_{t_k^N}\right)^4\right] \\
&\le M^2 P_\gamma^4 \overline{\sigma}^8 N \sum_{k=0}^{N-1} \left(t_{k+1}^N - t_k^N\right)^4
\end{aligned}$$

$$
\begin{aligned}
&= M^2 P_\gamma^4 \overline{\sigma}^8 \frac{(t-s)^4}{N^2} \\
&\to 0, \\
&\quad \hat{\mathbb{E}}\left[\Big| \sum_{k=0}^{N-1} \partial_{xx}^2 \ell\left(X_{t_k^N}\right) \alpha\beta \left(t_{k+1}^N - t_k^N\right) \left(B_{t_{k+1}^N} - B_{t_k^N}\right) \Big|^2\right] \\
&\le N \hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} \left|\partial_{xx}^2 \ell\left(X_{t_k^N}\right)\right|^2 \alpha^2\beta^2 \left(t_{k+1}^N - t_k^N\right)^2 \left(B_{t_{k+1}^N} - B_{t_k^N}\right)^2\right] \\
&\le M^2 P_\gamma^4 \frac{(t-s)^2}{N} \sum_{k=0}^{N-1} \hat{\mathbb{E}}\left[\left(B_{t_{k+1}^N} - B_{t_k^N}\right)^2\right] \\
&\le M^2 P_\gamma^4 \overline{\sigma}^2 \frac{(t-s)^2}{N} \sum_{k=0}^{N-1} \left(t_{k+1}^N - t_k^N\right) \\
&= M^2 P_\gamma^4 \overline{\sigma}^2 \frac{(t-s)^3}{N} \\
&\to 0, \\
&\quad \hat{\mathbb{E}}\left[\Big| \sum_{k=0}^{N-1} \partial_{xx}^2 \ell\left(X_{t_k^N}\right) \zeta\beta \left(C_{t_{k+1}^N} - C_{t_k^N}\right) \left(B_{t_{k+1}^N} - B_{t_k^N}\right) \Big|^2\right] \\
&\le N \hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} \left|\partial_{xx}^2 \ell\left(X_{t_k^N}\right)\right|^2 \zeta^2\beta^2 \left(C_{t_{k+1}^N} - C_{t_k^N}\right)^2 \left(B_{t_{k+1}^N} - B_{t_k^N}\right)^2\right] \\
&\le M^2 P_\gamma^4 K_\gamma^2 \frac{(t-s)^2}{N} \sum_{k=0}^{N-1} \hat{\mathbb{E}}\left[\left(B_{t_{k+1}^N} - B_{t_k^N}\right)^2\right] \\
&\le M^2 P_\gamma^4 K_\gamma^2 \overline{\sigma}^2 \frac{(t-s)^2}{N} \sum_{k=0}^{N-1} \left(t_{k+1}^N - t_k^N\right) \\
&= M^2 P_\gamma^4 K_\gamma^2 \overline{\sigma}^2 \frac{(t-s)^3}{N} \\
&\to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\Big| \sum_{k=0}^{N-1} \partial_{xx}^2 \ell\left(X_{t_k^N}\right) \rho\beta \left(\langle B\rangle_{t_{k+1}^N} - \langle B\rangle_{t_k^N}\right) \left(B_{t_{k+1}^N} - B_{t_k^N}\right) \Big|^2\right] \\
&\le N \hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} \left|\partial_{xx}^2 \ell\left(X_{t_k^N}\right)\right|^2 \zeta^2\beta^2 \left(\langle B\rangle_{t_{k+1}^N} - \langle B\rangle_{t_k^N}\right)^2 \left(B_{t_{k+1}^N} - B_{t_k^N}\right)^2\right] \\
&\le M^2 P_\gamma^4 N \sum_{k=0}^{N-1} \hat{\mathbb{E}}\left[\left(\langle B\rangle_{t_{k+1}^N} - \langle B\rangle_{t_k^N}\right)^2 \left(B_{t_{k+1}^N} - B_{t_k^N}\right)^2\right] \\
&\le M^2 P_\gamma^4 \overline{\sigma}^6 N \sum_{k=0}^{N-1} \left(t_{k+1}^N - t_k^N\right)^3
\end{aligned}
$$

$$
\begin{aligned}
&= M^2 P_\gamma^4 \overline{\sigma}^6 \frac{(t-s)^3}{N} \\
&\to 0.
\end{aligned}
$$

Thus, $\nu_k^N \to 0$ as $N \to \infty$ in $\mathbb{L}^2(\Gamma \times \Omega_t)$. The proof is completed. □

Theorem 3.1. (G-Itô-Liu formula) *Consider an uncertain random process with the form*

$$
X_t = X_0 + \int_0^t \alpha_s \mathrm{d}s + \int_0^t \zeta_s \mathrm{d}C_s + \int_0^t \rho_s \mathrm{d}\left\langle B \right\rangle_s + \int_0^t \beta_s \mathrm{d}B_s, \quad 0 \le t \le T, \tag{3.5}
$$

where $\{\alpha_t\}_{0\le t\le T}, \{\zeta_t\}_{0\le t\le T}, \{\rho_t\}_{0\le t\le T}$ and $\{\beta_t\}_{0\le t\le T}$ belong to $\mathbb{M}^2(0,T)$ *and are bounded with respect to* $\omega \in \Omega$*. Let* $\ell \in C^2(\mathbb{R})$ *such that* $\partial_{xx}^2 \ell$ *satisfying the polynomial growth condition, i.e., there exist constants* $c_1 > 0$ *and* $k_1 \in \mathbb{N}$ *such that* $|\partial_{xx}^2 \ell(x)|^2 \le c_1(1+|x|^{k_1})$*, for all* $x \in \mathbb{R}$*. Then in* $\mathbb{L}^2(\Gamma \times \Omega_t)$*, for* $0 \le s \le t \le T$*, we have*

$$
\begin{aligned}
\ell(X_t) - \ell(X_s) = &\int_s^t \partial_x \ell(X_u) \alpha_u \mathrm{d}u + \int_s^t \partial_x \ell(X_u) \zeta_u \mathrm{d}C_u \\
&+ \int_s^t \partial_x \ell(X_u) \beta_u \mathrm{d}B_u + \int_s^t \left[\partial_x \ell(X_u)\rho_u + \frac{1}{2}\partial_{xx}^2 \ell(X_u) \beta_u^2 \right] \mathrm{d}\langle B \rangle_u.
\end{aligned} \tag{3.6}
$$

Proof. The proof can be divided into three steps.

Step 1. Suppose that $\ell \in C^2(\mathbb{R})$ with $\partial_x \ell, \partial_{xx}^2 \ell \in C_{b,\mathrm{Lip}}(\mathbb{R})$, and that $\{\alpha_t\}_{0\le t\le T}$, $\{\zeta_t\}_{0\le t\le T}$, $\{\rho_t\}_{0\le t\le T}$ and $\{\beta_t\}_{0\le t\le T}$ are simple process with the forms

$$
\begin{aligned}
\alpha_t &= \sum_{k=0}^{N-1} \hat{\alpha}_k \mathbf{1}_{[t_k, t_{k+1})}(t), \quad & \zeta_t &= \sum_{k=0}^{N-1} \hat{\zeta}_k \mathbf{1}_{[t_k, t_{k+1})}(t), \\
\rho_t &= \sum_{k=0}^{N-1} \hat{\rho}_k \mathbf{1}_{[t_k, t_{k+1})}(t), \quad & \beta_t &= \sum_{k=0}^{N-1} \hat{\beta}_k \mathbf{1}_{[t_k, t_{k+1})}(t),
\end{aligned}
$$

where $\hat{\alpha}_k,\ \hat{\zeta}_k,\ \hat{\rho}_k,\ \hat{\beta}_k$ belong to $\mathbb{L}^2(\Gamma \times \Omega_{t_k})$ and are bounded with respect to $\omega \in \Omega$, $k = 0, 1, \cdots, N-1$. Applying Lemma 3.2, one can easily obtain that

$$
\begin{aligned}
\ell(X_{t_{i+1}}) - \ell(X_{t_i}) = &\int_{t_i}^{t_{i+1}} \partial_x \ell(X_u) \hat{\alpha}_i \mathrm{d}u + \int_{t_i}^{t_{i+1}} \partial_x \ell(X_u) \hat{\zeta}_i \mathrm{d}C_u \\
&+ \int_{t_i}^{t_{i+1}} \partial_x \ell(X_u) \hat{\beta}_i \mathrm{d}B_u + \int_{t_i}^{t_{i+1}} \left[\partial_x \ell(X_u)\hat{\rho}_i + \frac{1}{2}\partial_{xx}^2 \ell(X_u) \hat{\beta}_i^2 \right] \mathrm{d}\langle B \rangle_u.
\end{aligned}
$$

Sum over i from 0 to $N-1$, we obtain the desired result (3.6).

Step 2. Next we consider that $\ell \in C^2(\mathbb{R})$ with $\partial_x \ell, \partial_{xx}^2 \ell \in C_{b,\mathrm{Lip}}(\mathbb{R})$, and that $\{\alpha_t\}_{0\le t\le T}$, $\{\zeta_t\}_{0\le t\le T}$, $\{\rho_t\}_{0\le t\le T}$ and $\{\beta_t\}_{0\le t\le T}$ belong to $\mathbb{M}^2(0,T)$ and are bounded with respect to $\omega \in \Omega$. Obviously, there exist four sequences of simple processes $\{\alpha_t^N\}_{0\le t\le T}$, $\{\zeta_t^N\}_{0\le t\le T}$, $\{\rho_t^N\}_{0\le t\le T}$ and $\{\beta_t^N\}_{0\le t\le T}$ satisfying uniformly bounded with respect to $\omega \in \Omega$ such that

$$
\begin{aligned}
&\left\|\alpha_t^N - \alpha_t\right\|_{\mathbb{M}^2(0,T)} \to 0, \quad \left\|\zeta_t^N - \zeta_t\right\|_{\mathbb{M}^2(0,T)} \to 0, \\
&\left\|\rho_t^N - \rho_t\right\|_{\mathbb{M}^2(0,T)} \to 0, \quad \left\|\beta_t^N - \beta_t\right\|_{\mathbb{M}^2(0,T)} \to 0,
\end{aligned} \tag{3.7}
$$

as $N \to \infty$. Denote

$$X_t^N = X_0 + \int_0^t \alpha_s^N \mathrm{d}s + \int_0^t \zeta_s^N \mathrm{d}C_s + \int_0^t \rho_s^N \mathrm{d}\left\langle B\right\rangle_s + \int_0^t \beta_s^N \mathrm{d}B_s, \quad 0 \leq t \leq T.$$

It follwos from step 1 that

$$\begin{aligned} \ell\left(X_t^N\right) - \ell\left(X_s^N\right) = &\int_s^t \partial_x \ell\left(X_u^N\right) \alpha_u^N \mathrm{d}u + \int_s^t \partial_x \ell\left(X_u^N\right) \zeta_u^N \mathrm{d}C_u + \int_s^t \partial_x \ell\left(X_u^N\right) \beta_u^N \mathrm{d}B_u \\ &+ \int_s^t \left[\partial_x \ell\left(X_u^N\right) \rho_u^N + \frac{1}{2}\partial_{xx}^2 \ell\left(X_u^N\right)\left(\beta_u^N\right)^2\right] \mathrm{d}\langle B\rangle_u, \quad 0 \leq s \leq t \leq T. \end{aligned} \tag{3.8}$$

Note that for $t \in [0,T]$,

$$\begin{aligned} \hat{\mathbb{E}}\left[\left|X_t^N - X_t\right|^2\right] \leq &4\hat{\mathbb{E}}\left[\left|\int_0^t \left(\alpha_s^N - \alpha_s\right) \mathrm{d}s\right|^2\right] + 4\hat{\mathbb{E}}\left[\left|\int_0^t \left(\zeta_s^N - \zeta_s\right) \mathrm{d}C_s\right|^2\right] \\ &+ 4\hat{\mathbb{E}}\left[\left|\int_0^t \left(\rho_s^N - \rho_s\right) \mathrm{d}\left\langle B\right\rangle_s\right|^2\right] + 4\hat{\mathbb{E}}\left[\left|\int_0^t \left(\beta_s^N - \beta_s\right) \mathrm{d}B_s\right|^2\right]. \end{aligned}$$

By the Hölder inequality, Proposition 3.1 and (3.7), when $N \to \infty$, we obtain

$$\begin{aligned} &\hat{\mathbb{E}}\left[\left|\int_0^t \left(\alpha_s^N - \alpha_s\right) \mathrm{d}s\right|^2\right] \leq \hat{\mathbb{E}}\left[t \int_0^t \left|\alpha_s^N - \alpha_s\right|^2 \mathrm{d}s\right] \leq T\hat{\mathbb{E}}\left[\int_0^T \left|\alpha_s^N - \alpha_s\right|^2 \mathrm{d}s\right] \to 0, \\ &\hat{\mathbb{E}}\left[\left|\int_0^t \left(\zeta_s^N - \zeta_s\right) \mathrm{d}C_s\right|^2\right] \leq \hat{\mathbb{E}}\left[tK_\gamma^2 \int_0^t \left|\zeta_s^N - \zeta_s\right|^2 \mathrm{d}s\right] \leq TK_\gamma^2\hat{\mathbb{E}}\left[\int_0^T \left|\zeta_s^N - \zeta_s\right|^2 \mathrm{d}s\right] \to 0, \\ &\hat{\mathbb{E}}\left[\left|\int_0^t \left(\rho_s^N - \rho_s\right) \mathrm{d}\left\langle B\right\rangle_s\right|^2\right] \leq \overline{\sigma}^4 t\hat{\mathbb{E}}\left[\int_0^t \left|\rho_s^N - \rho_s\right|^2 \mathrm{d}s\right] \leq \overline{\sigma}^4 T\hat{\mathbb{E}}\left[\int_0^T \left|\rho_s^N - \rho_s\right|^2 \mathrm{d}s\right] \to 0, \end{aligned}$$

and

$$\hat{\mathbb{E}}\left[\left|\int_0^t \left(\beta_s^N - \beta_s\right) \mathrm{d}B_s\right|^2\right] \leq \overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^t \left|\beta_s^N - \beta_s\right|^2 \mathrm{d}s\right] \leq \overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^T \left|\beta_s^N - \beta_s\right|^2 \mathrm{d}s\right] \to 0.$$

Hence for $t \in [0,T]$, when $N \to \infty$,

$$\hat{\mathbb{E}}\left[\left|X_t^N - X_t\right|^2\right] \to 0. \tag{3.9}$$

From the Hölder inequality, Proposition 3.1, the boundedness of $\{\zeta_t\}_{0\leq t\leq T}$, $\{\beta_t\}_{0\leq t\leq T}$ with respect to $\omega \in \Omega$, $\partial_x \ell \in C_{b,\mathrm{Lip}}(\mathbb{R})$, (3.7) and (3.9), when $N \to \infty$, one can show that

$$\begin{aligned} &\hat{\mathbb{E}}\left[\left|\int_s^t \left(\partial_x \ell\left(X_u^N\right)\zeta_u^N - \partial_x \ell\left(X_u\right)\zeta_u\right) \mathrm{d}C_u\right|^2\right] \\ &\leq \hat{\mathbb{E}}\left[tK_\gamma^2 \int_0^t \left|\partial_x \ell\left(X_u^N\right)\zeta_u^N - \partial_x \ell\left(X_u\right)\zeta_u\right|^2 \mathrm{d}u\right] \\ &\leq TK_\gamma^2\hat{\mathbb{E}}\left[\int_0^T \left|\partial_x \ell\left(X_u^N\right)\zeta_u^N - \partial_x \ell\left(X_u\right)\zeta_u^N + \partial_x \ell\left(X_u\right)\zeta_u^N - \partial_x \ell\left(X_u\right)\zeta_u\right|^2 \mathrm{d}u\right] \end{aligned}$$

$$\begin{aligned}
&\leq 2TK_\gamma^2\hat{\mathbb{E}}\left[\int_0^T \left|\zeta_u^N\right|^2 \left|\partial_x \ell\left(X_u^N\right) - \partial_x \ell\left(X_u\right)\right|^2 \mathrm{d}u\right] + 2TK_\gamma^2\hat{\mathbb{E}}\left[\int_0^T \left|\partial_x \ell(X_u)\right|^2 \left|\zeta_u^N - \zeta_u\right|^2 \mathrm{d}u\right] \\
&\leq 2TK_\gamma^2\hat{\mathbb{E}}\left[\int_0^T M^2 \left|\zeta_u^N\right|^2 \left|X_u^N - X_u\right|^2 \mathrm{d}u\right] + 2TK_\gamma^2\hat{\mathbb{E}}\left[\int_0^T M^2 \left|\zeta_u^N - \zeta_u\right|^2 \mathrm{d}u\right] \\
&\leq M_\gamma \left(\int_0^T \hat{\mathbb{E}}\left[\left|X_u^N - X_u\right|^2\right] \mathrm{d}u + \hat{\mathbb{E}}\left[\int_0^T \left|\zeta_u^N - \zeta_u\right|^2 \mathrm{d}u\right]\right) \\
&\to 0,
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_0^t \left(\partial_x \ell\left(X_u^N\right) \beta_u^N - \partial_x \ell\left(X_u\right) \beta_u\right) \mathrm{d}B_u\right|^2\right] \\
&\leq \overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T \left|\partial_x \ell\left(X_u^N\right) \beta_u^N - \partial_x \ell\left(X_u\right) \beta_u\right|^2 \mathrm{d}u\right] \\
&= \overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T \left|\partial_x \ell\left(X_u^N\right) \beta_u^N - \partial_x \ell\left(X_u\right) \beta_u^N + \partial_x \ell\left(X_u\right) \beta_u^N - \partial_x \ell\left(X_u\right) \beta_u\right|^2 \mathrm{d}u\right] \\
&\leq 2\overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T \left|\beta_u^N\right|^2 \left|\partial_x \ell\left(X_u^N\right) - \partial_x \ell\left(X_u\right)\right|^2 \mathrm{d}u\right] + 2\overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T \left|\partial_x \ell(X_u)\right|^2 \left|\beta_u^N - \beta_u\right|^2 \mathrm{d}u\right] \\
&\leq 2\overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T M^2 \left|\beta_u^N\right|^2 \left|X_u^N - X_u\right|^2 \mathrm{d}u\right] + 2\overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T M^2 \left|\beta_u^N - \beta_u\right|^2 \mathrm{d}u\right] \\
&\leq M_\gamma \left(\int_0^T \hat{\mathbb{E}}\left[\left|X_u^N - X_u\right|^2\right] \mathrm{d}u + \hat{\mathbb{E}}\left[\int_0^T \left|\beta_u^N - \beta_u\right|^2 \mathrm{d}u\right]\right) \\
&\to 0,
\end{aligned} \tag{3.11}$$

where M_γ is a positive constant for each fixed $\gamma \in \Gamma$ that is independent of N. Similarly, we can also prove when $N \to \infty$,

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_s^t \left(\partial_x \ell\left(X_u^N\right) \alpha_u^N - \partial_x \ell\left(X_u\right) \alpha_u\right) \mathrm{d}u\right|^2\right] \\
&\leq T\hat{\mathbb{E}}\left[\int_0^T \left|\partial_x \ell\left(X_u^N\right) \alpha_u^N - \partial_x \ell\left(X_u\right) \alpha_u\right|^2 \mathrm{d}u\right] \\
&\to 0,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_s^t \left(\partial_x \ell\left(X_u^N\right) \rho_u^N - \partial_x \ell\left(X_u\right) \rho_u\right) \mathrm{d}\left\langle B\right\rangle_u\right|^2\right] \\
&\leq \overline{\sigma}^4 T\hat{\mathbb{E}}\left[\int_0^T \left|\partial_x \ell\left(X_u^N\right) \rho_u^N - \partial_x \ell(X_u)\rho_u\right|^2 \mathrm{d}u\right] \\
&\to 0,
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_0^t \left(\partial_{xx}^2 \ell\left(X_u^N\right) \left(\beta_u^N\right)^2 - \partial_{xx}^2 \ell\left(X_u\right) \beta_u^2\right) \mathrm{d}\left\langle B\right\rangle_u\right|^2\right] \\
&\leq \overline{\sigma}^4 T\hat{\mathbb{E}}\left[\int_0^T \left|\partial_{xx}^2 \ell\left(X_u^N\right) \left(\beta_u^N\right)^2 - \partial_{xx}^2 \ell\left(X_u\right) \beta_u^2\right|^2 \mathrm{d}u\right] \to 0.
\end{aligned} \tag{3.14}$$

Therefore, from (3.9)-(3.14), taking the limit in both sides of (3.8) as $N \to \infty$ in $\mathbb{L}^2(\Gamma \times \Omega_t)$, we arrive at (3.6).

Step 3. Finally, we consider that $\ell \in C^2(\mathbb{R})$ such that $\partial^2_{xx}\ell$ satisfying the polynomial growth condition, and that $\{\alpha_t\}_{0\le t\le T}$, $\{\zeta_t\}_{0\le t\le T}$, $\{\rho_t\}_{0\le t\le T}$ and $\{\beta_t\}_{0\le t\le T}$ belong to $\mathbb{M}^2(0,T)$ and are bounded with respect to $\omega \in \Omega$. By the assumptions of ℓ, we can easily choose a sequence of $\ell^N \in C_0^2(\mathbb{R})$ satisfying the conditions in step 2 such that

$$\left|\ell^N(x) - \ell(x)\right| + \left|\partial_x \ell^N(x) - \partial_x \ell(x)\right| + \left|\partial^2_{xx}\ell^N(x) - \partial^2_{xx}\ell(x)\right| \le \frac{c_2}{N}\left(1 + |x|^{k_2}\right), \quad x \in \mathbb{R}, \tag{3.15}$$

where $C_0^2(\mathbb{R})$ denotes the space of twice continuously differentiable functions with compact supports, and the constant $c_2 > 0$ and the integer $k_2 \in \mathbb{N}$ independent of N. By step 2, for $0 \le s \le t \le T$, one can obtain that

$$\begin{aligned}\ell^N(X_t) - \ell^N(X_s) = &\int_s^t \partial_x \ell^N(X_u)\alpha_u \mathrm{d}u + \int_s^t \partial_x \ell^N(X_u)\zeta_u \mathrm{d}C_u \\ &+ \int_s^t \partial_x \ell^N(X_u)\beta_u \mathrm{d}B_u + \int_s^t \left[\partial_x \ell^N(X_u)\rho_u + \frac{1}{2}\partial^2_{xx}\ell^N(X_u)\beta_u^2\right]\mathrm{d}\langle B\rangle_u.\end{aligned} \tag{3.16}$$

It is easy to verify that there exist a positive constant c_3 such that $\hat{\mathbb{E}}[|X_t|^{2k_2}] \le c_3$ for $t \in [0,T]$. Hence by the Hölder inequality, Proposition 3.1, the boundedness of $\{\zeta_t\}_{0\le t\le T}$, $\{\rho_t\}_{0\le t\le T}$ with respect to $\omega \in \Omega$ and (3.15), when $N \to \infty$, it follows that

$$\begin{aligned}\hat{\mathbb{E}}\left[\left|\int_s^t \left(\partial_x \ell^N(X_u)\zeta_u - \partial_x \ell(X_u)\zeta_u\right)\mathrm{d}C_u\right|^2\right] &\le \hat{\mathbb{E}}\left[K_\gamma^2 t\int_s^t \left|\partial_x \ell^N(X_u)\zeta_u - \partial_x \ell(X_u)\zeta_u\right|^2 \mathrm{d}u\right] \\ &\le K_\gamma^2 T\hat{\mathbb{E}}\left[\int_0^T |\zeta_u|^2 \left|\partial_x \ell^N(X_u) - \partial_x \ell(X_u)\right|^2 \mathrm{d}u\right] \\ &\le K_\gamma^2 T\hat{\mathbb{E}}\left[\int_0^T |\zeta_u|^2 \frac{c_2^2}{N^2}\left(1 + |X_u|^{k_2}\right)^2 \mathrm{d}u\right] \\ &\le \frac{c_\gamma}{N^2}\hat{\mathbb{E}}\left[\int_0^T \left(1 + |X_u|^{k_2}\right)^2 \mathrm{d}u\right] \\ &\to 0,\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\hat{\mathbb{E}}\left[\left|\int_s^t \left(\partial_x \ell^N(X_u)\rho_u - \partial_x \ell(X_u)\rho_u\right)\mathrm{d}\left\langle B\right\rangle_u\right|^2\right] &\le \overline{\sigma}^4 t\hat{\mathbb{E}}\left[\int_s^t \left|\partial_x \ell^N(X_u)\rho_u - \partial_x \ell(X_u)\rho_u\right|^2 \mathrm{d}u\right] \\ &\le \overline{\sigma}^4 T\hat{\mathbb{E}}\left[\int_0^T |\rho_u|^2 \left|\partial_x \ell^N(X_u) - \partial_x \ell(X_u)\right|^2 \mathrm{d}u\right] \\ &\le \overline{\sigma}^4 T\hat{\mathbb{E}}\left[\int_0^T |\rho_u|^2 \frac{c_2^2}{N^2}\left(1 + |X_u|^{k_2}\right)^2 \mathrm{d}u\right] \\ &\le \frac{c_\gamma}{N^2}\hat{\mathbb{E}}\left[\int_0^T \left(1 + |X_u|^{k_2}\right)^2 \mathrm{d}u\right] \\ &\to 0,\end{aligned} \tag{3.18}$$

where c_γ is a positive constant for each fixed $\gamma \in \Gamma$ that is independent of N. Similarly, we can also prove when $N \to \infty$,

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_s^t \left(\partial_x \ell^N(X_u)\alpha_u - \partial_x \ell(X_u)\alpha_u\right) \mathrm{d}u\right|^2\right] \\
\leq &\hat{\mathbb{E}}\left[t\int_s^t \left|\partial_x \ell^N(X_u)\alpha_u - \partial_x \ell(X_u)\alpha_u\right|^2 \mathrm{d}u\right] \\
\to &0,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_s^t \left(\partial_x \ell^N(X_u)\beta_u - \partial_x \ell(X_u)\beta_u\right) \mathrm{d}B_u\right|^2\right] \\
\leq &\bar{\sigma}^2\hat{\mathbb{E}}\left[\int_s^t \left|\partial_x \ell^N(X_u)\beta_u - \partial_x \ell(X_u)\beta_u\right|^2 \mathrm{d}u\right] \\
\to &0,
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_s^t \left(\partial_{xx}^2 \ell^N(X_u)\beta_u - \partial_{xx}^2 \ell(X_u)\beta_u^2\right) \mathrm{d}\left\langle B\right\rangle_u\right|^2\right] \\
\leq &\bar{\sigma}^4 T\hat{\mathbb{E}}\left[\int_0^T \left|\partial_{xx}^2 \ell^N(X_u)\beta_u - \partial_{xx}^2 \ell(X_u)\beta_u^2\right|^2 \mathrm{d}u\right] \\
\to &0.
\end{aligned} \tag{3.21}$$

From (3.17)-(3.21), taking the limit in both sides of (3.16) as $N \to \infty$ in $\mathbb{L}^2(\Gamma \times \Omega_t)$, we arrive at (3.6). The proof of Theorem 3.1 is completed. □

Remark 3.3. Under general conditions, the G-Itô-Liu formula (3.6) still holds, i.e., when $\ell \in C^2(\mathbb{R})$ and $\{\alpha_t\}_{0\leq t\leq T}$, $\{\zeta_t\}_{0\leq t\leq T}$, $\{\rho_t\}_{0\leq t\leq T}$, $\{\beta_t\}_{0\leq t\leq T} \in \mathbb{M}^2(0,T)$, for $0 \leq s \leq t \leq T$, we have

$$\begin{aligned}
\ell(X_t) - \ell(X_s) = &\int_s^t \partial_x \ell(X_u)\alpha_u \mathrm{d}u + \int_s^t \partial_x \ell(X_u)\zeta_u \mathrm{d}C_u \\
&+ \int_s^t \partial_x \ell(X_u)\beta_u \mathrm{d}B_u + \int_s^t \left[\partial_x \ell(X_u)\rho_u + \frac{1}{2}\partial_{xx}^2 \ell(X_u)\beta_u^2\right]\mathrm{d}\langle B\rangle_u.
\end{aligned}$$

The detailed proof is basically the same as that in [34], and we will not elaborate further here.

Remark 3.4. The G-Itô-Liu formula is essentially a chain rule that describes the evolution laws of uncertain stochastic dynamic systems. Its unification lies in the ability to simultaneously handle the joint disturbance of human uncertainty noise and random noise with ambiguous probability distributions on system state. $\int_s^t \partial_x \ell(X_u)\alpha_u \mathrm{d}u$ represents the deterministic drift of the system. $\int_s^t \partial_x \ell(X_u)\zeta_u \mathrm{d}C_u$ and $\int_s^t \partial_x \ell(X_u)\beta_u \mathrm{d}B_u$ represent the influence of human uncertainty noise and random noise, respectively. Different from classical Brownian motion, the quadratic variation process $\langle B\rangle_t$ of G-Brownian motion is not a deterministic function of time t, but rather a stochastic process with inherent indeterminacy. This means that we not only acknowledge the existence of randomness in the system, but also recognize that its volatility itself is indeterminate. $\int_s^t \partial_x \ell(X_u)\rho_u \mathrm{d}\langle B\rangle_u$ and $\int_s^t \frac{1}{2}\partial_{xx}^2 \ell(X_u)\beta_u^2 \mathrm{d}\langle B\rangle_u$ together reflect the cumulative effect of stochastic volatility.

Remark 3.5. The G-Itô-Liu formula (3.6) is equivalent to

$$\begin{aligned}\mathrm{d}\ell(X_t) =&\partial_x\ell(X_t)\alpha_t\mathrm{d}t + \partial_x\ell(X_t)\zeta_t\mathrm{d}C_t + \partial_x\ell(X_t)\beta_t\mathrm{d}B_t \\ &+ \left[\partial_x\ell(X_t)\rho_t + \frac{1}{2}\partial^2_{xx}\ell(X_t)\beta_t^2\right]\mathrm{d}\langle B\rangle_t, \quad 0 \le s \le t \le T.\end{aligned}$$

Remark 3.6. When $\beta_t = \rho_t = 0$, Theorem 3.1 is equivalent to Theorem A.2. When $\zeta_t = 0$, G-Itô-Liu formula degenerates to G-Itô formula (Theorem B.3). When $\overline{\sigma} = \underline{\sigma} > 0$, G-Brownian motion $\{B_t\}_{t\ge 0}$ degenerates to classical Brownian motion, then G-Itô-Liu formula degenerates to Itô-Liu formula in [10].

Example 3.1. Compute

$$\int_0^t B_s\mathrm{d}B_s \text{ and } \int_0^t C_s\mathrm{d}C_s, \quad t\in[0,T].$$

Let $\ell(x) = x^2$, $X_t = B_t$ ($X_t = C_t$, respectively). From G-Itô-Liu formula, we have

$$B_t^2 = \int_0^t 2B_s\mathrm{d}B_s + \int_0^t \mathrm{d}\left\langle B\right\rangle_s \text{ and } C_t^2 = \int_0^t 2C_s\mathrm{d}C_s,$$

which implies that

$$\int_0^t B_s\mathrm{d}B_s = \frac{B_t^2 - \langle B\rangle_t}{2} \text{ and } \int_0^t C_s\mathrm{d}C_s = \frac{C_t^2}{2}, \quad t\in[0,T].$$

Example 3.2. Compute

$$\int_0^t B_s\mathrm{d}C_s + \int_0^t C_s\mathrm{d}B_s, \quad t\in[0,T].$$

Let $\ell(x) = x^2$, $X_t = C_t + B_t$. It follows from G-Itô-Liu formula that

$$\begin{aligned}(C_t+B_t)^2 &= \int_0^t 2(C_s+B_s)\mathrm{d}C_s + \int_0^t 2(C_s+B_s)\mathrm{d}B_s + \int_0^t \mathrm{d}\left\langle B\right\rangle_S \\ &= 2\int_0^t C_s\mathrm{d}C_s + 2\int_0^t B_s\mathrm{d}C_s + 2\int_0^t C_s\mathrm{d}B_s + 2\int_0^t B_s\mathrm{d}B_s + \left\langle B\right\rangle_t,\end{aligned}$$

which implies that

$$\int_0^t B_s\mathrm{d}C_s + \int_0^t C_s\mathrm{d}B_s = \frac{(C_t+B_t)^2}{2} - \int_0^t C_s\mathrm{d}C_s - \int_0^t B_s\mathrm{d}B_s - \frac{\langle B\rangle_t}{2}, \quad t\in[0,T].$$

From Example 3.1, we have

$$\int_0^t B_s\mathrm{d}C_s + \int_0^t C_s\mathrm{d}B_s = \frac{(C_t+B_t)^2}{2} - \frac{B_t^2 - \langle B\rangle_t}{2} - \frac{C_t^2}{2} - \frac{\langle B\rangle_t}{2} = C_tB_t, \quad t\in[0,T].$$

Example 3.3. Consider an uncertain random process with the form

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t + cX_t\mathrm{d}\left\langle B\right\rangle_t + eX_t\mathrm{d}B_t, \quad t\in[0,T]. \tag{3.22}$$

where a, b, c and e are given constants. From G-Itô-Liu formula, for $t\in[0,T]$,

$$\ln(X_t) - \ln(X_0) = \int_0^t \frac{1}{X_s}aX_s\mathrm{d}s + \int_0^t \frac{1}{X_s}bX_s\mathrm{d}C_s + \int_0^t \frac{1}{X_s}cX_s\mathrm{d}\left\langle B\right\rangle_s$$

$$+\int_0^t \frac{1}{X_s} eX_s \mathrm{d}B_s + \frac{1}{2}\int_0^t -\frac{1}{X_s^2} e^2 X_s^2 \mathrm{d}\left\langle B\right\rangle_s$$
$$= at + bC_t + eB_t + \left(c - \frac{e^2}{2}\right)\left\langle B\right\rangle_t.$$

Thus

$$X_t = X_0 \exp\left(at + bC_t + eB_t + \left(c - \frac{e^2}{2}\right)\left\langle B\right\rangle_t\right), \quad t \in [0,T].$$

In Example 3.3, (3.22) can be seen as an uncertain stochastic differential equation driven by canonical Liu process and G-Brownian motion (LG-USDE), and $\{X_t\}_{0\le t\le T}$ is a solution to (3.22). The next section will focus on studying some the solvability of this class of differential equations.

4. Uncertain stochastic differential equation driven by canonical Liu process and G-Brownian motion

When a dynamic system exhibits both human uncertainty and probability distribution with ambiguity, the "noise" can be modeled by the canonical Liu process and G-Brownian motion simultaneously. In this case, the evolution of dynamic system can be characterized by a LG-USDE. This section presents the notion of LG-USDE and then studies the well-posedness to LG-USDE.

Definition 4.1. Let $\{C_t\}_{t\ge0}$ be a canonical Liu process, $\{B_t\}_{t\ge0}$ be a G-Brownian motion, and $\{\left\langle B\right\rangle_t\}_{t\ge0}$ be the quadratic variation process of G-Brownian motion. Then an uncertain stochastic differential equation driven by canonical Liu process and G-Brownian motion (LG-USDE) is defined by

$$X_t = X_0 + \int_0^t b(s,X_s)\mathrm{d}s + \int_0^t f(s,X_s)\mathrm{d}C_s + \int_0^t h(s,X_s)\mathrm{d}\left\langle B\right\rangle_s + \int_0^t g(s,X_s)\mathrm{d}B_s, \tag{4.1}$$

where $t \in [0,T]$, the initial condition $X_0 \in \mathbb{R}$ is a given constant, and $b,f,g,h : [0,T]\times\mathbb{R}\times\Omega\times\Gamma \mapsto \mathbb{R}$ are given functions with $b(\cdot,x), f(\cdot,x), g(\cdot,x), h(\cdot,x) \in \mathbb{M}^2(0,T)$ for each $x \in \mathbb{R}$.

We now discuss the well-posedness to the LG-USDE (4.1). Assume that the functions b,f,g and h satisfy the following conditions.

Assumption 4.1. *Assume that* b,f,g and h *satisfy the standard Lipschitz condition and linear growth condition, i.e., there exist positive constants* L *and* K *such that for each* $t \in [0,T]$, $x,y \in \mathbb{R}$, $\omega \in \Omega$, $\gamma \in \Gamma$,
(H1) $|b(t,x)-b(t,y)| + |f(t,x)-f(t,y)| + |g(t,x)-g(t,y)| + |h(t,x)-h(t,y)| \le L|x-y|$;
(H2) $|b(t,x)| + |f(t,x)| + |g(t,x)| + |h(t,x)| \le K(1+|x|)$.

Theorem 4.1. *Under the Assumption 4.1, there exists a unique solution* $\{X_t\}_{0\le t\le T} \in \mathbb{M}^2(0,T)$ *to the* LG-USDE (4.1).

Proof. We start with the proof of uniqueness. Suppose that $\{X_t\}_{0\le t\le T}$, $\{Y_t\}_{0\le t\le T} \in \mathbb{M}^2(0,T)$ are two solutions to the LG-USDE (4.1) with initial values $X_0 = Y_0$, then for $t \in [0,T]$,

$$\hat{\mathbb{E}}\left[|X_t - Y_t|^2\right] \le 4\Bigg[\hat{\mathbb{E}}\left|\int_0^t (b(s,X_s)-b(s,Y_s))\,\mathrm{d}s\right|^2 + \hat{\mathbb{E}}\left|\int_0^t (f(s,X_s)-f(s,Y_s))\,\mathrm{d}C_s\right|^2$$

$$
+\hat{\mathbb{E}}\left|\int_0^t (h(s,X_s)-h(s,Y_s))\,\mathrm{d}\langle B\rangle_s\right|^2+\hat{\mathbb{E}}\left|\int_0^t (g(s,X_s)-g(s,Y_s))\,\mathrm{d}B_s\right|^2\Bigg].
$$

By the Hölder inequality, Proposition 3.1 and **(H1)** of Assumption 4.1, for $t\in[0,T]$, it follows that

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\left|\int_0^t (b(s,X_s)-b(s,Y_s))\,\mathrm{d}s\right|^2\right] &\le \hat{\mathbb{E}}\left[t\int_0^t |b(s,X_s)-b(s,Y_s)|^2\mathrm{d}s\right]\\
&\le \hat{\mathbb{E}}\left[t\int_0^t L^2|X_s-Y_s|^2\mathrm{d}s\right]\\
&\le L^2T\int_0^t \hat{\mathbb{E}}\left[|X_s-Y_s|^2\right]\mathrm{d}s,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\left|\int_0^t (f(s,X_s)-f(s,Y_s))\,\mathrm{d}C_s\right|^2\right] &\le K_\gamma^2 t\hat{\mathbb{E}}\left[\int_0^t |f(s,X_s)-f(s,Y_s)|^2\mathrm{d}s\right]\\
&\le K_\gamma^2 t\hat{\mathbb{E}}\left[\int_0^t L^2|X_s-Y_s|^2\mathrm{d}s\right]\\
&\le L^2K_\gamma^2T\int_0^t \hat{\mathbb{E}}\left[|X_s-Y_s|^2\right]\mathrm{d}s,
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\left|\int_0^t (h(s,X_s)-h(s,Y_s))\,\mathrm{d}\langle B\rangle_s\right|^2\right] &\le \overline{\sigma}^4 t\hat{\mathbb{E}}\left[\int_0^t |h(s,X_s)-h(s,Y_s)|^2\mathrm{d}s\right]\\
&\le \overline{\sigma}^4 t\hat{\mathbb{E}}\left[\int_0^t L^2|X_s-Y_s|^2\mathrm{d}s\right]\\
&\le L^2\overline{\sigma}^4T\int_0^t \hat{\mathbb{E}}\left[|X_s-Y_s|^2\right]\mathrm{d}s,
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\left|\int_0^t (g(s,X_s)-g(s,Y_s))\,\mathrm{d}B_s\right|^2\right] &\le \overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^t |g(s,X_s)-g(s,Y_s)|^2\mathrm{d}s\right]\\
&\le \overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^t L^2|X_s-Y_s|^2\mathrm{d}s\right]\\
&\le L^2\overline{\sigma}^2\int_0^t \hat{\mathbb{E}}\left[|X_s-Y_s|^2\right]\mathrm{d}s.
\end{aligned}
\tag{4.5}
$$

Hence we conclude that

$$
\hat{\mathbb{E}}\left[|X_t-Y_t|^2\right]\le 4L^2(T+\overline{\sigma}^4T+\overline{\sigma}^2+K_\gamma^2T)\int_0^t \hat{\mathbb{E}}\left[|X_s-Y_s|^2\right]\mathrm{d}s,\quad t\in[0,T].
$$

The Gronwall inequality yields for $t\in[0,T]$,

$$
\hat{\mathbb{E}}\left[|X_t-Y_t|^2\right]=0.
$$

Thus the uniqueness is proved.

We now begin to prove the existence. Define $X_t^0 = X_0$, and

$$\begin{aligned} X_t^n =& X_0 + \int_0^t b\left(s, X_s^{n-1}\right) \mathrm{d}s + \int_0^t f\left(s, X_s^{n-1}\right) \mathrm{d}C_s \\ &+ \int_0^t h\left(s, X_s^{n-1}\right) \mathrm{d}\langle B\rangle_s + \int_0^t g\left(s, X_s^{n-1}\right) \mathrm{d}B_s, \quad n = 1, 2, \cdots, \end{aligned}$$

for $t \in [0, T]$. It is easy to check that the sequence $\{X_t^n\}_{n=1}^{\infty}$ is well defined. It is claimed that

$$\hat{\mathbb{E}}\left[\left|X_t^{j+1} - X_t^j\right|^2\right] \leq \frac{A_\gamma \bar{A}_\gamma^j t^{j+1}}{(j+1)!}, \text{ for each } j \geq 0, t \in [0, T], \tag{4.6}$$

where A_γ and $\bar{A}_\gamma$ are two positive constants for each fixed $\gamma \in \Gamma$. In fact, we can prove (4.6) by mathematical induction. When $j = 0$, for $t \in [0, T]$,

$$\begin{aligned} \hat{\mathbb{E}}\Big[\left|X_t^1 - X_t^0\right|^2\Big] =& \hat{\mathbb{E}}\left[\left|\int_0^t b(s, X_0)\mathrm{d}s + \int_0^t f(s, X_0)\mathrm{d}C_s + \int_0^t h(s, X_0)\mathrm{d}\left\langle B\right\rangle_s + \int_0^t g(s, X_0)\mathrm{d}B_s\right|^2\right] \\ \leq& 4\left(\hat{\mathbb{E}}\left[\left|\int_0^t b(s, X_0)\mathrm{d}s\right|^2\right] + \hat{\mathbb{E}}\left[\left|\int_0^t f(s, X_0)\mathrm{d}C_s\right|^2\right]\right. \\ &\left. + \hat{\mathbb{E}}\left[\left|\int_0^t h(s, X_0)\mathrm{d}\left\langle B\right\rangle_s\right|^2\right] + \hat{\mathbb{E}}\left[\left|\int_0^t g(s, X_0)\mathrm{d}B_s\right|^2\right]\right). \end{aligned}$$

The Hölder inequality, Proposition 3.1 and **(H2)** of Assumption 4.1 imply that

$$\begin{aligned} \hat{\mathbb{E}}\left[\left|\int_0^t b(s, X_0)\mathrm{d}s\right|^2\right] &\leq \hat{\mathbb{E}}\left[t\int_0^t |b(s, X_0)|^2\mathrm{d}s\right] \\ &\leq \hat{\mathbb{E}}\left[t\int_0^t K^2(1 + |X_0|)^2\mathrm{d}s\right] \\ &\leq 2K^2 T\left(1 + |X_0|^2\right)t, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \hat{\mathbb{E}}\left[\left|\int_0^t f(s, X_0)\mathrm{d}C_s\right|^2\right] &\leq K_\gamma^2 t\hat{\mathbb{E}}\left[\int_0^t |f(s, X_0)|^2\mathrm{d}s\right] \\ &\leq K_\gamma^2 t\hat{\mathbb{E}}\left[\int_0^t K^2(1 + |X_0|)^2\mathrm{d}s\right] \\ &\leq 2K^2 K_\gamma^2 T\left(1 + |X_0|^2\right)t, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \hat{\mathbb{E}}\left[\left|\int_0^t h(s, X_0)\mathrm{d}\left\langle B\right\rangle_s\right|^2\right] &\leq \overline{\sigma}^4 t\hat{\mathbb{E}}\left[\int_0^t |h(s, X_0)|^2\mathrm{d}s\right] \\ &\leq \overline{\sigma}^4 t\hat{\mathbb{E}}\left[\int_0^t K^2\left(1 + |X_0|\right)^2\mathrm{d}s\right] \\ &\leq 2K^2\overline{\sigma}^4 T\left(1 + |X_0|^2\right)t, \end{aligned} \tag{4.9}$$

and

$$\hat{\mathbb{E}}\left[\left|\int_0^t g(s, X_0)\mathrm{d}B_s\right|^2\right] \leq \overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^t |g(s, X_0)|^2\mathrm{d}s\right]$$

$$
\begin{aligned}
&\leq \overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^t K^2(1+|X_0|)^2\mathrm{d}s\right]\\
&\leq 2K^2\overline{\sigma}^2\left(1+|X_0|^2\right)t. \qquad (4.10)
\end{aligned}
$$

Hence

$$
\hat{\mathbb{E}}\left[\left|X_t^1-X_t^0\right|^2\right]\leq A_\gamma t, \quad t\in[0,T],
$$

where $A_\gamma=8K^2(T+\overline{\sigma}^4T+\overline{\sigma}^2+K_\gamma^2T)(1+|X_0|^2)$ depends on γ. Assume that (4.6) holds for $j=n-1$, that is,

$$
\hat{\mathbb{E}}\left[\left|X_t^n-X_t^{n-1}\right|^2\right]\leq \frac{A_\gamma\bar{A}_\gamma^{n-1}t^n}{n!}, \quad t\in[0,T].
$$

When $j=n$,

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\left|X_t^{n+1}-X_t^n\right|^2\right]\leq 4\Bigg(&\hat{\mathbb{E}}\left[\left|\int_0^t\left(b(s,X_s^n)-b(s,X_s^{n-1})\right)\mathrm{d}s\right|^2\right]\\
&+\hat{\mathbb{E}}\left[\left|\int_0^t\left(f(s,X_s^n)-f(s,X_s^{n-1})\right)\mathrm{d}C_s\right|^2\right]\\
&+\hat{\mathbb{E}}\left[\left|\int_0^t\left(h(s,X_s^n)-h(s,X_s^{n-1})\right)\mathrm{d}\left\langle B\right\rangle_s\right|^2\right]\\
&+\hat{\mathbb{E}}\left[\left|\int_0^t\left(g(s,X_s^n)-g(s,X_s^{n-1})\right)\mathrm{d}B_s\right|^2\right]\Bigg), \quad t\in[0,T].
\end{aligned}
$$

By the similar proofs of (4.2)-(4.5), from the Hölder inequality, Proposition 3.1 and **(H1)** of Assumption 4.1, one can show that

$$
\hat{\mathbb{E}}\left[\left|X_t^{n+1}-X_t^n\right|^2\right]\leq 4L^2(T+\overline{\sigma}^4T+\overline{\sigma}^2+K_\gamma^2T)\int_0^t\hat{\mathbb{E}}\left[\left|X_s^n-X_s^{n-1}\right|^2\right]\mathrm{d}s, \quad t\in[0,T].
$$

Let $\bar{A}_\gamma=4L^2(T+\overline{\sigma}^4T+\overline{\sigma}^2+K_\gamma^2T)$, then

$$
\hat{\mathbb{E}}\left[\left|X_t^{n+1}-X_t^n\right|^2\right]\leq \bar{A}_\gamma\int_0^t\frac{A_\gamma\bar{A}_\gamma^{n-1}s^n}{n!}\mathrm{d}s=\frac{A_\gamma\bar{A}_\gamma^n t^{n+1}}{(n+1)!}, \quad t\in[0,T].
$$

For each $m\geq n>0$,

$$
\begin{aligned}
\|X_t^m-X_t^n\|_{\mathbb{M}^2(0,T)}&=\left\|\sum_{j=n+1}^m\left(X_t^j-X_t^{j-1}\right)\right\|_{\mathbb{M}^2(0,T)}\\
&\leq\sum_{j=n+1}^m\left\|X_t^j-X_t^{j-1}\right\|_{\mathbb{M}^2(0,T)}\\
&\leq\sum_{j=n+1}^m\left(\int_0^T\hat{\mathbb{E}}\left[\left|X_s^j-X_s^{j-1}\right|^2\right]\mathrm{d}s\right)^{\frac{1}{2}}\\
&\leq\sum_{j=n+1}^m\left(\int_0^T\frac{A_\gamma\bar{A}_\gamma^{j-1}s^j}{j!}\mathrm{d}s\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=n+1}^{m} \left(\frac{A_\gamma \bar{A}_\gamma^{j-1} T^{j+1}}{(j+1)!} \right)^{\frac{1}{2}} \\
&\to 0
\end{aligned}
$$

as $m, n \to \infty$, which implies that $\{X_t^n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{M}^2(0,T)$. Thus, there exists a process $\{X_t\}_{0 \le t \le T} \in \mathbb{M}^2(0,T)$ such that $X_t^n \to X_t$ in $\mathbb{M}^2(0,T)$, which is a solution to the LG-USDE (4.1). □

Example 4.1. Let a, b, c, e be constants, the LG-USDE

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t + c\mathrm{d}\langle B\rangle_t + e\mathrm{d}B_t, \quad t \in [0,T]$$

has a unique solution

$$X_t = X_0 + at + bC_t + c\langle B\rangle_t + eB_t, \quad t \in [0,T].$$

Example 4.2. The LG-USDE

$$\mathrm{d}X_t = a(t)X_t\mathrm{d}t + b(t)X_t\mathrm{d}C_t + c(t)X_t\mathrm{d}\langle B\rangle_t + e(t)X_t\mathrm{d}B_t, \quad t \in [0,T] \tag{4.11}$$

exists a unique solution

$$X_t = X_0 \exp\left(\int_0^t a(s)\mathrm{d}s + \int_0^t b(s)\mathrm{d}C_s + \int_0^t e(s)\mathrm{d}B_s + \int_0^t \frac{2c(s) - e^2(s)}{2}\mathrm{d}\langle B\rangle_s \right), \quad t \in [0,T],$$

where $a(t)$, $b(t)$, $c(t)$, $e(t)$: $[0,T] \mapsto \mathbb{R}$ are continuous functions. In fact, we have

$$|\mu(t)y - \mu(t)x| \le \max_{t\in[0,T]} |\mu(t)||y-x|, \quad t \in [0,T], x, y \in \mathbb{R}, \tag{4.12}$$

and

$$|\mu(t)y| \le \max_{t\in[0,T]} |\mu(t)||y| \le \max_{t\in[0,T]} |\mu(t)|(1+|y|), \quad t \in [0,T], y \in \mathbb{R}, \tag{4.13}$$

where $\mu(t) = a(t), b(t), c(t), e(t)$, respectively. Hence the LG-USDE (4.11) has a unique solution $\{X_t\}_{0\le t\le T} \in \mathbb{M}^2(0,T)$ by Theorem 4.1. From G-Itô-Liu formula,

$$
\begin{aligned}
\ln(X_t) - \ln(X_0) &= \int_0^t \frac{1}{X_s} a(s)X_s\mathrm{d}s + \int_0^t \frac{1}{X_s} b(s)X_s\mathrm{d}C_s + \int_0^t \frac{1}{X_s} c(s)X_s\mathrm{d}\langle B\rangle_s \\
&\quad + \int_0^t \frac{1}{X_s} e(s)X_s\mathrm{d}B_s + \frac{1}{2}\int_0^t -\frac{1}{X_s^2} e^2(s)X_s^2\mathrm{d}\langle B\rangle_s \\
&= \int_0^t a(s)\mathrm{d}s + \int_0^t b(s)\mathrm{d}C_s + \int_0^t e(s)\mathrm{d}B_s + \int_0^t \frac{2c(s) - e^2(s)}{2}\mathrm{d}\langle B\rangle_s, \quad t \in [0,T].
\end{aligned}
$$

Thus the solution is

$$X_t = X_0 \exp\left(\int_0^t a(s)\mathrm{d}s + \int_0^t b(s)\mathrm{d}C_s + \int_0^t e(s)\mathrm{d}B_s + \int_0^t \frac{2c(s) - e^2(s)}{2}\mathrm{d}\langle B\rangle_s \right), \quad t \in [0,T].$$

Example 4.3. Let $b(t)$, $f(t)$, $h(t)$, $g(t)$: $[0,T] \mapsto \mathbb{R}$ be continuous functions. Then the LG-USDE

$$\mathrm{d}X_t = b(t)X_t\mathrm{d}t + f(t)\mathrm{d}C_t + h(t)\mathrm{d}\left\langle B\right\rangle_t + g(t)\mathrm{d}B_t, \quad t \in [0,T] \tag{4.14}$$

has a unique solution

$$\begin{aligned} X_t =& X_0 \exp\left(\int_0^t b(s)\mathrm{d}s\right) + \int_0^t f(u)\exp\left(\int_u^t b(s)\mathrm{d}s\right)\mathrm{d}C_u \\ &+ \int_0^t h(u)\exp\left(\int_u^t b(s)\mathrm{d}s\right)\mathrm{d}\left\langle B\right\rangle_u + \int_0^t g(u)\exp\left(\int_u^t b(s)\mathrm{d}s\right)\mathrm{d}B_u, \quad t \in [0,T]. \end{aligned}$$

Obviously, from (4.12) and (4.13), the functions $b(t)x$, $f(t)$, $h(t)$ and $g(t)$ satisfy **(H1)** and **(H2)** of Assumption 4.1, which implies that there exists a unique solution $\{X_t\}_{0\le t\le T}$ to the LG-USDE (4.14) by Theorem 4.1. Multiplying both sides of (4.14) by $\exp\left(-\int_0^t b(s)\mathrm{d}s\right)$, we have

$$\begin{aligned} &\exp\left(-\int_0^t b(s)\mathrm{d}s\right)\mathrm{d}X_t - b(t)X_t\exp\left(-\int_0^t b(s)\mathrm{d}s\right)\mathrm{d}t \\ =&\big(f(t)\mathrm{d}C_t + h(t)\mathrm{d}\left\langle B\right\rangle_t + g(t)\mathrm{d}B_t\big)\exp\left(-\int_0^t b(s)\mathrm{d}s\right), \quad t \in [0,T]. \end{aligned}$$

That is for $t \in [0,T]$,

$$\begin{aligned} \mathrm{d}\left(X_t\exp\left(-\int_0^t b(s)\mathrm{d}s\right)\right) =& f(t)\exp\left(-\int_0^t b(s)\mathrm{d}s\right)\mathrm{d}C_t + h(t)\exp\left(-\int_0^t b(s)\mathrm{d}s\right)\mathrm{d}\left\langle B\right\rangle_t \\ &+ g(t)\exp\left(-\int_0^t b(s)\mathrm{d}s\right)\mathrm{d}B_t. \end{aligned} \tag{4.15}$$

Then integrating both sides of (4.15) from 0 to t, it yields

$$\begin{aligned} X_t\exp\left(-\int_0^t b(s)\mathrm{d}s\right) =& \int_0^t f(u)\exp\left(-\int_0^u b(s)\mathrm{d}s\right)\mathrm{d}C_u \\ &+ \int_0^t h(u)\exp\left(-\int_0^u b(s)\mathrm{d}s\right)\mathrm{d}\left\langle B\right\rangle_u \\ &+ \int_0^t g(u)\exp\left(-\int_0^u b(s)\mathrm{d}s\right)\mathrm{d}B_u + X_0, \quad t \in [0,T]. \end{aligned}$$

Thus the solution of LG-USDE (4.14) is

$$\begin{aligned} X_t =& X_0 \exp\left(\int_0^t b(s)\mathrm{d}s\right) + \int_0^t f(u)\exp\left(\int_u^t b(s)\mathrm{d}s\right)\mathrm{d}C_u \\ &+ \int_0^t h(u)\exp\left(\int_u^t b(s)\mathrm{d}s\right)\mathrm{d}\left\langle B\right\rangle_u + \int_0^t g(u)\exp\left(\int_u^t b(s)\mathrm{d}s\right)\mathrm{d}B_u, \quad t \in [0,T]. \end{aligned}$$

5. Stability analysis for the uncertain stochastic system

Studying the stability of dynamic system is crucial for understanding the long-term behavior of solutions and determining the reliability and predictability of models in diverse applications ranging from economics and physics to biology and engineering. A stable system might exhibit solutions that are closer together as the initial conditions become closer. In contrast, solutions can be highly sensitive to initial conditions, showing substantial divergence in behavior even when initial conditions are very close. In recent years, the research on stabilities of stochastic dynamic systems and uncertain dynamic systems has attracted a large number of scholars' attention (see, e.g., Mao [28], Yao et al. [42]). In this section, we study the stability for LG-USDE, and three different kinds of concepts about stabilities for LG-USDE are given.

Definition 5.1. Let $\{X_t\}_{0\le t\le T}$ and $\{X_t^\varepsilon\}_{0\le t\le T}$ be two solutions to the LG-USDE (4.1) with the initial conditions X_0 and X_0^ε. Then the LG-USDE (4.1) is called stable in $\mathbb{L}^2(\Gamma\times\Omega)$, if

$$\lim_{|X_0-X_0^\varepsilon|\to 0}\hat{\mathbb{E}}\left[|X_t-X_t^\varepsilon|^2\right]=0,\quad t\in[0,T].$$

Example 5.1. The LG-USDE (3.22)

$$\mathrm{d}X_t=aX_t\mathrm{d}t+bX_t\mathrm{d}C_t+cX_t\mathrm{d}\left\langle B\right\rangle_t+eX_t\mathrm{d}B_t,\quad t\in[0,T]$$

of Example 3.3 is stable in $\mathbb{L}^2(\Gamma\times\Omega)$. In fact, let $\{X_t\}_{0\le t\le T}$ and $\{X_t^\varepsilon\}_{0\le t\le T}$ be two solutions with the initial conditions X_0 and X_0^ε, then from Example 3.3, we have

$$X_t=X_0\exp\left(at+bC_t+eB_t+\left(c-\frac{e^2}{2}\right)\left\langle B\right\rangle_t\right),\quad t\in[0,T],$$

and

$$X_t^\varepsilon=X_0^\varepsilon\exp\left(at+bC_t+eB_t+\left(c-\frac{e^2}{2}\right)\left\langle B\right\rangle_t\right),\quad t\in[0,T].$$

Hence for $t\in[0,T]$,

$$\begin{aligned}\hat{\mathbb{E}}\left[|X_t-X_t^\varepsilon|^2\right]&=\hat{\mathbb{E}}\left[\left|(X_0-X_0^\varepsilon)\exp\left(at+bC_t+eB_t+\left(c-\frac{e^2}{2}\right)\left\langle B\right\rangle_t\right)\right|^2\right]\\&=|X_0-X_0^\varepsilon|^2\exp(bC_t)\hat{\mathbb{E}}\left[\left|\exp\left(at+eB_t+\left(c-\frac{e^2}{2}\right)\left\langle B\right\rangle_t\right)\right|^2\right]\\&\to 0\end{aligned}$$

as $|X_0-X_0^\varepsilon|\to 0$.

Theorem 5.1. *Under Assumption 4.1, the* LG-USDE (4.1) *is stable in* $\mathbb{L}^2(\Gamma\times\Omega)$ *for* $t\in[0,T]$.

Proof. Let $\{X_t\}_{0\le t\le T}$ and $\{X_t^\varepsilon\}_{0\le t\le T}$ be two solutions to the LG-USDE (4.1) with the initial values X_0 and X_0^ε, then we have for $t\in[0,T]$,

$$\begin{aligned}\hat{\mathbb{E}}\left[|X_t-X_t^\varepsilon|^2\right]\le 5\Bigg(&\hat{\mathbb{E}}\left[|X_0-X_0^\varepsilon|^2\right]+\hat{\mathbb{E}}\left[\left|\int_0^t b(s,X_s)-b(s,X_s^\varepsilon)\mathrm{d}s\right|^2\right]\\&+\hat{\mathbb{E}}\left[\left|\int_0^t f(s,X_s)-f(s,X_s^\varepsilon)\mathrm{d}C_s\right|^2\right]\\&+\hat{\mathbb{E}}\left[\left|\int_0^t h(s,X_s)-h(s,X_s^\varepsilon)\mathrm{d}\left\langle B\right\rangle_s\right|^2\right]\end{aligned}$$

$$+\hat{\mathbb{E}}\left[\left|\int_0^t g(s,X_s)-g(s,X_s^\varepsilon)\mathrm{d}B_s\right|^2\right]\Bigg).$$

By the similar proofs of (4.2)-(4.5), from the Hölder inequality, Proposition 3.1 and **(H1)** of Assumption 4.1, one can show that

$$\hat{\mathbb{E}}\left[|X_t-X_t^\varepsilon|^2\right]\leq 5\,|X_0-X_0^\varepsilon|^2+\lambda_\gamma\int_0^t\hat{\mathbb{E}}\left[|X_s-X_s^\varepsilon|^2\right]\mathrm{d}s,\quad t\in[0,T],$$

where $\lambda_\gamma=5L^2(T+\overline{\sigma}^4T+\overline{\sigma}^2+K_\gamma^2T)$. The Gronwall inequality yields

$$\hat{\mathbb{E}}\left[|X_t-X_t^\varepsilon|^2\right]\leq 5\,|X_0-X_0^\varepsilon|^2\exp\Big(\lambda_\gamma t\Big),\quad t\in[0,T].$$

Thus for $t\in[0,T]$, $\hat{\mathbb{E}}\left[|X_t-X_t^\varepsilon|^2\right]\to 0$ as $|X_0-X_0^\varepsilon|\to 0$. □

Theorem 5.2. *Let $\{X_t\}_{0\leq t\leq T}$ and $\{X_t^\varepsilon\}_{0\leq t\leq T}$ be solutions to the LG-USDE* (4.1) *with initial conditions X_0 and X_0^ε. Then Under Assumption 4.1, there exists a sequence $\{\varepsilon_n\}_{n\geq 1}$ satisfying $\sum_{n=1}^\infty\varepsilon_n<\infty$, and $|X_0^{\varepsilon_n}-X_0|^2<\varepsilon_n^3$, such that for each fixed $\gamma\in\Gamma$,*

$$v\left\{\lim_{n\to\infty}|X_t-X_t^{\varepsilon_n}|=0\right\}=1,\quad t\in[0,T].$$

Proof. From Theorem 5.1, there exist a sequence $\{X_t^{\varepsilon_n}\}$ and a constant c_γ such that $\hat{\mathbb{E}}[|X_t-X_t^{\varepsilon_n}|^2]<c_\gamma\varepsilon_n^3$, for each fixed $\gamma\in\Gamma$. Let $A_n=\{|X_t-X_t^{\varepsilon_n}|\geq\varepsilon_n\}$. From Markov inequality, we have

$$\sum_{n=1}^\infty\mathbb{V}\{A_n\}\leq\sum_{n=1}^\infty\frac{\hat{\mathbb{E}}[|X_t-X_t^{\varepsilon_n}|^2]}{{\varepsilon_n}^2}<\sum_{n=1}^\infty c_\gamma\varepsilon_n<\infty.$$

It follows from Lemma B.2 that $\mathbb{V}\left(\bigcap_{n=1}^\infty\bigcup_{k=n}^\infty A_k\right)=0$, i.e., $v\left(\bigcup_{n=1}^\infty\bigcap_{k=n}^\infty A_k\right)=1$. Then for each fixed $\gamma\in\Gamma$,

$$v\left\{\lim_{n\to\infty}|X_t-X_t^{\varepsilon_n}|=0\right\}=1,\quad t\in[0,T].$$

□

Definition 5.2. Let $\{X_t\}_{0\leq t\leq T}$ and $\{X_t^\varepsilon\}_{0\leq t\leq T}$ be two solutions to the LG-USDE (4.1) with the initial conditions X_0 and X_0^ε. Then the LG-USDE (4.1) is called almost surely stable, if

$$\tilde{ch}\left\{\lim_{|X_0-X_0^\varepsilon|\to 0}|X_t-X_t^\varepsilon|=0\right\}=1,\quad t\in[0,T].$$

Remark 5.1. From Proposition 2.1, $\tilde{ch}\Big\{\lim_{|X_0-X_0^\varepsilon|\to 0}|X_t-X_t^\varepsilon|=0\Big\}=1$ implies that $CH\Big\{\lim_{|X_0-X_0^\varepsilon|\to 0}|X_t-X_t^\varepsilon|=0\Big\}=1$. Hence for each $t\in[0,T]$, we only need to focus on $\tilde{ch}\Big\{\lim_{|X_0-X_0^\varepsilon|\to 0}|X_t-X_t^\varepsilon|=0\Big\}=1$.

Example 5.2. The LG-USDE (3.22)

$$\mathrm{d}X_t=aX_t\mathrm{d}t+bX_t\mathrm{d}C_t+cX_t\mathrm{d}\langle B\rangle_t+eX_t\mathrm{d}B_t,\quad t\in[0,T]$$

of Example 3.3 is almost surely stable. In fact, from Example 3.3, the two solutions $\{X_t\}_{0\le t\le T}$ and $\{X_t^\varepsilon\}_{0\le t\le T}$ with the initial values X_0 and X_0^ε are

$$X_t = X_0 \exp\left(at + bC_t + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right), \quad t \in [0,T],$$

and

$$X_t^\varepsilon = X_0^\varepsilon \exp\left(at + bC_t + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right), \quad t \in [0,T].$$

Then for $t \in [0,T]$, we obtain

$$\begin{aligned}&\tilde{ch}\left\{\lim_{|X_0 - X_0^\varepsilon|\to 0} |X_t - X_t^\varepsilon| = 0\right\}\\ =& \tilde{ch}\left\{\lim_{|X_0 - X_0^\varepsilon|\to 0} |X_0 - X_0^\varepsilon| \exp\left(at + bC_t + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right) = 0\right\}\\ =& 1.\end{aligned}$$

Thus for each $t \in [0,T]$, the LG-USDE (3.22) is almost surely stable.

Definition 5.3. Let $\{X_t\}_{0\le t\le T}$ and $\{X_t^\varepsilon\}_{0\le t\le T}$ be two solutions to the LG-USDE (4.1) with the initial conditions X_0 and X_0^ε. The LG-USDE (4.1) is called stable in U-S chance measures, if

$$\lim_{|X_0 - X_0^\varepsilon|\to 0} \tilde{ch}\left\{|X_t - X_t^\varepsilon| < \delta\right\} = 1,$$

for each $\delta > 0$, $t \in [0,T]$.

Example 5.3. The LG-USDE (3.22)

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t + cX_t\mathrm{d}\langle B\rangle_t + eX_t\mathrm{d}B_t, \quad t \in [0,T]$$

of Example 3.3 is stable in U-S chance measures. Without loss of generality, we only consider the following case. Let $\{X_t\}_{0\le t\le T}$ and $\{X_t^n\}_{0\le t\le T}$ be two solutions to the LG-USDE (3.22) with the initial values X_0 and X_0^n satisfying $X_0^n \downarrow X_0$ as $n \to \infty$. It follows from Example 3.3 that

$$X_t = X_0 \exp\left(at + bC_t + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right), \quad t \in [0,T],$$

and

$$X_t^n = X_0^n \exp\left(at + bC_t + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right), \quad t \in [0,T].$$

From Example 5.2, the LG-USDE (3.22) is almost surely stable, then

$$\tilde{ch}\left\{\lim_{X_0^n\downarrow X_0} |X_t - X_t^n| = 0\right\} = 1, \quad t \in [0,T],$$

i.e., for any $\varepsilon > 0$,

$$\tilde{ch}\left\{\bigcap_{\varepsilon>0}\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\{|X_t - X_t^m| < \varepsilon\}\right\} = 1, \quad t \in [0,T].$$

Hence for $t \in [0, T]$,

$$
\begin{aligned}
&\tilde{ch}\left\{\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\{|X_t - X_t^m| < \varepsilon\}\right\} \\
&= \int_0^1 v\left\{\omega \in \Omega \,\middle|\, \mathcal{M}\left\{\gamma \in \Gamma \,\middle|\, \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\{|X_t - X_t^m| < \varepsilon\}\right\} \geq y\right\} \mathrm{d}y \\
&= \int_0^1 v\left\{\omega \in \Omega \,\middle|\, \mathcal{M}\left\{\gamma \in \Gamma \,\middle|\, \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\left\{|X_0 - X_0^m| \exp\left(at + bC_t + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right) < \varepsilon\right\}\right\} \geq y\right\} \mathrm{d}y \\
&= \int_0^1 v\left\{\omega \in \Omega \,\middle|\, \mathcal{M}\left\{\gamma \in \Gamma \,\middle|\, \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\left\{C_t < \ln\frac{\varepsilon b^{-1}}{|X_0 - X_0^m|} - b^{-1}\left(at + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right)\right\}\right\} \geq y\right\} \mathrm{d}y \\
&= \int_0^1 v\left\{\omega \in \Omega \,\middle|\, \mathcal{M}\left\{\gamma \in \Gamma \,\middle|\, \bigcup_{n=1}^{\infty}\left\{C_t < \ln\frac{\varepsilon b^{-1}}{|X_0 - X_0^n|} - b^{-1}\left(at + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right)\right\}\right\} \geq y\right\} \mathrm{d}y \\
&= 1.
\end{aligned}
$$

From Definition A.6, for $t \in [0, T]$, the uncertainty distribution of C_t is

$$
\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1} = \mathcal{M}\{C_t \leq x\}, \quad x \in \mathbb{R}.
$$

By Definition A.3, we know that $\Phi(x)$ is a regular uncertainty distribution. Hence from the regularity of $\Phi(x)$, and the continuity of v, we obtain that

$$
\begin{aligned}
&\int_0^1 v\left\{\omega \in \Omega \,\middle|\, \bigvee_{n\geq 1}\mathcal{M}\left\{\gamma \in \Gamma \,\middle|\, \left\{C_t < \ln\frac{\varepsilon b^{-1}}{|X_0 - X_0^n|} - b^{-1}\left(at + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right)\right\}\right\} > y\right\} \mathrm{d}y \\
&= \int_0^1 v\left\{\omega \in \Omega \,\middle|\, \sup_{n\geq 1}\mathcal{M}\left\{\gamma \in \Gamma \,\middle|\, \left\{C_t < \ln\frac{\varepsilon b^{-1}}{|X_0 - X_0^n|} - b^{-1}\left(at + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right)\right\}\right\} > y\right\} \mathrm{d}y \\
&= \int_0^1 v\left\{\omega \in \Omega \,\middle|\, \bigcup_{n=1}^{\infty}\mathcal{M}\left\{\gamma \in \Gamma \,\middle|\, \left\{C_t < \ln\frac{\varepsilon b^{-1}}{|X_0 - X_0^n|} - b^{-1}\left(at + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right)\right\}\right\} > y\right\} \mathrm{d}y \\
&= \int_0^1 \lim_{n\to\infty} v\left\{\omega \in \Omega \,\middle|\, \mathcal{M}\left\{\gamma \in \Gamma \,\middle|\, \left\{C_t < \ln\frac{\varepsilon b^{-1}}{|X_0 - X_0^n|} - b^{-1}\left(at + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right)\right\}\right\} \geq y\right\} \mathrm{d}y \\
&= \lim_{n\to\infty}\int_0^1 v\left\{\omega \in \Omega \,\middle|\, \mathcal{M}\left\{\gamma \in \Gamma \,\middle|\, \left\{C_t < \ln\frac{\varepsilon b^{-1}}{|X_0 - X_0^n|} - b^{-1}\left(at + eB_t + \left(c - \frac{e^2}{2}\right)\langle B\rangle_t\right)\right\}\right\} \geq y\right\} \mathrm{d}y \\
&= \lim_{n\to\infty}\tilde{ch}\{|X_t - X_t^n|\} \\
&= 1.
\end{aligned}
$$

Thus

$$
\lim_{X_0^n \downarrow X_0}\tilde{ch}\{|X_t - X_t^n| < \varepsilon\} = 1, \quad t \in [0, T].
$$

Remark 5.2. $\mathbb{L}^2$ stability means that after a disturbance, $\hat{\mathbb{E}}[|X_t - X_t^{\varepsilon}|^2]$ decays to 0, and this indicates the worst-case average deviation of the system tends to 0. Almost sure stability means that after a disturbance, the error tends to 0 almost surely under U-S chance measures. Almost every possible trajectory eventually returns to the equilibrium point, which means the system will not lose control regardless of how the indeterminacy materializes. U-S chance measure stability means that the belief degree of the state deviating from equilibrium beyond a given threshold tends to 0, which ensures the belief degree of exceeding the error bound becomes increasingly small. In general, the three stabilities mentioned above have no mutual implication in this paper.

Remark 5.3. Due to the lack of relevant research in existing literature on integrability of Lipschitz constant K_γ of Liu process $\{C_t\}_{t\geq 0}$ with respect to uncertain measure $\mathcal{M}$ as well as on uncertainty distribution, it is difficult to derive almost surely stable and stable in U-S chance measures for LG-USDEs with a general form.

6. Euler-Maruyama numerical approximate solutions

In many cases, the exact solutions of LG-USDEs are not easy to obtain. Therefore, it is necessary to establish numerical methods to approximate the exact solutions of LG-USDEs. In the study of SDEs, scholars have developed several different numerical methods, and widely acknowledge that the Euler-Maruyama method (Maruyama [29]) as one of the simplest and most intuitive. This section investigates the convergence and applicability of Euler-Maruyama method when applied to the LG-USDE (4.1).

6.1. Euler-Maruyama method

For each integer $n \geq 1$, let $x_n(0) = X_0$, and denote

$$\begin{aligned} x_n(t) :=& x_n\Big(\frac{k}{n}\Big) + \int_{\frac{k}{n}}^t b\bigg(s, x_n\Big(\frac{k}{n}\Big)\bigg)\mathrm{d}s + \int_{\frac{k}{n}}^t f\bigg(s, x_n\Big(\frac{k}{n}\Big)\bigg)\mathrm{d}C_s \\ &+ \int_{\frac{k}{n}}^t h\bigg(s, x_n\Big(\frac{k}{n}\Big)\bigg)\mathrm{d}\left\langle B\right\rangle_s + \int_{\frac{k}{n}}^t g\bigg(s, x_n\Big(\frac{k}{n}\Big)\bigg)\mathrm{d}B_s, \end{aligned} \tag{6.1}$$

for $k/n \leq t \leq (k+1)/n \wedge T$, $k = 0, 1, 2, \cdots$. We define a simple process:

$$\hat{x}_n(t) := \sum_{k\geq 0} x_n\Big(\frac{k}{n}\Big)\mathbf{1}_{[\frac{k}{n},\frac{k+1}{n})}(t), \quad t \in [0, T].$$

Hence it follows from (6.1) that

$$\begin{aligned} x_n(t) =& X_0 + \int_0^t b\left(s, \hat{x}_n(s)\right)\mathrm{d}s + \int_0^t f\left(s, \hat{x}_n(s)\right)\mathrm{d}C_s \\ &+ \int_0^t h\left(s, \hat{x}_n(s)\right)\mathrm{d}\left\langle B\right\rangle_s + \int_0^t g\left(s, \hat{x}_n(s)\right)\mathrm{d}B_s, \quad t \in [0, T]. \end{aligned} \tag{6.2}$$

Lemma 6.1. *Under* **(H2)** *of Assumption 4.1, the sequence of processes* $\{x_n(t), 0 \leq t \leq T\}_{n=0}^\infty$ *defined by* (6.1) *satisfy*

$$\hat{\mathbb{E}}\left[\sup_{0\leq s\leq t}|x_n(s)|^2\right] \leq C_1^\gamma \mathrm{e}^{C_2^\gamma t}, \quad t \in [0, T],$$

where $C_1^\gamma = 5|X_0|^2 + 10K^2T(T + K_\gamma^2T + \overline{\sigma}^4T + 4\overline{\sigma}^2)$ *and* $C_2^\gamma = 10K^2(T + K_\gamma^2T + \overline{\sigma}^4T + 4\overline{\sigma}^2)$ *are positive constants for each fixed* $\gamma \in \Gamma$.

Proof. From the Hölder inequality and Proposition 3.1, it follows that

$$\hat{\mathbb{E}}\left[\sup_{0\le s\le t}|x_n(s)|^2\right] \le 5\,|X_0|^2 + 5T\int_0^t \hat{\mathbb{E}}\left[|b\,(s,\hat{x}_n(s))|^2\right]\mathrm{d}s + 5K_\gamma^2 T\int_0^t \hat{\mathbb{E}}\left[|f\,(s,\hat{x}_n(s))|^2\right]\mathrm{d}s$$
$$+ 5\overline{\sigma}^4 T\int_0^t \hat{\mathbb{E}}\left[|h\,(s,\hat{x}_n(s))|^2\right]\mathrm{d}s + 20\overline{\sigma}^2\int_0^t \hat{\mathbb{E}}\left[|g\,(s,\hat{x}_n(s))|^2\right]\mathrm{d}s.$$

By **(H2)** of Assumption 4.1 and the definition of $x_n(t)$, for $t\in[0,T]$, one can show that

$$\begin{aligned}\int_0^t \hat{\mathbb{E}}\left[|\mu\,(s,\hat{x}_n(s))|^2\right]\mathrm{d}s &\le \int_0^t \hat{\mathbb{E}}\left[K^2\,(1+|\hat{x}_n(s)|)^2\right]\mathrm{d}s\\ &\le 2K^2 t + 2K^2\int_0^t \hat{\mathbb{E}}\left[|\hat{x}_n(s)|^2\right]\mathrm{d}s \qquad (6.3)\\ &\le 2K^2 T + 2K^2\int_0^t \hat{\mathbb{E}}\left[\sup_{0\le r\le s}|x_n(r)|^2\right]\mathrm{d}s,\end{aligned}$$

where $\mu = b, f, h, g$, respectively. Then for $t\in[0,T]$, we have

$$\hat{\mathbb{E}}\left[\sup_{0\le s\le t}|x_n(s)|^2\right] \le 5|X_0|^2 + 10K^2T(T+K_\gamma^2T+\overline{\sigma}^4T+4\overline{\sigma}^2)$$
$$+ 10K^2(T+K_\gamma^2T+\overline{\sigma}^4T+4\overline{\sigma}^2)\int_0^t \hat{\mathbb{E}}\left[\sup_{0\le r\le s}|x_n(r)|^2\right]\mathrm{d}s.$$

Let $C_1^\gamma = 5|X_0|^2 + 10K^2T(T+K_\gamma^2T+\overline{\sigma}^4T+4\overline{\sigma}^2)$, and $C_2^\gamma = 10K^2(T+K_\gamma^2T+\overline{\sigma}^4T+4\overline{\sigma}^2)$. The above inequality yields

$$\hat{\mathbb{E}}\left[\sup_{0\le s\le t}|x_n(s)|^2\right] \le C_1^\gamma + C_2^\gamma\int_0^t \hat{\mathbb{E}}\left[\sup_{0\le r\le s}|x_n(r)|^2\right]\mathrm{d}s, \quad t\in[0,T].$$

It follows from the Gronwall inequality that

$$\hat{\mathbb{E}}\left[\sup_{0\le s\le t}|x_n(s)|^2\right] \le C_1^\gamma \mathrm{e}^{C_2^\gamma t}, \quad t\in[0,T].$$

□

Lemma 6.2. *Let $0\le r\le t\le T$. Under* **(H2)** *of Assumption 4.1, the sequence of processes $\{x_n(t), 0\le t\le T\}_{n=0}^\infty$ defined by* (6.1) *satisfy*

$$\hat{\mathbb{E}}\left[\sup_{r\le u\le v\le t}|x_n(v)-x_n(u)|^2\right] \le \left(D_1^\gamma + D_1^\gamma C_1^\gamma \mathrm{e}^{C_2^\gamma T}\right)(t-r), \quad 0\le r\le t\le T,$$

where C_1^γ and C_2^γ are defined in Lemma 6.1, and $D_1^\gamma = 8K^2[(t-r)+K_\gamma^2(t-r)+\overline{\sigma}^4(t-r)+4\overline{\sigma}^2]$ is a positive constant for each fixed $\gamma\in\Gamma$.

Proof. By the Hölder inequality and Proposition 3.1, we obtain

$$\hat{\mathbb{E}}\left[\sup_{r\le u\le v\le t}|x_n(v)-x_n(u)|^2\right]$$

$$
\begin{aligned}
&\leq 4\hat{\mathbb{E}}\left[\sup_{r\leq u\leq v\leq t}\left|\int_u^v b\left(s,\hat{x}_n(s)\right)\mathrm{d}s\right|^2\right]+4\hat{\mathbb{E}}\left[\sup_{r\leq u\leq v\leq t}\left|\int_u^v f\left(s,\hat{x}_n(s)\right)\mathrm{d}C_s\right|^2\right]\\
&\quad+4\hat{\mathbb{E}}\left[\sup_{r\leq u\leq v\leq t}\left|\int_u^v h\left(s,\hat{x}_n(s)\right)\mathrm{d}\left\langle B\right\rangle_s\right|^2\right]+4\hat{\mathbb{E}}\left[\sup_{r\leq u\leq v\leq t}\left|\int_u^v g\left(s,\hat{x}_n(s)\right)\mathrm{d}B_s\right|^2\right]\\
&\leq 4(t-r)\int_r^t\hat{\mathbb{E}}\left[\left|b\left(s,\hat{x}_n(s)\right)\right|^2\right]\mathrm{d}s+4K_\gamma^2(t-r)\int_t^t\hat{\mathbb{E}}\left[\left|f\left(s,\hat{x}_n(s)\right)\right|^2\right]\mathrm{d}s\\
&\quad+4\overline{\sigma}^4(t-r)\int_t^t\hat{\mathbb{E}}\left[\left|h\left(s,\hat{x}_n(s)\right)\right|^2\right]\mathrm{d}s+16\overline{\sigma}^2\int_t^t\hat{\mathbb{E}}\left[\left|g\left(s,\hat{x}_n(s)\right)\right|^2\right]\mathrm{d}s.
\end{aligned}
$$

Similar to the proof of (6.3), it is easy to check that

$$
\int_r^t\hat{\mathbb{E}}\left[\left|\mu\left(s,\hat{x}_n(s)\right)\right|^2\right]\mathrm{d}s\leq 2K^2(t-r)+2K^2\int_r^t\hat{\mathbb{E}}\left[\sup_{0\leq z\leq s}|x_n(z)|^2\right]\mathrm{d}s,\quad 0\leq r\leq t\leq T,
$$

where $\mu=b,f,h,g$, respectively. Hence

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{r\leq u\leq v\leq t}|x_n(v)-x_n(u)|^2\right]\\
&\leq 8K^2(t-r)\left[(t-r)+K_\gamma^2(t-r)+\overline{\sigma}^4(t-r)+4\overline{\sigma}^2\right]\\
&\quad+8K^2\left[(t-r)+K_\gamma^2(t-r)+\overline{\sigma}^4(t-r)+4\overline{\sigma}^2\right]\int_r^t\hat{\mathbb{E}}\left[\sup_{0\leq z\leq s}|x_n(z)|^2\right]\mathrm{d}s\\
&=D_1^\gamma(t-r)+D_1^\gamma\int_r^t\hat{\mathbb{E}}\left[\sup_{0\leq z\leq s}|x_n(z)|^2\right]\mathrm{d}s,
\end{aligned}
$$

where $D_1^\gamma=8K^2\left[(t-r)+K_\gamma^2(t-r)+\overline{\sigma}^4(t-r)+4\overline{\sigma}^2\right]$. It follows from Lemma 6.1 that

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\sup_{r\leq u\leq v\leq t}|x_n(v)-x_n(u)|^2\right]&\leq D_1^\gamma(t-r)+D_1^\gamma\int_r^t C_1^\gamma\mathrm{e}^{C_2^\gamma s}\mathrm{d}s\\
&\leq\left(D_1^\gamma+D_1^\gamma C_1^\gamma\mathrm{e}^{C_2^\gamma T}\right)(t-r),\quad 0\leq r\leq t\leq T.
\end{aligned}
$$

□

Theorem 6.1. *Under the Assumption 4.1, the sequence of numerical approximate solutions* $\{x_n(t),0\leq t\leq T\}_{n=0}^\infty$ *defined by* (6.1) *and the exact solution* $\{X_t\}_{0\leq t\leq T}$ *to the* LG-USDE (4.1) *have the property*

$$
\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}|x_n(t)-X_t|^2\right]\leq\frac{2D_2^\gamma\left(D_1^\gamma+D_1^\gamma C_1^\gamma\mathrm{e}^{C_2^\gamma T}\right)T}{n}\mathrm{e}^{2D_2^\gamma T},
$$

where $D_2^\gamma=4L^2(T+K_\gamma^2T+\overline{\sigma}^4T+4\overline{\sigma}^2)$ *is a positive constant for each fixed* $\gamma\in\Gamma$, C_1^γ,C_2^γ *and* D_1^γ *are defined in Lemma 6.1 and Lemma 6.2.*

Proof. Applying the Hölder inequality and Proposition 3.1 yields that for $t\in[0,T]$,

$$
\hat{\mathbb{E}}\left[\sup_{0\leq u\leq t}|x_n(u)-X_u|^2\right]\leq 4\hat{\mathbb{E}}\left[\sup_{0\leq u\leq t}\left|\int_0^u b(s,\hat{x}_n(s))-b(s,X_s)\mathrm{d}s\right|^2\right]
$$

$$
\begin{aligned}
&+ 4\hat{\mathbb{E}}\left[\sup_{0\le u\le t}\left|\int_0^u f(s,\hat{x}_n(s)) - f(s,X_s)\mathrm{d}C_s\right|^2\right]\\
&+ 4\hat{\mathbb{E}}\left[\sup_{0\le u\le t}\left|\int_0^u h(s,\hat{x}_n(s)) - h(s,X_s)\mathrm{d}\left\langle B\right\rangle_s\right|^2\right]\\
&+ 4\hat{\mathbb{E}}\left[\sup_{0\le u\le t}\left|\int_0^u g(s,\hat{x}_n(s)) - g(s,X_s)\mathrm{d}B_s\right|^2\right]\\
\le{}& 4T\int_0^t \hat{\mathbb{E}}\left[|b(s,\hat{x}_n(s)) - b(s,X_s)|^2\right]\mathrm{d}s\\
&+ 4K_\gamma^2 T\int_0^t \hat{\mathbb{E}}\left[|f(s,\hat{x}_n(s)) - f(s,X_s)|^2\right]\mathrm{d}s\\
&+ 4\overline{\sigma}^4 T\int_0^t \hat{\mathbb{E}}\left[|h(s,\hat{x}_n(s)) - h(s,X_s)|^2\right]\mathrm{d}s\\
&+ 16\overline{\sigma}^2\int_0^t \hat{\mathbb{E}}\left[|g(s,\hat{x}_n(s)) - g(s,X_s)|^2\right]\mathrm{d}s.
\end{aligned}
$$

It follows from **(H1)** of Assumption 4.1 that

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\sup_{0\le u\le t}|x_n(u)-X_u|^2\right] &\le 4TL^2\int_0^t \hat{\mathbb{E}}\left[|\hat{x}_n(s)-X_s|^2\right]\mathrm{d}s + 4K_\gamma^2 TL^2\int_0^t \hat{\mathbb{E}}\left[|\hat{x}_n(s)-X_s|^2\right]\mathrm{d}s\\
&\quad + 4\overline{\sigma}^4 TL^2\int_0^t \hat{\mathbb{E}}\left[|\hat{x}_n(s)-X_s|^2\right]\mathrm{d}s + 16\overline{\sigma}^2 L^2\int_0^t \hat{\mathbb{E}}\left[|\hat{x}_n(s)-X_s|^2\right]\mathrm{d}s\\
&\le D_2^\gamma\int_0^t \hat{\mathbb{E}}\left[|\hat{x}_n(s)-X_s|^2\right]\mathrm{d}s, \quad t\in[0,T],
\end{aligned}
$$

where $D_2^\gamma = 4L^2(T + K_\gamma^2 T + \overline{\sigma}^4 T + 4\overline{\sigma}^2)$. By using the inequality $|\hat{x}_n(s) - X_s|^2 \le 2|\hat{x}_n(s) - x_n(s)|^2 + 2|x_n(s) - X_s|^2$, we obtain

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\sup_{0\le u\le t}|x_n(u)-X_u|^2\right] &\le 2D_2^\gamma\int_0^t \hat{\mathbb{E}}\left[|\hat{x}_n(s)-x_n(s)|^2\right]\mathrm{d}s\\
&\quad + 2D_2^\gamma\int_0^t \hat{\mathbb{E}}\left[|x_n(s)-X_s|^2\right]\mathrm{d}s, \quad t\in[0,T].
\end{aligned}
$$

The definition of $\hat{x}_n(s)$ yields

$$
\hat{x}_n(s) = \sum_{k\ge 0} x_n\Big(\frac{k}{n}\Big)\mathbf{1}_{[\frac{k}{n},\frac{k+1}{n})}(s) = x_n\Big(\frac{[sn]}{n}\Big).
$$

Hence

$$
\begin{aligned}
\hat{\mathbb{E}}\left[\sup_{0\le u\le t}|x_n(u)-X_u|^2\right] &\le 2D_2^\gamma\int_0^t \hat{\mathbb{E}}\left[\left|x_n(s) - x_n\Big(\frac{[sn]}{n}\Big)\right|^2\right]\mathrm{d}s\\
&\quad + 2D_2^\gamma\int_0^t \hat{\mathbb{E}}\left[|x_n(s)-X_s|^2\right]\mathrm{d}s, \quad t\in[0,T].
\end{aligned}
$$

From Lemma 6.2, one can show that

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{0\le u\le t}|x_n(u)-X_u|^2\right]\\
&\le 2D_2^\gamma\int_0^t \frac{D_1^\gamma + D_1^\gamma C_1^\gamma \mathrm{e}^{C_2^\gamma T}}{n}\mathrm{d}s + 2D_2^\gamma\int_0^t \hat{\mathbb{E}}\left[\sup_{0\le u\le s}|x_n(u)-X_u|^2\right]\mathrm{d}s\\
&\le \frac{2D_2^\gamma\left(D_1^\gamma + D_1^\gamma C_1^\gamma \mathrm{e}^{C_2^\gamma T}\right)T}{n} + 2D_2^\gamma\int_0^t \hat{\mathbb{E}}\left[\sup_{0\le u\le s}|x_n(u)-X_u|^2\right]\mathrm{d}s, \quad t\in[0,T].
\end{aligned}$$

From the Gronwall inequality, it follows that

$$\hat{\mathbb{E}}\left[\sup_{0\le u\le t}|x_n(u)-X_u|^2\right] \le \frac{2D_2^\gamma\left(D_1^\gamma + D_1^\gamma C_1^\gamma \mathrm{e}^{C_2^\gamma T}\right)T}{n}\mathrm{e}^{2D_2^\gamma t}, \quad t\in[0,T].$$

Thus

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le T}|x_n(t)-X_t|^2\right] \le \frac{2D_2^\gamma\left(D_1^\gamma + D_1^\gamma C_1^\gamma \mathrm{e}^{C_2^\gamma T}\right)T}{n}\mathrm{e}^{2D_2^\gamma T}.$$

The proof of Theorem 6.1 is completed. □

Remark 6.1. For each $t\in[0,T]$, we have

$$\hat{\mathbb{E}}\left[|x_n(t)-X_t|^2\right] \le \hat{\mathbb{E}}\left[\sup_{0\le t\le T}|x_n(t)-X_t|^2\right] \le \frac{2D_2^\gamma\left(D_1^\gamma + D_1^\gamma C_1^\gamma \mathrm{e}^{C_2^\gamma T}\right)T}{n}\mathrm{e}^{2D_2^\gamma T} \to 0$$

as $n\to\infty$, which implies that the sequence of numerical approximate solutions

$$\{x_n(t), 0\le t\le T\}_{n=0}^{\infty}$$

converges the exact solution $\{X_t\}_{0\le t\le T}$ to the LG-USDE (4.1), when the step size is small enough.

6.2. Numerical example

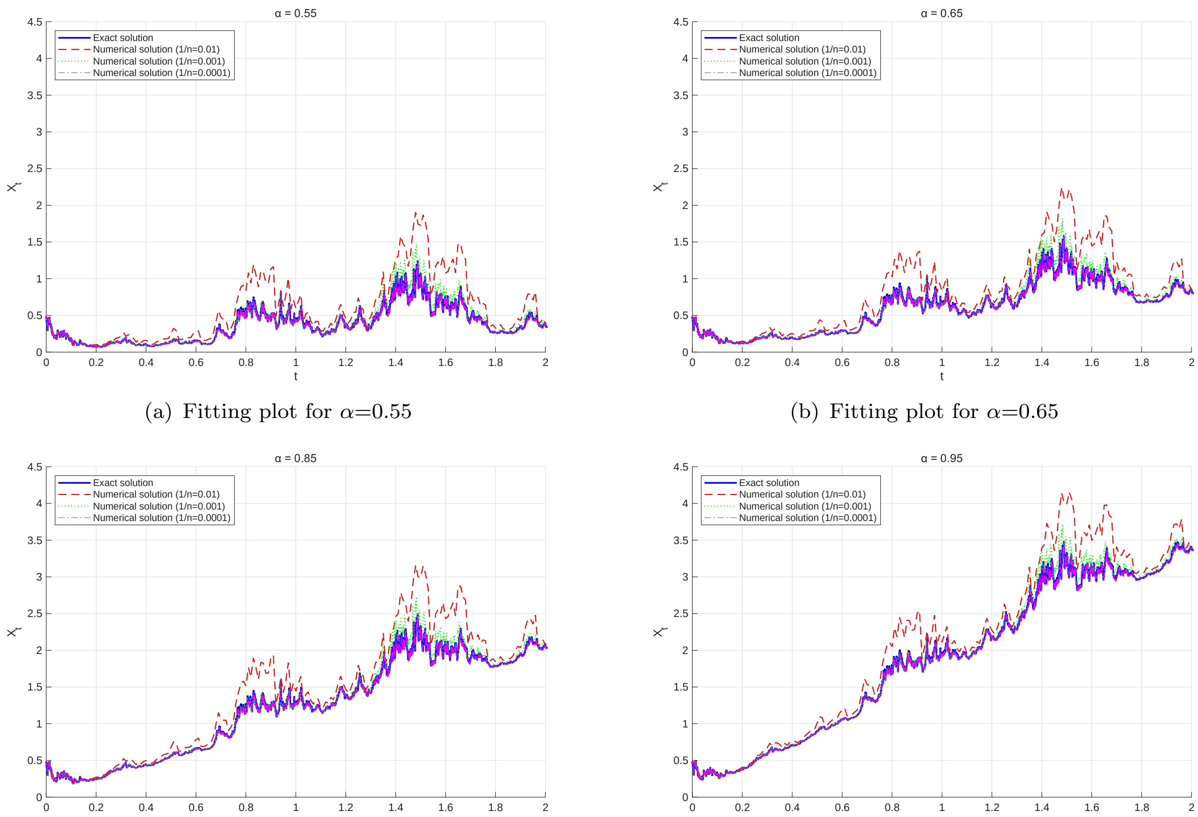
We verify the effectiveness of proposed Euler-Maruyama method through the following numerical experiment. Let $\{B_t\}_{t\ge0}$ be a G-Brownian motion with $B_t \overset{d}{=} N(\{0\}\times[0.53t, 0.81t])$ and $\{C_t\}_{t\ge0}$ be a canonical Liu process. Consider an equation

$$\mathrm{d}(X_t - C_t) = \sin t\cdot(X_t - C_t)\mathrm{d}t + c(X_t - C_t)\mathrm{d}\langle B\rangle_t + e(X_t - C_t)\mathrm{d}B_t, \quad t\in[0,T].$$

Then from Example 4.2, the exact solution is

$$X_t = C_t + X_0\exp\left(1-\cos t + eB_t + (c - e/2)\langle B\rangle_t\right), \quad t\in[0,T].$$

It is known that increment of Liu process $\{C_t\}_{t\ge0}$ follows normal uncertainty distribution with inverse uncertainty distribution $\Phi^{-1}(\alpha) = \frac{\sqrt{3}t}{\pi}\ln\frac{\alpha}{1-\alpha}$, where $\alpha\in(0,1)$ is quantile parameter under uncertainty measure. Therefore we use a method of taking α-quantile to approximate Liu process, i.e., for each fixed α, we take corresponding α-quantile value. This is consistent with the theoretical framework of this paper, as all theoretical analysis is performed for each fixed $\gamma \in \Gamma$, and the α-quantile corresponds to the realization of $\{C_t\}_{t\geq 0}$ under some specific γ. Let $X_0 = 0.45$, $c = 0.2$, $e = 3.5$, $T = 2$. The numerical solution is obtained by using the proposed Euler-Maruyama method, with three different step sizes set as $\frac{1}{n} = 0.01$, $\frac{1}{n} = 0.001$, and $\frac{1}{n} = 0.0001$. The fitting effects of numerical solutions and exact solution under different step sizes can be compared. From the final fitting plots (Figure 1(a)-1(d)), one can observe that as the step size decreases, numerical solution curves significantly improve in adherence to exact solution. This demonstrates the effectiveness of the proposed numerical method.



(a) Fitting plot for α=0.55

(b) Fitting plot for α=0.65

(c) Fitting plot for α=0.85

(d) Fitting plot for α=0.95

Figure 1. Fitting plots of numerical and exact solutions.

Remark 6.2. It should be noted that there are currently no research results in existing literature that can simulate the paths of canonical Liu process $\{C_t\}_{t\geq 0}$, and there is a lack of corresponding software packages. We think this is mainly due to the essential difference between the concept of independence in uncertainty theory and that in probability theory. Therefore, we apply use a method of taking α-quantile to approximate Liu process in above numerical experiment. This is consistent with idea in our theoretical framework that fixes each $\gamma \in \Gamma$.

7. Conclusion

The main contribution of this paper is to provide a theoretical framework for studying dynamic systems with both human uncertainty and probability distribution with ambiguity. We rigorously define the G-Itô-Liu integral to quantify the cumulative effects of dual indeterminacy in uncertain stochastic systems, and we prove the corresponding G-Itô-Liu formula. Based on the theoretical foundation discussed above, we creatively introduce the LG-USDE to enable accurate modeling of the uncertain stochastic system. Subsequently, the well-posedness of this system is proved, three different types of stabilities are proposed, and the Euler-Maruyama numerical method is employed to solve this system.

On the theoretical side, future work might focus on establishing the solvability to LG-USDE (4.1) under more general conditions than those presented here. Further research is also warranted to investigate an uncertain stochastic functional differential equation driven by canonical Liu process and G-Brownian motion (LG-USFDE), as well as other types of equations. In terms of applications, future efforts could focus on the uncertain stochastic optimal control problems of uncertain stochastic dynamic systems, such as that in Example 2.4.

Acknowledgements

The authors would like to thank the Associate Editor and anonymous referees for their constructive suggestions and valuable comments that greatly improved this article.

A. Uncertainty theory

This section introduces some fundamental concepts of uncertainty theory, and further details can be found in Chen and Liu [1], Liu [21, 24].

Definition A.1. (Liu [21]) Let $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. Each element $\mathbb{A}$ in $\mathcal{L}$ is called an event. An uncertain measure $\mathcal{M} : \mathcal{L} \mapsto [0,1]$ is a set function which satisfies the following axioms:
1. (Normality axiom) $\mathcal{M}\{\Gamma\} = 1$.
2. (Duality axiom) $\mathcal{M}\{\mathbb{A}\} + \mathcal{M}\{\mathbb{A}^c\} = 1$, for $\mathbb{A} \in \mathcal{L}$.
3. (Sub-additivity axiom) For every countable sequence of $\{\mathbb{A}_i\}_{i=1}^{\infty} \subseteq \mathcal{L}$,

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \mathbb{A}_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\mathbb{A}_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Moreover, the product uncertain measure is defined by the following axiom:
4. (Product axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure that satisfies

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \mathbb{A}_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\mathbb{A}_k\},$$

for each $\mathbb{A}_k \in \mathcal{L}_k, k = 1, 2, \cdots$.

Yang et al. [40] studied continuities of uncertain measure $\mathcal{M}$, and they proved some equivalent continuous conditions for uncertain measure $\mathcal{M}$.

Lemma A.1. (Yang et al. [40]) *Let $\mathcal{M}$ be an uncertain measure on Γ, and $\{\mathbb{A}_n\}$ be a sequence of events. Then the following conditions are equivalent:*

(a) *Strongly order continuous: If* $\mathbb{A}_n \searrow \Lambda$, *and* $\mathcal{M}\{\mathbb{A}\} = 0$, *then* $\lim_{n\to\infty} \mathcal{M}\{\mathbb{A}_n\} = 0$.
(b) *Strongly continuous: If* $\mathbb{A}_n \nearrow \mathbb{A}$, *and* $\mathcal{M}\{\mathbb{A}\} = 1$, *then* $\lim_n \to \infty\mathcal{M}\{\mathbb{A}_n\} = 1$.
(c) *Continuous from above: If* $\mathbb{A}_n \searrow \mathbb{A}$, *then* $\lim_{n\to\infty} \mathcal{M}\{\mathbb{A}_n\} = \mathcal{M}\{\mathbb{A}\}$.
(d) *Continuous from below: If* $\mathbb{A}_n \nearrow \mathbb{A}$, *then* $\lim_{n\to\infty} \mathcal{M}\{\mathbb{A}_n\} = \mathcal{M}\{\mathbb{A}\}$.
(e) *Continuous from above at* $\emptyset$*: If* $\mathbb{A}_n \searrow \emptyset$, *then* $\lim_{n\to\infty} \mathcal{M}\{\mathbb{A}_n\} = 0$.
(f) *Continuous from below at* Γ*: If* $\mathbb{A}_n \nearrow \Gamma$, *then* $\lim_{n\to\infty} \mathcal{M}\{\mathbb{A}_n\} = 1$.

Definition A.2. (Liu [21]) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be uncertainty space. A measurable function $\tau : \Gamma \mapsto \mathbb{R}$ is called an uncertain variable if $\{\tau \in B\} = \{\gamma \in \Gamma | \tau(\gamma) \in B\} \in \mathcal{L}$ for each $B \in \mathcal{B}(\mathbb{R})$. The uncertainty distribution of τ is defined by

$$\Phi(x) = \mathcal{M}\{\tau \le x\}, \quad x \in \mathbb{R}.$$

Definition A.3. (Liu [21]) An uncertainty distribution $\Phi(x)$ is said to be regular, if it is continuous and strictly increasing with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty} \Phi(x) = 0, \quad \lim_{x\to+\infty} \Phi(x) = 1.$$

Definition A.4. (Liu [21]) Uncertain variables $\zeta_1, \zeta_2, \cdots, \zeta_n$ is said to be independent, if for any Borel sets $B_1, B_2, \cdots, B_n$,

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\tau_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{\tau_i \in B_i\right\}.$$

Definition A.5. (Liu [24]) (Uncertain process) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and T be an index set. A measurable function $X_t(\gamma) : T \times \Gamma \mapsto \mathbb{R}$ is called an uncertain process if $\{X_t \in B\} = \{\gamma \in \Gamma | X_t(\gamma) \in B\} \in \mathcal{L}$ for each $B \in \mathcal{B}(\mathbb{R})$.

Definition A.6. (Liu [24]) (Canonical Liu process) An uncertain process $\{C_t\}_{t\ge0}$ is defined as a canonical Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous.
(ii) The increments of C_t are independent and stationary.
(iii) $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2 for each $t, s \ge 0$, and uncertainty distribution is

$$\Phi(y) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad y \in \mathbb{R}.$$

Remark A.1. From Definition A.6, there exists a Lipschitz constant K_γ such that the canonical Liu process $\{C_t(\gamma)\}_{t\ge0}$ satisfies

$$|C_{t_{i+1}}(\gamma) - C_{t_i}(\gamma)| \le K_\gamma |t_{i+1} - t_i|,$$

where the Lipschitz constant K_γ is regarded as different values of an uncertain variable.

Definition A.7. (Liu [24]) (Liu integral) Let $\{X_t\}_{t\ge0}$ be an uncertain process, $\{C_t\}_{t\ge0}$ be a canonical Liu process, and the mesh be $\Delta = \max_{1\le i\le k} |t_{i+1} - t_i|$ for any partition of closed interval $[0, T]$ with $0 = t_1 < t_2 < \cdots < t_{k+1} = T$. Then Liu integral of $\{X_t\}_{t\ge0}$ is defined by

$$\int_0^T X_t \mathrm{d}C_t = \lim_{\Delta\to0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}),$$

if the limit exists almost surely and is finite.

Theorem A.1. (Liu [24]) *Let* $\{X_t\}_{t\geq 0}$ *and* $\{Y_t\}_{t\geq 0}$ *be integrable uncertain processes on* $[0,\mathrm{T}]$, *then we have*

$$\int_0^T (\mu X_t + \nu Y_t)\mathrm{d}C_t = \mu \int_0^T X_t \mathrm{d}C_t + \nu \int_0^T Y_t \mathrm{d}C_t, \quad \mu,\nu \in \mathbb{R}.$$

Theorem A.2. (Liu [24]) *Let* $h(t,c)$ *be a continuously differentiable function, and* $\{C_t\}_{t\geq 0}$ *be a canonical Liu process. Then the uncertain process* $\{Z_t = h(t,C_t)\}_{t\geq 0}$ *has an uncertain differential*

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t,C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t,C_t)\mathrm{d}C_t.$$

Lemma A.2. (Chen and Liu [1]) *Let* $\{C_t\}_{t\geq 0}$ *be a canonical Liu process, and* $\{X_t\}_{t\geq 0}$ *be an integrable uncertain process on* $[0,\mathrm{T}]$. *Then there exists a Lipschitz constant* K_γ *of* $\{C_t(\gamma)\}_{t\geq 0}$ *such that*

$$\left|\int_0^T X_t(\gamma)\mathrm{d}C_t(\gamma)\right| \leq K_\gamma \int_0^T |X_t(\gamma)|\mathrm{d}t.$$

Definition A.8. (Liu [24]) Let $\{C_t\}_{t\geq 0}$ be a canonical Liu process, and f and g be two functions. Then an uncertain differential equation (UDE) is defined by

$$\mathrm{d}X_t = f(t,X_t)\mathrm{d}t + g(t,X_t)\mathrm{d}C_t. \tag{A.1}$$

Theorem A.3. (Liu [24]) *There exists a unique solution to UDE* (A.1) *if* $f(t,x)$ *and* $g(t,x)$ *satisfy the standard Lipschitz condition*

$$|f(t,x)-f(t,y)| + |g(t,x)-g(t,y)| \leq L|x-y|, \quad \forall x,y\in\mathbb{R}, t\geq 0,$$

and linear growth condition

$$|f(t,x)| + |g(t,x)| \leq L(1+|x|), \quad \forall x\in\mathbb{R}, t\geq 0,$$

for some constant L

Example A.1. Let $\{u_t\}_{t\geq 0}$ and $\{v_t\}_{t\geq 0}$ be two integrable uncertain processes, then the solution to UDE

$$\mathrm{d}X_t = u_t X_t \mathrm{d}t + v_t X_t \mathrm{d}C_t$$

is

$$X_t = X_0 \exp\left(\int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s\right).$$

B. Theory of sublinear expectation and related stochastic calculus

This section will introduce some basic definitions and results under sublinear expectations. More details are referred to Chen et al. [7], Gao [12] and Peng [32–34]. Let $(\Omega,\mathcal{F})$ be a given measurable space, and $\mathcal{H}$ be a linear space of real-valued functions defined on Ω such that
(i) if $X_1,\cdots,X_n \in \mathcal{H}$, then $\varphi(X_1,\cdots,X_n)\in\mathcal{H}$ for each $\varphi\in C_{l.\mathrm{Lip}}(\mathbb{R}^n)$, where $C_{l.\mathrm{Lip}}(\mathbb{R}^n)$ denotes the linear space of functions φ satisfying the following local Lipschitz condition:

$$|\varphi(x)-\varphi(y)| \leq C(1+|x|^m+|y|^m)|x-y| \text{ for } x,y\in\mathbb{R}^n,$$

where the constant $C>0$ and the integer $m\in\mathbb{N}$ depend on φ.
(ii) $I_A\in\mathcal{H}$, for each $A\in\mathcal{F}$, where I_A is the indicator function of event A.

Definition B.1. (Peng [34]) (Sublinear expectation) A functional $\hat{\mathbb{E}} : \mathcal{H} \mapsto \mathbb{R}$ is a sublinear expectation if it satisfies
(i) **Monotonicity:** $\hat{\mathbb{E}}[X] \le \hat{\mathbb{E}}[Y]$, if $X \le Y$.
(ii) **Constant preserving:** $\hat{\mathbb{E}}[c] = c$, for $c \in \mathbb{R}$.
(iii) **Sub-additivity:** $\hat{\mathbb{E}}[X + Y] \le \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$, for each $X, Y \in \mathcal{H}$.
(iv) **Positive homogeneity:** $\hat{\mathbb{E}}[\lambda X] = \lambda\hat{\mathbb{E}}[X]$, for $\lambda \ge 0$.
The triplet $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space, and the conjugate expectation of $\mathbb{E}$ is defined by $\mathcal{E}[X] := -\hat{\mathbb{E}}[-X], \forall X \in \mathcal{H}$.

Proposition B.1. (Peng [34]) *Given the sublinear expectation $\hat{\mathbb{E}}$, for any $X, Y \in \mathcal{H}$,we have*

$$\hat{\mathbb{E}}[|X + Y|^\alpha] \le \max\{1, 2^{\alpha-1}\}(\hat{\mathbb{E}}[|X|^\alpha] + \hat{\mathbb{E}}[|Y|^\alpha]), \text{ for } \alpha > 0.$$

Definition B.2. (Chen et al. [7], Choquet, [8]) (Non-additive probability) Let Ω be a non-empty set, and $\mathcal{F}$ be a σ-algebra on Ω. A non-additive probability is a set function $V : \mathcal{F} \mapsto [0, 1]$ satisfies
(i) $V(\emptyset) = 0, V(\Omega) = 1$;
(ii) if $A_1, A_2 \in \mathcal{F}$ and $A_1 \subseteq A_2$, then $V(A_1) \le V(A_2)$.
The triplet $(\Omega, \mathcal{F}, V)$ is called a non-additive probability space. Furthermore, a non-additive probability V is called lower continuous or upper continuous, if it satisfies (iii) or (iv):
(iii) $V(A_n) \uparrow V(A)$, whenever $A_n \uparrow A$, for $A_n, A \in \mathcal{F}$;
(iv) $V(A_n) \downarrow V(A)$, whenever $A_n \downarrow A$, for $A_n, A \in \mathcal{F}$.

The following theorem is a bridge between sublinear expectations and non-additive probabilities.

Theorem B.1. (Peng [34]) (Robust Daniell-Stone theorem) *Assume that $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is a sublinear expectation space satisfying $\hat{\mathbb{E}}[X_i] \to 0$, as $i \to \infty$, for each sequence $\{X_i\}_{i=1}^\infty$ of random variables in $\mathcal{H}$ such that $X_i(\omega) \downarrow 0$ for each $\omega \in \Omega$. Then there exists a family of probability measures $\{P_\theta\}_{\theta\in\Theta}$ defined on the measurable space $(\Omega, \sigma(\mathcal{H}))$ such that*

$$\hat{\mathbb{E}}[X] = \max_{\theta\in\Theta} \int_\Omega X(\omega)\mathrm{d}P_\theta, \text{ for each } X \in \mathcal{H}.$$

Here $\sigma(\mathcal{H})$ is the smallest σ-algebra generated by $\mathcal{H}$.

Definition B.3. (Chen et al. [7]) Let $(\Omega, \mathcal{F})$ be a given measurable space, and $\mathbb{M}$ be the set of all probability measures on Ω. For any nonempty subset $\{P_\theta\}_{\theta\in\Theta} \subseteq \mathbb{M}$ and $A \in \mathcal{F}$, we define lower probability $v(A) := \inf\limits_{\theta\in\Theta} P_\theta(A)$, upper probability $\mathbb{V}(A) := \sup_{\theta\in\Theta} P_\theta(A)$, lower expectation $\hat{\mathcal{E}}[X] := \inf_{\theta\in\Theta} E_{P_\theta}[X]$, upper expectation $\hat{E}[X] := \sup\limits_{\theta\in\Theta} E_{P_\theta}[X]$.

The continuity of lower probability v and upper probability $\mathbb{V}$ was studied by Chen et al. [7].

Lemma B.1. (Chen et al. [7]) *The lower probability v is upper continuous and the upper probability $\mathbb{V}$ is lower continuous. Moreover, v being lower/upper continuous is equivalent to $\mathbb{V}$ being upper/lower continuous.*

Lemma B.2. [6] (Borel-Cantelli lemma) *Let $\{A_n\}_{n=1}^\infty$ be a sequence of events on $\mathcal{F}$, and $(\mathbb{V}, v)$ be a pair of capacities generated by $\{P_\theta\}_{\theta\in\Theta}$.*
(i) *If $\sum_{n=1}^\infty \mathbb{V}(A_n) < \infty$, then $\mathbb{V}(\bigcap_{n=1}^\infty \bigcup_{i=n}^\infty A_i) = 0$.*

(ii) *Suppose that* $\{A_n^c\}_{n=1}^{\infty}$ *are mutually independent with respect to* $\mathbb{V}(\cdot)$. *If* $\sum_{n=1}^{\infty} v(A_n) = \infty$, *then*

$$v\left(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i\right) = 1.$$

Definition B.4. (Peng [34]) **Identical distribution:** Let X and Y be two random variables defined on sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. They are called identically distributed, denoted by $X \overset{d}{=} Y$, if

$$\hat{\mathbb{E}}\left[\varphi(X)\right] = \hat{\mathbb{E}}\left[\varphi(Y)\right], \text{ for all } \varphi \in C_{l.\mathrm{Lip}}(\mathbb{R}).$$

Independence: Let $X_1, X_2, \cdots, X_n$ be a sequence of random variables. In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random variable X_n is said to be independent to $X := (X_1, X_2, \cdots, X_{n-1})$ under $\hat{\mathbb{E}}$, if for each Borel measurable function φ on $\mathbb{R}^n$ with $\varphi(X, X_n) \in \mathcal{H}$ and $\varphi(x, X_n) \in \mathcal{H}$ for each $x \in \mathbb{R}^{n-1}$, we have

$$\hat{\mathbb{E}}\left[\varphi(X, X_n)\right] = \hat{\mathbb{E}}\left[\bar{\varphi}(X)\right],$$

where $\bar{\varphi}(x) := \hat{\mathbb{E}}\left[\varphi(x, X_n)\right]$ and $\bar{\varphi}(X) \in \mathcal{H}$.
IID random variables: A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ is said to be independent and identically distributed (IID), if $X_i \overset{d}{=} X_1$ and X_{i+1} is independent to $(X_1, \cdots, X_i)$ for each $i \geq 1$.

Definition B.5. (Peng [32]) (Maximal distribution) A random variable ξ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called maximally distributed, denoted as $\xi \overset{d}{=} N([\underline{\mu}, \overline{\mu}] \times \{0\})$, if for each $\varphi \in C_{l.\mathrm{Lip}}(\mathbb{R})$, there exists a subset $[\underline{\mu}, \overline{\mu}] \subseteq \mathbb{R}$ such that

$$\hat{\mathbb{E}}[\varphi(\xi)] = \max_{y \in [\underline{\mu}, \overline{\mu}]} \varphi(y),$$

where $\overline{\mu} = \hat{\mathbb{E}}[\xi]$, $\underline{\mu} = \mathcal{E}[\xi]$.

Definition B.6. (Peng [32]) (G-normal distribution) Let η be a random variable on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\overline{\sigma}^2 = \hat{\mathbb{E}}[\eta^2]$ and $\underline{\sigma}^2 = \mathcal{E}[\eta^2]$. η is called G-normally distributed, denoted as $\eta \overset{d}{=} N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$, if for each $\varphi \in C_{l.\mathrm{Lip}}(\mathbb{R})$,

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}\eta)], \quad (t, x) \in [0, \infty) \times \mathbb{R}$$

is the unique viscosity solution of the following parabolic partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - G(\partial_{xx}^2 u) = 0, \\ u|_{t=0} = \varphi, \end{cases}$$

where G is a function with the form

$$G(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \quad a \in \mathbb{R}.$$

Definition B.7. (Peng [34]) Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. $\{X_t\}_{t \geq 0}$ is called a stochastic process if for each $t \geq 0$, X_t is a random variable in $\mathcal{H}$.

Definition B.8. (Peng [34]) (G-Brownian motion) A stochastic process $\{B_t\}_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a G-Brownian motion if it satisfies
(i) $B_0(\omega) = 0$.
(ii) For each $t, s \geq 0$, $B_{t+s} - B_t$ is identically distributed with B_s, and is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.
(iii) For each $t, s \geq 0$, $B_{t+s} - B_t$ is $N(\{0\} \times [s\underline{\sigma}^2, s\overline{\sigma}^2])$-distributed.

Consider a canonical process $\{B_t(\omega)\}_{t\geq 0} = \{\omega_t\}_{t\geq 0}$, for $\omega \in \Omega$, and define

$$\mathrm{Lip}(\Omega_T) := \left\{\varphi(B_{t_1}, \cdots, B_{t_n}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0,T],\ \varphi \in C_{l.\mathrm{Lip}}(\mathbb{R}^n)\ \right\}$$

and

$$\mathrm{Lip}(\Omega) := \bigcup_{T>0} \mathrm{Lip}(\Omega_T).$$

Similar to the classical situation, Peng [32, 34] constructed a sublinear expectation called G-expectation on $(\Omega, \mathrm{Lip}(\Omega))$ such that the canonical process $\{B_t\}_{t\geq 0}$ is a G-Brownian motion. For $p \geq 1$, let $L_G^p(\Omega)$ and $L_G^p(\Omega_T)$ be the completions of the spaces $\mathrm{Lip}(\Omega)$ and $\mathrm{Lip}(\Omega_T)$ under the norm $\|X\|_p := \left(\hat{\mathbb{E}}[|X|^p]\right)^{1/p}$. According to Peng [34], $\hat{\mathbb{E}}[\cdot]$ can be continuously extended to a sublinear expectation on $(\Omega, L_G^p(\Omega))$ and still denoted by $\hat{\mathbb{E}}[\cdot]$.

Hu and Peng [17] proved that the G-expectation coincides with the upper expectation on $L_G^1(\Omega)$.

Theorem B.2. (Hu and Peng [17]) *Let $\mathcal{B}(\Omega) = \sigma(B_s, s \geq 0)$. For each $X \in L_G^1(\Omega)$, there exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that*

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathcal{P}} E_P[X].$$

Remark B.1. Let $\mathcal{B}(\Omega) = \sigma(B_s, s \geq 0)$, and $\hat{\mathbb{E}}$ be a G-expectation. Then for each $A \in \mathcal{B}(\Omega)$, there exists a lower probability $v(A) = \inf_{P\in\mathcal{P}} P(A)$, and an upper probability $\mathbb{V}(A) = \sup_{P\in\mathcal{P}} P(A)$ generated by $\hat{\mathbb{E}}$ under the G-framework. Thus $(\Omega, \mathcal{B}(\Omega), \mathbb{V})$ and $(\Omega, \mathcal{B}(\Omega), v)$ are a pair of non-additive probability spaces.

For $T \in \mathbb{R}^+$, let $\pi_T = \{t_0, t_1, \cdots, t_N\}$ be a partition of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_N = T$. For each $p \geq 1$, let $M_G^{p,0}(0,T)$ be the space of simple process $\eta_t = \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k, t_{k+1})}(t)$, $t \in [0,T]$, where $\xi_k \in L_G^p(\Omega_{t_k})$, $k = 0, 1, \cdots, N-1$, and the completion of $M_G^{p,0}(0,T)$ is denoted by $M_G^p(0,T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} := \left\{\hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p \mathrm{d}t\right]\right\}^{1/p}.$$

Definition B.9. (Peng [34]) (G-Itô integral) For each $\{\eta_t\}_{t\geq 0} \in M_G^{2,0}(0,T)$ of the form $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j \mathbf{1}_{[t_j, t_{j+1})}(t)$, we define the stochastic integral by

$$I(\eta_t) = \int_0^T \eta_t \mathrm{d}B_t := \sum_{j=0}^{N-1} \xi_j \left(B_{t_{j+1}} - B_{t_j}\right).$$

From Peng [34], the mapping $I : M_G^{2,0}(0,T) \mapsto L_G^2(\Omega_T)$ can be continuously extended to $I : M_G^2(0,T) \mapsto L_G^2(\Omega_T)$. Then for each $\eta \in M_G^2(0,T)$, the stochastic integral is defined by

$$\int_0^T \eta_t \mathrm{d}B_t := I(\eta).$$

For $T \in \mathbb{R}^+$, set $\mu(\pi_T^N) := \max\left\{\left|t_{i+1}^N - t_i^N\right| : i = 0, 1, \cdots, N-1\right\}$, where $\pi_T^N = \{t_0^N, t_1^N, \cdots, t_N^N\}$ is a partition of $[0,T]$ with $0 = t_0^N < t_1^N < \cdots < t_N^N = T$ and $\lim_{N\to\infty} \mu(\pi_T^N) = 0$. For a G-Brownian motion $\{B_t\}_{t\geq 0}$, its quadratic variation process is defined by

$$\langle B\rangle_t := \lim_{\mu(\pi_t^N)\to 0} \sum_{j=0}^{N-1} \left(B_{t_{j+1}^N} - B_{t_j^N}\right)^2 = B_t^2 - 2\int_0^t B_s \mathrm{d}B_s.$$

Moreover, from Peng [34], $\{\langle B\rangle_t\}_{t\geq 0}$ itself is a typical process with mean-uncertainty with $N([\underline{\sigma}^2 t, \overline{\sigma}^2 t] \times \{0\})$-distributed.

Proposition B.2. (Peng [34]) *Let $\{B_t\}_{t\geq 0}$ be a G-Brownian motion, and $\{\langle B\rangle_t\}_{t\geq 0}$ be the quadratic variation process of G-Brownian motion $\{B_t\}_{t\geq 0}$, then for each $t, s \geq 0, n \in \mathbb{N}$, we have the following properties:*

(i)

$$\hat{\mathbb{E}}\left[|B_{s+t} - B_t|^n\right] = 2(n-1)!!\overline{\sigma}^n s^{\frac{n}{2}}/\sqrt{2\pi}, \text{ when } n \text{ is odd},$$
$$\hat{\mathbb{E}}\left[|B_{s+t} - B_t|^n\right] = (n-1)!!\overline{\sigma}^n s^{\frac{n}{2}}, \text{ when } n \text{ is even};$$

(ii)

$$\hat{\mathbb{E}}\left[(\langle B\rangle_{s+t} - \langle B\rangle_t)^n\right] = \hat{\mathbb{E}}\left[\langle B\rangle_s^n\right] = \overline{\sigma}^{2n} s^n,$$
$$\hat{\mathbb{E}}\left[-(\langle B\rangle_{s+t} - \langle B\rangle_t)^n\right] = \hat{\mathbb{E}}\left[-\langle B\rangle_s^n\right] = -\underline{\sigma}^{2n} s^n.$$

Definition B.10. (Peng [34]) The integral of a process $\{\eta_t\}_{t\geq 0} \in M_G^{1,0}(0,T)$ with respect to $\{\langle B\rangle_t\}_{t\geq 0}$ is defined by

$$Q(\eta) = \int_0^T \eta_t \mathrm{d}\langle B\rangle_t := \sum_{j=0}^{N-1} \xi_j \left(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}\right).$$

From Peng [34], $Q : M_G^{1,0}(0,T) \mapsto L_G^1(\Omega_T)$ is a continuous linear mapping can be continuously extended to $M_G^1(0,T)$, and we still denote this mapping by

$$\int_0^T \eta_t \mathrm{d}\langle B\rangle_t := Q(\eta), \text{ for } \eta \in M_G^1(0,T).$$

The following BDG type inequalities were proved by Gao [12].

Lemma B.3. (Gao [12]) *For each $\eta \in M_G^2(0,T)$, we have*

(i)

$$\hat{\mathbb{E}}\left[\sup_{0\leq u\leq t}\left|\int_0^u \eta_s \mathrm{d}B_s\right|^2\right] \leq 4\hat{\mathbb{E}}\left[\left|\int_0^t \eta_s \mathrm{d}B_s\right|^2\right] \leq 4\overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^t |\eta_s|^2 \mathrm{d}s\right].$$

(ii)

$$\hat{\mathbb{E}}\left[\sup_{0\leq u\leq t}\left|\int_0^u \eta_s \mathrm{d}\langle B\rangle_s\right|^2\right] \leq \overline{\sigma}^4 t\hat{\mathbb{E}}\left[\int_0^t |\eta_s|^2 \mathrm{d}s\right].$$

The G-Itô formula was constructed by Peng [32], and further developed in Peng [33] under more general conditions.

Theorem B.3. (Peng [33]) (G-Itô formula) *Suppose* $\Phi \in C^2(\mathbb{R})$. *Let* $\alpha, \beta, \eta \in M_G^2(0,T)$ *and consider* $X_t = X_0 + \int_0^t \alpha_s \mathrm{d}s + \int_0^t \eta_s \mathrm{d}\langle B\rangle_s + \int_0^t \beta_s \mathrm{d}B_s$. *Then for* $t \in [0,T]$,

$$\begin{aligned}\Phi(X_t) - \Phi(X_0) = &\int_0^t \partial_x \Phi(X_s)\alpha_s \mathrm{d}s + \int_0^t \partial_x \Phi(X_s)\beta_s \mathrm{d}B_s \\ &+ \int_0^t \left(\partial_x \Phi(X_s)\eta_s + \frac{1}{2}\partial_{xx}^2 \Phi(X_s)\beta_s^2\right)\mathrm{d}\langle B\rangle_s.\end{aligned}$$

Peng [32] introduced the following stochastic differential equation driven by G-brownian motion (known as G-SDE) defined on $M_G^2(0,T)$:

$$X_t = X_0 + \int_0^t g(s, X_s)\mathrm{d}s + \int_0^t h(s.X_s)\mathrm{d}\left\langle B\right\rangle_s + \int_0^t f(s, X_s)\mathrm{d}B_s, \quad t \in [0,T], \tag{B.1}$$

where the initial condition $X_0 \in \mathbb{R}$ is given, and $g(\cdot, x)$, $h(\cdot, x)$, $f(\cdot, x) \in M_G^2(0,T)$ are given functions satisfying the standard Lipschitz condition, i.e., $|\sigma(t,x) - \sigma(t,x')| \leq K|x - x'|$ with some constant L, for each $x, x' \in \mathbb{R}$, $t \in [0,T]$, $\sigma = g, h$ and f, respectively. The existence and uniqueness of solution to (B.1) was proved in [32].

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Received August 2025; Accepted June 2026; Available online June 2026.