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GENERALIZED φ-FIXED POINT RESULTS FOR NONLINEAR MAPPINGS WITH AN APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract In the present work, we investigate the existence and uniqueness of φ-fixed points and common φ-fixed points for a class of nonlinear mappings defined on metric spaces. The results are obtained under a unified admissibility framework that encompasses several generalized admissibility notions. The contractive conditions are formulated by employing C-class functions together with altering distance functions. The established theorems extend and improve various existing results in the literature. To illustrate the applicability of the theoretical findings, appropriate examples are provided. Furthermore, as an application of the main results, the existence of solutions for a class of fractional differential equations is discussed.

Keywords φ-fixed point, common φ-fixed point, C-class function, fractional differential equation, α, η-admissibility.

MSC(2010) 54H25, 47H10.

1. Introduction

Fixed point (FP) theory has wide applications in various fields,including biology, physics, economics, engineering, and game theory. FP results are applied to solve various problems like functional equations [19], integral equations [17]. The Banach contraction principle [4] plays a fundamental role in this area and has been extended in several directions by many authors. In this context, Samet et al. [24] introduced α-ψ-contractive mappings and developed corresponding FP results, which motivated further studies based on α-admissible (α-adm) mappings. Subsequently, Salimi et al. [23] modified these notions, while Hussain et al. [8] obtained FP results for single and multivalued α-η-ψ contractions. More recently, Ansari and Kaewcharoen [2] proposed contractions involving α-admissibility with respect to another function η (α, η adm).

Another important contribution was made by Jleli et al. [11], who introduced the concept of φ-fixed points (φ-FP) by incorporating lower semi-continuous (LSC) auxiliary functions, along with (F, φ)-contractions. After this, a considerable amount of work has been carried out on φ-FP [3, 9, 14] and [15]. Moreover this approach was later extended to φ-coincidence points and common φ-FP in metric and partial metric spaces by [22] and [21].

Motivated by the above developments, in this paper we employ the concepts of α, η-admissibility, triangular α-admissibility, and C-class functions to establish φ-fixed point results

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for self-mappings in the setting of complete metric spaces. The aim is to provide a unified and flexible framework that accommodates these notions simultaneously, leading to a broader class of admissible mappings. The obtained results not only generalize and unify several known fixed point theorems, but also offer a systematic approach that enhances their applicability. The theoretical findings are supported by illustrative and nontrivial examples. Furthermore, as an application, the developed results are employed to investigate the existence of solutions for a class of fractional differential equations.

2. Preliminaries

In this section, some important definitions and results are provided for further explanation of the main results.

Definition 2.1 ([24])**.** Let $T : X \to X$ and $\mathfrak{a} : X \times X \to [0, \infty)$. Then T is called $\mathfrak{a}$-*admissible* ($\mathfrak{a}$-adm) if for all $x, y \in X$,

$$\mathfrak{a}(x, y) \geq 1$$

implies

$$\mathfrak{a}(Tx, Ty) \geq 1.$$

Definition 2.2 ([12])**.** Let $T : X \to X$ and $\mathfrak{a} : X \times X \to [0, \infty)$. Then T is called a *triangular* $\mathfrak{a}$-*admissible mapping* if:

(i) T is $\mathfrak{a}$-admissible;

(ii) $\mathfrak{a}(x, z) \geq 1$ and $\mathfrak{a}(z, y) \geq 1$ imply $\mathfrak{a}(x, y) \geq 1$.

Definition 2.3 ([23])**.** Let T be a self-mapping on X, and let $\mathfrak{a}, \mathfrak{n} : X \times X \to [0, +\infty)$ be two functions. We say that T is an $\mathfrak{a}$-*admissible mapping with respect to* $\mathfrak{n}$ if for all $x, y \in X$,

$$\mathfrak{a}(x, y) \geq \mathfrak{n}(x, y)$$

implies

$$\mathfrak{a}(Tx, Ty) \geq \mathfrak{n}(Tx, Ty).$$

Note that if $\mathfrak{n}(x, y) = 1$, then this definition reduces to Definition 2.2. Also, if we take $\mathfrak{a}(x, y) = 1$, then T is called an $\mathfrak{n}$-*sub admissible mapping.*

Definition 2.4 ([7])**.** Let (X, d) be a metric space and let $\mathfrak{a}, \mathfrak{n} : X \times X \to [0, \infty)$ be two functions. Then X is said to be an $\mathfrak{a}, \mathfrak{n}$-*complete metric space (CMS)* if every Cauchy sequence $\{x_n\}$ in X with

$$\mathfrak{a}(x_n, x_{n+1}) \geq \mathfrak{n}(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N}$$

converges in X.

Definition 2.5 ([7])**.** Let (X, d) be a metric space and let $\mathfrak{a}, \mathfrak{n} : X \times X \to [0, \infty)$ be two functions. A mapping $T : X \to X$ is said to be an $\mathfrak{a}, \mathfrak{n}$-*continuous mapping* if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$ and

$$\mathfrak{a}(x_n, x_{n+1}) \geq \mathfrak{n}(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N},$$

we have

$$Tx_n \to Tx \quad \text{as } n \to \infty.$$

Lemma 2.1 ([5]). *Suppose that (X,d) is a metric space and $\{x_n\}$ is a sequence in X such that*

$$d(x_n, x_{n+1}) \to 0 \quad \text{as } n \to \infty.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences of positive integers $\{n'(k)\}$ and $\{n(k)\}$ with $n(k) > n'(k) > k$ such that

$$d(x_{n'(k)}, x_{n(k)}) \geq \varepsilon, \quad d(x_{n'(k)-1}, x_{n(k)}) < \varepsilon,$$

and

(i) $\lim_{k\to\infty} d(x_{n'(k)}, x_{n(k)}) = 0$;

(ii) $\lim_{k\to\infty} d(x_{n'(k)-1}, x_{n(k)}) = 0$;

(iii) $\lim_{k\to\infty} d(x_{n'(k)}, x_{n(k)-1}) = 0$.

Khan et al. [13] introduced the notion of an altering distance function as follows.

Definition 2.6 ([13]). A mapping $D : [0,+\infty) \to [0,+\infty)$ is called an *altering distance function* if the following conditions are satisfied:

(i) D is monotone and non-decreasing;

(ii) $D(\tau) = 0$ if and only if $\tau = 0$.

Definition 2.7 ([20]). The set of all altering distance functions is denoted by D and set

$$\Theta^* = \{\Theta : [0,+\infty) \to [0,+\infty) \mid \Theta \text{ is continuous and } \Theta(\tau) = 0 \iff \tau = 0\}.$$

Definition 2.8 ([20]). Let $S,T : X \to X$ be two mappings and let $\mathfrak{a} : X \times X \to [0,+\infty)$ be a function such that the following conditions hold:

(i) If $\mathfrak{a}(x,y) \geq 1$, then $\mathfrak{a}(Sx,Ty) \geq 1$ and $\mathfrak{a}(TSx,STy) \geq 1$;

(ii) if $\mathfrak{a}(x,z) \geq 1$ and $\mathfrak{a}(z,y) \geq 1$, then $\mathfrak{a}(x,y) \geq 1$.

Then the pair (S,T) is said to be *triangular $\mathfrak{a}$-admissible.*

Definition 2.9 ([20]). Let $S,T : X \to X$ be two mappings and let $\mathfrak{a},\mathfrak{n} : X \times X \to [0,+\infty)$ be two functions such that the following conditions hold:

(i) If $\mathfrak{a}(x,y) \geq \mathfrak{n}(x,y)$, then

$$\mathfrak{a}(Sx,Ty) \geq \mathfrak{n}(Sx,Ty) \quad \text{and} \quad \mathfrak{a}(TSx,STy) \geq \mathfrak{n}(TSx,STy);$$

(ii) if $\mathfrak{a}(x,z) \geq \mathfrak{n}(x,z)$ and $\mathfrak{a}(z,y) \geq \mathfrak{n}(z,y)$, then

$$\mathfrak{a}(x,y) \geq \mathfrak{n}(x,y).$$

Then the pair (S,T) is said to be *triangular $\mathfrak{a}$-admissible with respect to $\mathfrak{n}$.*

Lemma 2.2 ([20]). *Let $S,T : X \to X$ be two mappings and let $\mathfrak{a},\mathfrak{n} : X \times X \to [0,+\infty)$ be two functions such that the pair (S,T) is triangular $\mathfrak{a}$-admissible with respect to $\mathfrak{n}$. Assume that there exists $x_0 \in X$ such that*

$$\mathfrak{a}(x_0, Sx_0) \geq \mathfrak{n}(x_0, Sx_0).$$

Define a sequence $\{x_n\}$ in X by

$$Sx_{2n} = x_{2n+1} \quad \text{and} \quad Tx_{2n+1} = x_{2n+2}, \quad n \in \mathbb{N}.$$

Then

$$\mathfrak{a}(x_n, x'_n) \geq \mathfrak{n}(x_n, x'_n) \quad \text{for all } n', n \in \mathbb{N} \text{ with } n < n'.$$

Definition 2.10 ([1]). A mapping $F^* : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ is called a *C-class function* if it is continuous and for all $\omega, \tau \in [0,+\infty)$, F^* satisfies the following conditions:

(i) $F^*(\omega,\tau) \leq \omega$;

(ii) $F^*(\omega,\tau) = \omega$ implies that either $\omega = 0$ or $\tau = 0$.

The family of all C-class functions is denoted by $\mathcal{C}$.

Example 2.1 ([1]). The following functions $F^* : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ belong to the class $\mathcal{C}$:

(i) $F^*(\omega,\tau) = \omega - \tau$, for all $\omega, \tau \in [0,+\infty)$;

(ii) $F^*(\omega,\tau) = k\omega$, for all $\omega, \tau \in [0,+\infty)$, where $0 < k < 1$;

(iii) $F^*(\omega,\tau) = \frac{\omega}{(1+\tau)^{\Bbbk}}$, for all $\omega, \tau \in [0,+\infty)$, where $\Bbbk \in [0,+\infty)$;

(iv) $F^*(\omega,\tau) = (\omega + l)^{(1+\tau)^{-\Bbbk}} - l$, for all $\omega, \tau \in [0,+\infty)$, where $\Bbbk \in (0,+\infty)$ and $l > 1$.

Definition 2.11 ([11]). Let $\varphi : E \to [0,\infty)$ be a lower semi continuous function and $T : E \to E$. A point $x \in E$ is called a φ-fixed point of T if $x = Tx$ and $\varphi(x) = 0$.

Definition 2.12 ([11]). Let $\varphi : E \to [0,\infty)$. A mapping $T : E \to E$ is called a φ-Picard mapping if the following conditions hold:

(i) $F(T) \cap Z_\varphi = \{x\}$;

(ii) $\lim_{n\to\infty} T^n x_0 = x$ for all $x_0 \in E$.

Clearly, if T is a φ-Picard mapping, then it has a unique φ-fixed point.

Definition 2.13 ([11]). A function $\varrho : [0,\infty)^3 \to [0,\infty)$ is said to be a control function if for $\theta_1, \theta_2, \theta_3 \in [0,\infty)$, it satisfies the following conditions:

(ϱ1) $\max\{\theta_1, \theta_2\} \leq \varrho(\theta_1, \theta_2, \theta_3)$;

(ϱ2) $\varrho(0,0,0) = 0$;

(ϱ3) ϱ is continuous.

3. Main results

Theorem 3.1. *Let $T : X \to X$ be an $\mathfrak{a}$-admissible ($\mathfrak{a}$-adm) mapping with respect to $\mathfrak{n}$ defined on (X,d), a CMS. Assume that there exists a function $\mathcal{B} : [0,\infty) \to [0,1)$ such that for any sequence $\{\breve{t_n}\} \in \mathbb{R}^+$, $\mathcal{B}(\breve{t_n}) \to 1$ implies that $\breve{t_n} \to 0$ for $n \in \mathbb{N}$ and for all $x, y \in X$, we have*

$$\mathfrak{a}(x,Tx)\mathfrak{a}(y,Ty) \geq \mathfrak{n}(x,Tx)\mathfrak{n}(y,Ty)$$

implies

$$\varrho(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leq \mathcal{B}(\varrho(d(x,y), \varphi(x), \varphi(y)))\mho(x,y), \tag{3.1}$$

where

$$\mho(x,y) = \max\{\varrho(d(x,y), \varphi(x), \varphi(y))), \min\{\varrho(d(x,Tx), \varphi(x), \varphi(Tx)), \varrho(d(y,Ty), \varphi(y), \varphi(Ty))\}\},$$

where ϱ is a control function and $\varphi : X \to [0,\infty)$ is a LSC function. Then, $T : X \to X$ has a unique φ-FP is it satisfies:

(i) *There exists* $x_0 \in X$ *such that* $\mathfrak{a}(x_0, Tx_0) \geq \mathfrak{n}(x_0, Tx_0)$

(ii) T *is continuous (CTS) or*

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\{x_n\} \to x$, $\mathfrak{a}(x_n, x_{n+1}) \geq \mathfrak{n}(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$, *then* $\mathfrak{a}(x, Tx) \geq \mathfrak{n}(x, Tx)$.

Proof. Let $x_0 \in X$ such that $\mathfrak{a}(x_0, Tx_0) \geq \mathfrak{n}(x_0, Tx_0)$ and let the sequence x_n be defined by $x_n = T(x_{n-1}) = T^n(x_0)$ for all $n \in \mathbb{N}$. If for any $n' \in \mathbb{N}$, $x_{n'} = x_{n'-1}$, then, $Tx_{n'} = Tx_{n'-1}$, that is $x_{n'+1} = x_{n'}$ and $x_{n'}$ is FP of T. Hence, we consider $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. As T is $\mathfrak{a}, \mathfrak{n}$-adm and $\mathfrak{a}(x_0, Tx_0) \geq \mathfrak{n}(x_0, Tx_0)$, we get $\mathfrak{a}(x_1, x_2) = \mathfrak{a}(Tx_0, T^2x_0) = \mathfrak{n}(x_1, x_2)$. Continuing like this, we have $\mathfrak{a}(x_n, Tx_n) \geq \mathfrak{n}(x_n, Tx_n)$. Then,

$$\mathfrak{a}(x_{n-1}, Tx_{n-1})\mathfrak{a}(x_n, Tx_n) \geq \mathfrak{n}(x_{n-1}, Tx_{n-1})\mathfrak{n}(x_n, Tx_n).$$

Now, for $x = x_{n-1}$ and $= y_{n-1}$ by given contractive condition, we get

$$\varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) \leq \mathcal{B}(\varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)))\mho(x_{n-1}, x_n).$$

Throughout the paper, for the sake of simplicity and to avoid repetition, we shall use the notations $\Delta_{n-1,n} = \varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))$ and $\Delta_{n,n+1} = \varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1}))$. Then

$$\begin{aligned}\mho(x_{n-1}, x_n) &= \max\{\Delta_{n-1,n}, \min\{\Delta_{n-1,n}, \Delta_{n,n+1}\}\} \\ &= \Delta_{n-1,n} \\ &= \varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)).\end{aligned}$$

Consequently

$$\Delta_{n,n+1} \leq \mathcal{B}(\Delta_{n-1,n})\Delta_{n-1,n},$$

which follows that

$$\frac{\Delta_{n,n+1}}{\Delta_{n-1,n}} \leq \mathcal{B}(\Delta_{n-1,n}).$$

Hence

$$\frac{\varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1}))}{\varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)))} \leq \mathcal{B}(\varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))),$$

implies

$$\frac{\varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1}))}{\varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)))} < 1.$$

Therefore

$$\varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) < \varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))).$$

This yields $\Delta_{n-1,n} \geq \Delta_{n,n+1}$ that is $\Delta_{n-1,n}$ is a decreasing sequence and as it is bounded, hence converges to some d^*. Now we will prove that d^*=0. Therefore for this, we see

$$\lim_{n\to\infty} \frac{\varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1}))}{\varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)))} \leq \lim_{n\to\infty} \mathcal{B}(\varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))).$$

From this, we get

$$1 \leq \lim_{n\to\infty} \mathcal{B}(\varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))) < 1,$$

accordingly

$$\lim_{n\to\infty} \mathcal{B}(\varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))) = 1.$$

Therefore

$$\lim_{n\to\infty} \varrho(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))) = 0,$$

which implies

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0, \quad \lim_{n\to\infty} \varphi(x_{n-1}) = 0 \quad \text{(by } (\varrho 1)).$$

Now, we will prove that x_n is a Cauchy sequence. Let x_n is not Cauchy. That means there exists some $\epsilon > 0$ and sequences $\{n'(k)\}$ and $\{n(k)\}$ such that $\forall\, k \in Z^+$, we have $\{n(k)\} > \{n'(k)\} > k$, $d(x_{n(k)}, x_{n'(k)}) \geq \epsilon$ and $d(x_{n(k)}, x_{n'(k)-1}) < \epsilon$. Then by triangle inequality, we get

$$\begin{aligned}
\epsilon &\leq d(x_{n(k)}, x_{n'(k)}) \\
&\leq d(x_{n(k)}, x_{n'(k)-1})) + d(x_{n'(k)-1}, x_{n'(k)}) \\
&< \epsilon + d(x_{n'(k)-1}, x_{n'(k)}),
\end{aligned}$$

implies

$$\lim_{k\to\infty} d(x_{n(k)}, x_{n'(k)}) = \epsilon.$$

Now

$$d(x_{n(k)}, x_{n'(k)}) \leq d(x_{n'(k)}, x_{n'(k)+1}) + d(x_{n'(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{n'(k)+1}, x_{n(k)+1}) \leq d(x_{n(k)+1}, x_{n(k)}) + d(x_{n(k)}, x_{n'(k)}) + d(x_{n'(k)}, x_{n'(k)+1}).$$

Taking limit $k \to \infty$, we have

$$\lim_{k\to\infty} d(x_{n'(k)+1}, x_{n(k)+1}) = \epsilon.$$

Here by using

$$\mathfrak{a}(x_{n(k)}, Tx_{n(k)})\mathfrak{a}(x_{n'(k)}, Tx_{n'(k)}) \geq \mathfrak{n}(x_{n(k)}, Tx_{n(k)})\mathfrak{n}(x_{n'(k)}, Tx_{n'(k)}),$$

we obtain

$$\begin{aligned}
&\varrho(d(x_{n(k)+1}, x_{n'(k)+1}), \varphi(x_{n(k)+1}), \varphi(x_{n'(k)+1})) \\
\leq &\mathcal{B}(\varrho(d(x_{n(k)}, x_{n'(k)}), \varphi(x_{n(k)}), \varphi(x_{n'(k)}))\mho(x_{n(k)}, x_{n'(k)}),
\end{aligned}$$

where

$$\begin{aligned}&\mho(x_{n(k)}, x_{n'(k)})\\&= \max\{\varrho(d(x_{n(k)}, x_{n'(k)}), \varphi(x_{n(k)}), \varphi(x_{n'(k)})), \varrho(d(x_{n(k)+1}, x_{n'(k)+1}), \varphi(x_{n(k)+1}), \varrho(x_{n'(k)+1}))\}.\end{aligned}$$

Taking $k \to \infty$, we get

$$\varrho(\epsilon, 0, 0) \leq \mathcal{B}(\varrho(d(x_{n(k)}, x_{n'(k)}), \varphi(x_{n(k)})) \times \varrho(\epsilon, 0, 0),$$

which gives

$$\lim_{k\to\infty} \mathcal{B}(\varrho(d(x_{n(k)}, x_{n'(k)}), \varphi(x_{n(k)}), \varphi(x_{n'(k)})) = 1,$$

consequently

$$\varrho(d(x_{n(k)}, x_{n'(k)}), \varphi(x_{n(k)}, \varphi(x_{n'(k)})) = 0.$$

Hence

$$\lim_{k\to\infty} d(x_{n(k)}, x_{n'(k)}) = 0, \lim_{k\to\infty} \varphi(x_{n(k)}) = 0,$$

which is a contradiction. Therefore our assumption is wrong and sequence x_n is Cauchy. Now by completeness of space there exists some $z \in X$ to which sequence x_n converges. As mapping T is CTS, so $Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z$. Also as function φ is LSC, then $\lim_{n\to\infty} \varphi(x_{n+1}) = \varphi(z) = 0$. Therefore by uniqueness of limit z is unique φ-FP of T.

If we assume there is no condition of continuity over T but condition 3. holds, then

$$\mathfrak{a}(z, Tz) \geq \mathfrak{n}(z, Tz),$$

which implies

$$\mathfrak{a}(x_n, Tx_n)\mathfrak{a}(z, Tz) \geq \mathfrak{n}(x_n, Tx_n)\mathfrak{n}(z, Tz).$$

Now by given condition of contraction, we get

$$\varrho(d(Tz, x_{n+1}), \varphi(Tz), \varphi(x_{n+1})) \leq \mathcal{B}(\varrho(d(z, x_n), \varphi(z), \varphi(x_n)))\mho(z, x_n),$$

where

$$\mho(z, x_n) = \max\{\varrho(d(z, x_n), \varphi(z), \varphi(x_n)), \varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1}))\}.$$

Letting $n \to \infty$, we have

$$\varrho(d(Tz, z), \varphi(Tz), \varphi(z)) \leq \mathcal{B}(\varrho(d(z, x_n), \varphi(z), \varphi(x_n))) \times \max\{\varrho(0, 0, 0), \varrho(0, 0, 0)\}.$$

This gives

$$\varrho(d(Tz, z), \varphi(Tz), \varphi(z)) \leq 0,$$

thus

$$d(Tz, z) = 0,$$

which implies

$$Tz = z, \varphi(Tz) = \varphi(z) = 0.$$

We will prove now that z is unique φ-FP of T. For this, consider z and z' be two φ-FP of T. Then by given condition

$$\mathfrak{a}(z, z)\mathfrak{a}(z', z') \geq \mathfrak{n}(z, z)\mathfrak{n}(z', z').$$

Therefore

$$\varrho(d(z, z'), \varphi(z), \varphi(z')) \leq \mathcal{B}(\varrho(d(z, z'), \varphi(z), \varphi(z')))\mho(z, z'),$$

which implies that

$$\varrho(d(z, z'), 0, 0) \leq \mathcal{B}(\varrho(d(z, z'), 0, 0))\mho(z, z')$$

where

$$\begin{aligned}\mho(z, z') &= \{\varrho(d(z, z'), 0, 0), \min\{\varrho(d(z, Tz), 0, 0), \varrho(d(z', Tz'), 0, 0)\}\} \\ &= \{\varrho(d(z, z'), 0, 0), \min\{\varrho(d(z, z), 0, 0), \varrho(d(z', z'), 0, 0)\}\} \\ &= \{\varrho(d(z, z'), 0, 0), \varrho(0, 0, 0)\},\end{aligned}$$

which follows that

$$\varrho(d(z, z'), 0, 0) \leq \mathcal{B}(\varrho(d(z, z'), 0, 0)\varrho(d(z, z'), 0, 0).$$

This shows that

$$\mathcal{B}(\varrho(d(z, z'), 0, 0)) = 1,$$

implies

$$\varrho(d(z, z'), 0, 0) = 0.$$

So, we have

$$d(z, z') = 0,$$

which gives

$$z = z'.$$

Hence z is unique φ-FP of T. □

In all the examples we will consider $\varrho(\theta_1, \theta_2, \theta_3) = \theta_1 + \theta_2 + \theta_3$ and $\varphi(x) = x$.

Example 3.1. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Consider $T : X \to X$ be defined by

$$Tx = \begin{cases} \dfrac{x}{4}, & \text{if } x \in [0, 1], \\ \ln(x^2 + x + 3), & \text{if } x \in (1, \infty). \end{cases}$$

Also let $\mathfrak{a}, \mathfrak{n} : X \to X$ be defined as

$$\mathfrak{a}(x,y) = \begin{cases} 6, & \text{if } x, y \in [0,1], \\ 1, & \text{if } x, y \in (1,\infty), \end{cases}$$
$$\mathfrak{n}(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1], \\ 2, & \text{if } x, y \in (1,\infty). \end{cases}$$

Now we will show that for $\mathcal{B}(\theta) = \frac{1}{2}$, all the conditions of Theorem (3.1) are satisfied.

If $x, y \in [0,1]$, then

$$\mathfrak{a}(x,y) \geq \mathfrak{n}(x,y) \quad \text{and} \quad \mathfrak{a}(Tx,Ty) = \mathfrak{a}\left(\frac{x}{4}, \frac{y}{4}\right) = 6 \geq 1 = \mathfrak{n}\left(\frac{x}{4}, \frac{y}{4}\right) = \mathfrak{n}(Tx,Ty).$$

Therefore T is $\mathfrak{a}, \mathfrak{n}$-adm mapping for $x, y \in [0,1]$. Now we show that T is satisfying contractive condition (3.1). For this let $x, y \in [0,1]$, then

$$\mathfrak{a}(x,Tx)\mathfrak{a}(y,Ty) = \mathfrak{a}\left(x, \frac{x}{4}\right)\mathfrak{a}\left(y, \frac{y}{4}\right) = 36 \quad \text{and} \quad \mathfrak{n}(x,Tx)\mathfrak{n}(y,Ty) = \mathfrak{a}\left(x, \frac{x}{4}\right)\mathfrak{a}\left(y, \frac{y}{4}\right) = 1.$$

Hence we get

$$\mathfrak{a}(x,Tx)\mathfrak{a}(y,Ty) \geq \mathfrak{n}(x,Tx)\mathfrak{n}(y,Ty).$$

Now for $x, y \in [0,1]$, we have

$$\begin{aligned}
\varrho(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) &= \left|\frac{x-y}{4}\right| + \frac{x}{4} + \frac{y}{4}, \\
\varrho(d(x,Tx), \varphi(x), \varphi(Tx)) &= \frac{3x}{4} + x + \frac{x}{4} = 2x, \\
\varrho(d(y,Ty), \varphi(y), \varphi(Ty)) &= \frac{3y}{4} + y + \frac{y}{4} = 2y, \\
\varrho(d(x,y), \varphi(x), \varphi(Ty)) &= |x-y| + x + y.
\end{aligned}$$

Then clearly

$$\min\{\varrho(d(x,Tx), \varphi(x), \varphi(Tx)), \varrho(d(y,Ty), \varphi(y), \varphi(Ty))\} = \min\{2x, 2y\}$$

and

$$\begin{aligned}
\max\{\varrho(d(x,y), \varphi(x), \varphi(y))), \min\{2x, 2y\}\} &= \max\{|x-y| + x + y, \min\{2x, 2y\}\} \\
&= 2\max\{x, y\}.
\end{aligned}$$

Now, we have

$$\frac{1}{2}\max\{x,y\} \leq \max\{x,y\}$$

implies

$$\left|\frac{x-y}{4}\right| + \frac{x}{4} + \frac{y}{4} \leq \max\{x,y\}.$$

Hence

$$\begin{aligned}&\varrho(d(Tx,Ty),\varphi(Tx),\varphi(Ty))\\ \leq&\mathcal{B}(\varrho(d(x,y),\varphi(x),\varphi(y)))\max\{\varrho(d(x,y),\varphi(x),\varphi(y))),\min\{2x,2y\}\}.\end{aligned}$$

Therefore condition (3.1) is satisfied. Moreover condition (i) of Theorem (3.1) is satisfied for $x_0 = 0$ and if $\mathfrak{a}(x_n, x_{n+1}) \geq \mathfrak{n}(x_n, x_{n+1})$ it implies that $x_n \in [0,1]$ for all $n \in \mathbb{N}$. Now due to completeness of $[0,1]$ there exists $x \in [0,1]$, such that x_n converges to x. Hence $\mathfrak{a}(x,Tx) \geq \mathfrak{n}(x,Tx)$. Then the above Theorem is true here and $z' = 0$ is the unique φ-FP of mapping T.

Now we consider two different choices of $\mathcal{B}$ in the same example, where function $\mathcal{B}$ is not CTS.

1: If we consider

$$\mathcal{B}(\theta) = \begin{cases} e^{-\theta}, & \text{if } \theta > 0,\\ 0, & \text{if } \theta = 0\end{cases}$$

and modify mapping T as follows:

$$Tx = \begin{cases} \dfrac{x}{16}, & \text{if } x \in [0,1],\\ \ln(x^2+x+3), & \text{if } x \in (1,\infty).\end{cases}$$

Then, clearly the result holds trivially if both x and y are zero. But if anyone of them is non-zero then for contractive condition (3.1) to be satisfied, we need to prove

$$\frac{1}{8}\max\{x,y\} \leq 2e^{-\max\{2x,2y\}}\max\{x,y\}$$

implies

$$e^{-\max\{2x,2y\}} \geq \frac{1}{16}.$$

Here it can be seen easily that minimum value of $e^{-\max\{2x,2y\}} = e^{-2}$ in $[0,1]$ and $e^{-2} > \frac{1}{16}$. Hence Theorem (3.1) is true here.

2: If we consider

$$\mathcal{B}(\theta) = \begin{cases} \dfrac{1}{1+\theta}, & \text{if } \theta > 0,\\ 0, & \text{if } \theta = 0\end{cases}$$

and do not change mapping T, then again for contractive condition (3.1) to be satisfied, we have to prove

$$\frac{1}{2}\max\{x,y\} \leq \frac{2}{1+\max\{2x,2y\}}\max\{x,y\}$$

implies

$$\frac{1}{1+\max\{2x,2y\}} \geq \frac{1}{4}.$$

Now minimum value of $\frac{1}{1+\max\{2x,2y\}} = \frac{1}{3}$ in $[0,1]$ and $\frac{1}{3} > \frac{1}{4}$. Therefore Theorem (3.1) is true in this case also.

Definition 3.2. Let (X,d) be a metric space, $T,S : X \to X$ be two mappings and $\mathfrak{a},\mathfrak{n} : X \times X \to [0,\infty)$ are two functions. Then the pair (T,S) is called generalized F^*-φ-Contraction if $\mathfrak{a}(x,y) \geq \mathfrak{n}(x,y)$, then

$$D(\varrho(d(Tx,Sy),\varphi(Tx),\varphi(Sy))) \leq F^*(D(M^*(x,y)),\Theta(M^*(x,y))),$$

where

$$M^*(x,y) = \max\{\varrho(d(x,y),\varphi(x),\varphi(y)),\varrho(d(x,Tx),\varphi(x),\varphi(Tx)),\varrho(d(y,Sy),\varphi(y),\varphi(Sy))\}.$$

Theorem 3.3. *Let (X,d) be a metric space and T,S be two self-mappings on X. Assume that the following conditions are satisfied:*

(i) *(X,d) is an $\mathfrak{a},\mathfrak{n}$-CMS;*

(ii) *(T,S) is generalized F^*-φ-Contraction;*

(iii) *(T,S) is triangular $\mathfrak{a},\mathfrak{n}$-adm.*

(iv) *There exists $x_0 \in X$ such that $\mathfrak{a}(x_0,Tx_0) \geq \mathfrak{n}(x_0,Tx_0)$;*

(v) *(T,S) is $\mathfrak{a},\mathfrak{n}$-CTS mappings or*

(vi) *If $\{x_n\}$ is a sequence in X such that $\mathfrak{a}(x_n,x_{n+1}) \geq \mathfrak{n}(x_n,x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to \infty$, then there exist a subsequence $\{x_n(k)\}$ of $\{x_n\}$ such that $\mathfrak{a}(x_{n(k)},z) \geq \mathfrak{n}(x_{n(k)},z)$ for all $k \in \mathbb{N}$.*

Then (T,S) have a common φ-FP.

Proof. Let $x_0 \in X$ be such that $\mathfrak{a}(x_0,Tx_0) \geq \mathfrak{n}(x_0,Tx_0)$. Define a sequence $\{x_n\}$ in X such that $x_{2n+1} = T_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for all $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then clearly x_{n_0} is the required common φ-FP of (T,S). As (T,S) is $\mathfrak{a},\mathfrak{n}$-adm, hence

$$\mathfrak{a}(x_1,x_2) = \mathfrak{a}(Tx_0,Sx_1) \geq \mathfrak{n}(Tx_0,Sx_1) = \mathfrak{n}(x_1,x_2)$$

and

$$\mathfrak{a}(x_2,x_1) = \mathfrak{a}(Sx_1,Tx_0) \geq \mathfrak{n}(Sx_1,Tx_0) = \mathfrak{n}(x_2,x_1).$$

If we use $\mathfrak{a},\mathfrak{n}$-adm property again, we get

$$\mathfrak{a}(x_3,x_2) = \mathfrak{a}(Tx_2,Sx_1) \geq \mathfrak{n}(Tx_2,Sx_1) = \mathfrak{n}(x_3,x_2)$$

and

$$\mathfrak{a}(x_2,x_3) = \mathfrak{a}(Sx_1,Tx_2) \geq \mathfrak{n}(Sx_1,Tx_2) = \mathfrak{n}(x_2,x_3).$$

By repetition of this process n-times, we get $\mathfrak{a}(x_n,x_{n+1}) \geq \mathfrak{n}(x_n,x_{n+1})$ and similarly we have $\mathfrak{a}(x_{n+1},x_n) \geq \mathfrak{n}(x_{n+1},x_n)$. Now as (T,S) is triangular $\mathfrak{a},\mathfrak{n}$-adm, therefore we can deduce that $\mathfrak{a}(x_n,x_{n'}) \geq \mathfrak{n}(x_n,x_{n'})$ and $\mathfrak{a}(x_{n'},x_n) \geq \mathfrak{n}(x_{n'},x_n)$ for any $n,n' \in \mathbb{N}$ with $n' > n$.

Let $x_{2n} \neq x_{2n+1}$ and $x_{2n+1} \neq x_{2n+2}$ for all $n \in \mathbb{N}$, then by Lemma (2.2), we get $\mathfrak{a}(x_{2n},x_{2n+1}) \geq \mathfrak{n}(x_{2n},x_{2n+1})$ for all $n \in \mathbb{N}$. Then as (T,S) is generalized F^*-φ-Contraction mapping, we have

$$D(\varrho(d(x_{2n+1},x_{2n+2}),\varphi(x_{2n+1}),\varphi(x_{2n+1}))) = D(\varrho(d(Tx_{2n},Sx_{2n+1}),\varphi(Tx_{2n}),\varphi(Sx_{2n+1})))$$

$$\leq F^*(D(M^*(x_{2n}, x_{2n+1})), \Theta(M^*(x_{2n}, x_{2n+1}))),$$

where

$$\begin{aligned}
&M^*(x_{2n}, x_{2n+1}) \\
=& \max\{\varrho(d(x_{2n}, x_{2n+1}), \varphi(x_{2n}), \varphi(x_{2n+1})), \varrho(d(x_{2n}, Tx_{2n}), \varphi(x_{2n}), \varphi(Tx_{2n})), \\
&\varrho(d(x_{2n+1}, Sx_{2n+1}), \varphi(x_{2n+1}), \varphi(Sx_{2n+1}))\} \\
=& \max\{\varrho(d(x_{2n}, x_{2n+1}), \varphi(x_{2n}), \varphi(x_{2n+1})), \varrho(d(x_{2n}, x_{2n+1}), \varphi(x_{2n}), \varphi(x_{2n+1})), \\
&\varrho(d(x_{2n+1}, x_{2n+2}), \varphi(x_{2n+1}), \varphi(x_{2n+2}))\} \\
&\max\{\varrho(d(x_{2n}, x_{2n+1}), \varphi(x_{2n}), \varphi(x_{2n+1})), \varrho(d(x_{2n+1}, x_{2n+2}), \varphi(x_{2n+1}), \varphi(x_{2n+2}))\},
\end{aligned}$$

which clearly gives a contradiction to our assumption. Hence we obtain

$$M^*(x_{2n}, x_{2n+1}) = \max\{\varrho(d(x_{2n}, x_{2n+1}), \varphi(x_{2n}), \varphi(x_{2n+1}))\}.$$

Therefore, we have

$$\begin{aligned}
D(\varrho(d(x_{2n+1}, x_{2n+2}), \varphi(x_{2n+1}), \varphi(x_{2n+2}))) &\leq F^*(D(M^*(x_{2n}, x_{2n+1})), \Theta(M^*(x_{2n}, x_{2n+1}))) \\
&\leq D(M^*(x_{2n}, x_{2n+1})) \\
&\leq D(\varrho(d(x_{2n}, x_{2n+1}), \varphi(x_{2n}), \varphi(x_{2n+1}))).
\end{aligned}$$

Here, as D is non-decreasing, hence

$$\varrho(d(x_{2n+1}, x_{2n+2}), \varphi(x_{2n+1}), \varphi(x_{2n+2})) \leq \varrho(d(x_{2n}, x_{2n+1}), \varphi(x_{2n}), \varphi(x_{2n+1})).$$

Now, if we consider

$$\begin{aligned}
d_n &= \varrho(d(x_{2n}, x_{2n+1}), \varphi(x_{2n}), \varphi(x_{2n+1})), \\
d_{n+1} &= \varrho(d(x_{2n+1}, x_{2n+2}), \varphi(x_{2n+1}), \varphi(x_{2n+2})),
\end{aligned}$$

then $d_n \geq d_{n+1}$, that is d_n is a decreasing sequence. There must exists some $\Bbbk$ such that

$$\lim_{n\to\infty} \varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = \Bbbk.$$

Now, we have

$$\begin{aligned}
&D(\varrho(d(x_{n+1}, x_{n+2}), \varphi(x_{n+1}), \varphi(x_{n+2}))) \\
\leq& F^*(D(\varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1}))), \Theta(\varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})))).
\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$D(\Bbbk) \leq F^*(D(\Bbbk), \Theta(\Bbbk)),$$

which yields

$$D(\Bbbk) = 0 or \Theta(\Bbbk) = 0.$$

Hence

$$\lim_{n\to\infty} \varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = 0$$

implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \quad \lim_{n\to\infty} \varphi(x_n) = 0 \quad (\text{by } (\varrho 1)). \tag{3.2}$$

Now, the next step is to prove that sequence $\{x_n\}$ is Cauchy. For this, we use Lemma 2.1 by which there exists $\epsilon > 0$ and two subsequences $\{x_{n'(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > n(k) > k$ such that:

$$d(x_{n(k)}, x_{n'(k)}) \geq \epsilon, d(x_{n(k)}, x_{n'(k)-1}) < \epsilon.$$

By using triangular inequality here, we have

$$\begin{aligned}\epsilon &\leq d(x_{n(k)}, x_{n'(k)}) \\ &\leq d(x_{n(k)}, x_{n'(k)-1}) + d(x_{n'(k)-1}, x_{n'(k)}) \\ &< \epsilon + d(x_{n'(k)-1}, x_{n'(k)}).\end{aligned}$$

Now if we take $k \to \infty$ and use (3.2) in the above inequality, we get

$$\lim_{k\to\infty} d(x_{n(k)}, x_{n'(k)}) = \epsilon.$$

Since

$$|d(x_{n(k)}, x_{n'(k)-1}) - d(x_{n(k)}, x_{n'(k)})| \leq d(x_{n'(k)}, x_{n'(k)-1}).$$

We obtain $\lim\limits_{k\to\infty} d(x_{n(k)}, x_{n'(k)-1}) = \epsilon$. Similarly, we have

$$\lim_{k\to\infty} d(x_{n(k)}, x_{n'(k)-1}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{n'(k)-1}) = \epsilon.$$

For $n'(k), n(k) \in \mathbb{N}$ with $n(k) > n(k)$, we discuss four cases.

Case 1. Consider $n(k) = 2k+1$ and $n'(k) = 2k+s+1$ that is $n(k)$ is odd and $n'(k)$ is even for some $k, s \in \mathbb{N}$, where s is odd. As $\mathfrak{a}(x_{2k+1}, x_{2k+1+s}) \geq \mathfrak{n}(x_{2k+1}, x_{2k+1+s})$, therefore we conclude that

$$\begin{aligned}&D(\varrho(d(x_{n(k)}, x_{n'(k)}), \varphi(x_{n(k)}), \varphi(x_{n'(k)}) \\ &= D(\varrho(d(Tx_{n(k)-1}, Sx_{n'(k)-1}), \varphi(Tx_{n(k)-1}), \varphi(Sx_{n'(k)-1}))) \\ &= D(\varrho(d(Tx_{2k}, Sx_{2k+s}), \varphi(Tx_{2k}), \varphi(Sx_{2k+s}))) \\ &\leq F^*(D(M^*(x_{2k}, x_{2k+s}), \Theta(M^*(x_{2k}, x_{2k+s}))),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}M^*(x_{2k}, x_{2k+s}) &= \max\{\varrho(d(x_{2k}, x_{2k+s}), \varphi(x_{2k}), \varphi(x_{2k+s})), \varrho(d(x_{2k}, Tx_{2k}), \varphi(x_{2k}), \varphi(Tx_{2k})), \\ &\quad \varrho(d(x_{2k+s}, Sx_{2k+s}), \varphi(x_{2k+s}), \varphi(Sx_{2k+s}))\} \\ &= \max\{\varrho(d(x_{2k}, x_{2k+s}), \varphi(x_{2k}), \varphi(x_{2k+s})), \varrho(d(x_{2k}, x_{2k+1}), \varphi(x_{2k}), \varphi(x_{2k+1})), \\ &\quad \varrho(d(x_{2k+s}, x_{2k+s+1}), \varphi(x_{2k+s}), \varphi(x_{2k+s+1}))\}.\end{aligned}$$

Hence

$$\lim_{k\to\infty} M^*(x_{2k}, x_{2k+s}) = \max\{\varrho(\epsilon, 0, 0), \varrho(0, 0, 0), \varrho(0, 0, 0)\} = \varrho(\epsilon, 0, 0). \tag{3.4}$$

Now, by (3.3) and (3.4), we get

$$D(\varrho(\epsilon,0,0)) \leq F^*(D(\varrho(\epsilon,0,0)),\Theta(\varrho(\epsilon,0,0))).$$

Here, by Property (ii) of F^*, we conclude that $\varrho(\epsilon,0,0) = 0$, hence $\epsilon = 0$, which gives contradiction to the condition $\epsilon > 0$. Therefore our assumption is wrong and sequence $\{x_n\}$ is Cauchy sequence in this case.

Case 2. Consider $n(k)$ and $n'(k)$ both are odd. Then by triangle inequality and Case 1, we have

$$d(x_{n(k)},x_{n'(k)}) \leq d(x_{n(k)},x_{n'(k)-1}) + d(x_{n'(k)-1},x_{n'(k)})$$

implies

$$\lim_{k\to\infty} d(x_{n(k)},x_{n'(k)}) \leq \lim_{k\to\infty} d(x_{n(k)},x_{n'(k)-1}) + \lim_{k\to\infty} d(x_{n'(k)-1},x_{n'(k)}) \leq 0.$$

Consequently

$$\lim_{k\to\infty} d(x_{n(k)},x_{n'(k)}) = 0.$$

Case 3. Now, let n and n' both are even, then again by triangular inequality, we get

$$d(x_{n(k)},x_{n'(k)}) \leq d(x_{n(k)},x_{n(k)+1}) + d(x_{n(k)+1},x_{n'(k)}).$$

Here again by (3.2) and Case 1, we have

$$\lim_{k\to\infty} d(x_{n(k)},x_{n'(k)}) = 0.$$

Case 4. In the last case, suppose that n is even and n' is odd. Using triangular inequality, we have

$$d(x_{n(k)},x_{n'(k)}) \leq d(x_{n(k)},x_{n(k)+1}) + d(x_{n(k)+1},x_{n'(k)-1}) + d(x_{n'(k)-1},x_{n'(k)}),$$

which clearly gives $\lim_{k\to\infty} d(x_{n(k)},x_{n'(k)}) = 0$. Now by combining all these cases together, it is concluded that sequence x_n is a Cauchy sequence in X. Then by completeness of space X, there exists $z \in X$ such that sequence x_n converges to z. Since T, S are CTS mappings, therefore we have $Tx_{2n} \to Tz$ and $Sx_{2n+1} \to Sz$. This implies $x_{2n+1} = Tx_{2n} \to z$. By uniqueness of limit, we get

$$Tz = z.$$

Similarly, we have

$$Sz = z.$$

Hence z is the common FP of mappings T and S. Moreover as φ is lower semi CTS, hence $\lim_{n\to\infty} \varphi(x_n) = \varphi(z) = 0$. Hence z is common φ-FP of mappings T and S.

If we assume there is no condition of continuity over T but condition 6. holds, then by using above proof, a sequence $\{x_n\} \in X$ can be constructed which is defined as $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for all $n \in \mathbb{N}$, such that the sequence converges to $z \in X$. Then by condition

6, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\mathfrak{a}(x_{n(k)}, z) \geq \mathfrak{n}(x_{n(k)}, z)$ for all $k \in \mathbb{N}$. Hence, we get

$$\begin{aligned} D(\varrho(d(x_{2n(k)+1}, Sz), \varphi(x_{2n(k)+1}), \varphi(Sz))) &= D(\varrho(d(Tx_{2n(k)}, Sz), \varphi(Tx_{2n(k)}), \varphi(Sz))) \\ &\leq F^*(D(M^*(x_{2n(k)}, z)), \Theta(M^*(x_{2n(k)}, z))), \end{aligned}$$

where

$$\begin{aligned} M^*(x_{2n(k)}, z) &= \max\{\varrho(d(x_{2n(k)}, z), \varphi(x_{2n(k)}), \varphi(z)), \\ &\qquad \varrho(d(x_{2n(k)}, Tx_{2n(k)}), \varphi(x_{2n(k)}), \varphi(Tx_{2n(k)})), \varrho(d(z, Sz), \varphi(z), \varphi(Sz))\} \\ &= \max\{\varrho(d(x_{2n(k)}, z), \varphi(x_{2n(k)}), \varphi(z)), \\ &\qquad \varrho(d(x_{2n(k)}, x_{2n(k)+1}), \varphi(x_{2n(k)}), \varphi(x_{2n(k)+1})), \varrho(d(z, Sz), \varphi(z), \varphi(Sz))\} \end{aligned}$$

implies

$$\begin{aligned} \lim_{k\to\infty} M^*(x_{2n(k)}, z) &= \max\{\varrho(0,0,0), \varrho(0,0,0), \varrho(d(z, Sz), \varphi(z), \varphi(Sz))\} \\ &= \max\{\varrho(0,0,0), \varrho(d(z, Sz), \varphi(z), \varphi(Sz))\} \\ &= \varrho(d(z, Sz), \varphi(z), \varphi(Sz)). \end{aligned}$$

Consequently

$$\begin{aligned} D(\varrho(d(z, Sz), \varphi(z), \varphi(Sz))) &= D(\varrho(d(Tx_{2n(k)}, Sz), \varphi(Tx_{2n(k)}), \varphi(Sz))) \\ &\leq F^*(D(\varrho(d(z, Sz), \varphi(z), \varphi(Sz)), \Theta(\varrho(d(z, Sz), \varphi(z), \varphi(Sz)))). \end{aligned}$$

Hence

$$\varrho(d(z, Sz), \varphi(z), \varphi(Sz)) = 0,$$

which follows

$$Sz = z.$$

In the same way we can obtain $Tz = z$. $\square$

Example 3.2. Let $X = (-1, 1]$ and $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $D, \Theta : [0, \infty) \to [0, \infty)$ by $D(x) = \frac{x}{3}$ and $\Theta(x) = \frac{x}{3}$. Define the mappings $T, S : X \to X$ by

$$Tx = \begin{cases} \dfrac{x}{3}, & \text{if } x \in (-1, 0), \\ \dfrac{x}{9}, & \text{if } x \in [0, 1] \end{cases}$$

and

$$Sx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in (-1, 0), \\ \dfrac{x}{3}, & \text{if } x \in [0, 1]. \end{cases}$$

Also let $\mathfrak{a}, \mathfrak{n} : X \to X$ be defined as

$$\mathfrak{a}(x,y) = \begin{cases} 2, & \text{if } x, y \in [0,1], \\ \dfrac{1}{5}, & \text{otherwise,} \end{cases}$$

$$\mathfrak{n}(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1], \\ \dfrac{1}{3}, & \text{otherwise.} \end{cases}$$

Now for $F^*(s,t) = s - t$, first of all we prove that (T,S) is generalized F^*-φ contraction.

If $\mathfrak{a}(x,y) \geq \mathfrak{n}(x,y)$, then $x, y \in [0,1]$. Therefore

$$D(\varrho(d(Tx,Sy), \varphi(Tx), \varphi(Sy)) = \left|\frac{x}{9} - \frac{y}{3}\right| + \frac{x}{9} + \frac{y}{3}$$

and

$$\begin{aligned} M^*(x,y) &= \max\left\{|x-y| + x + y, \left|x - \frac{x}{9}\right| + x + \frac{x}{9}, \left|y - \frac{y}{9}\right| + y + \frac{y}{9}\right\} \\ &= \max\{|x-y| + x + y, 2x, 2y\} \\ &= \max\{2x, 2y\}. \end{aligned}$$

Hence

$$F^*(D(M^*(x,y)), \Theta(M^*(x,y))) = \max\{2x, 2y\} - \frac{\max\{2x,2y\}}{3} = \frac{\max\{4x,4y\}}{3}.$$

Clearly $\max\left\{\frac{2x}{9}, \frac{2y}{3}\right\} \leq \frac{\max\{4x,4y\}}{3}$. Therefore the pair (T,S) is a generalized F^*-φ contraction type mapping. Moreover condition (iv) of Theorem 3.3 is satisfied for $x_0 = 0$ and if $\mathfrak{a}(x_n, x_{n+1}) \geq \mathfrak{n}(x_n, x_{n+1})$ it implies that $x_n \in [0,1]$ for all $n \in \mathbb{N}$. Now due to completeness of $[0,1]$ there exists $x \in [0,1]$, such that x_n converges to x and $Tx, Sy, STx, TSx \in [0,1]$. Then it can be observed that (T,S) is $\mathfrak{a}, \mathfrak{n}$-adm. Now if $\{x_n\} \subseteq [0,1]$ for all $n \in \mathbb{N}$, then we have

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \frac{x_n}{9} = \frac{x}{9} = Tx,$$

and

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} \frac{x_n}{3} = \frac{x}{3} = Sx.$$

This implies (T,S) is an $\mathfrak{a}, \mathfrak{n}$-CTS. Hence all assumptions of above theorem are true and $z' = 0$ is the unique Common φ-FP of mapping T and S.

Definition 3.4. Let (X,d) be a metric space, $T, S : X \to X$ be two mappings and $\mathfrak{a}, \mathfrak{n} : X \times X \to [0,\infty)$ are two functions. Let $\mathcal{B} : [0,\infty) \to [0,1)$ is a function such that for any bounded sequence $\{\breve{t_n}\}$ of positive reals, $\mathcal{B}(\breve{t_n}) \to 1$ implies that $\{\breve{t_n}\} \to 0$. Then the pair (T,S) is called $\varrho - \mathcal{B} - \varphi$-Contraction if

$$\mathfrak{a}(x,y) \geq \mathfrak{n}(x,y)$$

implies

$$\varrho(d(Tx,Sy), \varphi(Tx), \varphi(Sy)) \leq \mathcal{B}(\varrho(d(x,y), \varphi(x), \varphi(y)))\mho(x,y), \tag{3.5}$$

where

$$\mho(x,y) = \max\{\varrho(d(x,y), \varphi(x), \varphi(y))), \min\{\varrho(d(x,Tx), \varphi(x), \varphi(Tx)), \varrho(d(y,Sy), \varphi(y), \varphi(Sy))\}\}.$$

Theorem 3.5. *Let (X,d) be a metric space and T,S be two self-mappings on X. Assume that $\mathfrak{a},\mathfrak{n}: X\times X\to[0,\infty)$ are two functions and $\mathcal{B}:[0,\infty)\to[0,1)$ is a function such that for any bounded sequence $\{\breve{t_n}\}$ of positive reals, $\mathcal{B}(\breve{t_n})\to 1$ implies that $\{\breve{t_n}\}\to 0$. Then (T,S) have a common φ-FP if it satisfies the following conditions:*

(i) *(X,d) is $\mathfrak{a},\mathfrak{n}$-CMS;*

(ii) *(T,S) is $\varrho-\mathcal{B}-\varphi$-Contraction;*

(iii) *(T,S) is triangular $\mathfrak{a},\mathfrak{n}$-adm.*

(iv) *There exists $x_0\in X$ such that $\mathfrak{a}(x_0,Tx_0)\ge\mathfrak{n}(x_0,Tx_0)$;*

(v) *(T,S) is $\mathfrak{a},\mathfrak{n}$-CTS mappings or*

(vi) *If $\{x_n\}$ is a sequence in X such that $\mathfrak{a}(x_n,x_{n+1})\ge\mathfrak{n}(x_n,x_{n+1})$ for all $n\in\mathbb{N}$ and $x_n\to z\in X$ as $n\to\infty$, then there exist a subsequence $\{x_n(k)\}$ of $\{x_n\}$ such that $\mathfrak{a}(x_{n(k)},z)\ge\mathfrak{n}(x_{n(k)},z)$ for all $k\in\mathbb{N}$.*

Then (T,S) have a common φ-FP.

Proof. Let $x_0\in X$ be such that $\mathfrak{a}(x_0,Tx_0)\ge\mathfrak{n}(x_0,Tx_0)$. Define a sequence $\{x_n\}$ in X such that $x_{2n+1}=T_{2n}$ and $x_{2n+2}=Sx_{2n+1}$ for all $n\in\mathbb{N}$. If $x_{n_0}=x_{n_0+1}$ for some $n_0\in\mathbb{N}$, then clearly x_{n_0} is the required common φ-FP of (T,S). As (T,S) is $\mathfrak{a},\mathfrak{n}$-adm, hence

$$\mathfrak{a}(x_1,x_2)=\mathfrak{a}(Tx_0,Sx_1)\ge\mathfrak{n}(Tx_0,Sx_1)=\mathfrak{n}(x_1,x_2)$$

and

$$\mathfrak{a}(x_2,x_1)=\mathfrak{a}(Sx_1,Tx_0)\ge\mathfrak{n}(Sx_1,Tx_0)=\mathfrak{n}(x_2,x_1).$$

If we use $\mathfrak{a}$-adm property with respect to $\mathfrak{n}$ again, we get

$$\mathfrak{a}(x_3,x_2)=\mathfrak{a}(Tx_2,Sx_1)\ge\mathfrak{n}(Tx_2,Sx_1)=\mathfrak{n}(x_3,x_2)$$

and

$$\mathfrak{a}(x_2,x_3)=\mathfrak{a}(Sx_1,Tx_2)\ge\mathfrak{n}(Sx_1,Tx_2)=\mathfrak{n}(x_2,x_3).$$

By repetition of this process n-times, we get $\mathfrak{a}(x_n,x_{n+1})\ge\mathfrak{n}(x_n,x_{n+1})$ and $\mathfrak{a}(x_{n+1},x_n)\ge\mathfrak{n}(x_{n+1},x_n)$. Now as (T,S) is triangular $\mathfrak{a},\mathfrak{n}$-adm, therefore we can deduce that $\mathfrak{a}(x_n,x_{n'})\ge\mathfrak{n}(x_n,x_{n'})$ and $\mathfrak{a}(x_{n'},x_n)\ge\mathfrak{n}(x_{n'},x_n)$ for any $n,n'\in\mathbb{N}$ with $n'>n$.

Let $x_{2n}\ne x_{2n+1}$ and $x_{2n+1}\ne x_{2n+2}$ for all $n\in\mathbb{N}$, then by lemma (2.2), we get $\mathfrak{a}(x_{2n},x_{2n+1})\ge\mathfrak{n}(x_{2n},x_{2n+1})$ for all $n\in\mathbb{N}$. Then as (T,S) is generalized F^*-φ-Contraction mapping, we have

$$\varrho(d(Tx_{2n},Sx_{2n+1}),\varphi(Tx_{2n}),\varphi(Sx_{2n+1}))\le\mathcal{B}(\varrho(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1})))\mho(x_{2n},x_{2n+1})$$

implies

$$\varrho(d(x_{2n+1},x_{2n+2}),\varphi(x_{2n+1}),\varphi(x_{2n+2}))\le\mathcal{B}(\varrho(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1})))\mho(x_{2n},x_{2n+1}),$$

where

$$\mho(x_{2n},x_{2n+1})=\max\{\varrho(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1}))),$$

$$\begin{aligned}&\min\{\varrho(d(x_{2n},Tx_{2n}),\varphi(x_{2n}),\varphi(Tx_{2n})),\\&\varrho(d(x_{2n+1},Sx_{2n+1}),\varphi(x_{2n+1}),\varphi(Sx_{2n+1}))\}\}\\=\ &\max\{\Delta_{2n,2n+1},\min\{\Delta_{2n,2n+1},\Delta_{2n+1,2n+2}\}\}\end{aligned}$$

implies

$$\Delta_{2n+1,2n+2}\leq\mathcal{B}(\Delta_{2n,2n+1})\Delta_{2n,2n+1}.$$

Now if we consider

$$d_n=\varrho(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1}))),d_{n+1}=\varrho(d(x_{2n+1},x_{2n+2}),\varphi(x_{2n+1}),\varphi(x_{2n+2})),$$

then sequence d_n is monotonically decreasing in $[0,\infty)$, hence converges to some $\Bbbk\in[0,\infty)$. Then by letting $n\to\infty$ in above inequality, we have

$$\lim_{n\to\infty}\frac{\varrho(d(x_{2n+1},x_{2n+2}),\varphi(x_{2n+1}),\varphi(x_{2n+2}))}{\varrho(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1}))}\leq\lim_{n\to\infty}\mathcal{B}(\varrho(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1}))),$$

which implies

$$\lim_{n\to\infty}\mathcal{B}(\varrho(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1})))\geq1.$$

Therefore

$$\lim_{n\to\infty}\mathcal{B}(\varrho(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1})))=1.$$

This yields

$$\lim_{n\to\infty}\varrho(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1}))=0.$$

Accordingly

$$\lim_{n\to\infty}d(x_{2n},x_{2n+1})=0,\quad\lim_{n\to\infty}\varphi(x_{2n})=0.$$

Now assume that $\lim_{n\to\infty}d(x_n,x_{n'})=\epsilon$, where $\epsilon>0$. Then depending on n and n', four cases arise:

Case 1. Consider $n(k)=2k+1$ and $n(k)=2k+s+1$ that is $n(k)$ is odd and $n(k)$ is even for some $k,s\in\mathbb{N}$, where s is odd. As $\mathfrak{a}(x_{2k+1},x_{2k+1+s})\geq\mathfrak{n}(x_{2k+1},x_{2k+1+s})$, hence we conclude that

$$\begin{aligned}\varrho(d(x_{n(k)},x_{n'(k)}),\varphi(x_{n(k)}),\varphi(x_{n'(k)}))&=\varrho(d(Tx_{n(k)-1},Sx_{n'(k)-1}),\varphi(Tx_{n(k)-1}),\varphi(Sx_{n'(k)-1}))\\&=\varrho(d(Tx_{2k},Sx_{2k+s}),\varphi(Tx_{2k}),\varphi(Sx_{2k+s}))\\&\leq\mathcal{B}(\varrho(d(x_{2k},x_{2k+s}),\varphi(Tx_{2k}),\varphi(x_{2k+s})))\mho(x_{2k},x_{2k+s}),\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}\mho(x_{2k},x_{2k+s})&=\max\{\varrho(d(x_{2k},x_{2k+s}),\varphi(x_{2k}),\varphi(x_{2k+s})),\min\{\varrho(d(x_{2k},Tx_{2k}),\varphi(x_{2k}),\varphi(Tx_{2k})),\\&\quad\varrho(d(x_{2k+s},Sx_{2k+s}),\varphi(x_{2k+s}),\varphi(Sx_{2k+s}))\}\}\\&=\max\{\varrho(d(x_{2k},x_{2k+s}),\varphi(x_{2k}),\varphi(x_{2k+s})),\min\{\varrho(d(x_{2k},x_{2k+1}),\varphi(x_{2k}),\varphi(x_{2k+1})),\end{aligned}$$

$$\varrho(d(x_{2k+s}, x_{2k+s+1}), \varphi(x_{2k+s}), \varphi(x_{2k+s+1}))\}\},$$

which follows that

$$\begin{aligned}\lim_{k\to\infty} \mho(x_{2k}, x_{2k+s}) = &\lim_{k\to\infty} \max\{\varrho(d(x_{2k}, x_{2k+s}), \varphi(x_{2k}), \varphi(x_{2k+s})),\\ &\min\{\varrho(d(x_{2k}, x_{2k+1}), \varphi(x_{2k}), \varphi(x_{2k+1})),\\ &\varrho(d(x_{2k+s}, x_{2k+s+1}), \varphi(x_{2k+s}), \varphi(x_{2k+s+1}))\}\}\\ = &\max\{\varrho(\epsilon, 0, 0), \min\{\varrho(0, 0, 0), \varrho(0, 0, 0)\}\}.\end{aligned}$$

Hence

$$\lim_{k\to\infty} \mho(x_{2k}, x_{2k+s}) = \varrho(\epsilon, 0, 0). \tag{3.7}$$

Now, by (3.6) and (3.7), we get

$$\varrho(\epsilon, 0, 0) \leq \mathcal{B}(\varrho(d(x_{2k}, x_{2k+s}), \varphi(Tx_{2k}), \varphi(x_{2k+s})))\varrho(\epsilon, 0, 0)$$

implies

$$1 \leq \mathcal{B}(\varrho(d(x_{2k}, x_{2k+s}), \varphi(Tx_{2k}), \varphi(x_{2k+s}))) < 1.$$

Therefore

$$\lim_{k\to\infty} \mathcal{B}(\varrho(d(x_{2k}, x_{2k+s}), \varphi(Tx_{2k}), \varphi(x_{2k+s}))) = 1,$$

which gives

$$\lim_{k\to\infty} \varrho(d(x_{2k}, x_{2k+s}), \varphi(Tx_{2k}), \varphi(x_{2k+s})) = 0.$$

Consequently

$$\lim_{k\to\infty} d(x_{2k}, x_{2k+s}) = 0,$$

and

$$\lim_{k\to\infty} d(x_{2k+s}, x_{2k}) = 0.$$

From here, we conclude that $\epsilon = 0$, which gives contradiction to the condition $\epsilon > 0$. Therefore our assumption is wrong and sequence $\{x_n\}$ is Cauchy sequence in this case.

Case 2. Consider $n(k)$ and $n(k)$ both are odd. Then by triangle inequality and Case 1, we have

$$d(x_{n(k)}, x_{n'(k)}) \leq d(x_{n(k)}, x_{n'(k)-1}) + d(x_{n'(k)-1}, x_{n'(k)})$$

implies

$$\begin{aligned}\lim_{k\to\infty} d(x_{n(k)}, x_{n'(k)}) &\leq \lim_{k\to\infty} d(x_{n(k)}, x_{n'(k)-1}) + \lim_{k\to\infty} d(x_{n'(k)-1}, x_{n'(k)})\\ &\leq 0,\end{aligned}$$

hence

$$\lim_{k\to\infty} d(x_{n(k)}, x_{n'(k)}) = 0.$$

Case 3. Now, let n and n' both are even, then again by triangular inequality, we get

$$d(x_{n(k)}, x_{n'(k)}) \leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n'(k)}).$$

Here again by (3.2) and Case 1, we have $\lim\limits_{k\to\infty} d(x_{n(k)}, x_{n'(k)}) = 0$.

Case 4. In the last case, suppose that n is even and n' is odd. Using triangular inequality, we have

$$d(x_{n(k)}, x_{n'(k)}) \leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n'(k)-1}) + d(x_{n'(k)-1}, x_{n'(k)}),$$

which clearly gives $\lim\limits_{k\to\infty} d(x_{n(k)}, x_{n'(k)}) = 0$. Now by combining all these cases together, it is concluded that sequence x_n is a Cauchy sequence in X. Then by completeness of space X, there exists $z \in X$, such that sequence x_n converges to z. Since T, S are CTS mappings, therefore we have $Tx_{2n} \to Tz$ and $Sx_{2n+1} \to Sz$. This implies $x_{2n+1} = Tx_{2n} \to z$. By uniqueness of limit, we get

$$Tz = z.$$

Similarly, we have

$$Sz = z.$$

Hence z is the common FP of mappings T and S. Moreover as φ is lower semi CTS, hence $\lim\limits_{n\to\infty} \varphi(x_n) = \varphi(z) = 0$. Hence z is common φ-FP of mappings T and S. If we assume there is no condition of continuity over T but condition 6. holds, then by using above proof, a sequence $\{x_n\} \in X$ can be constructed which is defined as $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for all $n \in \mathbb{N}$, such that the sequence converges to $z \in X$. Then by condition 6, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\mathfrak{a}(x_{n(k)}, z) \geq \mathfrak{n}(x_{n(k)}, z)$ for all $k \in \mathbb{N}$. Hence, we get

$$\begin{aligned}\varrho(d(x_{2n(k)+1}, Sz), \varphi(x_{2n(k)+1}), \varphi(Sz)) &= \varrho(d(Tx_{2n(k)}, Sz), \varphi(Tx_{2n(k)}), \varphi(Sz))\\ &\leq \mathcal{B}(\varrho(d(x_{2n(k)}, z), \varphi(x_{2n(k)}), \varphi(z)))\mho(x_{2n(k)}, z),\end{aligned}$$

where

$$\begin{aligned}&\mho(x_{2n(k)}, z)\\ &= \max\{\varrho(d(x_{2n(k)}, z), \varphi(x_{2n(k)}), \varphi(z)),\\ &\quad \min\{\varrho(d(x_{2n(k)}, Tx_{2n(k)}), \varphi(x_{2n(k)}), \varphi(Tx_{2n(k)})), \varrho(d(z, Sz), \varphi(z), \varphi(Sz))\}\}\\ &= \max\{\varrho(d(x_{2n(k)}, z), \varphi(x_{2n(k)}), \varphi(z)),\\ &\quad \min\{\varrho(d(x_{2n(k)}, x_{2n(k)+1}), \varphi(x_{2n(k)}), \varphi(x_{2n(k)+1})), \varrho(d(z, Sz), \varphi(z), \varphi(Sz))\}\},\end{aligned}$$

implies

$$\lim_{k\to\infty} x(x_{2n(k)}, z) = \max\{\varrho(0,0,0), \min\{\varrho(0,0,0), \varrho(d(z, Sz), \varphi(z), \varphi(Sz))\}\}$$

$$
\begin{aligned}
&= \varrho(0,0,0)\\
&= 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\varrho(d(z,Sz),\varphi(z),\varphi(Sz))) &= \varrho(d(Tx_{2n(k)},Sz),\varphi(Tx_{2n(k)}),\varphi(Sz))\\
&\leq \mathcal{B}(\varrho(d(x_{2n(k)},z),\varphi(x_{2n(k)}),\varphi(z))) \times 0,
\end{aligned}
$$

which yields

$$\varrho(d(z,Sz),\varphi(z),\varphi(Sz)) = 0.$$

Hence

$$Sz = z.$$

In the same way we can obtain $Tz = z$. By uniqueness of limit z is unique common φ-FP of mappings Γ and S. $\square$

Example 3.3. Let $X = (-1,1]$ and $d(x,y) = |x-y|$ for all $x,y \in X$. Consider $T,S : X \to X$ be two mappings defined by

$$
Tx = \begin{cases} \dfrac{x}{3}, & \text{if } x \in (-1,0),\\ \dfrac{x}{4}, & \text{if } x \in [0,1] \end{cases}
$$

and

$$
Sx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in (-1,0),\\ \dfrac{x}{8}, & \text{if } x \in [0,1]. \end{cases}
$$

Also let $\mathfrak{a},\mathfrak{n} : X \to X$ be defined as same as in Example (3.1). Now we will show that for $\mathcal{B}(\theta) = \frac{1}{2}$, all the conditions of Theorem (3.5) are satisfied.

If $x,y \in [0,1]$, then

$$\mathfrak{a}(x,y) \geq \mathfrak{n}(x,y)$$

Now we show that T is satisfying contractive condition (3.5), for this take $x,y \in [0,1]$, we have

$$
\begin{aligned}
&\varrho(d(Tx,Sy),\varphi(Tx),\varphi(Sy)) = \left|\frac{x}{4} - \frac{y}{8}\right| + \frac{x}{4} + \frac{y}{8},\\
&\varrho(d(x,Tx),\varphi(x),\varphi(Tx)) = \frac{3x}{4} + x + \frac{x}{4} = 2x,\\
&\varrho(d(y,Sy),\varphi(y),\varphi(Sy)) = \frac{7y}{8} + y + \frac{y}{8} = 2y,\\
&\varrho(d(x,y),\varphi(x),\varphi(Ty)) = |x-y| + x + y.
\end{aligned}
$$

Then, clearly

$$\min\{\varrho(d(x,Tx),\varphi(x),\varphi(Tx)),\varrho(d(y,Ty),\varphi(y),\varphi(Ty))\} = \min\{2x,2y\}$$

and

$$\max\{\varrho(d(x,y),\varphi(x),\varphi(y))),\min\{2x,2y\}\} = \max\{|x-y|+x+y,\min\{2x,2y\}\}$$
$$= 2\max\{x,y\}$$

implies

$$\mathcal{B}(\varrho(d(x,y),\varphi(x),\varphi(y)))\max\{\varrho(d(x,y),\varphi(x),\varphi(y))),\min\{2x,2y\}\} = \max\{x,y\}.$$

Now, we have

$$\left|\frac{x}{4}-\frac{y}{8}\right|+\frac{x}{4}+\frac{y}{8}=\frac{x}{2} \text{ or } \frac{y}{4}.$$

Clearly, $\max\{\frac{x}{2},\frac{y}{4}\} \leq \max\{x,y\}$. Therefore inequality (3.5) is satisfied. Moreover condition (iv) of Theorem (3.5) is satisfied for $x_0 = 0$ and if $\mathfrak{a}(x_n,x_{n+1}) \geq \mathfrak{n}(x_n,x_{n+1})$ it implies that $x_n \in [0,1]$ for all $n \in \mathbb{N}$. Now due to completeness of $[0,1]$, there exists $x \in [0,1]$, such that x_n converges to x and $Tx, Sy, STx, TSx \in [0,1]$. Then it can be observed that (T,S) is $\mathfrak{a},\mathfrak{n}$-adm. Therefore above Theorem is true here and $z' = 0$ is the unique Common φ-FP of mapping T and S.

Proceeding as in Example (3.1), we consider the same two choices of the function $\mathcal{B}$ and discuss the result in two cases.

1: Let

$$\mathcal{B}(\theta) = \begin{cases} e^{-\theta}, & \text{if } \theta > 0,\\ 0, & \text{if } \theta = 0 \end{cases}$$

and refine T and S as follows:

$$Tx = \begin{cases} \dfrac{x}{15}, & \text{if } x \in (-1,0),\\ \dfrac{x}{16}, & \text{if } x \in [0,1], \end{cases}$$
$$Sx = \begin{cases} \dfrac{x}{14}, & \text{if } x \in (-1,0),\\ \dfrac{x}{32}, & \text{if } x \in [0,1]. \end{cases}$$

Then as in Example (3.1), we will prove

$$\left|\frac{x}{16}-\frac{y}{32}\right|+\frac{x}{16}+\frac{y}{32}=\frac{x}{8} \text{ or } \frac{y}{16} \leq 2e^{-\max\{2x,2y\}}\max\{x,y\}$$

implies

$$\max\left\{\frac{x}{8},\frac{y}{16}\right\} \leq 2e^{-\max\{2x,2y\}}\max\{x,y\}.$$

Now, $\max\left\{\frac{x}{8},\frac{y}{16}\right\} \leq \max\left\{\frac{x}{8},\frac{y}{8}\right\}$ and $\max\left\{\frac{x}{8},\frac{y}{8}\right\} \leq 2e^{-\max\{2x,2y\}}\max\{x,y\}$ because $e^{-\max\{2x,2y\}} \geq \frac{1}{16}$ for each $x,y \in [0,1]$.Therefore given example supports Theorem (3.5).

2: Let

$$\mathcal{B}(\theta) = \begin{cases} \dfrac{1}{1+\theta}, & \text{if } \theta > 0,\\ 0, & \text{if } \theta = 0 \end{cases}$$

and consider T and S in the same way as in Case 1. Then similar to Case 1, we aim to prove that

$$\left|\frac{x}{16}-\frac{y}{32}\right|+\frac{x}{16}+\frac{y}{32}=\frac{x}{8} \text{ or } \frac{y}{16}\leq\frac{2}{1+\max\{2x,2y\}}\max\{x,y\}$$

implies

$$\max\left\{\frac{x}{8},\frac{y}{16}\right\}\leq\frac{2}{1+\max\{2x,2y\}}\max\{x,y\}.$$

Proceeding in the same manner as in the above calculations, we obtain $\frac{2}{1+\max\{2x,2y\}}\max\{x,y\}\geq \max\left\{\frac{x}{8},\frac{y}{8}\right\}$ as $\frac{1}{1+\max\{2x,2y\}}\geq\frac{1}{16}$ for each $x,y\in[0,1]$. Hence, Theorem (3.5) is satisfied in this case as well.

Theorem 3.6. *Let (X,d) be a CMS and $T:X\to X$ be an $\mathfrak{a}$-adm mapping. Suppose that the following condition is satisfied:*

$$\begin{aligned}&(D(\varrho(d(Tx,Ty),\varphi(Tx),\varphi(Ty)))+l)^{\mathfrak{a}(x,Tx)\mathfrak{a}(y,Ty)}\\ \leq&F^*(D(\varrho(d(x,y),\varphi(x),(y))),\Theta(\varrho(d(x,y),\varphi(x),(y))))+l\end{aligned}\tag{3.8}$$

for all $x,y\in X$ and $l\geq 1$. Suppose that either,

(i) *T is CTS;*

(ii) *if $\{x_n\}$ is a sequence in X such that $x_n\to x,\mathfrak{a}(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$, then $\mathfrak{a}(x,Tx)\geq 1$.*
If there exists $x_0\in X$ such that $\mathfrak{a}(x_0,Tx_0)\geq 1$.

Then T has a φ-FP.

Proof. Following the same approach as in Theorem 3.1, we consider a sequence $\{x_n\}\in X$ and using the $\mathfrak{a}$-admissibility condition, we can prove that $\mathfrak{a}(x_n,Tx_n)\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$. Then by inequality (3.8), we have

$$\begin{aligned}&D(\varrho(d(Tx_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(Tx_n)))+l\\ &\leq D(\varrho(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1}))+l)^{\mathfrak{a}(x_{n-1},x_n)\mathfrak{a}(x_n,x_{n+1})}\\ &\leq F^*(D(\Delta_{n-1,n}),\Theta(\Delta_{n-1,n}))+l\end{aligned}$$

implies

$$D(\Delta_{n,n+1})\leq F^*(D(\Delta_{n-1,n}),\Theta(\Delta_{n-1,n}))\leq D(\Delta_{n-1,n}).$$

Using properties of D, we have

$$\Delta_{n,n+1}\leq\Delta_{n-1,n}$$

implies

$$\varrho(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1}))\leq\varrho(d(x_{n-1},x_n),\varphi(x_{n-1}),(x_n)).$$

Using the same arguments as employed in the earlier theorems, we obtain that the sequence $\Delta_{n,n+1}$ is monotonically decreasing and converges to a limit $\Bbbk \in \mathbb{R}^+$. Now we will prove that $\Bbbk = 0$. For this we consider

$$\lim_{n\to\infty} D(\Delta_{n,n+1}) \leq \lim_{n\to\infty} \leq F^*(D(\Delta_{n-1,n}), \Theta(\Delta_{n-1,n}))$$

implies

$$D(\Bbbk) \leq F^*(D(\Bbbk), \Theta(\Bbbk)) \leq D(\Bbbk),$$

which gives

$$F^*(D(\Bbbk), \Theta(\Bbbk)) = D(\Bbbk).$$

Therefore

$$D(\Bbbk) = 0$$

implies

$$\Bbbk = 0.$$

Hence

$$\lim_{n\to\infty} \varrho(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) = 0$$

implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} \varphi(x_n) = 0. \quad \text{(By property } (\varrho1)).$$

Now we will prove that the sequence x_n is Cauchy. Before this proceeding similarly to the above theorem, we can easily prove that

$$\lim_{n,n'\to\infty} d(x_{n'+1}, x_{n+1}) = \lim_{n,n'\to\infty} d(x_{n'}, x_n) = \epsilon.$$

We also have

$$\begin{aligned}
&D(\varrho(d(x_{n'+1}, x_{n+1}), \varphi(x_{n'+1}), \varphi(x_{n+1})))\\
\leq\ &F^*(D(\varrho(d(x_{n'}, x_n), \varphi(x_{n'}), (x_n)), \Theta(\varrho(d(x_{n'}, x_n), \varphi(x_{n'}), (x_n)))\\
\leq\ &D(\varrho(d(x_{n'}, x_n), \varphi(x_{n'}), (x_n))).
\end{aligned}$$

Taking limits on n' and n, we have

$$D(\varrho(\epsilon, 0, 0)) \leq F^*(D(\varrho(\epsilon, 0, 0)), \Theta(\varrho(\epsilon, 0, 0)) \leq D(\varrho(\epsilon, 0, 0)).$$

This shows that

$$F^*(D(\varrho(\epsilon, 0, 0)), \Theta(\varrho(\epsilon, 0, 0)) = D(\epsilon),$$

which gives

$$D(\varrho(\epsilon, 0, 0)) = 0.$$

Hence, we get

$$\epsilon = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence and by completeness of X, there exits $z \in X$ such that $d(x_n, z) = 0$ and by continuity of T, we get $Tz = z$. Moreover by LSC property of φ function we get

$$\lim_{n\to\infty} \varphi(x_n) = \varphi(z) = 0.$$

But if we suppose that we have only condition (2), then we get

$$\begin{aligned}
&D(\varrho(d(Tx_n, Tz), \varphi(Tx_n), (Tz))) + l \\
&\le D(\varrho(d(x_{n+1}, z), \varphi(x_{n+1}), (z)) + l)^{\mathfrak{a}(x_n,x_{n+1})\mathfrak{a}(z,z)} \\
&\le F^*(D(\varrho(d(x_n, \varphi(z)), \varphi(x_n), (z))), \Theta(\varrho(d(x_n, z), \varphi(x_n), \varphi(z)))) + l \\
&\le D(\varrho(d(x_n, z), \varphi(x_n), \varphi(z))) + l.
\end{aligned}$$

Letting $n \to \infty$ in above expression and using uniqueness of limits, we have

$$\lim_{n\to\infty} D(\varrho(d(Tx_n, Tz), \varphi(Tx_n), (Tz))) \le F^*(D(0), \Theta(0)) \le D(0) = 0,$$

that implies

$$\lim_{n\to\infty} \varrho(d(Tx_n, Tz), \varphi(Tx_n), (Tz)) = 0,$$

hence

$$d(Tx_n, Tz) = 0.$$

Therefore, we obtain

$$Tz = z.$$

Hence z is a φ-FP of mapping T. □

Theorem 3.7. *Let all the assumptions of the preceding theorem hold, except that the contraction condition is replaced by the following:*

$$(\mathfrak{a}(x, Tx)\mathfrak{a}(y, Ty) + 1)^{D(\varrho(d(Tx,Ty),\varphi(Tx),(Ty)))} \le 2^{F^*(D(\varrho(d(x,y),\varphi(x),(y))),\Theta(\varrho(d(x,y),\varphi(x),(y))))} \tag{3.9}$$

Then the mapping admits a FP.

Proof. Following the same step as in above theorem, be inequality (3.9), we have

$$\begin{aligned}
&2^{D(\varrho(d(Tx_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(Tx_n)))} \\
&\le (\mathfrak{a}(x_{n-1}, Tx_{n-1})\mathfrak{a}(x_n, Tx_n) + 1)^{D(\varrho(d(Tx_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(Tx_n)))} \\
&\le 2^{F^*(D(\varrho(d(x_{n-1},x_n),\varphi(x_{n-1}),(x_n))),\Theta(\varrho(d(x_{n-1},x_n),\varphi(x_{n-1}),(x_n))))}
\end{aligned}$$

implies

$$2^{D(\Delta_{n,n+1})} \le 2^{F^*(D(\Delta_{n-1,n},\Theta(\Delta_{n-1,n})))},$$

which follows that

$$D(\Delta_{n,n+1}) \leq F^*(D(\Delta_{n-1,n}, \Theta(\Delta_{n-1,n}))) \leq D(\Delta_{n-1,n}).$$

Now, following the same arguments as in the above theorems, we can easily prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \lim_{n\to\infty} \varphi(x_n) = 0$$

and for $m, n \in \mathbb{N}$

$$\lim_{n,n'\to\infty} d(x_{n'}, x_n) = \epsilon, \lim_{n,n'\to\infty} d(x_{n'+1}, x_{n+1}) = \epsilon.$$

Now using these results in inequality (3.9), we get

$$\begin{aligned} 2^{D(\varrho(d(x_{n'+1},x_{n+1}),\varphi(x_{n'+1}),\varphi(x_{n+1})))} &\leq (\mathfrak{a}(x_{n'}, Tx_{n'})\mathfrak{a}(x_n, Tx_n) + 1)^{D(\varrho(d(x_{n'+1},x_{n+1}),\varphi(x_{n'+1}),\varphi(x_{n+1})))} \\ &\leq 2^{F^*(D(\varrho(d(x_{n'},x_n),\varphi(x_{n'}),\varphi(x_n))),\Theta(\varrho(d(x_{n'},x_n),\varphi(x_{n'}),\varphi(x_n)))} \end{aligned}$$

implies

$$2^{D(\Delta_{n'+1,n+1})} \leq 2^{F^*(D(\Delta_{n',n}),\Theta(\Delta_{n',n}))}.$$

Hence

$$D(\Delta_{n'+1,n+1}) \leq F^*(D(\Delta_{n',n}), \Theta(\Delta_{n',n})) \leq D(\Delta_{n',n}).$$

It follows that

$$\lim_{n,n'\to\infty} D(\Delta_{n'+1,n+1}) \leq \lim_{n,n'\to\infty} F^*(D(\Delta_{n',n}), \Theta(\Delta_{n',n})).$$

Therefore

$$D(\varrho(\epsilon, 0, 0)) \leq F^*(D(\varrho(\epsilon, 0, 0)), \Theta(\varrho(\epsilon, 0, 0))) \leq D(\varrho(\epsilon, 0, 0)).$$

Accordingly

$$D(\varrho(\epsilon, 0, 0)) or \Theta(\varrho(\epsilon, 0, 0)) = 0,$$

which follows that

$$\varrho(\epsilon, 0, 0) = 0.$$

Hence,

$$\epsilon = 0.$$

Now proceeding similarly and using continuity of T, we can prove that the mapping T admits a φ-FP $z \in X$.

If instead of continuity, we assume only Condition (2), then as z is limit of sequence $\{x_n\}$, we have $\varphi(z) = 0$ by LSC property of function φ. Hence using all these condition we get

$$2^{D(\varrho(d(Tx_n,Tz),\varphi(Tx_n),\varphi(Tz)))} \leq (\mathfrak{a}(x_n, Tx_n)\mathfrak{a}(z, Tz) + 1)^{D(\varrho(d(Tx_n,Tz),\varphi(Tx_n),\varphi(Tz)))}$$

$$\leq 2^{F^*(D(\varrho(d(x_n,z),\varphi(x_n),\varphi(z)))},$$

implies

$$\lim_{n\to\infty} 2^{D(\varrho(d(Tx_n,Tz),\varphi(Tx_n),\varphi(Tz)))} \leq \lim_{n\to\infty} 2^{F^*(D(\varrho(d(x_n,z),\varphi(x_n),\varphi(z)))},$$

hence

$$\lim_{n\to\infty} D(\varrho(d(Tx_n,Tz),\varphi(Tx_n),\varphi(Tz))) \leq F^*(D(0),\Theta(0)) \leq D(0)=0$$

implies

$$\lim_{n\to\infty} D(\varrho(d(Tx_n,Tz),\varphi(Tx_n),\varphi(Tz))) \leq 0.$$

Clearly it gives

$$\lim_{n\to\infty} d(Tx_n,Tz)=0,$$

consequently

$$\lim_{n\to\infty} Tx_n = Tx_{n+1} = Tz = z.$$

By uniqueness of limit of sequence $\{x_n\}$, we have $Tz=z$. Hence z is unique φ-FP of mapping T. □

Theorem 3.8. *Let all the assumptions of the preceding Theorem 3.6 hold, except that the contraction condition is replaced by the following:*

$$\begin{aligned}&\mathfrak{a}(x,Tx)\mathfrak{a}(y,Ty)D(\varrho(d(Tx,Ty),\varphi(Tx),(Ty)))\\&\leq F^*(D(\varrho(d(x,y),\varphi(x),(y))),\Theta(\varrho(d(x,y),\varphi(x),(y)))).\end{aligned}\tag{3.10}$$

Then the mapping admits a FP.

Proof. Following the same step as in above theorem, be inequality (3.9), we have

$$\begin{aligned}&D(\varrho(d(Tx_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(Tx_n)))\\&\leq \mathfrak{a}(x_{n-1},Tx_{n-1})\mathfrak{a}(x_n,Tx_n)D(\varrho(d(Tx_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(Tx_n)))\\&\leq F^*(D(\varrho(d(x_{n-1},x_n),\varphi(x_{n-1}),(x_n))),\Theta(\varrho(d(x_{n-1},x_n),\varphi(x_{n-1}),(x_n)))\end{aligned}$$

implies

$$D(\Delta_{n,n+1}) \leq F^*(D(\Delta_{n-1,n}),\Theta(\Delta_{n-1,n})).$$

Therefore

$$D(\Delta_{n,n+1}) \leq F^*(D(\Delta_{n-1,n},\Theta(\Delta_{n-1,n})) \leq D(\Delta_{n-1,n}).$$

Now, following the same arguments as in the above theorems, we can easily prove that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0, \quad \lim_{n\to\infty} \varphi(x_n)=0$$

and for $m, n \in \mathbb{N}$

$$\lim_{n,n' \to \infty} d(x_{n'}, x_n) = \epsilon, \qquad \lim_{n,n' \to \infty} d(x_{n'+1}, x_{n+1}) = \epsilon.$$

Now using these results in inequality (3.10), we get

$$\begin{aligned}
&D(\varrho(d(x_{n'+1}, x_{n+1}), \varphi(x_{n'+1}), \varphi(x_{n+1}))) \\
&\leq \mathfrak{a}(x_{n'}, Tx_{n'})\mathfrak{a}(x_n, Tx_n) D(\varrho(d(x_{n'+1}, x_{n+1}), \varphi(x_{n'+1}), \varphi(x_{n+1}))) \\
&\leq F^*(D(\varrho(d(x_{n'}, x_n), \varphi(x_{n'}), \varphi(x_n))), \Theta(\varrho(d(x_{n'}, x_n), \varphi(x_{n'}), \varphi(x_n)))
\end{aligned}$$

implies

$$D(\Delta_{n'+1,n+1}) \leq F^*(D(\Delta_{n',n}), \Theta(\Delta_{n',n})).$$

Hence

$$D(\Delta_{n'+1,n+1}) \leq F^*(D(\Delta_{n',n}), \Theta(\Delta_{n',n})) \leq D(\Delta_{n',n})$$

implies

$$\lim_{n',n \to \infty} D(\Delta_{n'+1,n+1}) \leq \lim_{n,n' \to \infty} F^*(D(\Delta_{n',n}), \Theta(\Delta_{n',n})).$$

Clearly it follows that

$$D(\varrho(\epsilon, 0, 0)) \leq F^*(D(\varrho(\epsilon, 0, 0)), \Theta(\varrho(\epsilon, 0, 0))) \leq D(\varrho(\epsilon, 0, 0)).$$

Therefore

$$D(\varrho(\epsilon, 0, 0)) or \Theta(\varrho(\epsilon, 0, 0)) = 0,$$

which yields

$$\varrho(\epsilon, 0, 0) = 0.$$

Hence

$$\epsilon = 0.$$

Now proceeding similarly and using continuity of T, we can prove that the mapping T admits a φ-FP $z \in X$.

If instead of continuity, we assume only Condition (2), then as z is limit of sequence $\{x_n\}$, we have $\varphi(z) = 0$ by LSC property of function φ. Hence using all these condition we get

$$\begin{aligned}
D(\varrho(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz))) &\leq \mathfrak{a}(x_n, Tx_n)\mathfrak{a}(z, Tz) D(\varrho(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz))) \\
&\leq F^*(D(\varrho(d(x_n, z), \varphi(x_n), \varphi(z))), \Theta(\varrho(d(x_n, z), \varphi(x_n), \varphi(z)))
\end{aligned}$$

implies

$$\begin{aligned}
&\lim_{n \to \infty} D(\varrho(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz))) \\
&\leq \lim_{n \to \infty} F^*(D(\varrho(d(x_n, z), \varphi(x_n), \varphi(z))), \Theta(\varrho(d(x_n, z), \varphi(x_n), \varphi(z))).
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} D(\varrho(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz))) \leq F^*(D(0), \Theta(0)) \leq D(0) = 0.$$

This yields

$$\lim_{n\to\infty} D(\varrho(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz))) \leq 0.$$

Then clearly

$$\lim_{n\to\infty} d(Tx_n, Tz) = 0.$$

Hence

$$\lim_{n\to\infty} Tx_n = Tx_{n+1} = Tz = z.$$

By uniqueness of limit of sequence $\{x_n\}$, we have $Tz = z$. Hence z is unique φ-FP of mapping T. □

Theorem 3.9. *Under the assumption that the conditions of Theorems 3.6, 3.7, and 3.8 hold, we introduce the following additional constraint:*

(iii) *If $Tx = x$, then $\mathfrak{a}(x, Tx) \geq 1$,*

then φ-FP of T is unique.

Proof. Let z and z' be two φ-FPs of mapping T that is $Tz = z, \varphi(z) = 0$ and $Tz' = z', \varphi(z') = 0$.

Then by Theorem 3.6, we have

$$\begin{aligned} D(\varrho(d(Tz, Tz'), \varphi(Tz), \varphi(Tz'))) + l &\leq (D(\varrho(d(Tz, Tz'), \varphi(Tz), (Tz')) + l)^{\mathfrak{a}(z,Tz)\mathfrak{a}(z',Tz')} \\ &\leq F^*(D(\Delta_{z,z'}), \Theta(\Delta_{z,z'})) + l \end{aligned}$$

where $\Delta_{z,z'} = \varrho(d(z, z'), \varphi(z), \varphi(z'))$. Then we get

$$\begin{aligned} D(\varrho(d(z, z'), 0, 0)) + l &\leq (D(\varrho(d(z, z'), 0, 0) + l)^{\mathfrak{a}(z,z)\mathfrak{a}(z',z')} \\ &\leq F^*(D(\varrho(d(z, z'), 0, 0)), \Theta(\varrho(d(z, z'), 0, 0))) + l \\ &\leq D(\varrho(d(z, z'), 0, 0)) + l, \end{aligned}$$

which follows that

$$F^*(D(\varrho(d(z, z'), 0, 0)), \Theta(\varrho(d(z, z'), 0, 0))) = D(\varrho(d(z, z'), 0, 0)$$

implies

$$D(\varrho(d(z, z'), 0, 0)) or \Theta(\varrho(d(z, z'), 0, 0)) = 0,$$

therefore

$$\varrho(d(z, z'), 0, 0) = 0.$$

This shows that

$$d(z, z') = 0,$$

hence

$$z = z'.$$

For Theorem 3.7, we have

$$\begin{aligned}2^{D(\varrho(d(Tz,Tz'),\varphi(Tz),\varphi(Tz')))} &\leq (\mathfrak{a}(z,Tz)\mathfrak{a}(z',Tz')+1)^{D(\varrho(d(Tz,Tz'),\varphi(Tz),\varphi(Tz')))}\\ &\leq 2^{F^*(D(\varrho(d(z,z'),\varphi(z),\varphi(z'))),\Theta(\varrho(d(z,z'),\varphi(z),\varphi(z'))))},\end{aligned}$$

implies

$$D(\varrho(d(z,z'),0,0)) \leq F^*(D(\varrho(d(z,z'),0,0)),\Theta(\varrho(d(z,z'),0,0))) \leq D(\varrho(d(z,z'),0,0)).$$

In a similar manner to the proof above, we can demonstrate that

$$z = z'.$$

By Theorem 3.8, we have

$$\begin{aligned}D(\varrho(d(Tz,Tz'),\varphi(Tz),\varphi(Tz'))) &\leq \mathfrak{a}(z,Tz)\mathfrak{a}(z',Tz')D(\varrho(d(Tz,Tz'),\varphi(Tz),\varphi(Tz')))\\ &\leq F^*(\Delta_{z,z'},\Delta_{z,z'})\\ &\leq D(\varrho(d(z,z'),\varphi(z),\varphi(z')))\end{aligned}$$

implies

$$F^*(D(\varrho(d(z,z'),0,0)),\Theta(\varrho(d(z,z'),0,0))) = D(\varrho(d(z,z'),0,0)).$$

Applying the same logic as the preceding proof, we can establish that

$$z = z'.$$

□

Example 3.4. Let $X = [1, 5]$ and $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $D, \Theta : [0, \infty) \to [0, \infty)$ by $D(x) = x$ and $\Theta(x) = \frac{x}{3}$. Define the mappings $T : X \to X$ by

$$Tx = \begin{cases} \dfrac{x+2}{2}, & \text{if } x \in [1,3],\\ 1, & \text{if } x \in (3,5]\end{cases}$$

and

$$\mathfrak{a}(x,y) = \begin{cases} 1, & \text{if } x,y \in [1,3],\\ \dfrac{1}{5}, & \text{otherwise.}\end{cases}$$

Here, instead of assuming the function φ to be CTS, we consider φ to be a LSC defined as follows:

$$\varphi(x) = \begin{cases} |x-2|, & \text{if } x \in [1,3], \\ x+1, & \text{if } x \in (3,5]. \end{cases}$$

Now for $F^*(s,t) = s - t$, and $l = 1$, we prove that (T,S) supports Theorems 3.6, 3.7 and 3.8.

Clearly here three cases arise.

Case 1. $x, y \in [1,2)$

$$\begin{aligned} D(\varrho(d(Tx,Ty),\varphi(Tx),\varphi(Ty))) &= \left|\frac{x+2}{2} - \frac{y+2}{2}\right| + \left|\frac{x+2}{2} - 2\right| + \left|\frac{y+2}{2} - 2\right| \\ &= \left|\frac{x-y}{2}\right| + \left|\frac{x-2}{2}\right| + \left|\frac{y-2}{2}\right| \\ &= \left|\frac{x-y}{2}\right| - \left(\frac{x-2}{2}\right) - \left(\frac{y-2}{2}\right) \\ &= \left|\frac{x-y}{2}\right| + \frac{4-x-y}{2} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \varrho(d(x,y),\varphi(x),(y))) &= |x-y| + |x-2| + |y-2| \\ &= |x-y| - (x-2) - (y-2) \\ &= |x-y| + (4-x-y) \end{aligned}$$

implies

$$F^*(D(\varrho(d(x,y),\varphi(x),(y))),\Theta(\varrho(d(x,y),\varphi(x),(y)))) = \frac{2(|x-y| + (4-x-y))}{3}. \tag{3.12}$$

Here $\mathfrak{a}(x,Tx) = \mathfrak{a}(y,Ty) = 1$. Therefore it can be verified easily that inequality (3.8) is satisfied in this case.

Now using (3.11), (3.12), we get

$$(\mathfrak{a}(x,Tx)\mathfrak{a}(y,Ty)+1)^{D(\varrho(d(Tx,Ty),\varphi(Tx),(Ty)))} = 2^{\left|\frac{x-y}{2}\right| + \frac{4-x-y}{2}}$$

and

$$2^{F^*(D(\varrho(d(x,y),\varphi(x),(y))),\Theta(\varrho(d(x,y),\varphi(x),(y))))} = 2^{\frac{2(|x-y|+(4-x-y))}{3}}.$$

Hence inequality (3.9) will also be satisfied here. In view of the above two inequalities, the third inequality (3.10) holds immediately.

Case 2. $x, y \in [2,3]$

$$\begin{aligned} D(\varrho(d(Tx,Ty),\varphi(Tx),\varphi(Ty))) &= \left|\frac{x+2}{2} - \frac{y+2}{2}\right| + \left|\frac{x+2}{2} - 2\right| + \left|\frac{y+2}{2} - 2\right| \\ &= \left|\frac{x-y}{2}\right| + \left|\frac{x-2}{2}\right| + \left|\frac{y-2}{2}\right| \\ &= \left|\frac{x-y}{2}\right| + \frac{x+y-4}{2} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\varrho(d(x,y),\varphi(x),(y))) &= |x-y|+|x-2|+|y-2| \\ &= |x-y|+(x+y-4)\end{aligned}$$

implies

$$F^*(D(\varrho(d(x,y),\varphi(x),(y))),\Theta(\varrho(d(x,y),\varphi(x),(y)))) = \frac{2(|x-y|+(x+y-4))}{3}. \tag{3.14}$$

Now using (3.13), (3.14), we get

$$(\mathfrak{a}(x,Tx)\mathfrak{a}(y,Ty)+1)^{D(\varrho(d(Tx,Ty),\varphi(Tx),(Ty)))} = 2^{\left|\frac{x-y}{2}\right|+\frac{x+y-4}{2}}$$

and

$$2^{F^*(D(\varrho(d(x,y),\varphi(x),(y))),\Theta(\varrho(d(x,y),\varphi(x),(y))))} = 2^{\frac{2(|x-y|+(x+y-4))}{3}}.$$

Thus inequalities (3.8), (3.9) and (3.10) hold here.

Case 3. $x \in [1,2), y \in [2,3]$

$$\begin{aligned}D(\varrho(d(Tx,Ty),\varphi(Tx),\varphi(Ty))) &= \left|\frac{x+2}{2}-\frac{y+2}{2}\right|+\left|\frac{x+2}{2}-2\right|+\left|\frac{y+2}{2}-2\right| \\ &= \left|\frac{x-y}{2}\right|+\left|\frac{x-2}{2}\right|+\left|\frac{y-2}{2}\right| \\ &= \left|\frac{x-y}{2}\right|+\frac{y-x}{2}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\varrho(d(x,y),\varphi(x),(y))) &= |x-y|+|x-2|+|y-2| \\ &= |x-y|+(y-x)\end{aligned}$$

implies

$$F^*(D(\varrho(d(x,y),\varphi(x),(y))),\Theta(\varrho(d(x,y),\varphi(x),(y)))) = \frac{2(|x-y|+(y-x))}{3}. \tag{3.16}$$

By (3.15), (3.16), we get

$$(\mathfrak{a}(x,Tx)\mathfrak{a}(y,Ty)+1)^{D(\varrho(d(Tx,Ty),\varphi(Tx),(Ty)))} = 2^{\left|\frac{x-y}{2}\right|+\frac{y-x}{2}}$$

and

$$2^{F^*(D(\varrho(d(x,y),\varphi(x),(y))),\Theta(\varrho(d(x,y),\varphi(x),(y))))} = 2^{\frac{2(|x-y|+(y-x))}{3}}.$$

This completes the verification of all three inequalities in the last case as well. With all three inequalities holding, the example falls within the scope of all three Theorems 3.6, 3.7 and 3.8 and clearly $x = 2$ is unique φ-FP of mapping T.

In this example, the mapping is discontinuous and the function φ is assumed to be LSC rather than CTS, showing that the results hold under weaker assumptions.

4. An application to fractional differential equations

Preliminaries

In this section, we present some fundamental concepts and definitions from fractional calculus which are essential for the analysis of the proposed problem.

Definition 4.1 ([16]). Let $f \in C([0,\infty),\mathbb{R})$ and $\mathfrak{n} > 0$. The Riemann-Liouville fractional integral of order $\mathfrak{n}$ is defined as

$$(I^{\mathfrak{n}} f)(\tau) = \frac{1}{\Gamma(\mathfrak{n})} \int_0^t (\tau-\omega)^{\mathfrak{n}-1} f(\omega)\, d\omega, \quad \tau > 0, \tag{4.1}$$

provided the integral exists.

Definition 4.2 ([6]). Let $j-1 < \mathfrak{n} < j$, where $j \in \mathbb{N}$. The Caputo fractional derivative of order $\mathfrak{n}$ of a function $\gamma : [0,1]\times\mathbb{R} \to [0,\infty)$ is given by

$$({}^C D^{\mathfrak{n}} \gamma)(\tau) = \frac{1}{\Gamma(j-\mathfrak{n})} \int_0^t (\tau-\omega)^{j-\mathfrak{n}-1} \gamma^{(j)}(\omega)\, d\omega, \quad \tau > 0. \tag{4.2}$$

Fractional derivatives play a significant role in modeling real-world phenomena where memory and hereditary properties are present. Unlike classical integer-order derivatives, fractional derivatives allow the current state of a system to depend on its entire history. Due to this nonlocal property, fractional operators have been widely employed in various theoretical frameworks.

In physics, fractional derivatives are used to describe anomalous diffusion and viscoelastic materials, where the stress–strain relationship depends on past deformations. In control theory, fractional-order systems provide better stability and robustness compared to integer-order models. In biology, they are applied to population dynamics and biological tissues exhibiting memory effects.

From a mathematical viewpoint, FDE offer a generalized setting for classical differential equations and allow the development of more flexible analytical techniques. In economics and finance, fractional derivatives are utilized to capture long-term dependence and persistent memory effects in economic growth models, financial markets, and investment dynamics.

Definition 4.3. Let $X = \{x : x \in C([0,1],\mathbb{R})\}$ be endowed with the supremum norm

$$\|x\|_\infty = \sup_{\tau\in[0,1]} |x(\tau)|. \tag{4.3}$$

Then $(X, \|\cdot\|_\infty)$ is a Banach space with metric d given by

$$d(x,y) = \|x-y\|_\infty = \sup_{\tau\in[0,1]} |x(\tau)-y(\tau)| \quad \forall\, x,y \in X. \tag{4.4}$$

Theorem 4.4. *Consider the nonlinear FDE*

$${}^C D^n x(\tau) = \gamma(\tau, x(\tau)), \quad 0 < \tau < 1,\ 1 < n \le 2, \tag{4.5}$$

subject to the integral boundary conditions

$$x(0) = 0, \quad Ix(1) = x'(0).$$

Let $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a given function. Let the following conditions be fulfilled:

(i) *The function* $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is CTS.*

(ii) *Suppose that mapping* γ *satisfies*

$$|\gamma(\tau, x) - \gamma(\tau, y)| \leq \frac{\Gamma(n+1)}{8}|x-y|,$$

for all $x, y \in X$ *and for all* $\tau \in [0,1]$ *with* $d(x,y) > 0$.

(iii) *There exists* $x_0 \in X$ *such that* $d(x_0(\tau), Tx_0(\tau)) > 0$ *for all* $t \in [0,1]$, *where operator* $T : X \to X$ *is defined by*

$$(Tx)(\tau) = \frac{1}{\Gamma(n)} \int_0^t (\tau - \omega)^{n-1} \gamma(\omega, x(\omega))\, d\omega + \frac{2\tau}{\Gamma(n)} \int_0^1 \left(\int_0^\omega (\omega - \upsilon)^{n-1} \gamma(\upsilon, x(\upsilon)) d\upsilon \right) d\omega,$$

for all $x, y \in X$.

(iv) *For each* $\tau \in [0,1]$ *and* $x, y \in X$, $d(x(\tau), Tx(\tau)) > 0$ *implies that* $d(Tx(\tau), Ty(\tau)) > 0$.

(v) *For* $\{x_n\} \subseteq X$ *such that* $x_n \to x \in X$ *and* $d(x_n, x_{n+1}) > 0$ *for all* $n \in \mathbb{N}$ *implies that* $d(x_n, x) > 0$ *for all* $n \in \mathbb{N}$.

(vi) *Suppose* γ *is considered in such a way that* T *satisfies* $\|T(x) - T(T(x))\|_\infty \leq \frac{3}{4}\|(x) - T(x)\|_\infty$ *for all* $x \in X$;
then (4.5) *has at least one solution.*

Then problem admits at least one solution.

Proof. Now by condition (ii), for $x, y \in X$, we obtain

$$\begin{aligned}
|Tx(\tau) - Ty(\tau)| \leq & \left| \frac{1}{\Gamma(n)} \int_0^\tau (\tau - \omega)^{n-1} (\gamma(\omega, x(\omega)) - \gamma(\omega, y(\omega))) d\omega \right| \\
& + \left| \frac{2\tau}{\Gamma(n)} \int_0^1 \left(\int_0^\omega (\omega - \upsilon)^{n-1} (\gamma(\upsilon, x(\upsilon)) - \gamma(\upsilon, y(\upsilon))) d\upsilon \right) d\omega \right| \\
\leq & \frac{1}{\Gamma(n)} \int_0^\tau |(\tau - \omega)^{n-1}| |(\gamma(\omega, x(\omega)) - \gamma(\omega, y(\omega)))| d\omega \\
& + \frac{2\tau}{\Gamma(n)} \int_0^1 \left(\int_0^\omega (\omega - \upsilon)^{n-1} |\gamma(\upsilon, x(\upsilon)) - \gamma(\upsilon, y(\upsilon))| d\upsilon \right) d\omega
\end{aligned}$$

implies

$$\begin{aligned}
|Tx(\tau) - Ty(\tau)| \leq & \frac{1}{\Gamma(n)} \int_0^\tau |(\tau - \omega)^{n-1}| \frac{\Gamma(n+1)}{8} |x(\omega) - y(\omega)| d\omega \\
& + \frac{2\omega}{\Gamma(n)} \int_0^1 \left(\int_0^\omega (\omega - \upsilon)^{n-1} \frac{\Gamma(n+1)}{8} |x(\upsilon) - y(\upsilon)| d\upsilon \right) ds \\
\leq & \frac{\Gamma(n)\Gamma(n+1)}{8\Gamma(n)\Gamma(n+1)} d(x,y) + \frac{\Gamma(n)\Gamma(n+1)}{4\Gamma(n)\Gamma(n+1)} d(x,y) \\
\leq & \frac{3}{8} d(x,y) \\
\leq & \frac{3}{4} d(x,y).
\end{aligned}$$

Clearly, by choosing

$$\varrho(\theta_1, \theta_2, \theta_3) = \theta_1 + \theta_2 + \theta_3, \quad \varphi(x) = \|x - T(x)\|_\infty \text{ for all } x \in X,$$
$$F^*(\omega - \tau) = \omega - \tau, \quad \mathfrak{D}(\tau) = \tau, \quad \Theta(\tau) = \frac{\tau}{4},$$

and

$$\mathfrak{a}(x, y) = \begin{cases} 1, & \text{if } X(x(\tau), y(\tau)) \in X, t \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly T is an $\mathfrak{a}$-adm and all the assumptions of Theorem (3.8) are satisfied. Hence, the mapping T admits a unique φ-FP, and consequently, the given problem has a solution. Consequently, problem (4.5) has a solution. □

5. Conclusion

The results obtained in this work offer a flexible and effective framework for the study of φ-FP and common φ-FP problems beyond classical settings. By relaxing the usual continuity requirements and employing generalized admissibility conditions, the proposed approach considerably broadens the applicability of FP techniques. The examples clearly demonstrate that the developed results are valid for both CTS and discontinuous mappings. Furthermore, the application to FDE confirms that the theoretical findings are not only mathematically consistent but also relevant to practical models.

6. Future scope

The established results can be effectively applied to study the existence of solutions for various classes of nonlinear integral and FDE. The proposed framework is particularly useful for models involving non-smooth or discontinuous phenomena arising in applied sciences. Moreover, the present approach can be adapted to investigate solution existence for coupled and system-type problems occurring in mathematical physics and engineering.

References

[1] A. H. Ansari, *Note on φ-ψ-contractive type mappings and related fixed point*, 2nd Reg. Conf. Math. Appl., Payame Noor Univ., 2014, 11, 377–380.

[2] A. H. Ansari and A. Kaewcharoen, *C-class functions and fixed point theorems for generalized α-η-ψ-φ-f-contraction type mappings in α-η-complete metric spaces*, J. Nonlinear Sci. Appl., 2016, 9(6), 4170–4178.

[3] M. Asadi, *Discontinuity of control function in the (f, ϕ, θ)-contraction in metric spaces*, Filomat, 2017, 31(17), 5427–5433.

[4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., 1922, 3(1), 133–181.

[5] P. Chuadchawna, A. Kaewcharoen and S. Plubtieng, *Fixed point theorems for generalized α-η-ψ-Geraghty contraction type mappings in α-η-complete metric spaces*, J. Nonlinear Sci. Appl., 2016, 9, 471–485.

[6] K. Diethelm, *General theory of caputo-type fractional differential equations*, Fractional Differential Equations, 2019, 1–20.

[7] N. Hussain, M. A. Kutbi and P. Salimi, *Fixed point theory in α-complete metric spaces with applications*, Abstr. Appl. Anal., 2014, 2014, 11 pp.

[8] N. Hussain, P. Salimi and A. Latif, *Fixed point results for single and set-valued α-η-ψ-contractive mappings*, Fixed Point Theory Appl., 2013, 2013(1), 212.

[9] M. Imdad, A. R. Khan, H. N. Saleh and W. M. Alfaqih, *Some φ-fixed point results for $(f, \varphi, \theta$-$\psi)$-contractive type mappings with applications*, Mathematics, 2019, 7(2), 1–16.

[10] G. Janardhanan, G. Mani, S. D. Rajkumar, et al., *Solving boundary value problem and integral equation via bipolar $\mathcal{R}$-metric space*, Filomat, 2025, 39(33), 11919–11929.

[11] M. Jleli, B. Samet and C. Vetro, *Fixed point theory in partial metric spaces via φ-fixed point's concept in metric spaces*, J. Inequal. Appl., 2014, 2014, 1–9.

[12] E. Karapinar, P. Kumam and P. Salimi, *On α-ψ-meir-keeler contractive mappings*, Fixed Point Theory Appl., 2013, 2013(1), 94.

[13] M. S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc., 1984, 30(1), 1–9.

[14] P. Kumrod and W. Sintunavarat, *A new contractive condition approach to φ-fixed point results in metric spaces and its applications*, J. Comput. Appl. Math., 2017, 311, 194–204.

[15] P. Kumrod and W. Sintunavarat, *On new fixed point results in various distance spaces via φ-fixed point theorems in d-generalized metric spaces with numerical results*, J. Fixed Point Theory Appl., 2019, 21, 86.

[16] J. Liouville, *Mémoire sur le calcul des différentielles à indices quelconques*, J. École Polytechnique, 1832, 13, 71–162.

[17] G. Mani, A. J. Gnanaprakasam, O. Ege, et al., *Solution of fredholm integral equation via common fixed point theorem on bicomplex valued b-metric space*, Symmetry, 2023, 15(2), 297.

[18] G. Mani, S. Haque, A. J. Gnanaprakasam, et al., *The study of bicomplex-valued controlled metric spaces with applications to fractional differential equations*, Mathematics, 2023, 11(12), 2742.

[19] M. Nazam, M. Arshad, C. Park and H. Mahmood, *On a fixed point theorem with application to functional equations*, Open Math., 2019, 17(1), 1724–1736.

[20] H. Qawaqneh, M. S. M. Noorani, W. Shatanawi and H. Alsamir, *Common fixed points for pairs of triangular α-admissible mappings*, J. Nonlinear Sci. Appl., 2017, 10(12), 6192–6204.

[21] S. Rathee, D. Kaushik, D. R. Bansal and R. Gupta, *φ-fixed point and common φ-fixed point results in dislocated metric space*, Nomerical Algebra, Control and Optimization, 2025, 15(4), 1105–1115.

[22] H. N. Saleh, M. Imdad and E. Karapinar, *A study of common fixed points that belong to zeros of a certain given function with applications*, Nonlinear Anal. Model. Control, 2021, 26(5), 781–800.

[23] P. Salimi, A. Latif and N. Hussain, *Modified α-ψ-contractive mappings with applications*, Fixed Point Theory Appl., 2013, 2013(1), 151.

[24] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. Theory Methods Appl., 2012, 75(4), 2154–2165.

[25] K. Zhang, M. E. E. Ege, A. J. Gnanaprakasam, et al., *New α-ε-suzuki-type contraction mapping methods on fractional differential and integral equations*, Fractal Fract., 2025, 9(11), 692.

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