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A NEW MULTIVALENT ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS IN DISTORTION BOUNDS AND ITS EXTREMAL PROPERTIES

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Abstract This work investigates a new subclass $\mathcal{SM}^{n}_{b,\beta,\tau,p}(\mu)$ of multivalent analytic functions with negative coefficients, characterized by a generalized multiplier transformation operator $\mathcal{M}^{n}_{\beta,\tau,p}$. Various properties of this subclass are investigated, including coefficient bounds, extreme points of extremal functions, distortion bounds, radii of convexity and starlikeness, and sharp lower bounds for partial sums. Additionally, graphical representations of extremal functions are provided to visually illustrate the theoretical results. These findings contribute to the ongoing development of geometric function theory.

Keywords Multivalent functions, negative coefficient multivalent functions, convolution, derivative operators.

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1. Introduction

In complex analysis, the terms univalent and multivalent refer to functions that are investigated in the complex plane [41–43]. These functions are especially appealing due to their distinct geometric characteristics and their significance in complex analysis and conformal mapping. Univalent and multivalent functions are important for a variety of reasons, serving as a foundational idea in many fields of mathematics, including differential equation analysis, complex analysis, and geometric function theory. Investigating univalent and multivalent functions provide important insights into the behavior of analytic functions in geometric situations.

Consider that an open unit disc $\mathbb{U} : |\xi| < 1$ which is, connected and open nonempty subset

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of $\mathbb{C}$ and the analytical function $R(\xi)$, which is provided by the series expansion

$$R(\xi) = \sum_{k=0}^{\infty} b_n \xi^n = b_0 + b_1\xi + b_2\xi^2 + ... \tag{1.1}$$

The sequence $\{b_n\}$ of coefficients in equation (1.1) acts as a crucial part of the function f, which transforms a Riemann surfaces universal cover $\mathbb{U}$ into its corresponding co-domain $\mathbb{S}$. The statement that a geometric property of $\mathbb{S}$ is characterized by the univalent function f in $\mathbb{U}$.

A function $f : \mathbb{D} \to \mathbb{C}$ is called univalent in $\mathbb{U}$ if it is one-to-one (injective). That is, f does not map two different points to the same value. Mathematically, this means:

$$f(\xi_1) \neq f(\xi_2) \quad \text{for all} \quad \xi_1, \xi_2 \in \mathbb{U}, \quad \text{such that} \quad \xi_1 \neq \xi_2.$$

This property ensures that f does not fold or overlap different parts of $\mathbb{D}$ onto the same region in $\mathbb{C}$. Instead, f preserves distinctness, which is crucial in conformal mappings. To simplify the study of analytic univalent functions on any simply connected domain, it is common to restrict attention to functions defined in the open unit disk $\mathbb{U}$. For this purpose, the class of functions $\mathcal{S}$ was introduced, consisting of univalent functions that satisfy the normalization conditions: $f(0) = 0,\ f'(0) - 1 = 0$.

The Riemann mapping theorem has had a significant impact on the development of this concept.

In the open unit disk $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$, $\mathcal{A}$ often denotes the class of normalized and analytic functions $f(\xi)$ of the form

$$f(\xi) = \xi + \sum_{k=2}^{\infty} a_k \xi^k \tag{1.2}$$

and $\mathcal{S}$ act as a subclass of all functions in $\mathcal{A}$.

The multivalent function follows logically from the idea of a univalent function. If the number of roots of the equation $R(\xi) = w$ in $\mathbb{U}$, for any w, does not exceed p, then it can accept each of its values at most p times. Understanding the characteristics and behavior of multivalent functions in greater detail and refinement requires an understanding of their subclasses. The multivalent function class has subclasses that are defined by additional requirements or standards. Let $\mathcal{A}_p$ described as a class of multivalent functions of the form

$$f(\xi) = \xi^p + \sum_{k=p+1}^{\infty} a_k \xi^k, \tag{1.3}$$

which is lies in $\mathbb{U} = \{\xi : |\xi| < 1\}$.

Univalent functions are generally thought to have originated in 1907 with the work of Koebe [24]. The Koebe coefficient, k, was established in 1907 by Koebe. This coefficient ensures that the boundary of the image of $\mathbb{U}$, under any function $R(\xi) \in \mathbb{S}$, is always at least k away from $w = 0$.

The study of analytic univalent functions was greatly influenced by the famous Bieberbach [9] hypothesis from 1916. According to this conjecture, for each function $f \in \mathbb{S}$, we have $|a_n| \leq n$ for each n. Given that f is normalized by $f(0) = 0$ and $f'(0) = 1$, the second coefficient $|a_2| \leq 2$ if and only if the Koebe function's rotation is $R(\xi)$. In addition, he made the generally true hypothesis that $|a_n| \leq n, (n = 2, 3, ...)$. For many years, this conjecture has remained challenge for Mathematicians. This condition $|a_n| \leq n, (n = 2, 3, ...)$ was positively settled by Louis de Branges [11] in 1984.

Biernacki [10] and Montel [28] conducted a comprehensive analysis of univalent and multivalent functions in 1938. However, the field has since grown significantly, making it difficult for researchers to evaluate the present state of specific topics. However, recently, this projected direction of events has erratically reversed. Books by Jenkins [21], Schaeffer, and Spencer [35], among others, go into extensive detail on particular aspects of the subject. The works of Goluzin [17] and Hayman [19] provide an extensive overview, and they provide sufficient unsolved problems. However, the idea of univalent as well as multivalent functions remains fascinated. The studies by Bernardi [7] and Hayman [20] provide further guidance in the domain.

In the study of complex functions, various types of operators have been employed from the beginning. Using operators can facilitate the creation of fresh, original research while also making it possible to provide easier proofs of conclusions that are previously known. Operators that are most well-known are differential and integral operators. In 1915, Alexander [2] presented the first integral operator. Libera [25] introduced an integral operator that is widely recognised, and Bernardi [8] generalised it in 1969. This paper's original results were obtained using some of the most well-known differential operators. It has been quite fashionable lately to use the integral and differential operators of normalised analytic functions. A large number of articles discuss operators and new generalisations proposed by different writers. To the best of our knowledge, Salagean [33] introduced the differential operator in 1985 after Ruscheweyh [32] introduced it in 1975. The two above operators were employed for a very long period to look into different subclasses of univalent function characteristics and problems.Following the expansion of the Salagean operator by Al-Oboudi [30] in 2004, Shaqsi and Darus in [37,38] redefined the Salagean and Ruscheweyh operations in 2008.

In their own unique ways, a number of writers started to create new subclasses for univalent and multivalent functions associated with Ruscheweyh and Salagean operators. For example, see ([3,4,12,14,18,22,23,29,31,34,36,37,40] and [44]).

For $f(\xi) \in \mathcal{A}_p$,the following multiplier transformation operator was examined by R. Aghalary et al. [1].

$$\mathcal{I}_p(n,\beta) = \xi^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\beta}{p+\beta}\right)^n a_k \xi^k, \quad (\beta \geq 0). \tag{1.4}$$

For $f(\xi) \in \mathcal{A}_p$, the operator $\mathcal{M}^n_{\beta,\tau,p}(R(\xi)) = \mathcal{I}_p(n,\beta) * f(\xi)$ is defined as

$$\begin{aligned}
\mathcal{M}^0_{\beta,\tau,p} &= \xi^p + \sum_{k=p+1}^{\infty} a_k \xi^k, \\
\mathcal{M}^1_{\beta,\tau,p} &= (1-\tau)\,\mathcal{I}_p(1,\beta) + \frac{\tau\xi}{p}\left(\mathcal{I}_p(1,\beta)\right)' = \xi^p + \sum_{k=p+1}^{\infty} \left[\frac{p+(k-p)\,\tau}{p}\right]\left[\frac{k+\beta}{p+\beta}\right] a_k \xi^k, \\
\mathcal{M}^2_{\beta,\tau,p} &= \mathcal{M}_{\beta,\tau,p}\left(\mathcal{M}^1_{\beta,\tau,p}\right).
\end{aligned}$$

Similarly,

$$\mathcal{M}^n_{\beta,\tau,p} = \mathcal{M}_{\beta,\tau,p}\left(\mathcal{M}^{n-1}_{\beta,\tau,p}\right) = \xi^p + \sum_{k=p+1}^{\infty} \left[\frac{p+(k-p)\,\tau}{p}\right]^n \left[\frac{k+\beta}{p+\beta}\right]^n a_k \xi^k, (\beta, \tau \geq 0, n \in N_0). \tag{1.5}$$

We further let $\mathcal{SM}^n_{\beta,\tau,p} = \mathcal{M}^n_{\beta,\tau,p} \cap \mathcal{T}$, where $\mathcal{T} = f \in \mathcal{A} : f(\xi) = \xi - \sum_{k=p+1}^{\infty} a_k \xi^k$, which is lies in $\mathbb{U} = \{\xi : |\xi| < 1\}$ is asubclass of $\mathcal{A}$ introduced and studied by Silverman [39].

$$\mathcal{SM}^n_{\beta,\tau,p} = \xi^p - \sum_{k=p+1}^{\infty} \left[\frac{p+(k-p)\,\tau}{p}\right]^n \left[\frac{k+\beta}{p+\beta}\right]^n a_k \xi^k, (\beta, \tau \geq 0, n \in N_0). \tag{1.6}$$

Remark 1.1. The multiplier transformation operator $\mathcal{M}^n_{\beta,\tau,p}$ is obtained for positive coefficient of multivalent function class $\mathcal{S}^n_{b,\lambda,\delta,p}(\phi(\zeta))$; Byeon et al. [13] explored this transformation (1.6).

The operator $\mathcal{I}^n_\beta$ was explored by Cho and Srivastava [15] for $\tau = 0$, and $p = 1$ in (1.6).

According to Uralegaddi et al. [45], the differential operator for $\tau = 0, p = 1$, and $\beta = 1$ in (1.6) is $\mathcal{I}^n$.

The operator $\mathcal{D}^n$, which was stated by Salagean [33], is found for $\beta = 0, \tau = 0$, and $p = 1$ in (1.6).

I^{-n}, which was suggested by Flett [16], is obtained for $\beta = 0, \tau = 0, p = 1, n = -n$ in (1.6).

Definition 1.1. Let $\mathcal{T}^n_{b,\beta,\tau,p}(\mu)$ represents the subclass of $\mathcal{A}_p$, which consists of the functions $f(\xi)$ satisfying the condition

$$Re\left(1 + \frac{1}{b}\left(\frac{\frac{\xi}{p}(\mathcal{SM}^n_{\beta,\tau,p})'}{\mathcal{SM}^n_{\beta,\tau,p}} - 1\right)\right) > \mu, \tag{1.7}$$

where $n \in N_0, 0 \leq \mu < 1, \forall \xi \in \mathbb{U}$ and $\beta, \tau \geq 0, b \in C - \{0\}$.

The motivation behind this study stems from the ongoing interest in characterizing multivalent analytic functions with negative coefficients, particularly those governed by transformation operators that generalize classical forms. While various subclasses have been introduced through differential and integral operators, the incorporation of the generalized multiplier transformation operator $\mathcal{M}^n_{\beta,\tau,p}$ provides a novel analytical pathway. This operator introduces additional flexibility through its parameters, allowing for a more refined control over function behavior. The novelty of this work lies in the definition and systematic investigation of the subclass $\mathcal{SM}^n_{b,\beta,\tau,p}(\mu)$, which has not been previously studied in the literature. By deriving sharp coefficient estimates, distortion theorems, geometric radii, and partial sum bounds, this research fills a gap in geometric function theory and establishes a foundation for further analytical and graphical explorations in complex analysis.

2. Estimation of coefficients

The concept of multivalent functions is crucial for complex analysis, which is usually defined on a complex plane. It has certain characteristics that make it intriguing for a variety of reasons. In this case, estimating the coefficients of such functions is traditional, or more precisely, their inequalities. This can be crucial when working with functions such as the Riemann surface. It is also useful in deciphering how branch points behave.

Theorem 2.1. *Let $f(\xi) \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, then*

$$\sum_{k=p+1}^{\infty} \left|\left(\frac{(1-\mu)pb + (k-p)}{p}\right)\right| C_k \,|a_k| \leq (1-\mu)\,|b|\,, \text{where } C_k = \left[\frac{p+(k-p)\,\tau}{p}\right]^n \left[\frac{k+\beta}{p+\beta}\right]^n. \tag{2.1}$$

Proof. Let

$$R(\xi)=1+\frac{1}{b}\left(\frac{\frac{\xi}{p}(\mathcal{SM}^n_{\beta,\tau,p})'}{\mathcal{SM}^n_{\beta,\tau,p}}-1\right)-\mu$$
$$=1+\frac{\frac{\xi}{p}(\mathcal{SM}^n_{\beta,\tau,p})'-(1+\mu b)\mathcal{SM}^n_{\beta,\tau,p}}{b\mathcal{SM}^n_{\beta,\tau,p}}.$$

By the well known condition of the class,

$$R(\xi)\prec\frac{1+\xi}{1-\xi}. \tag{2.2}$$

There exists a Schwarz function $w(\xi)$ satisfying $w(0)=0$. Specifically, $|w(\xi)|<1$, ensuring that

$$R(\xi)=\frac{1+w(\xi)}{1-w(\xi)}.$$

This states that

$$w(\xi)=\frac{R(\xi)-1}{R(\xi)+1}.$$

Then

$$|w(\xi)|=\left|\frac{R(\xi)-1}{R(\xi)+1}\right|<1. \tag{2.3}$$

Substitute (2.2) in (2.3), we get

$$\begin{aligned}
\left|\frac{R(\xi)-1}{R(\xi)+1}\right| &= \left|\frac{\frac{\xi}{p}(\mathcal{SM}^n_{\beta,\tau,p})'-(1+\mu b)\mathcal{SM}^n_{\beta,\tau,p}}{\frac{\xi}{p}(\mathcal{SM}^n_{\beta,\tau,p})'-(1+\mu b-2b)\mathcal{SM}^n_{\beta,\tau,p}}\right|\\
&= \left|\frac{\xi^p-\sum_{k=p+1}^{\infty}\frac{k}{p}C_k a_k\xi^k-(1+\mu b)\xi^p+\sum_{k=p+1}^{\infty}(1+\mu b)C_k a_k\xi^k}{\xi^p-\sum_{k=p+1}^{\infty}\frac{k}{p}C_k a_k\xi^k-(1+\mu b-2b)\xi^p+\sum_{k=p+1}^{\infty}(1+\mu b-2b)C_k a_k\xi^k}\right|\\
&= \left|\frac{-\mu b+\sum_{k=p+1}^{\infty}(1+\mu b-\frac{k}{p})C_k a_k\xi^{k-p}}{(2-\mu)b+\sum_{k=p+1}^{\infty}(1+\mu b-2b-\frac{k}{p})C_k a_k\xi^{k-p}}\right|\\
&\le \frac{\mu\left|b\right|+\sum_{k=p+1}^{\infty}\left|(1+\mu b-\frac{k}{p})\right|C_k\left|a_k\right|\left|\xi^{k-p}\right|}{(2-\mu)\left|b\right|-\sum_{k=p+1}^{\infty}\left|(1+\mu b-2b-\frac{k}{p})\right|C_k\left|a_k\right|\left|\xi^{k-p}\right|}.
\end{aligned}$$

The expression is bounded by 1, provided that

$$\mu\left|b\right|+\sum_{k=p+1}^{\infty}\left|(1+\mu b-\frac{k}{p})\right|C_k\left|a_k\right|\le(2-\mu)\left|b\right|-\sum_{k=p+1}^{\infty}\left|(1+\mu b-2b-\frac{k}{p})\right|C_k\left|a_k\right|.$$

Then,

$$\sum_{k=p+1}^{\infty}\left|\left(\frac{(1-\mu)pb+(k-p)}{p}\right)\right|C_k\left|a_k\right|\le(1-\mu)\left|b\right|.$$

Hence, equation (2.1) holds. □

Corollary 2.1. *Let* $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, *then*

$$|a_k| \leq \frac{(1-\mu)\,|b|}{\left|\left(\frac{(1-\mu)pb+(k-p)}{p}\right)\right| C_k},$$

and the equality is estimated for the function $f(\xi)$, *which is given by*

$$f(\zeta) = \zeta^p - \frac{(1-\mu)b}{\left(\frac{(1-\mu)bp+(k-p)}{p}\right)\left(\frac{p+(k-p)\tau}{p}\right)^n\left(\frac{\beta+k}{p+\beta}\right)^n} \cdot \zeta^k, \;\; k \geq p+1. \tag{2.4}$$

3. Extreme points

The extreme points of the class $\mathcal{T}^n_{b,\beta,\tau,p}(\mu)$ are determined.

Theorem 3.1. *Let*

$$f_p(\xi) = \xi^p, f_k(\xi) = \xi^p - \frac{(1-\mu)\,|b|}{C_\beta}\xi^k, k = p+1, p+2, ...,$$

where

$$C_\beta = \sum_{k=p+1}^{\infty} \left|\left(\frac{(1-\mu)pb+(k-p)}{p}\right)\right| C_k.$$

Then, $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$ *only when it is in the form*

$$f(\xi) = \rho_p f_p(\xi) + \sum_{k=p+1}^{\infty} \rho_k f_k(\xi),$$

where $\rho_k \geq 0$ *and* $\rho_p = 1 - \sum_{k=p+1}^{\infty} \rho_k$.

Proof. Assume that

$$f(\xi) = \rho_p f_p(\xi) - \sum_{k=p+1}^{\infty} \rho_k f_k(\xi).$$

Then

$$\begin{aligned} f(\xi) &= \left(1 - \sum_{k=p+1}^{\infty} \rho_k\right)\xi^p + \sum_{k=p+1}^{\infty} \rho_k \left(\xi^p - \frac{(1-\mu)\,|b|}{C_\beta}\xi^k\right) \\ &= \xi^p - \sum_{k=p+1}^{\infty} \rho_k \frac{(1-\mu)\,|b|}{C_\beta}\xi^k \\ &= \sum_{k=p+1}^{\infty} \rho_k \frac{(1-\mu)\,|b|}{C_\beta} C_\beta \\ &= (1-\mu)\,|b| \sum_{k=p+1}^{\infty} \rho_k \\ &= (1-\mu)\,|b|\,(1-a_p) \\ &< (1-\mu)\,|b|\,, \end{aligned}$$

which implies that

$$f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu).$$

Conversely, consider this

$$f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu),$$

while

$$|a_k| \leq \frac{(1-\mu)\,|b|}{C_\beta}, k = p+1, p+2, ...$$

Let

$$\rho_k \leq \frac{C_\beta}{(1-\mu)\,|b|} a_k, \rho_p = 1 - \sum_{k=p+1}^{\infty} \rho_k.$$

Then

$$\begin{aligned} f(\xi) &= \xi^p - \sum_{k=p+1}^{\infty} a_k \xi^k, \\ f(\xi) &= (\rho_p + \sum_{k=p+1}^{\infty} \rho_k)\xi^p - \sum_{k=p+1}^{\infty} \rho_k \frac{(1-\mu)\,|b|}{C_\beta}\xi^k \\ &= \rho_p f_p(\xi) + \sum_{k=p+1}^{\infty} \rho_k \{\xi^k - \frac{(1-\mu)\,|b|}{C_\beta}\xi^k\} \\ &= \rho_p f_p(\xi) + \sum_{k=p+1}^{\infty} \rho_k f_k(\xi). \end{aligned}$$

Hence the result is sharp. □

Corollary 3.1. *The extremal points of $\mathcal{T}^n_{b,\beta,\tau,p}(\mu)$ correspond to the functions $f_p(\xi) = \xi^p$ and*

$$f(\xi) = \xi^p - \frac{(1-\mu)|b|}{\left|\frac{(1-\mu)pb+(k-p)}{p}\right| C_k}\xi^k, \quad k \geq p+1.$$

4. Growth and distortion theorems

Both growth and distortion theorems are fundamental in analyzing, describing, and connecting the behavior of multivalent functions to the geometric properties of the complex plane. These theorems provide valuable insights for mathematicians and researchers exploring complex analytic functions, particularly in understanding their effects on the size and shape of curves and regions within the complex plane. In this section, the growth and distortion bounds for $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$ has been obtained.

Theorem 4.1. *If $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$,then*

$$\rho^p - \frac{(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right| \left[\frac{p+\tau}{p}\right]^n \left[\frac{p+1+\beta}{p+\beta}\right]^n} \rho^{p+1} \leq |f(\xi)| \leq \rho^p + \frac{(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right| \left[\frac{p+\tau}{p}\right]^n \left[\frac{p+1+\beta}{p+\beta}\right]^n} \rho^{p+1} \tag{4.1}$$

$|z| = \rho < 1$, *provided* $k \geq p+1$. *The result becomes sharp for*

$$R(\xi) = \xi^p - \frac{(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right| \left[\frac{p+\tau}{p}\right]^n \left[\frac{p+1+\beta}{p+\beta}\right]^n}\xi^{p+1}.$$

Proof. By exploiting inequalities (2.1) for $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$ together with

$$\begin{aligned} &\left|\frac{1+\mu bp-bp}{p}\right| \left[\frac{p+\tau}{p}\right]^n \left[\frac{p+1+\beta}{p+\beta}\right]^n \\ &\leq \left|\left(\frac{(1-\mu)pb+(k-p)}{p}\right)\right| \left(\frac{p(\beta+k)+(k-p)(\beta+k)\tau}{p(p+\beta)}\right)^n, \end{aligned}$$

equation (4.2) is obtained

$$\begin{aligned} &\left|\frac{1+\mu bp-bp}{p}\right| \left[\frac{p+\tau}{p}\right]^n \left[\frac{p+1+\beta}{p+\beta}\right]^n \sum_{k=p+1}^{\infty} a_k \\ &\leq \left|\left(\frac{(1-\mu)pb+(k-p)}{p}\right)\right| \left(\frac{p(\beta+k)+(k-p)(\beta+k)\tau}{p(p+\beta)}\right)^n \sum_{k=p+1}^{\infty} a_k \\ &\leq (1-\mu)\,|b|\,, \\ &\sum_{k=p+1}^{\infty} a_k \leq \frac{(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right| \left[\frac{p+\tau}{p}\right]^n \left[\frac{p+1+\beta}{p+\beta}\right]^n}. \end{aligned} \tag{4.2}$$

By employing (4.2) for the function $R(\xi) = \xi^p - \sum_{k=p+1}^{\infty} a_k \xi^k \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, with $|\xi| = \rho$, implies

$$\begin{aligned} |R(\xi)| &= \rho^p + \sum_{k=p+1}^{\infty} a_k \rho^k \\ &\leq \rho^p + \rho^{p+1} \sum_{k=p+1}^{\infty} a_k \\ &\leq \rho^p + \frac{(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right| \left[\frac{p+\tau}{p}\right]^n \left[\frac{p+1+\beta}{p+\beta}\right]^n}\rho^{p+1}, \end{aligned}$$

and similarly,

$$|R(\xi)| \geq \rho^p - \frac{(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right| \left[\frac{p+\tau}{p}\right]^n \left[\frac{p+1+\beta}{p+\beta}\right]^n}\rho^{p+1},$$

which proves assertion (4.1) of theorem (4.1).

Hence, the bound in theorem (4.1) is attained for the extremal function given by

$$f(\xi) = \xi^p - \frac{(1-\mu)\,|b|}{\left|\left(\frac{(1-\mu)pb+(k-p)}{p}\right)\right| C_k}\xi^k, \quad k \geq p+1.$$

□

Theorem 4.2. *If* $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$,*then*

$$p\rho^{p-1} - \frac{(p+1)(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right|\left[\frac{p+\tau}{p}\right]^n\left[\frac{p+1+\beta}{p+\beta}\right]^n}\rho^p \le \left|f'(\xi)\right| \le p\rho^{p-1} + \frac{(p+1)(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right|\left[\frac{p+\tau}{p}\right]^n\left[\frac{p+1+\beta}{p+\beta}\right]^n}\rho^p, \quad (4.3)$$

$|z| = \rho < 1$, *provided* $k \ge p+1$. *It is obvious that the final result becomes sharp for*

$$R(\xi) = \xi^p - \frac{(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right|\left[\frac{p+\tau}{p}\right]^n\left[\frac{p+1+\beta}{p+\beta}\right]^n}\xi^{p+1}.$$

Proof. From employing an inequality (2.1) for $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, the following relation has been obtained

$$\sum_{k=p+1}^{\infty} a_k \le \frac{(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right|\left[\frac{p+\tau}{p}\right]^n\left[\frac{p+1+\beta}{p+\beta}\right]^n}.$$

Equation (4.2), gives

$$\sum_{k=p+1}^{\infty} k a_k \le \frac{(p+1)(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right|\left[\frac{p+\tau}{p}\right]^n\left[\frac{p+1+\beta}{p+\beta}\right]^n}.$$

For the function $R(\xi) = \xi^p - \sum_{k=p+1}^{\infty} a_k \xi^k \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, then

$$\begin{aligned}
|f'(\xi)| &= p\rho^{p-1} + \sum_{k=p+1}^{\infty} k a_k \rho^{k-1} \\
&\le p\rho^{p-1} + \rho^p \sum_{k=p+1}^{\infty} k a_k \\
&\le p\rho^{p-1} + \frac{(p+1)(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right|\left[\frac{p+\tau}{p}\right]^n\left[\frac{p+1+\beta}{p+\beta}\right]^n}\rho^p,
\end{aligned}$$

and similarly,

$$|f'(\xi)| \ge p\rho^{p-1} - \frac{(p+1)(1-\mu)\,|b|}{\left|\frac{1+\mu bp-bp}{p}\right|\left[\frac{p+\tau}{p}\right]^n\left[\frac{p+1+\beta}{p+\beta}\right]^n}\rho^p.$$

which proves that the assertion (4.3) of theorem (4.2).

Thus, for the extremal function provided by, the bound in theorem (4.2) is reached

$$f(\xi) = \xi^p - \frac{(1-\mu)\,|b|}{\left|\left(\frac{(1-\mu)pb+(k-p)}{p}\right)\right| C_k}\xi^k, \quad k \ge p+1.$$

□

5. Convexity and starlikeness

A function is considered convex if it maps the unit disk onto a convex domain [5, 6, 26, 27]. A function is considered starlike when all line segments traced from the origin to any point in the image remain completely inside the image. This property is crucial for a number of complex analysis and conformal mapping applications. Convexity is important for multivalent functions because it guarantees some geometric aspects of the image domain that can be used to generalize the behavior of these functions. In terms of growth, distortion, and covering theorems, these stronger conditions frequently produce higher estimates and crisper results.

Theorem 5.1. *Let $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, then the class consists of convex functions.*

Proof. Assume that

$$f_j(\xi) = \xi^p - \sum_{k=p+1}^{\infty} a_{k,j}\xi^k, a_{k,j} \geq 0, j = 1, 2,$$

with $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$.

It must be demonstrated that

$$h(\xi) = (\beta + 1)f_1(\xi) - \beta f_2(\xi), 0 \leq \beta \leq 1,$$

while

$$h(\xi) = \xi^p - \sum_{k=p+1}^{\infty} \left[(1 + \beta)a_{k,1} - \beta a_{k,2}\right] \xi^k,$$

this leads to

$$\begin{aligned}&\sum_{k=p+1}^{\infty} \left|\left(\frac{(1-\mu)pb + (k-p)}{p}\right)\right| C_k(1+\beta)a_{k,1} \\ &- \left|\left(\frac{(1-\mu)pb + (k-p)}{p}\right)\right| C_k\beta a_{k,2} \\ &\leq (1+\beta)(1-\mu)\left|b\right| - \beta(1-\mu)\left|b\right| \\ &\leq (1-\mu)\left|b\right|.\end{aligned}$$

Thus

$$h \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu).$$

Hence $\mathcal{T}^n_{b,\beta,\tau,p}(\mu)$ is called convex. □

Theorem 5.2. *If $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, then f is p-valently convex of order γ in the disk. $|\xi| < \rho_2$, where*

$$\rho_2 := inf\left(\frac{p(p-\gamma)\left|\left(\frac{(1-\mu)pb+(k-p)}{p}\right)\right| C_k}{k(k-\gamma)(1-\mu)\left|b\right|}\right)^{\frac{1}{k-p}}, (k \geq p+1).$$

For every k, the maximum value for $|z|$ is sharp, and the extreme function is of the form (2.4).

Proof. Assuming f is an orderly convex of γ and that $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, proof of this is needed.

$$\left|1 + \frac{\xi f''(\xi)}{f'(\xi)} - p\right| < p - \gamma \quad |\xi| < \rho_2. \tag{5.1}$$

Now, the equation (5.1) gives

$$\left|1 + \frac{\xi f''(\xi)}{f'(\xi)} - p\right| = \left|\frac{f'(\xi) + \xi f''(\xi) - pf'(\xi)}{f'(\xi)}\right| \leq \frac{\sum_{k=p+1}^{\infty} k(k-p)a_k\left|\xi\right|^{k-p}}{p - \sum_{k=p+1}^{\infty} ka_k\left|\xi\right|^{k-p}}. \tag{5.2}$$

From (5.1) and (5.2), then

$$\sum_{k=p+1}^{\infty} \frac{k(k-\gamma)}{p(p-\gamma)} a_k \left|\xi\right|^{k-p} \leq 1. \tag{5.3}$$

Hence, (5.1) implies that (5.3) is true if

$$|z| \leq \left(\frac{p(p-\gamma) \left| \left(\frac{(1-\mu)pb+(k-p)}{p} \right) \right| C_k}{k(k-\gamma)(1-\mu)\left|b\right|} \right)^{\frac{1}{k-p}}, (k \geq p+1). \tag{5.4}$$

Setting $|\xi| = \rho_2$ in (5.4), the result follows. The sharpness can be verified. □

Theorem 5.3. *If $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, then the extremal function f said to be p-valently starlike of order γ $(0 \leq \gamma < p)$ in the disc $|\xi| < \rho_3$, where*

$$\rho_3 := inf \left(\frac{(p-\gamma) \left| \left(\frac{(1-\mu)pb+(k-p)}{p} \right) \right| C_k}{(k-\gamma)(1-\mu)\left|b\right|} \right)^{\frac{1}{k-p}}, (k \geq p+1).$$

For every k, the bound for $|z|$ is sharp, and the extreme function is of the type (2.4).

Proof. It is required to demonstrate that f is described as an organized starlike structure of γ if $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$,

$$\left| \frac{\xi f'(\xi)}{R(\xi)} - p \right| < p - \gamma \quad ; |\xi| < \rho_3. \tag{5.5}$$

Now, the LHS of the equation (5.5) derived as

$$\left| \frac{\xi f'(\xi)}{R(\xi)} - p \right| = \left| \frac{\xi f'(\xi) - pR(\xi)}{R(\xi)} \right| \leq \frac{\sum_{k=p+1}^{\infty} (k-p) a_k \left|\xi\right|^{k-p}}{1 - \sum_{k=p+1}^{\infty} a_k \left|\xi\right|^{k-p}}. \tag{5.6}$$

From (5.5) and (5.6), the equation becomes

$$\sum_{k=p+1}^{\infty} \frac{(k-\gamma)}{(p-\gamma)} a_k \left|\xi\right|^{k-p} \leq 1. \tag{5.7}$$

According to (5.5), it follows that (5.7) is true if

$$|z| \leq \left(\frac{(p-\gamma) \left| \left(\frac{(1-\mu)pb+(k-p)}{p} \right) \right| C_k}{(k-\gamma)(1-\mu)\left|b\right|} \right)^{\frac{1}{k-p}}, (k \geq p+1). \tag{5.8}$$

The outcome is as follows when $|\xi| = \rho_3$ in (5.8) is defined. It provides an evidence of the sharpness. □

6. Partial sums

For estimating, examining, and analysing multivalent functions, partial sums are a helpful tool. They shed light on boundary behaviour, convergence, and coefficient estimations, particularly in the vicinity of the unit disc or other domains. These sums aid in the identification of extreme circumstances and the approximation of boundary behaviour, such as the argument's and modulus' expansion. Partial sums aid in demonstrating uniform convergence on certain domains by examining the rate of convergence of series for multivalent functions, guaranteeing that characteristics such as analyticity are maintained in the limit. The connection between the form (1.3) and its sequence of partial sums is examined in this section.

$$f(\xi) = \xi^p \tag{6.1}$$

and

$$f_k(\xi) = \xi^p - \sum_{k=p+1}^{n} a_k \xi^k, k = p+1, p+2, p+3, ..., \tag{6.2}$$

when the coefficients meet the corresponding criteria by being sufficiently small.

$$\sum_{k=p+1}^{\infty} \left| \left(\frac{(1-\mu)pb + (k-p)}{p} \right) \right| C_k \left| a_k \right| \leq (1-\mu) \left| b \right|. \tag{6.3}$$

It can be written as

$$\sum_{k=p+1}^{\infty} \Upsilon_k \left| a_k \right| \leq 1, \tag{6.4}$$

where

$$\Upsilon_k = \frac{\left(\left| \left(\frac{(1-\mu)pb+(k-p)}{p} \right) \right| C_k \right)}{(1-\mu) \left| b \right|}. \tag{6.5}$$

Then $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$.

Theorem 6.1. *If $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, satisfying (6.4), then*

$$Re\left(\frac{f(\xi)}{f_k(\xi)} \right) \geq 1 - \frac{1}{\Upsilon_{n+1}}.$$

Proof. Evidently, $\Upsilon_{k+1} > \Upsilon_k > 1, k = p+1, p+2, p+3, ...,$

By using (1.3),

$$\sum_{k=p+1}^{n} \left| a_k \right| + \Upsilon_{n+1} \sum_{k=n+1}^{\infty} \left| a_k \right| \leq \sum_{k=p+1}^{\infty} \Upsilon_k \left| a_k \right| \leq 1.$$

Let

$$\Omega_1(\xi) = \Upsilon_{n+1} \left[\frac{f(\xi)}{f_k(\xi)} - \left(1 - \frac{1}{\Upsilon_{n+1}} \right) \right].$$

Through straightforward calculation,

$$\left| \frac{\Omega_1(\xi) - 1}{\Omega_1(\xi) + 1} \right| \leq \frac{\Upsilon_{n+1} \sum_{k=n+1}^{\infty} \left| a_k \right|}{2 - 2\sum_{k=p+1}^{n} \left| a_k \right| - \Upsilon_{n+1} \sum_{k=n+1}^{\infty} \left| a_k \right|} \leq 1, \tag{6.6}$$

which gives,

$$Re\left(\frac{f(\xi)}{f_k(\xi)}\right) \geq 1 - \frac{1}{\Upsilon_{n+1}}.$$

□

Theorem 6.2. *If $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$ and satisfies* (6.4)*. Then*

$$Re\left(\frac{f_k(\xi)}{f(\xi)}\right) \geq \frac{\Upsilon_{n+1}}{1+\Upsilon_{n+1}}.$$

Proof. It is obvious that $\Upsilon_{k+1} > \Upsilon_k > 1, k = p+1, p+2, p+3, ...$

Let

$$\Omega_2(\xi) = (1+\Upsilon_{n+1})\left[\frac{f_k(\xi)}{f(\xi)} - \left(\frac{\Upsilon_{n+1}}{1+\Upsilon_{n+1}}\right)\right].$$

By using simple calculations,

$$\left|\frac{\Omega_2(\xi)-1}{\Omega_2(\xi)+1}\right| \leq \frac{(1+\Upsilon_{n+1})\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=p+1}^{n}|a_k| - (1+\Upsilon_{n+1})\sum_{k=n+1}^{\infty}|a_k|} \leq 1. \tag{6.7}$$

Hence, the result

$$Re\left(\frac{f_k(\xi)}{f(\xi)}\right) \geq \frac{\Upsilon_{n+1}}{1+\Upsilon_{n+1}}.$$

□

Theorem 6.3. *If $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, satisfying* (6.4)*,then*

$$Re\left(\frac{f'(\xi)}{f'_k(\xi)}\right) \geq 1 - \frac{n+1}{\Upsilon_{n+1}}.$$

Proof. Certainly, $\Upsilon_{k+1} > \Upsilon_k > 1, k = p+1, p+2, p+3, ...$

Let

$$\Omega_3(\xi) = \Upsilon_{n+1}\left[\frac{f'(\xi)}{f'_k(\xi)} - \left(1 - \frac{n+1}{\Upsilon_{n+1}}\right)\right].$$

By using simple calculation, we obtain

$$\left|\frac{\Omega_3(\xi)-1}{\Omega_3(\xi)+1}\right| \leq \frac{\frac{\Upsilon_{n+1}}{n+1}\sum_{k=n+1}^{\infty}k|a_k|}{2-2\sum_{k=p+1}^{n}k|a_k| - \frac{\Upsilon_{n+1}}{n+1}\sum_{k=n+1}^{\infty}k|a_k|} \leq 1, \tag{6.8}$$

which gives,

$$Re\left(\frac{f'(\xi)}{f'_k(\xi)}\right) \geq 1 - \frac{n+1}{\Upsilon_{n+1}}.$$

□

Theorem 6.4. *If $f \in \mathcal{T}^n_{b,\beta,\tau,p}(\mu)$, satisfying* (6.4)*, then*

$$Re\left(\frac{f'_k(\xi)}{f'(\xi)}\right) \geq \frac{\Upsilon_{n+1}}{n+1+\Upsilon_{n+1}}.$$

Proof. Certainly, $\Upsilon_{k+1} > \Upsilon_k > 1, k = p+1, p+2, p+3, ...$ Let

$$\Omega_4(\xi) = ((n+1) + \Upsilon_k)\left[\frac{f_k'(\xi)}{f'(\xi)} - \left(\frac{\Upsilon_{n+1}}{n+1+\Upsilon_{n+1}}\right)\right].$$

By using simple calculation, is obtained

$$\left|\frac{\Omega_4(\xi)-1}{\Omega_4(\xi)+1}\right| \leq \frac{\left(1+\frac{\Upsilon_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty} k\,|a_k|}{2-2\sum_{k=p+1}^{n} k\,|a_k| - \left(1+\frac{\Upsilon_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty} k\,|a_k|} \leq 1, \tag{6.9}$$

which gives,

$$Re\left(\frac{f_k'(\xi)}{f'(\xi)}\right) \geq \frac{\Upsilon_{n+1}}{n+1+\Upsilon_{n+1}}.$$

□

7. The geometrical projection and discussion

7.1. Geometric interpretation of modulus and phase

Complex functions are mathematical functions that operate on complex numbers, which consist of both a real and an imaginary part. When we input a complex number into such a function, it returns another complex number. These functions are difficult to graph because they involve two input and two output dimensions.

One important application of complex functions is conformal mapping, which preserves angles while possibly changing other properties like distances. This is often illustrated with phase (angle) and absolute value (distance) diagrams. The absolute value of a complex number shows its distance from the origin in the complex plane, while the phase indicates the angle it makes with the positive real axis. In various fields like signal processing, control theory, wave propagation, and fluid dynamics, the magnitude (absolute value) and phase of complex functions are important. The magnitude tells us about signal strength or field strength, and the phase tells us about timing or position in a wave cycle. For multivalent functions, the phase and absolute value help understand geometric properties, such as convexity and branching. These diagrams are useful for visualizing the function's behavior, especially for identifying key characteristics, singularities, and transformations. In the context of the equation (2.4), the phase and absolute value of the function are shown in diagrams (Figures 1-3), which help better understand its behavior when different parameters are applied.

7.2. Discussion results

The behavior of the extremal function

$$f(\zeta) = \zeta^p - \frac{(1-\mu)b}{\left(\frac{(1-\mu)bp+(k-p)}{p}\right)\left(\frac{p+(k-p)\tau}{p}\right)^n\left(\frac{\beta+k}{p+\beta}\right)^n}\cdot\zeta^k, \quad k \geq p+1,$$

is illustrated for three representative parameter sets.

In the first case, depicted in Figure 1, we consider relatively small parameter values: $\mu = 0.1$, $\tau = 1$, $\beta = 1$, with degrees $p = 1$ and $k = 2$. Under this configuration, the function retains a predominantly quadratic profile. The correction term involving ζ^k contributes moderately, and the corresponding denominator factor remains small. As a result, the higher-order correction is perceptible but not dominant, yielding a smooth surface where the underlying monomial structure is only slightly modified.

In the second case, illustrated in Figure 2, the parameters are increased to $\mu = 0.5$, $\tau = 3$, $\beta = 5$, $p = 3$, and $k = 4$, introducing greater complexity. The larger values of τ and β amplify the denominator, thereby diminishing the coefficient of the ζ^k term. Consequently, the correction term exerts a weaker influence, and the function exhibits a more pronounced cubic shape with only mild curvature contributed by the quartic component.

The third scenario, shown in Figure 3, explores a high-degree case with $p = 10$, $k = 11$, $\mu = 0.5$, $\tau = 3$, $\beta = 2$, and $b = 5$. Here, the dominant behavior is governed by the base term ζ^{10}, while the correction term ζ^{11} introduces noticeable variations, particularly near the complex boundary values of ζ. The increased degrees and intricate coefficient interplay lead to pronounced oscillations or ripples in both the real and imaginary components of the function, underscoring the sensitivity of its structure to parameter choices in high-degree settings.

Overall, the evolution of the function's shape with respect to the parameters $\mu, \tau, \beta, k, n, b$, and p confirms the theoretical expectation that the extremal function's complexity and curvature grow as the relative difference between k and p increases, and as the weighting structure becomes more intricate.

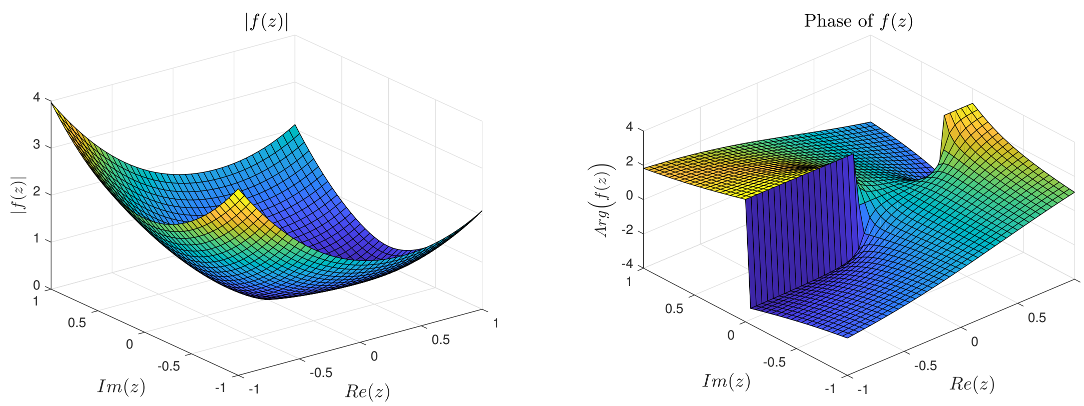

Figure 1. For $\mu = 0.1; \tau = 1; \beta = 1;\ k = 2; n = 1; b = 1; p = 1$.

8. Conclusions

In conclusion, this study introduces the subclass $\mathcal{SM}^n_{b,\beta,\tau,p}(\mu)$ of multivalent analytic functions with negative coefficients, defined via the generalized multiplier transformation operator $\mathcal{SM}^n_{\beta,\tau,p}$. The obtained results, including coefficient bounds, extreme points, distortion bounds, radii of convexity and starlikeness, and sharp lower bounds on partial sums, provide a comprehensive mathematical framework for this subclass. The integration of graphical methods further enhances the understanding of extremal functions by offering visual insights into the theoretical constructs. These findings contribute to the broader field of geometric function theory and open avenues for future exploration, such as extending the analysis to other transformation operators or applying these results to problems in mathematical modeling and complex analysis.

Author contributions. Conceptualization, D.B., M.A., and M.A.Y.; Funding acquisition, D.B. and M.A.; Investigation, M.A.Y. and P.O.M.; Methodology, S.T.; Project administration,

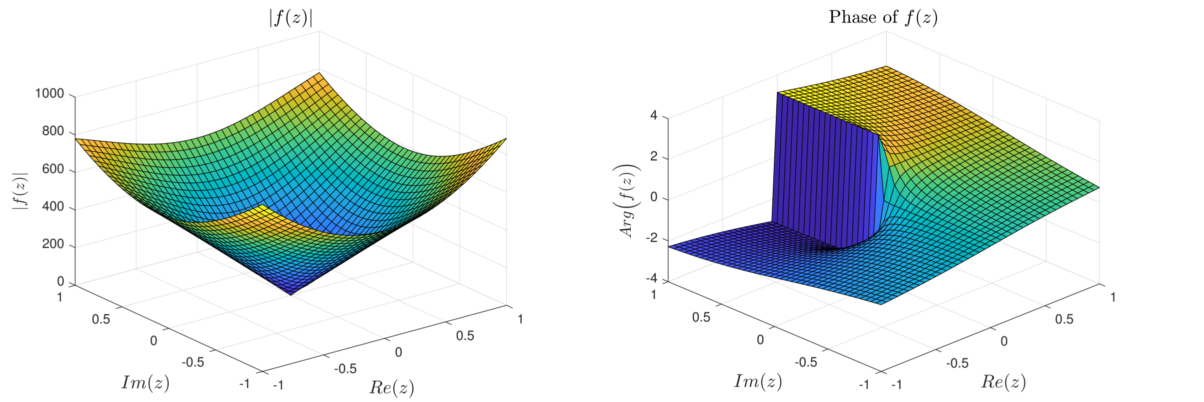

Figure 2. For $\mu = 0.5; \tau = 3; \beta = 5; k = 4; n = 4; b = 2; p = 3$.

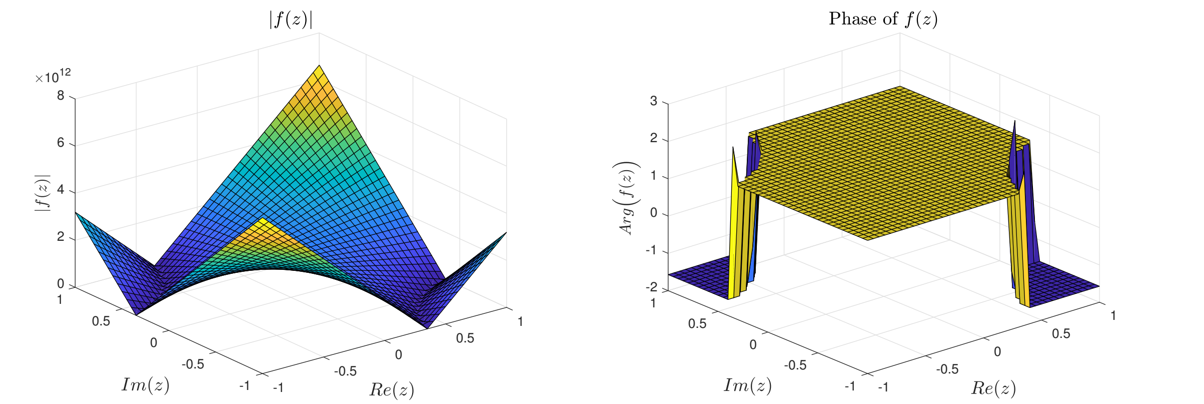

Figure 3. For $\mu = 0.5; \tau = 3; \beta = 2; k = 11; n = 4; b = 5; p = 10$.

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