Journal of Applied Analysis and Computation
Volume 16, Number 6, December 2026, 3232–3243
Website:http://www.jaac-online.com
DOI:10.11948/20250363

EXTENSIONS OF FRACTIONAL INTEGRAL INEQUALITIES VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract The paper introduces new generalizations of well-known inequalities for convex functions using the Katugampola fractional integral, a powerful extension of the Riemann–Liouville fractional integral. Based on basic inequalities, we establish new estimates within the context of Katugampola fractional operators. The results are proven for convex and symmetric convex functions, with proofs that make use of integral representations, kernel properties, and estimations based on convexity. In addition, we have improved estimates for differentiable functions with convex-in-modulus derivatives. These findings offer a unifying and broad framework for fractional integral inequalities, complementing the current literature and paving the way for potential applications in fractional differential equations and functional analysis.

Keywords Fractional integrals, Hermite-Hadamard inequality, convex functions.

MSC(2010) 26A24, 26A33, 26A51, 26B15, 33E12.

1. Introduction

Fractional calculus has profoundly impacted nearly every discipline of modern engineering and science. In recent years, numerous researchers have contributed novel findings to this rapidly evolving field. The applications of fractional calculus are now vast and encompass areas such as viscoelasticity, control theory, signal processing, and complex systems modeling. At the same time, convex analysis and optimization have emerged as essential tools across a wide range of disciplines, including control systems, estimation theory, and signal processing. The role of convex functions is particularly critical in formulating and proving many foundational inequalities, especially in the domain of mathematical optimization. The theory of inequalities is deeply influenced by the properties of convex functions, which provide structure and tractability to otherwise complex mathematical problems. Notably, the concept of convex functions was introduced over a century ago by Johan Jensen, whose celebrated inequality remains a cornerstone

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in convex analysis [3, 5, 9, 11].

A real-valued function ξ with domain $A \subseteq R^n$ being a convex set, such that

$$\xi(wp + (1-w)q) \leq w\xi(p) + (1-w)\xi(q) \tag{1.1}$$

is called a convex function where $p, q \in A$ and $0 \leq \omega \leq 1$. Convex functions are a crucial tool in a broad spectrum of pure and applied mathematical issues. Convex functions are often used to establish and generalize results, methods, and algorithms. Specifically, convex functions are at the core of the mathematical inequalities theory, where they give the structural basis of generalizations of classical outcomes [1]. Conversely, fractional calculus is an extension of standard calculus emphasizing the differentiation and integration of non-integer (fractional) orders. Historically, the differentiation and integration are considered to be discrete processes—executed once, twice, or any integer number of times. However, fractional calculus provides a continuum of orders, significantly expanding the analytical scheme [10]. The present contribution uses some well-known classes of convex functions in relation with classical definitions of fractional integrals, specially, Riemann–Liouville fractional integral (RLFI), which is a basic operator in the theory of fractional integration [8]. The RLFI is initial expression for a non-integral order integral operator. We define RLFI of order $\alpha > 0$ with $v \geq 0$ in the following

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a \tag{1.2}$$

and

$$I^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, x < b. \tag{1.3}$$

Another generalization of RLFI is Katugampola fractional integral, which is defined as follows:

Consider the finite interval $[a, b]$, then for real value function f, the Katugampola fractional integrals of order $\alpha > 0$ are defined by

$$\rho I^{\alpha}_{a+} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x t^{\rho-1}(x^{\rho} - t^{\rho})^{\alpha-1} f(t)dt \tag{1.4}$$

and

$$\rho I^{\alpha}_{b-} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_b^b t^{\rho-1}(t^{\rho} - x^{\rho})^{\alpha-1} f(t)dt. \tag{1.5}$$

Well known inequalities were obtained by Chen and Katugampola in [2], for example convex functions satisfy the following Hermite-Hadamard inequality:

$$f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \leq \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})^{\alpha}} \left(\rho I^{\alpha}_{a+} f(b^{\rho}) + \rho I^{\alpha}_{b-} f(a^{\rho})\right) \leq \frac{f(a^{\rho}) + f(b^{\rho})}{2}.$$

Also, a similar inequality was proved in [2, Theorem 3.1]. They also proved Hermite-Hadamard-Fejér inequality in [2, Theorem 3.8]. Similar inequalities for Katugampola fractional integrals were studied in [6]. Convexity has emerged as a key principle in contemporary mathematical analysis, especially because of its usefulness for developing inequalities and minimizing functions under disciplined conditions. Generalization of convex functions like $(h, g; m)$-convex, s-convex, and (α, m)-convex classes has helped to establish sharper inequalities, generalizing classical ones such as the Hermite–Hadamard, Ostrowski, and Grüss inequalities. Recent studies of Fahad et al. [3] and Özcan et al. [9] demonstrate the strong use of such generalized convexity in fractional contexts, where both operator bounds and numerical approximations are enhanced. Such advances are imperative in fields where stability, boundedness, and convergence of integral operators are key concerns, particularly when tied with fractional differential models.

Meanwhile, the development of fractional calculus remains to enrich theoretical and applied mathematics. Among the impressive generalizations of traditional fractional operators is the Katugampola fractional integral, which encompasses and generalizes various integral operators, such as Riemann–Liouville and Hadamard types. Its versatile form, controlled by the order parameter α and the scale index ρ, enables it to describe systems with variable scaling and non-local memory effects. Researchers such as Jarad et al. [7] and Chen and Katugampola [2] have illustrated the utility of Katugampola integrals for improving inequality structures and enriching analytic techniques in functional analysis. Inspired by these developments, our manuscript targets the extension of RLFI-type inequalities by means of Katugampola-type operators and convexity conditions, thus enriching the current body of literature on fractional integral inequalities with generalized convex frameworks.

2. Main results

This section introduces the main contributions of the paper, which involve new inequalities obtained through Katugampola fractional integrals for convex functions and their counterparts. These results generalize classical inequalities such as Ostrowski, Hadamard, and Grüss to fractional orders, presenting bounds that can be used to analyze fractional differential equations and optimization problems.

First theorem gives a basic inequality for the total of normalized left as well as right Katugampola integral operators of a positive convex function. It makes use of the convexity condition in order to obtain upper bounds in terms of simple algebraic expressions, giving a generalization of Riemann-Liouville-type estimates within a more flexible framework determined by parameters ρ, α, and β.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a positive convex function. Then, for $a, b \in I$, with $a < b$ and $\rho > 0, \alpha, \beta \geq 1$, we have the inequality for Katugampola fractional integrals as follows:*

$$\begin{aligned}&\frac{\Gamma(\alpha)\, {}^{\rho}I^{\alpha}_{a+} f(x)}{\rho^{1-\alpha}(x^\rho - a^\rho)^{\alpha-2}} + \frac{\Gamma(\beta)\rho I^{\beta}_{b-} f(x)}{\rho^{1-\beta}(b^\rho - x^\rho)^{\beta-2}} \\ \leq & \frac{(x^\rho - a^\rho)(x^\rho f(a^\rho) - a^\rho f(x^\rho)) + (b^\rho - x^\rho)\big(b^\rho f(x^\rho) - x^\rho f(b^\rho)\big)}{\rho} \\ &+ \frac{(x^{\rho+1} - a^{\rho+1})(f(x^\rho) - f(a^\rho)) + (b^{\rho+1} - x^{\rho+1})\big(f(b^\rho) - f(x^\rho)\big)}{\rho+1}.\end{aligned} \tag{2.1}$$

Proof. Since f is convex on $[a, b]$, if we consider f on the interval $[a, x]$, $x \in (a, b]$, then, for $t \in [a, x]$ and $\rho > 0$, $\alpha \geq 1$, the following two inequalities hold:

$$t^{\rho-1}(x^\rho - t^\rho)^{\alpha-1} \leq (x^\rho - a^\rho)^{\alpha-1} t^{\rho-1}, \tag{2.2}$$

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} f(t) \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{x^\rho - t}{x^\rho - a^\rho} f(a^\rho) + \frac{t - a^\rho}{x^\rho - a^\rho} f(x^\rho) \right). \tag{2.3}$$

Multiplying inequalities (2.2) and (2.3), and integrating on $[a, x]$, one has:

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x t^{\rho-1}(x^\rho - t^\rho)^{\alpha-1} f(t)\, dt \tag{2.4}$$

$$\leq \frac{\rho^{1-\alpha}(x^\rho - a^\rho)^{\alpha-1}}{\Gamma(\alpha)(x^\rho - a^\rho)} \left(f(a^\rho) \int_a^x t^{\rho-1}(x^\rho - t)\, dt + f(x^\rho) \int_a^x t^{\rho-1}(t - a^\rho)\, dt \right).$$

By computing integrals on the right hand side we have

$$\begin{aligned} &\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x t^{\rho-1}(x^\rho - t^\rho)^{\alpha-1} f(t) dt \\ \leq &\frac{\rho^{1-\alpha}(x^\rho - a^\rho)^{\alpha-2}}{\Gamma(\alpha)} \left(f(a^\rho) \left(\frac{x^\rho(x^\rho - a^\rho)}{\rho} - \frac{x^{\rho+1} - a^{\rho+1}}{\rho+1} \right) \right. \\ &\left. + f(x^\rho) \left(\frac{x^{\rho+1} - a^{\rho+1}}{\rho+1} - \frac{a^\rho(x^\rho - a^\rho)}{\rho} \right) \right). \end{aligned} \tag{2.5}$$

This further takes the following form

$$\frac{\Gamma(\alpha)\, {}^\rho I_{a^+}^\alpha f(x)}{\rho^{1-\alpha}(x^\rho - a^\rho)^{\alpha-2}} \leq \frac{(x^\rho - a^\rho)(x^\rho f(a^\rho) - a^\rho f(x^\rho))}{\rho} + \frac{(x^{\rho+1} - a^{\rho+1})(f(x^\rho) - f(a^\rho))}{\rho+1}. \tag{2.6}$$

On the other hand, since f is convex on $[a, b]$, if we consider f on the interval $[x, b]$, $x \in [a, b)$, then, for $t \in [x, b]$ and $\rho > 0$, $\beta \geq 1$, the following two inequalities hold:

$$t^{\rho-1}(t^\rho - x^\rho)^{\beta-1} \leq (b^\rho - x^\rho)^{\beta-1} t^{\rho-1} \tag{2.7}$$

and

$$\frac{\rho^{1-\beta}}{\Gamma(\beta)} f(t) \leq \frac{\rho^{1-\beta}}{\Gamma(\beta)} \left(\frac{t - x^\rho}{b^\rho - x^\rho} f(b^\rho) + \frac{b^\rho - t}{b^\rho - x^\rho} f(x^\rho) \right). \tag{2.8}$$

Multiplying inequalities (2.7) and (2.8), and integrating on $[x, b]$, one has:

$$\begin{aligned} &\frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_x^b t^{\rho-1}(t^\rho - x^\rho)^{\beta-1} f(t) dt \\ \leq &\frac{\rho^{1-\beta}(b^\rho - x^\rho)^{\beta-1}}{\Gamma(\beta)(b^\rho - x^\rho)} \left(f(b^\rho) \int_x^b (t - x^\rho) t^{\rho-1} dt + f(x^\rho) \int_x^b (b^\rho - t) t^{\rho-1} dt \right). \end{aligned} \tag{2.9}$$

By computing integrals on the right hand side we have

$$\begin{aligned} &\frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_x^b t^{\rho-1}(t^\rho - x^\rho)^{\beta-1} f(t) dt \\ \leq &\frac{\rho^{1-\beta}(b^\rho - x^\rho)^{\beta-2}}{\Gamma(\beta)} \left(f(b^\rho) \left(\frac{b^{\rho+1} - x^{\rho+1}}{\rho+1} - \frac{b^\rho(b^\rho - x^\rho)}{\rho} \right) \right. \\ &\left. + f(x^\rho) \left(\frac{b^\rho(b^\rho - x^\rho)}{\rho} - \frac{b^{\rho+1} - x^{\rho+1}}{\rho+1} \right) \right), \end{aligned} \tag{2.10}$$

then it takes the following form

$$\frac{\Gamma(\beta) \rho I_{b_-}^\beta f(x)}{\rho^{1-\beta}(b^\rho - x^\rho)^{\beta-2}} \leq \frac{(b^{\rho+1} - x^{\rho+1})\big(f(b^\rho) - f(x^\rho)\big)}{\rho+1} + \frac{(b^\rho - x^\rho)\big(b^\rho f(x^\rho) - x^\rho f(b^\rho)\big)}{\rho}. \tag{2.11}$$

Adding (2.6) and (2.11), we get inequality (2.1). □

The above result reproduces [4, Theorem 1], if we set $\rho = 1$ and rearrange some terms in the proof. Building on Theorem 2.1, the forthcoming result offers a refined bound by taking the minimum of two expressions. This refinement sharpens the estimate, making it tighter in certain cases and more applicable for numerical approximations or when comparing different fractional orders.

Theorem 2.2. *Let all conditions imposed on functions and parameters are same as in Theorem 2.1. Then we have:*

$$\begin{aligned}
&\frac{\Gamma(\alpha)\,{}^{\rho}I^{\alpha}_{a+}f(x)}{\rho^{1-\alpha}(x^{\rho}-a^{\rho})^{\alpha-2}}+\frac{\Gamma(\beta)\rho I^{\beta}_{b-}f(x)}{\rho^{1-\beta}(b^{\rho}-x^{\rho})^{\beta-2}} \qquad (2.12)\\
\leq \min\Bigg\{&\frac{(x^{\rho}-a^{\rho})(x^{\rho}f(a^{\rho})-a^{\rho}f(x^{\rho}))+(b^{\rho}-x^{\rho})\big(b^{\rho}f(x^{\rho})-x^{\rho}f(b^{\rho})\big)}{\rho}\\
&+\frac{(x^{\rho+1}-a^{\rho+1})(f(x^{\rho})-f(a^{\rho}))+(b^{\rho+1}-x^{\rho+1})\big(f(b^{\rho})-f(x^{\rho})\big)}{\rho+1},\\
&\frac{x^{\rho}-a^{\rho}}{b^{\rho}-x^{\rho}}\left(\frac{(x^{\rho+1}-a^{\rho+1})\big(f(b^{\rho})-f(x^{\rho})\big)}{\rho+1}+\frac{(x^{\rho}-a^{\rho})\big(b^{\rho}f(x^{\rho})-x^{\rho}f(b^{\rho})\big)}{\rho}\right)\\
&+\frac{b^{\rho}-x^{\rho}}{x^{\rho}-a^{\rho}}\left(\frac{(b^{\rho+1}-x^{\rho+1})\big(f(x^{\rho})-f(a^{\rho})\big)}{\rho+1}+\frac{(b^{\rho}-x^{\rho})\big(x^{\rho}f(a^{\rho})-a^{\rho}f(x^{\rho})\big)}{\rho}\right)\Bigg\}.
\end{aligned}$$

Proof. By using the following inequality along with (2.2), the integral inequality (2.14) can be constituted:

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}f(t)\leq\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{t-x^{\rho}}{b^{\rho}-x^{\rho}}f(b^{\rho})+\frac{b^{\rho}-t}{b^{\rho}-x^{\rho}}f(x^{\rho})\right), \qquad (2.13)$$

$$\begin{aligned}
&\frac{\Gamma(\alpha)\,{}^{\rho}I^{\alpha}_{a+}f(x)}{\rho^{1-\alpha}(x^{\rho}-a^{\rho})^{\alpha-2}}\\
&\leq\frac{x^{\rho}-a^{\rho}}{b^{\rho}-x^{\rho}}\left\{f(b^{\rho})\int_a^x t^{\rho-1}(t-x^{\rho})\,dt+f(x^{\rho})\int_a^x t^{\rho-1}(b^{\rho}-t)\,dt\right\} \qquad (2.14)\\
&=\frac{x^{\rho}-a^{\rho}}{b^{\rho}-x^{\rho}}\left\{\frac{(x^{\rho+1}-a^{\rho+1})\big(f(b^{\rho})-f(x^{\rho})\big)}{\rho+1}+\frac{(x^{\rho}-a^{\rho})\big(b^{\rho}f(x^{\rho})-x^{\rho}f(b^{\rho})\big)}{\rho}\right\}.
\end{aligned}$$

Moreover, analyzing the forthcoming inequality along with (2.7), the integral inequality (2.33) can be constituted:

$$\frac{\rho^{1-\beta}}{\Gamma(\beta)}f(t)\leq\frac{\rho^{1-\beta}}{\Gamma(\beta)}\left(\frac{x^{\rho}-t}{x^{\rho}-a^{\rho}}f(a^{\rho})+\frac{t-a^{\rho}}{x^{\rho}-a^{\rho}}f(x^{\rho})\right),$$

$$\begin{aligned}
&\frac{\Gamma(\beta)\rho I^{\beta}_{b-}f(x)}{\rho^{1-\beta}(b^{\rho}-x^{\rho})^{\beta-2}}\\
&\leq\frac{b^{\rho}-x^{\rho}}{x^{\rho}-a^{\rho}}\left\{f(a^{\rho})\int_x^b t^{\rho-1}(x^{\rho}-t)\,dt+f(x^{\rho})\int_x^b t^{\rho-1}(t-a^{\rho})\,dt\right\} \qquad (2.15)\\
&=\frac{b^{\rho}-x^{\rho}}{x^{\rho}-a^{\rho}}\left\{\frac{(b^{\rho+1}-x^{\rho+1})\big(f(x^{\rho})-f(a^{\rho})\big)}{\rho+1}+\frac{(b^{\rho}-x^{\rho})\big(x^{\rho}f(a^{\rho})-a^{\rho}f(x^{\rho})\big)}{\rho}\right\}.
\end{aligned}$$

Inequality (2.34) is constituted from above inequalities (2.14) and (2.33).

$$\begin{aligned}
&\frac{\Gamma(\alpha)\,{}^{\rho}I^{\alpha}_{a+}f(x)}{\rho^{1-\alpha}(x^{\rho}-a^{\rho})^{\alpha-2}}+\frac{\Gamma(\beta)\rho I^{\beta}_{b-}f(x)}{\rho^{1-\beta}(b^{\rho}-x^{\rho})^{\beta-2}} \qquad (2.16)\\
\leq&\frac{x^{\rho}-a^{\rho}}{b^{\rho}-x^{\rho}}\left\{\frac{(x^{\rho+1}-a^{\rho+1})\big(f(b^{\rho})-f(x^{\rho})\big)}{\rho+1}+\frac{(x^{\rho}-a^{\rho})\big(b^{\rho}f(x^{\rho})-x^{\rho}f(b^{\rho})\big)}{\rho}\right\}
\end{aligned}$$

$$+\frac{b^\rho - x^\rho}{x^\rho - a^\rho}\left\{\frac{(b^{\rho+1}-x^{\rho+1})\big(f(x^\rho)-f(a^\rho)\big)}{\rho+1}+\frac{(b^\rho - x^\rho)\big(x^\rho f(a^\rho)-a^\rho f(x^\rho)\big)}{\rho}\right\}.$$

The required inequality is obtained from (2.34) and (2.1). □

The above result provides refinement of [4, Theorem 1], if we set $\rho = 1$ and rearrange some terms in the proof. The next theorem generalizes the inequalities to differentiable functions whose absolute value of the derivative is convex. It gives bounds on the absolute difference between normalized Katugampola fractional integrals and function values at endpoints, which is especially helpful for error estimates in fractional approximation theory.

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be a differentiable function. If $|f'|$ is convex, then for $a, b \in I$, with $0 \le a < b$, we have the inequality for Katugampola fractional integrals as follows:*

$$\begin{aligned}&\left|\frac{\rho^\alpha\Gamma(\alpha+1)\rho I^\alpha_{a+}f(x)}{(x^\rho - a^\rho)^\alpha}+\frac{\rho^\beta\Gamma(\beta+1)\,\rho I^\beta_{b-}f(x)}{(b^\rho - x^\rho)^\beta}\right| \\ &\le f(a)+f(x)+\frac{x-a}{x^\rho - a^\rho}\left(|f'(a^\rho)\left(x^\rho - \frac{a+x}{2}\right)+|f'(x^\rho)\left(\frac{a+x}{2}-x^\rho\right)\right) \\ &\quad+\frac{b-x}{b^\rho - x^\rho}\left(|f'(x^\rho)\left(b^\rho - \frac{x+b}{2}\right)+|f'(b^\rho)\left(\frac{x+b}{2}-x^\rho\right)\right).\end{aligned} \tag{2.17}$$

Proof. Since $|f'|$ is convex, therefore for $t \in [a, x]$, we can consider the inequality as follows:

$$f'(t) \le \frac{x^\rho - t}{x^\rho - a^\rho}|f'(a^\rho)|+\frac{t-a^\rho}{x^\rho - a^\rho}|f'(x^\rho)|. \tag{2.18}$$

Now, for $\alpha > 0$ we have the following inequality

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}(x^\rho - t^\rho)^\alpha \le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}(x^\rho - a^\rho)^\alpha, t \in [a, x]. \tag{2.19}$$

The product of above two inequalities yields the following integral inequality

$$\begin{aligned}&\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^\rho - t^\rho)^\alpha f'(t)dt \\ &\le \frac{\rho^{1-\alpha}(x^\rho - a^\rho)^{\alpha-1}}{\Gamma(\alpha)}\left(|f'(a^\rho)|\int_a^x (x^\rho - t)\,dt+|f'(x^\rho)|\int_a^x (t-a^\rho)\,dt\right).\end{aligned} \tag{2.20}$$

From the above inequality (2.20), after using integration by parts on the left hand side and computing integrals on the right hand side one can obtain the following inequality:

$$\begin{aligned}&\frac{\rho^\alpha\Gamma(\alpha+1)\rho I^\alpha_{a+}f(x)}{(x^\rho - a^\rho)^\alpha} \\ &\le f(a)+\frac{x-a}{x^\rho - a^\rho}\left(|f'(a^\rho)\left(x^\rho - \frac{a+x}{2}\right)+|f'(x^\rho)\left(\frac{a+x}{2}-x^\rho\right)\right).\end{aligned} \tag{2.21}$$

Convexity of $|f'|$ also provides the following inequality:

$$-f'(t) \le \frac{x^\rho - t}{x^\rho - a^\rho}|f'(a^\rho)|+\frac{t-a^\rho}{x^\rho - a^\rho}|f'(x^\rho)| \tag{2.22}$$

Repeating all steps of above scheme performed to obtain the inequality (2.21) from inequalities (2.18) and (2.19), one can obtain the inequality (2.23) from (2.19) and (2.22) as follows:

$$\begin{aligned}&\frac{\rho^{\alpha}\Gamma(\alpha+1)\rho I^{\alpha}_{a+}f(x)}{(x^{\rho}-a^{\rho})^{\alpha}} \\ &\geq -f(a)-\frac{x-a}{x^{\rho}-a^{\rho}}\left(|f'(a^{\rho})\left(x^{\rho}-\frac{a+x}{2}\right)+|f'(x^{\rho})\left(\frac{a+x}{2}-x^{\rho}\right)\right).\end{aligned} \tag{2.23}$$

Hence from inequalities (2.21) and (2.23), we get the bounding inequality (2.24) for left fractional integral operator:

$$\begin{aligned}&\left|\frac{\rho^{\alpha}\Gamma(\alpha+1)\,\rho I^{\alpha}_{a+}f(x)}{(x^{\rho}-a^{\rho})^{\alpha}}\right| \\ &\leq f(a)+\frac{x-a}{x^{\rho}-a^{\rho}}\left(|f'(a^{\rho})\left(x^{\rho}-\frac{a+x}{2}\right)+|f'(x^{\rho})\left(\frac{a+x}{2}-x^{\rho}\right)\right).\end{aligned} \tag{2.24}$$

On the other hand for $t\in[x,b]$ using convexity of $|f'|$ and for $\beta>0$, we have the following two inequalities (2.25) and (2.26). These inequalities can be used very likely as we have utilized (2.18), (2.19) and (2.22), the reader can develop the following bounding inequality (2.27) for right left fractional integral operator:

$$|f'(t)|\leq\frac{t-x^{\rho}}{b^{\rho}-x^{\rho}}(|f'(b^{\rho})|)+\frac{b^{\rho}-t}{b^{\rho}-x^{\rho}}(|f'(x^{\rho})|), \tag{2.25}$$

$$\frac{\rho^{1-\beta}}{\Gamma(\beta)}(t^{\rho}-x^{\rho})^{\beta}\leq\frac{\rho^{1-\beta}}{\Gamma(\beta)}(b^{\rho}-x^{\rho})^{\beta}, \tag{2.26}$$

$$\begin{aligned}&\left|\frac{\rho^{\beta}\Gamma(\beta+1)\,\rho I^{\beta}_{b-}f(x)}{(b^{\rho}-x^{\rho})^{\beta}}\right| \\ &\leq f(x)+\frac{b-x}{b^{\rho}-x^{\rho}}\left(|f'(x^{\rho})\left(b^{\rho}-\frac{x+b}{2}\right)+|f'(b^{\rho})\left(\frac{x+b}{2}-x^{\rho}\right)\right).\end{aligned} \tag{2.27}$$

By combining the inequalities (2.24) and (2.27) via triangular inequality, the inequality (2.17) is obtained. □

The above result reproduces [4, Theorem 2], if we set $\rho=1$ and rearrange some terms in the proof. As a refinement of Theorem 2.3, this result introduces a minimum operator to provide a sharper bound on the absolute value of the sum of normalized integrals. This allows for better control in applications involving convex-modulus derivatives, such as in stability analysis of fractional systems.

Theorem 2.4. *Let all conditions imposed on functions and parameters are same as in Theorem 2.3. Then we have:*

$$\begin{aligned}&\left|\frac{\rho^{\alpha}\Gamma(\alpha+1)\,{}^{\rho}I^{\alpha}_{a+}f(x)}{(x^{\rho}-a^{\rho})^{\alpha}}+\frac{\rho^{\beta}\Gamma(\beta+1)\rho I^{\beta}_{b-}f(x)}{(b^{\rho}-x^{\rho})^{\beta}}\right| \\ &\leq\min\left\{\frac{(x^{\rho}-a^{\rho})(x^{\rho}f(a^{\rho})-a^{\rho}f(x^{\rho}))+(b^{\rho}-x^{\rho})\big(b^{\rho}f(x^{\rho})-x^{\rho}f(b^{\rho})\big)}{\rho}\right. \\ &\quad+\frac{(x^{\rho+1}-a^{\rho+1})(f(x^{\rho})-f(a^{\rho}))+(b^{\rho+1}-x^{\rho+1})\big(f(b^{\rho})-f(x^{\rho})\big)}{\rho+1},\end{aligned} \tag{2.28}$$

$$\frac{x^\rho-a^\rho}{b^\rho-x^\rho}\left(\frac{(x^{\rho+1}-a^{\rho+1})\big(f(b^\rho)-f(x^\rho)\big)}{\rho+1}+\frac{(x^\rho-a^\rho)\big(b^\rho f(x^\rho)-x^\rho f(b^\rho)\big)}{\rho}\right)$$
$$+\frac{b^\rho-x^\rho}{x^\rho-a^\rho}\left(\frac{(b^{\rho+1}-x^{\rho+1})\big(f(x^\rho)-f(a^\rho)\big)}{\rho+1}+\frac{(b^\rho-x^\rho)\big(x^\rho f(a^\rho)-a^\rho f(x^\rho)\big)}{\rho}\right)\Bigg\}.$$

Proof. By using the following inequality along with (2.19), the integral inequality (2.30) can be constituted:

$$f'(t)\le\frac{t-x^\rho}{b^\rho-x^\rho}f(b^\rho)+\frac{b^\rho-t}{b^\rho-x^\rho}f(x^\rho), \tag{2.29}$$

$$\begin{aligned}&\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x(x^\rho-t^\rho)^\alpha f'(t)dt \qquad (2.30)\\ &\le\frac{\rho^{1-\alpha}(x^\rho-a^\rho)^\alpha}{\Gamma(\alpha)(b^\rho-x^\rho)}\left(|f'(b^\rho)|\int_a^x(t-x^\rho)\,dt+|f'(x^\rho)|\int_a^x(b^\rho-t)\,dt\right),\\ &\frac{\rho^\alpha\Gamma(\alpha+1)\,{}^\rho I^\alpha_{a^+}f(x)}{(x^\rho-a^\rho)^\alpha}\\ &\le\frac{x^\rho-a^\rho}{b^\rho-x^\rho}\left\{f(b^\rho)\int_a^x t^{\rho-1}(t-x^\rho)\,dt+f(x^\rho)\int_a^x t^{\rho-1}(b^\rho-t)\,dt\right\} \qquad (2.31)\\ &=\frac{x^\rho-a^\rho}{b^\rho-x^\rho}\left\{\frac{(x^{\rho+1}-a^{\rho+1})\big(f(b^\rho)-f(x^\rho)\big)}{\rho+1}+\frac{(x^\rho-a^\rho)\big(b^\rho f(x^\rho)-x^\rho f(b^\rho)\big)}{\rho}\right\}.\end{aligned}$$

Moreover, analyzing the forthcoming inequality (2.32) along with (2.7), the integral inequality (2.33) can be constituted:

$$\frac{\rho^{1-\beta}}{\Gamma(\beta)}f(t)\le\frac{\rho^{1-\beta}}{\Gamma(\beta)}\left(\frac{x^\rho-t}{x^\rho-a^\rho}f(a^\rho)+\frac{t-a^\rho}{x^\rho-a^\rho}f(x^\rho)\right), \tag{2.32}$$

$$\begin{aligned}&\frac{\Gamma(\beta)\rho I^\beta_{b_-}f(x)}{\rho^{1-\beta}(b^\rho-x^\rho)^{\beta-2}}\\ &\le\frac{b^\rho-x^\rho}{x^\rho-a^\rho}\left\{f(a^\rho)\int_x^b t^{\rho-1}(x^\rho-t)\,dt+f(x^\rho)\int_x^b t^{\rho-1}(t-a^\rho)\,dt\right\} \qquad (2.33)\\ &=\frac{b^\rho-x^\rho}{x^\rho-a^\rho}\left\{\frac{(b^{\rho+1}-x^{\rho+1})\big(f(x^\rho)-f(a^\rho)\big)}{\rho+1}+\frac{(b^\rho-x^\rho)\big(x^\rho f(a^\rho)-a^\rho f(x^\rho)\big)}{\rho}\right\}.\end{aligned}$$

Inequality (2.34) is constituted from above inequalities (2.14) and (2.33).

$$\begin{aligned}&\frac{\Gamma(\alpha)\,{}^\rho I^\alpha_{a^+}f(x)}{\rho^{1-\alpha}(x^\rho-a^\rho)^{\alpha-2}}+\frac{\Gamma(\beta)\rho I^\beta_{b_-}f(x)}{\rho^{1-\beta}(b^\rho-x^\rho)^{\beta-2}} \qquad (2.34)\\ &\le\frac{x^\rho-a^\rho}{b^\rho-x^\rho}\left\{\frac{(x^{\rho+1}-a^{\rho+1})\big(f(b^\rho)-f(x^\rho)\big)}{\rho+1}+\frac{(x^\rho-a^\rho)\big(b^\rho f(x^\rho)-x^\rho f(b^\rho)\big)}{\rho}\right\}\\ &\quad+\frac{b^\rho-x^\rho}{x^\rho-a^\rho}\left\{\frac{(b^{\rho+1}-x^{\rho+1})\big(f(x^\rho)-f(a^\rho)\big)}{\rho+1}+\frac{(b^\rho-x^\rho)\big(x^\rho f(a^\rho)-a^\rho f(x^\rho)\big)}{\rho}\right\}.\end{aligned}$$

The required inequality is obtained from (2.34) and (2.1). □

The above result provides refinement of [4, Theorem 2], if we set $\rho = 1$ and rearrange some terms in the proof. Upcoming lemma, cited from existing literature, establishes a basic property for symmetric convex functions around the midpoint of an interval. It serves as a foundational tool for deriving inequalities in symmetric settings, ensuring that the function value at the center is the minimum.

Lemma 2.1. [4] *Let $f : [a; b] \to R$ be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq f(x),\ x \in [a, b]. \tag{2.35}$$

Utilizing the symmetry from Lemma 2.1, this theorem derives a double-sided inequality for symmetric convex functions, bounding the fractional integrals between the midpoint function value and an upper expression involving endpoint values. This is especially relevant for applications where symmetry simplifies fractional models, such as in physics or signal processing.

Theorem 2.5. *Let $f : I \to R$ be a positive convex function. If f is symmetric about $\frac{a+b}{2}$, then for $a, b \in I$, $0 < a < b$, $\alpha, \beta > 0$, $\rho > 1$, the following inequality holds:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) \frac{(b^\rho - a^\rho)^2(\alpha+\beta+2)}{2\rho(\alpha+1)(\beta+1)} \\
\leq & \frac{\Gamma(\beta+1)\,{}^\rho I_{b-}^{\beta+1} f(a)}{2\rho^\beta (b^\rho - a^\rho)^{\beta-1}} + \frac{\Gamma(\alpha+1)\,{}^\rho I_{a+}^{\alpha+1} f(b)}{2\rho^\alpha (b^\rho - a^\rho)^{\alpha-1}} \\
\leq & f(b^\rho)\left(\frac{b^{\rho+1} - a^{\rho+1}}{\rho+1} - \frac{a^\rho(b^\rho - a^\rho)}{\rho}\right) + f(a^\rho)\left(\frac{b^\rho(b^\rho - a^\rho)}{\rho} - \frac{b^{\rho+1} - a^{\rho+1}}{\rho+1}\right).
\end{aligned}$$

Proof. Following inequalities hold for $x \in [a, b]$, by utilizing given conditions in the statement:

$$x^{\rho-1}(x^\rho - a^\rho)^\beta \leq x^{\rho-1}(b^\rho - a^\rho)^\beta,\ \beta > 0, \tag{2.36}$$

$$f(x) \leq \frac{x - a^\rho}{b^\rho - a^\rho} f(b^\rho) + \frac{b^\rho - x}{b^\rho - a^\rho} f(a^\rho). \tag{2.37}$$

Consequently, the forthcoming integral inequality is obtained from above inequalities

$$\begin{aligned}
& \int_a^b (x^\rho - a^\rho)^\beta x^{\rho-1} f(x)dx \\
\leq & (b^\rho - a^\rho)^{\beta-1}\left(f(b^\rho)\int_a^b (x - a^\rho)x^{\rho-1}dx + f(a^\rho)\int_a^b (b^\rho - x)x^{\rho-1}dx\right).
\end{aligned}$$

This takes the following form after applying definition of fractional integral and calculating the integrals on the right hand side

$$\begin{aligned}
& \frac{\Gamma(\beta+1)\rho I_{b-}^{\beta+1} f(a)}{\rho^\beta (b^\rho - a^\rho)^{\beta-1}} \\
\leq & f(b^\rho)\left(\frac{b^{\rho+1} - a^{\rho+1}}{\rho+1} - \frac{a^\rho(b^\rho - a^\rho)}{\rho}\right) + f(a^\rho)\left(\frac{b^\rho(b^\rho - a^\rho)}{\rho} - \frac{b^{\rho+1} - a^{\rho+1}}{\rho+1}\right).
\end{aligned} \tag{2.38}$$

On the other hand, we also have the following inequality holds for $x \in [a, b]$, under given conditions in the statement:

$$(b^\rho - x^\rho)^\alpha \leq (b^\rho - a^\rho)^\alpha. \tag{2.39}$$

The integral integral inequality (2.40) is obtained from above inequalities (2.37) and (2.39):

$$\begin{aligned}&\frac{\Gamma(\alpha+1)\rho I_{a+}^{\alpha+1}f(b)}{\rho^{\alpha}(b^{\rho}-a^{\rho})^{\alpha-1}} \\ \leq& f(b^{\rho})\left(\frac{b^{\rho+1}-a^{\rho+1}}{\rho+1}-\frac{a^{\rho}(b^{\rho}-a^{\rho})}{\rho}\right)+f(a^{\rho})\left(\frac{b^{\rho}(b^{\rho}-a^{\rho})}{\rho}-\frac{b^{\rho+1}-a^{\rho+1}}{\rho+1}\right).\end{aligned} \tag{2.40}$$

Adding the inequalities (2.38) and (2.40), we ge the forthcoming inequality

$$\begin{aligned}&\frac{\Gamma(\beta+1)\,{}^{\rho}I_{b-}^{\beta+1}f(a)}{2\rho^{\beta}(b^{\rho}-a^{\rho})^{\beta-1}}+\frac{\Gamma(\alpha+1)\,{}^{\rho}I_{a+}^{\alpha+1}f(b)}{2\rho^{\alpha}(b^{\rho}-a^{\rho})^{\alpha-1}} \\ \leq& f(b^{\rho})\left(\frac{b^{\rho+1}-a^{\rho+1}}{\rho+1}-\frac{a^{\rho}(b^{\rho}-a^{\rho})}{\rho}\right)+f(a^{\rho})\left(\frac{b^{\rho}(b^{\rho}-a^{\rho})}{\rho}-\frac{b^{\rho+1}-a^{\rho+1}}{\rho+1}\right).\end{aligned}$$

Using Lemma 2.1 and multiplying (2.35) with $x^{\rho-1}(x^{\rho}-a^{\rho})^{\beta}$, then integrating over $[a,b]$, one can have

$$f\left(\frac{a+b}{2}\right)\int_a^b x^{\rho-1}(x^{\rho}-a^{\rho})^{\beta}dx \leq \int_a^b x^{\rho-1}(x^{\rho}-a^{\rho})^{\beta}f(x)dx, \tag{2.41}$$

$$f\left(\frac{a+b}{2}\right)\frac{(b^{\rho}-a^{\rho})^{\beta+1}}{2\rho(\beta+1)(b^{\rho}-a^{\rho})^{\beta-1}} \leq \frac{\Gamma(\beta+1)\rho I_{b-}^{\beta+1}f(a)}{2\rho^{\beta}(b^{\rho}-a^{\rho})^{\beta-1}}, \tag{2.42}$$

$$f\left(\frac{a+b}{2}\right)\frac{(b^{\rho}-a^{\rho})^{2}}{2\rho(\beta+1)} \leq \frac{\Gamma(\beta+1)\rho I_{b-}^{\beta+1}f(a)}{2\rho^{\beta}(b^{\rho}-a^{\rho})^{\beta-1}}. \tag{2.43}$$

Again using Lemma 2.1 and multiplying (2.35) with $x^{\rho-1}(b^{\rho}-x^{\rho})^{\alpha}$, then integrating over $[a,b]$, one can have

$$f\left(\frac{a+b}{2}\right)\frac{(b^{\rho}-a^{\rho})^{2}}{2\rho(\alpha+1)} \leq \frac{\Gamma(\alpha+1)\rho I_{a+}^{\alpha+1}f(b)}{2\rho^{\alpha}(b^{\rho}-a^{\rho})^{\alpha-1}}. \tag{2.44}$$

Add inequalities (2.43) and (2.44) to get the next inequality

$$f\left(\frac{a+b}{2}\right)\frac{(b^{\rho}-a^{\rho})^{2}(\alpha+\beta+2)}{2\rho(\alpha+1)(\beta+1)} \leq \frac{\Gamma(\alpha+1)\rho I_{a+}^{\alpha+1}f(b)}{2\rho^{\alpha}(b^{\rho}-a^{\rho})^{\alpha-1}}+\frac{\Gamma(\beta+1)\rho I_{b-}^{\beta+1}f(a)}{2\rho^{\beta}(b^{\rho}-a^{\rho})^{\beta-1}}. \tag{2.45}$$

By combining inequalities (2.38) and (2.45), one can have the resulting inequality. □

The above result reproduces [4, Theorem 3], if we set $\rho=1$ and rearrange some terms in the proof.

3. Conclusion

The present paper has extended a number of traditional inequalities, such as Ostrowski, Hadamard, and Grüss type, to the context of Katugampola fractional integrals, which are generalizations of the Riemann-Liouville operators. Using rigorous arguments that involve convexity properties, integral representations, and kernel estimations, we derived new estimates for convex and symmetric convex functions and improvements for differentiable functions where the derivative is convex in modulus.

These findings present a more general and cohesive way of fractional integral inequalities, allowing variable scaling through the parameter ρ and fractional orders α, β. By including classical cases as special limits, our results enhance the analytical arsenal at the disposal of fractional calculus. The obtained inequalities have remarkable implications for use in fractional differential equations, where better bounds can improve solution stability and convergence analysis, and in functional analysis and optimization problems with the need for accurate estimations.

Potential future research avenues might be numerical simulations to confirm these inequalities, extensions to other classes of generalized convexity (for instance, quasi-convex or log-convex functions), or applications in real-world physics, engineering, and signal processing models. Further breakthroughs could also come from investigating the inequalities on the time scales or other fractional operators.

Authors contributions. Conceptualization: H.H.T., G.F., Y.R., S.U. and J.R. Validation: H.H.T., G.F., Y.R., S.U. and J.R. Writing manuscript: G.F. and Y.R. Editing and Review manuscript: H.H.T., G.F., Y.R., S.U. and J.R.

Acknowledgments. This work was supported by the Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R899), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (RS-2026-25471383).

Conflict of interest statement. The authors do not have any conflict of interest.

References

[1] M. Andrić and J. Pečarić, *On $(h, g; m)$-convexity and the Hermite–Hadamard inequality*, J. Convex Anal., 2022, 29, 257–268.

[2] H. Chen and U. N. Katugampola, *Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities for generalized fractional integrals*, J. Math. Anal. Appl., 2017, 448, 1092–1106. DOI: 10.1016/j.jmaa.2016.09.018.

[3] A. Fahad, Z. Ali, S. Furuichi and Y. Wang, *Novel fractional integral inequalities for GA-Cr-convex functions and connections with information systems*, Alexandria Eng. J., 2025, 113, 509–515. DOI: 10.1016/j.aej.2024.11.034.

[4] G. Farid, *Some Riemann–Liouville fractional integral inequalities for convex functions*, J. Anal., 2019, 27, 1095–1102. DOI: 10.1007/s41478-018-0079-4.

[5] F. Hezenci and H. Budak, *Fractional Euler–Maclaurin-type inequalities for various function classes*, Comput. Appl. Math., 2024, 43, 261. DOI: 10.1007/s40314-024-02766-8.

[6] S, Kermausuor, E. R. Nwaeze and A. M. Tameru, *New integral inequalities via the Katugampola fractional integrals for functions whose second derivatives are strongly η-convex*, Mathematics, 2019, 7, 183. DOI: 10.3390/math7020183.

[7] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math., 2014, 264, 65–70. DOI: 10.1016/j.cam.2014.01.002.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, First edition, North-Holland Mathematics Studies, 204, Elsevier, New York-London, 2013.

[9] S. Özcan, S. I. Butt, S. Tipurić-Spužević and B. B. Mohsin, *Construction of new fractional inequalities via generalized n-fractional polynomial s-type convexity*, Aims Math., 2024, 9(9), 23924–23944. DOI: 10.3934/math.20241163.

[10] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[11] A. Tassaddiq, C. Cattani, R. Alharbi, D. K. Almutairi and R. M. Kasmani, *New fractional integral inequalities involving the Fox-H and Meijer-G functions for convex and synchronous functions*, Fractal Fract., 2025, 9(4), 256. DOI: 10.3390/fractalfract9040256.

Received December 2025; Accepted June 2026; Available online June 2026.