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ON MULTIPLE POSITIVE SOLUTIONS TO HADAMARD FRACTIONAL BOUNDARY VALUE PROBLEMS ON AN INFINITE INTERVAL INVOLVING INTEGRAL AND INFINITE-POINT CONSTRAINTS

Limin Guo[1], Ying Wang[2,†], Yonghua Chen[3], Yuzhu Lei[1] and Yumei Zi[2]

Abstract In the fields of science and engineering, analyzing fractional-order differential equations over infinite intervals is crucial for modeling complex systems. These practical applications rely on the research results of the theory. Therefore, this paper conducts a theoretical analysis of boundary value problems over infinite intervals. This paper focuses on a class of Hadamard fractional equations with mixed integrals and infinite-point boundary constraints. By deriving the properties of the Green function, a suitable cone is established, and subsequently, the existence of multiple positive solutions is derived. Finally, a computational case study is conducted to verify the effectiveness of this theoretical framework.

Keywords Hadamard fractional derivative, multiple positive solutions, infinite domain.

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1. Introduction

While conventional integer-order differential equations remain foundational for modeling problems in physical chemistry, economics, and biomedical systems, their limitations become evident when addressing complex phenomena involving memory effects or long-range interactions. In such cases, integer-order frameworks often fail to reconcile theoretical predictions with experimental observations, requiring excessive parameters for system characterization. Fractional-order differential models, by contrast, overcome these shortcomings through their inherent nonlocality, enabling concise yet precise descriptions of intricate dynamics. Notably, fractional operators provide physically interpretable representations of anomalous diffusion and viscoelasticity, outperforming integer-order counterparts in both parametric efficiency and mechanistic clarity. Previous research predominantly focuses on Riemann-Liouville and Caputo derivatives as well as their theoretical foundations [6–10, 17, 19, 23, 27, 28], and related models and corresponding modified versions have been applied in many fields, such as ecosystem model [4], viscoelastic materials [3, 22], control system [15, 26, 29, 32], nonlinear dynamics [14, 16, 18, 33], and macroscopic or micro-structural vibration [12, 13], and the associated references therein.

Most studies in the relevant literature on boundary value problems for differential equations focus on finite intervals. Studying equations such as differential equations, difference equations

[†]The corresponding author.
[1]School of science, Changzhou Institute of Technology, Liaohe 213002, Changzhou, China
[2]School of Mathematicacs and Statistics, Linyi University, Linyi 276000, Shandong, China
[3]Soochow University, Library, Suzhou 215006, Jiangsu, China
Emails: guolimin811113@163.com(L. Guo), lywy1981@163.com(Y. Wang)
chenyonghua@suda.edu.cn(Y. Chen), leiyz@czu.cn(Y. Lei), ziyumei@lyu.edu.cn (Y. Zi)

or integral equations over an infinite interval have significant advantages compared to doing so over a regular closed interval. The main difference lies in that solutions over a closed interval typically focus on the behavior within a local or finite time frame, while solutions over an infinite interval must consider global properties, long-term evolution and asymptotic behaviors. In short, moving from a closed interval to an infinite interval is a leap from instant snapshots to eternal evolution. It enables us not only to understand how the equation changes at this moment, but also to grasp its ultimate destination - this is one of the ultimate goals of prediction and control in science. However, the theoretical framework for fractional differential equations on infinite intervals has seen limited advancement, which is caused by the complexity of the solution space and the difficulty of dealing with the compactness of the operator on infinite intervals. Therefore, it is difficult and promising to study the structure of boundary-value problems of fractional differential models with nonlinear derivative terms on infinite intervals. Although the structure of the solution of fractional differential equation in infinite interval develops slowly, some mathematical scholars have still made some research achievements [2, 21, 24, 34]. In [34], Zhou et al. studyed the existence and multiplicity of solutions of fractional differential equations on infinite intervals. In [1], Kausar, N. et al. conducted a study on the existence and uniqueness of solutions for fuzzy fractional differential equations. In [30], Zhang et al. conducted a study on the existence of solutions for Hadamard fractional differential equations on an infinite interval. In [2], Bouchelaghem et al. conducted a study on the existence of solutions for higher-order fractional differential equations. In [20], Nandhaprasadh et al. conducted a study on the existence of solutions for fractional-order stochastic differential equations on infinite intervals. In [24], Thiramanus investigated the existence criteria for solutions to integral boundary value problems involving Hadamard-type nonlinear fractional differential equations, and the study specifically established the existence of multiple nonnegative solutions under nonlocal fractional integral boundary conditions on unbounded domains, based on fixed point theorem methodologies, particularly leveraging both the Leggett-Williams fixed point theorem, to derive the multiplicity results. On infinite domain, Pei et al. [21] examined a class of nonlinear fractional differential equations, developing successive iterative techniques to construct positive extremal solutions in unbounded domain configurations. The authors systematically established the existence and convergence of these solutions using monotone iterative methodologies, addressing the analytical challenges posed by unbounded spatial domains. In [25], the authors investigated a nonlocal boundary value problem for Hadamard-type fractional differential equations, establishing not only the existence of twin positive solutions but also demonstrating the convergence of monotone iterative schemes to a unique positive solution. Their analysis leveraged the monotone iterative method to unify existence and computational convergence within a single framework. Separately, in [31], researchers explored similar Hadamard-type fractional systems, deriving novel existence criteria, uniqueness conditions, and multiplicity properties of positive solutions. This work systematically combined Schauder's fixed point theorem, Banach's contraction mapping principle, and complementary analytical tools to address both theoretical and computational challenges.

Building upon these foundational advancements, we establish existence criteria for multiple positive solutions to the fractional differential system (1.1)-(1.2). Drawing on methodological insights from prior studies, our work extends the analytical framework to rigorously address the nonlinear complexities inherent in such boundary value problems.

$$\Psi_p({}^H D^{\zeta}_{1^+}\varrho(\sigma)) + g(\sigma)f(\sigma, \varrho(\sigma), {}^H D^{\zeta-2}_{1^+}\varrho(\sigma), {}^H D^{\zeta-1}_{1^+}\varrho(\sigma)) = 0, \tag{1.1}$$

subject to nonlocal integral and infinite discrete boundary constraints

$$\begin{aligned}&\varrho(1)=\varrho'(1)=\cdots=\varrho^{(n-2)}(1)=0,\\&{}^HD_{1^+}^{\zeta-1}\varrho(+\infty)=\int_1^\infty h(\sigma)\varrho(\sigma)dB(\sigma)+\iota\sum_{i=1}^\infty\alpha_i\varrho(\eta_i),\end{aligned}\tag{1.2}$$

where $1<\sigma<\infty$, $\zeta\in\mathbb{R}_+^1=[0,\infty)$, $n-1<\zeta\le n$, ι is a constant, $\alpha_i\ge 0(i=1,2,\ldots,+\infty)$, p-Laplacian operator Ψ_p is defined as $\Psi_p(\upsilon)=|\upsilon|^{p-2}\upsilon$, $p,\ q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $1<\eta_i<+\infty(i=1,2,\ldots,\infty)$, ${}^HD_{1+}^{\zeta}$ is the standard Hadamard fractional derivative, B is function of bounded variation, and $\int_1^\infty h(t)\varrho(t)dB(t)$ denotes the Riemann-Stieltjes integral with respect to B, $h(t)\in L^1(1,\infty)$, $\int_1^\infty h(t)(\ln t)^{\zeta-1}dB(t)+\iota\sum_{i=1}^\infty\alpha_i(\ln\eta_i)^{\zeta-1}<\Gamma(\zeta)$, $g\in C[1,+\infty)$, $f(t,u,v,w):[1,+\infty)\times\mathbb{R}^3\to\mathbb{R}^+$. The multiple positive solutions for Hadamard fractional differential equation with integral and infinite-point boundary value conditions are obtained under some sufficient conditions on an infinite interval.

Compared with relevant references, motivated by the above, this study to investigated the existence of multiple positive solutions for Hadamard fractional equations featuring integral and infinite-point constraints over an infinite interval, three core advancements are presented in this research:

(1) The nonlinear term f contains derivative term, this will result in our having to study the issue in a complex space.

(2) The problem we studied over an infinite interval, which is with difficulties of proving equicontinuity and operator convergence.

(3) An infinite number of points and integral boundary values are simultaneously incorporated into the boundary conditions.

2. Preliminaries and lemmas

We introduce the Banach space E, which is characterized by

$$E=\left\{\varrho(\sigma),{}^HD_{1^+}^{\zeta-2}\varrho(\sigma),{}^HD_{1^+}^{\zeta-1}\varrho(\sigma)\in C([1,\infty)):\sup_{\sigma\in[1,\infty)}\frac{|\varrho(\sigma)|}{1+(\ln\sigma)^{\zeta-1}}<\infty,\right.$$
$$\left.\sup_{\sigma\in[1,\infty)}\frac{|{}^HD_{1^+}^{\zeta-2}\varrho(\sigma)|}{1+(\ln\sigma)}<\infty,\ \sup_{\sigma\in[1,\infty)}|{}^HD_{1^+}^{\zeta-1}\varrho(\sigma)|<\infty\right\},\tag{$\star$}$$

and E forms a Banach space under the norm

$$\|\varrho\|=\max\left\{\sup_{\sigma\in[1,\infty)}\frac{|\varrho(\sigma)|}{1+(\ln\sigma)^{\zeta-1}},\ \sup_{\sigma\in[1,\infty)}\frac{|{}^HD_{1^+}^{\zeta-2}\varrho(\sigma)|}{1+(\ln\sigma)},\ \sup_{\sigma\in[1,\infty)}|{}^HD_{1^+}^{\zeta-1}\varrho(\sigma)|\right\},$$

moreover, define a cone

$$P=\{\varrho\in E:\varrho(\sigma)\ge 0,\sigma\in[1,\infty)\}.$$

To establish foundational clarity, we first delineate essential definitions and preliminary lemmas critical to the subsequent analysis.

Definition 2.1. [11] The Hadamard fractional integral operator of order $\zeta>0$, acting on functions $\varrho:(0,\infty)\to\mathbb{R}_+^1$, is defined as

$${}^HI_{1^+}^{\zeta}\varrho(\sigma)=\frac{1}{\Gamma(\zeta)}\int_1^\sigma(\ln\frac{\sigma}{\upsilon})^{\zeta-1}\frac{\varrho(\upsilon)}{\upsilon}d\upsilon.$$

Definition 2.2. [11] The Hadamard fractional derivative of order $\zeta > 0$, acting on a continuous function $y : (0, \infty) \to \mathbb{R}^1_+$, is defined as

$$^{H}D^{\zeta}_{1+}\varrho(\sigma) = \frac{1}{\Gamma(n-\zeta)} \left(\sigma \frac{d}{d\sigma}\right)^n \int_1^{\sigma} \frac{\varrho(\upsilon)}{\upsilon \left(\ln \frac{\sigma}{\upsilon}\right)^{\zeta-n+1}} d\upsilon,$$

where $n = \lfloor \zeta \rfloor + 1$, with $\lfloor \zeta \rfloor$ denoting the integer floor function of ζ, under the condition that the right-hand expression converges pointwise on $(0, \infty)$.

Lemma 2.1. [11] *If $\zeta, \beta > 0$, then*

$$^{H}I^{\zeta}_{a} \left(\ln(\frac{\sigma}{a})^{\beta-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\beta+\zeta)} \left(\ln \frac{\sigma}{a}\right)^{\beta+\zeta-1},$$
$$^{H}D^{\zeta}_{a} \left(\ln(\frac{\sigma}{a})^{\beta-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta)} \left(\ln \frac{\sigma}{a}\right)^{\beta-\zeta-1}.$$

Lemma 2.2. [11] *Let $\zeta > 0$ and assume $\varrho \in \mathcal{C}[1, \infty) \cap L^1[1, \infty)$. The solution to the homogeneous Hadamard fractional differential equation*

$$^{H}\mathcal{D}^{\zeta}_{1+}\varrho(\sigma) = 0,$$

where $^{H}\mathcal{D}^{\zeta}_{1+}$ denotes the Hadamard fractional derivative of order ζ with lower limit 1^+, is given by

$$\varrho(\sigma) = c_1(\ln \sigma)^{\zeta-1} + c_2(\ln \sigma)^{\zeta-2} + \cdots + c_n(\ln \sigma)^{\zeta-n}, c_i \in R, i = 0, 1, \cdots, n, n = [\zeta] + 1.$$

Lemma 2.3. [11] *Let $\zeta > 0$ be a non-integer real number, and assume $\varrho \in \mathcal{C}[1, \infty) \cap L^1([1, \infty))$, where $\mathcal{C}[1, \infty)$ denotes the space of continuous functions and $L^1([1, \infty))$ the space of Lebesgue integrable functions on the interval $[1, \infty)$. Then,*

$$\varrho(\sigma) =^{H} I^{\zeta}_{1+} {}^{H}D^{\zeta}_{1+}\varrho(\sigma) + \sum_{k=1}^{n} c_k(\ln \sigma)^{\zeta-k},$$

for $\sigma \in (1, e]$, where $c_k \in \mathbb{R}(k = 1, 2, \cdots, n)$, and $n = [\zeta] + 1$.

Lemma 2.4. [5] *Let P be a cone in a Banach space $\mathcal{B}$, and let $\gamma : P \to \mathbb{R}_+$ be a nonnegative continuous concave functional on P. For positive constants a, b, c, d with $0 < a < b < d \le c$, we introduce the following subsets:*

$$P_c = \{\varrho \in P : \|\varrho\| < c\}, \quad P(\gamma, a, b) = \{\varrho \in P : a \le \gamma(\varrho), \|\varrho\| \le b\}.$$

Assume the operator $T : \overline{P_c} \to \overline{P_c}$ is completely continuous, and for all $\varrho \in \overline{P_c}$, the inequality $\gamma(\varrho) \le \|\varrho\|$ holds. Suppose further that the following conditions are satisfied:

(A_1) $\{\varrho \in P(\gamma, b, d) : \gamma(\varrho) > b\} \neq \varnothing$, and when $\varrho \in P(\gamma, b, d)$, we have $\gamma(T\varrho) > b$;

(A_2) when $\varrho \in \overline{P_a}$, we have $\|T\varrho\| < a$;

(A_3) when $\varrho \in P(\gamma, b, c), \|T\varrho\| > d$, we have $\gamma(T\varrho) > b$.

Then, the operator T admits at least three distinct fixed points $\varrho_1, \varrho_2, \varrho_3 \in \overline{P_c}$ satisfying:

$$\|\varrho_1\| < a, \quad \gamma(\varrho_2) > b, \quad \|\varrho_3\| > a \text{ with } \gamma(\varrho_3) < b.$$

Remark 2.1. If $d = c$, condition (A_3) follows directly from the preceding assumptions.

In the subsequent analysis, we derive the Green function corresponding to the linear boundary value problem.

Lemma 2.5. *For a function $y \in \mathcal{C}(1, +\infty)$, the linear boundary value problem under consideration is formulated as follows:*

$$^{H}D^{\zeta}_{1^+}\varrho(\sigma) + \Psi_q(y(\sigma)) = 0, 1 < \sigma < \infty, \tag{2.1}$$

combing boundary value problem (1.2) admits a unique solution that can be expressed by

$$\varrho(\sigma) = \int_1^{+\infty} \mathscr{Q}(\sigma, \upsilon)\frac{\Psi_q(y(\upsilon))}{\upsilon}d\upsilon, \tag{2.2}$$

where $\mathscr{Q}(\sigma, \upsilon) = \mathscr{Q}_1(\sigma, \upsilon) + \mathscr{Q}_2(\sigma, \upsilon)$, and

$$\mathscr{Q}_1(\sigma, \upsilon) = g(\sigma, \upsilon) + \frac{\iota\sum_{i=1}^{\infty}\alpha_i(\ln\sigma)^{\zeta-1}}{\Delta\Gamma(\zeta)}g(\eta_i, \upsilon), \tag{2.3}$$

$$\mathscr{Q}_2(\sigma, \upsilon) = \frac{(\ln\sigma)^{\zeta-1}}{\Delta_1}\int_1^{+\infty} h(\sigma)\mathscr{Q}_1(\sigma, \upsilon)dB(\sigma), \tag{2.4}$$

and

$$g(\sigma, \upsilon) = \begin{cases} \dfrac{(\ln\sigma)^{\zeta-1} - (\ln\frac{\sigma}{\upsilon})^{\zeta-1}}{\Gamma(\zeta)}, 1 \leq \upsilon \leq \sigma < +\infty, \\ \dfrac{(\ln\sigma)^{\zeta-1}}{\Gamma(\zeta)}, 1 \leq \sigma \leq \upsilon < +\infty, \end{cases} \tag{2.5}$$

$$\Delta_1 = \Delta - \int_1^{\infty} h(\sigma)(\ln\sigma)^{\zeta-1}dB(\sigma), \Delta = \Gamma(\zeta) - \iota\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\zeta-1}.$$

Proof. Using Lemma 2.3, the linear problem (2.1) can be reformulated as an equivalent integral equation

$$\varrho(\sigma) = -^{H}I^{\zeta}_{1^+}\Psi_q(y(\sigma)) + \Lambda_1(\ln\sigma)^{\zeta-1} + \Lambda_2(\ln\sigma)^{\zeta-2} + \cdots + \Lambda_n(\ln\sigma)^{\zeta-n},$$

for some $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathbb{R}^1$. Consequently, we get

$$\varrho(\sigma) = -\int_1^{\sigma}\frac{(\ln\frac{\sigma}{\upsilon})^{\zeta-1}}{\Gamma(\zeta)}\frac{\Psi_q(y(\upsilon))}{\upsilon}d\upsilon + \Lambda_1(\ln\sigma)^{\zeta-1} + \Lambda_2(\ln\sigma)^{\zeta-2} + \cdots + \Lambda_n(\ln\sigma)^{\zeta-n}.$$

From $\varrho(1) = 0$, we have $\Lambda_n = 0$, then

$$\varrho'(\sigma) = -^{H}I^{\zeta-1}_{1^+}\Psi_q(y(\sigma)) + \Lambda_1(\zeta-1)(\ln\sigma)^{\zeta-2}\frac{1}{\sigma} + \Lambda_2(\zeta-2)(\ln\sigma)^{\zeta-3}\frac{1}{\sigma} + \cdots,$$

by $\varrho'(1) = 0$, we get $\Lambda_{n-1} = 0$, the rest can be done in the same manner, one has $\Lambda_2 = \Lambda_3 = \ldots = \Lambda_{n-2} = 0$, then

$$\varrho(\sigma) = -\int_1^{\sigma}\frac{(\ln\frac{\sigma}{\upsilon})^{\zeta-1}}{\Gamma(\zeta)}\frac{\Psi_q(y(\upsilon))}{\upsilon}d\upsilon + \Lambda_1(\ln\sigma)^{\zeta-1},$$

$$ {}^{H}D_{1^+}^{\zeta-1}\varrho(\sigma) = \Lambda_1\Gamma(\zeta) - \int_1^{\sigma} \frac{\Psi_q(y(\upsilon))}{\upsilon} d\upsilon. $$

On the other hand, ${}^{H}D_{1^+}^{\zeta-1}\varrho(\infty) = \int_1^{\infty} h(\sigma)\varrho(\sigma)dB(\sigma) + \iota\sum_{i=1}^{\infty}\alpha_i\varrho(\eta_i)$, combining with

$$ {}^{H}D_{1^+}^{\zeta-1}\varrho(\infty) = -\int_1^{\infty} \Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \Gamma(\zeta)\Lambda_1, $$

we get

$$ \begin{aligned} \Lambda_1(\Gamma(\zeta) - \iota\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\alpha-1}) = &\int_1^{\infty} h(\sigma)\varrho(\sigma)dB(\sigma) + \int_1^{\infty}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} \\ &- \frac{\iota}{\Gamma(\zeta)}\sum_{i=1}^{\infty}\alpha_i\int_1^{\eta_i}(\ln\eta_i - \ln\upsilon)^{\zeta-1}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon}, \end{aligned} $$

then

$$ \begin{aligned} \Lambda_1 = &\frac{1}{\Delta}\left(\int_1^{\infty} h(\sigma)\varrho(\sigma)dB(\sigma) + \int_1^{\infty}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon}\right. \\ &\left. - \frac{\iota}{\Gamma(\zeta)}\sum_{i=1}^{\infty}\alpha_i\int_1^{\eta_i}(\ln\eta_i - \ln\upsilon)^{\zeta-1}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon}\right). \end{aligned} $$

So we have

$$ \begin{aligned} \varrho(\sigma) = &\frac{(\ln\sigma)^{\zeta-1}}{\Delta}\int_1^{\infty}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \frac{(\ln\sigma)^{\zeta-1}}{\Delta}\int_1^{\infty} h(\sigma)\varrho(\sigma)dB(\sigma) \\ &- \frac{1}{\Gamma(\zeta)}\int_1^{\sigma}(\ln\frac{\sigma}{s})^{\zeta-1}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} \\ &- \frac{(\ln\sigma)^{\zeta-1}\iota}{\Gamma(\zeta)\Delta}\sum_{i=1}^{\infty}\alpha_i\int_1^{\eta_i}(\ln\eta_i - \ln\upsilon)^{\zeta-1}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} \\ = &\frac{(\ln\sigma)^{\zeta-1}}{\Gamma(\zeta)}\int_1^{\infty}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \frac{(\Gamma(\zeta)-\Delta)(\ln\sigma)^{\zeta-1}}{\Delta\Gamma(\zeta)}\int_1^{\infty}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} \\ &+ \frac{(\ln\sigma)^{\zeta-1}}{\Delta}\int_1^{\infty} h(\sigma)\varrho(\sigma)dB(\sigma) \\ &- \frac{1}{\Gamma(\zeta)}\int_1^{\sigma}(\ln\frac{\sigma}{\upsilon})^{\zeta-1}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} - \frac{(\ln\sigma)^{\zeta-1}\iota}{\Gamma(\zeta)\Delta}\sum_{i=1}^{\infty}\alpha_i\int_1^{\eta_i}(\ln\eta_i - \ln\upsilon)^{\zeta-1}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} \\ = &\int_1^{+\infty} g(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \frac{\iota\sum_{i=1}^{\infty}\alpha_i(\ln\sigma)^{\zeta-1}}{\Delta\Gamma(\zeta)}\int_1^{\infty}(\ln\eta_i)^{\zeta-1}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} \\ &- \frac{(\ln\sigma)^{\zeta-1}\iota}{\Delta\Gamma(\alpha)}\sum_{i=1}^{\infty}\alpha_i\int_1^{\eta_i}(\ln\eta_i - \ln\upsilon)^{\zeta-1}\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \frac{(\ln\sigma)^{\zeta-1}}{\Delta}\int_1^{\infty} h(\sigma)\varrho(\sigma)dB(\sigma) \\ = &\int_1^{+\infty} g(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \frac{\iota\sum_{i=1}^{\infty}\alpha_i(\ln\sigma)^{\zeta-1}}{\Delta\Gamma(\alpha)}\int_1^{+\infty} g(\eta_i,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} \\ &+ \frac{(\ln\sigma)^{\zeta-1}}{\Delta}\int_1^{\infty} h(\sigma)\varrho(\sigma)dB(\sigma) \end{aligned} $$

$$= \int_1^\infty \mathscr{Q}_1(\sigma, \upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \frac{(\ln\sigma)^{\alpha-1}}{\Delta}\int_1^\infty h(\sigma)\varrho(\sigma)dB(\sigma). \tag{2.6}$$

Multiplying (2.6) with $h(\sigma)$ and performing integration from 1 to ∞ leads to

$$\begin{aligned}\int_1^\infty h(\sigma)\varrho(\sigma)dB(\sigma) = &\int_1^\infty h(\sigma)\int_1^\infty \mathscr{Q}_1(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon}dB(\upsilon)\\ &+ \frac{\int_1^\infty h(\sigma)(\ln\sigma)^{\zeta-1}dB(\sigma)}{\Delta}\int_1^\infty h(\sigma)\varrho(\sigma)dB(\sigma),\end{aligned}$$

then

$$\left(1 - \frac{\int_1^\infty h(\sigma)(\ln\sigma)^{\zeta-1}dB(\sigma)}{\Delta}\right)\int_1^\infty h(\sigma)\varrho(\sigma)dB(\sigma) = \int_1^\infty h(\sigma)\int_1^\infty \mathscr{Q}_1(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon}dB(\upsilon).$$

Let $\Delta_1 = \Delta - \int_1^\infty h(\sigma)(\ln\sigma)^{\zeta-1}dB(\sigma)$, then

$$\begin{aligned}\int_1^\infty h(\sigma)\varrho(\sigma)dB(\sigma) =&\frac{\Delta}{\Delta_1}\int_1^\infty h(\sigma)\int_1^\infty \mathscr{Q}_1(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon}dB(\sigma)\\ =&\frac{\Delta}{\Delta_1}\int_1^\infty \Psi_q(y(s))\int_1^\infty \mathscr{Q}_1(\sigma,\upsilon)h(\sigma)dB(\sigma)\frac{d\upsilon}{\upsilon}.\end{aligned}$$

Hence, we get

$$\begin{aligned}\varrho(\sigma) =& \int_1^\infty \mathscr{Q}_1(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \frac{(\ln\sigma)^{\zeta-1}}{\Delta}\int_1^\infty h(\sigma)\varrho(\sigma)dB(\sigma)\\ =& \int_1^\infty \mathscr{Q}_1(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \frac{(\ln\sigma)^{\zeta-1}}{\Delta}\left(\frac{\Delta}{\Delta_1}\int_1^\infty \Psi_q(y(\upsilon))\int_1^\infty \mathscr{Q}_1(\sigma,\upsilon)h(\sigma)dB(\sigma)\frac{d\upsilon}{\upsilon}\right)\\ =& \int_1^\infty \mathscr{Q}_1(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon} + \int_1^\infty \mathscr{Q}_2(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon}\\ =& \int_1^\infty \mathscr{Q}(\sigma,\upsilon)\Psi_q(y(\upsilon))\frac{d\upsilon}{\upsilon},\end{aligned}$$

where $\mathscr{Q}_1(\sigma,s), \mathscr{Q}_2(\sigma,s), \mathscr{Q}(\sigma,s)$ are as (2.2)-(2.4). □

Lemma 2.6. *Using (2.2), the elementary manipulations yield*

$${}^H D_{1^+}^{\alpha-1}\varrho(\sigma) = \int_1^\infty \overline{\mathscr{Q}}(\sigma,\upsilon)\frac{\Psi_q(y(\upsilon))}{\upsilon}d\upsilon, {}^H D_{1^+}^{\zeta-2}\varrho(\sigma) = \int_1^\infty \mathscr{Q}_0(\sigma,\upsilon)\frac{\Psi_q(y(\upsilon))}{\upsilon}d\upsilon,$$

where

$$\overline{\mathscr{Q}}(\sigma,\upsilon) = \vartheta(\sigma,\upsilon) + \frac{\iota\sum_{i=1}^\infty \alpha_i g(\eta_i,\upsilon)}{\Delta} + \frac{\Gamma(\zeta)}{\Delta_1}\int_1^\infty h(\sigma)\mathscr{Q}_1(\sigma,\upsilon)dB(\sigma), \tag{2.7}$$

$$\mathscr{Q}_0(\sigma,\upsilon) = {}^H D_{1^+}^{\zeta-2}g(\sigma,\upsilon) + \frac{\iota\sum_{i=1}^\infty \alpha_i g(\eta_i,\upsilon)\ln\sigma}{\Delta} + \frac{\Gamma(\zeta)\ln\sigma}{\Delta_1}\int_1^\infty h(\sigma)\mathscr{Q}_1(\sigma,\upsilon)dB(\sigma), \tag{2.8}$$

and

$$\vartheta(\sigma,\upsilon) = \begin{cases}0, 1 \le \upsilon \le \sigma < \infty,\\ 1, 1 \le \sigma \le \upsilon < \infty,\end{cases} \tag{2.9}$$

$$
{}^{H}D_{1^+}^{\zeta-2}g(\sigma,\upsilon)=\begin{cases}\ln\sigma-\ln\dfrac{\sigma}{\upsilon}, 1\le\upsilon\le\sigma<+\infty,\\ \ln\sigma, 1\le\sigma\le\upsilon<+\infty.\end{cases} \tag{2.10}
$$

Lemma 2.7. *The following properties hold for the Green function $\mathscr{Q}(\sigma,s)$:*

(A_1)*: $\mathscr{Q}(\sigma,\upsilon)$ has continuous properties and $\mathscr{Q}(\sigma,\upsilon)\ge 0$ for $(\sigma,\upsilon)\in[1,\infty)\times[1,\infty)$;*
(A_2): $\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}\le\Upsilon$, *where*

$$
\Upsilon=\left(\frac{1}{\Gamma(\zeta)}+\frac{\iota\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\zeta-1}}{\Delta\Gamma^2(\zeta)}\right)\left(1+\frac{1}{\Delta_1}\int_1^{\infty}h(\sigma)(\ln\sigma)^{\zeta-1}dB(\sigma)\right).
$$

Proof. (A_1) is obviously hold, we only need to prove (A_2). By (2.2-2.4), we have

$$
\begin{aligned}
&\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}\\
=&\frac{\mathscr{Q}_1(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}+\frac{\mathscr{Q}_2(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}\\
\le&\frac{1}{\Gamma(\zeta)}+\frac{\iota\sum_{i=1}^{\infty}\alpha_i(\ln\sigma)^{\zeta-1}}{(1+(\ln\sigma)^{\zeta-1})\Delta\Gamma(\zeta)}g(\eta_i,\upsilon)+\frac{(\ln\sigma)^{\zeta-1}}{\Delta_1(1+(\ln\sigma)^{\zeta-1}))}\int_1^{\infty}h(\sigma)\mathscr{Q}_1(\sigma,\upsilon)dB(\sigma)\\
\le&\frac{1}{\Gamma(\alpha)}+\frac{\iota\sum_{i=1}^{\infty}\alpha_i\ln\eta_i)^{\zeta-1}}{\Delta\Gamma^2(\zeta)}+\frac{1}{\Delta_1}\int_1^{\infty}h(\sigma)(\frac{(\ln\sigma)^{\zeta-1}}{\Gamma(\zeta)}+\frac{\iota\sum_{i=1}^{\infty}\alpha_i(\ln\sigma)^{\zeta-1}(\ln\eta_i)^{\zeta-1}}{\Delta\Gamma^2(\zeta)})dB(t)\\
=&\left(\frac{1}{\Gamma(\zeta)}+\frac{\iota\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\zeta-1}}{\Delta\Gamma^2(\zeta)}\right)\left(1+\frac{1}{\Delta_1}\int_1^{\infty}h(\sigma)(\ln\sigma)^{\zeta-1}dB(\sigma)\right)\\
=&\Upsilon,
\end{aligned}
$$

where

$$
\Upsilon=\left(\frac{1}{\Gamma(\zeta)}+\frac{\iota\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\zeta-1}}{\Delta\Gamma^2(\zeta)}\right)\left(1+\frac{1}{\Delta_1}\int_1^{\infty}h(\sigma)(\ln\sigma)^{\zeta-1}dB(\sigma)\right),
$$

hence, (A_2) holds. □

Remark 2.2. Defined through (2.2), the Green function $\mathscr{Q}(\sigma,\upsilon)$,$\mathscr{Q}_0(\sigma,\upsilon)$ and $\overline{\mathscr{Q}}(\sigma,s)$ retain the following properties:

(B_1): $\mathscr{Q}(\sigma,\upsilon)\le\Upsilon(\ln\sigma)^{\zeta-1}$ for $(\sigma,\upsilon)\in[1,\infty)\times[1,\infty)$;
(B_2): $\mathscr{Q}_0(\sigma,\upsilon)\le\Upsilon\Gamma(\zeta)\ln\sigma$ for $(\sigma,\upsilon)\in[1,\infty)\times[1,\infty)$;
(B_3): $\overline{\mathscr{Q}}(\sigma,\upsilon)\le\Upsilon\Gamma(\zeta)$, for $(\sigma,\upsilon)\in[1,\infty)\times[1,\infty)$.

Proof. By $(2.3-2.5)$ and (H_0), we easily get

$$
\begin{aligned}
\mathscr{Q}(\sigma,\upsilon)\le&\frac{(\ln\sigma)^{\zeta-1}}{\Gamma(\zeta)}+\frac{\iota\sum_{i=1}^{\infty}\alpha_i g(\eta_i,\upsilon)}{\Delta\Gamma(\zeta)}(\ln\sigma)^{\zeta-1}+\frac{(\ln\sigma)^{\zeta-1}}{\Delta_1}\int_1^{+\infty}h(\sigma)\mathscr{Q}_1(\sigma,\upsilon)dB(\sigma)\\
=&\left(\frac{1}{\Gamma(\zeta)}+\frac{\iota\sum_{i=1}^{\infty}\alpha_i g(\eta_i,\upsilon)}{\Delta\Gamma(\zeta)}+\frac{1}{\Delta_1}\int_1^{+\infty}h(\sigma)\mathscr{Q}_1(\sigma,s)dB(\sigma)\right)(\ln\sigma)^{\zeta-1}\\
\le&\left(\frac{1}{\Gamma(\zeta)}+\frac{\iota\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\zeta-1}}{\Delta\Gamma^2(\zeta)}\right)\left(1+\frac{1}{\Delta_1}\int_1^{\infty}h(\sigma)(\ln\sigma)^{\zeta-1}dB(\sigma)\right)(\ln\sigma)^{\zeta-1}\\
\le&\Upsilon(\ln\sigma)^{\zeta-1},
\end{aligned}
$$

so (B_1) holds. Moreover,

$$\begin{aligned}\mathscr{Q}_0(\sigma,\upsilon) \leq& \ln\sigma + \frac{\iota\sum_{i=1}^{\infty}\alpha_i g(\eta_i,\upsilon)\ln\sigma}{\Delta} + \frac{\Gamma(\zeta)\ln\sigma}{\Delta_1}\int_1^{\infty} h(\sigma)\mathscr{Q}_1(\sigma,\upsilon)dB(t)\\ =&\left(1+\frac{\iota\sum_{i=1}^{\infty}\alpha_i g(\eta_i,\upsilon)}{\Delta}+\frac{\Gamma(\zeta)}{\Delta_1}\int_1^{\infty} h(\sigma)\mathscr{Q}_1(\sigma,\upsilon)dB(\sigma)\right)\ln\sigma\\ \leq&\Upsilon\Gamma(\zeta).\end{aligned}$$

Hence, (B_2) holds, similarly, (B_3) is easily checked, we omit the detail here. □

Lemma 2.8. [31] *The functions $\mathscr{Q}(\sigma,\upsilon)$ and $g(\sigma,\upsilon)$, defined by equations (2.2) and (2.5) respectively, satisfy the following properties:*

(i) $\mathscr{Q}(\sigma,\upsilon)$ is non-negative and continuous over the domain $[1,\infty)\times[1,\infty)$.

(ii) $g(\sigma,\upsilon)$ is non-negative and continuous for all $(\sigma,\upsilon)\in[1,\infty)\times[1,\infty)$.

(iii) $\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}\leq\Upsilon,\quad\forall(\sigma,\upsilon)\in[1,\infty)\times[1,\infty)$.

(iv) The function $g(\sigma,\upsilon)$ is monotonically increasing with respect to the variable t.

(v) For any fixed constant $k>1$, the function $g(\sigma,\upsilon)$ satisfies:

$$\min_{\sigma\in[e^{\frac{1}{k}},e^k]}\frac{g(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}\geq\frac{1}{4k^2(1+k^{\zeta-1})}\sup_{\sigma\in J}\frac{g(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}.$$

Proof. The proof of (i,ii,iv,v) is from Lemma 3.2 (i,ii,iv,v) of [31], and the proof of (iii) is similar with Lemma 3.2 (iii) of [31]. □

Remark 2.3. Choose $k>1$ as a fixed parameter, then define

$$\mu(k)=\min\left\{\frac{1}{4k^2(1+k^{\zeta-1})},\frac{1}{k^{2\zeta-2}(1+k^{\zeta-1})}\right\},\tag{2.11}$$

then,

$$\min_{t\in[e^{\frac{1}{k}},e^k]}\frac{\mathscr{Q}(\sigma,s)}{1+(\ln t)^{\zeta-1}}\geq\mu(k)\sup_{\sigma\in J}\frac{\mathscr{Q}(\sigma,s)}{1+(\ln\sigma)^{\zeta-1}}.$$

Proof. Let

$$\mathscr{Q}(\sigma,\upsilon)=\frac{\varrho\sum_{i=1}^{\infty}\alpha_i(\ln t)^{\zeta-1}}{\Delta\Gamma(\zeta)}g(\eta_i,\upsilon)+\frac{(\ln\sigma)^{\zeta-1}}{\Delta_1}\int_1^{+\infty}h(\sigma)\mathscr{Q}_1(\sigma,\upsilon)dB(\upsilon).\tag{2.12}$$

Equation (2.12) leads directly to

$$\begin{aligned}\min_{\sigma\in[e^{\frac{1}{k}},e^k]}\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln t)^{\zeta-1}}\geq&\frac{1}{k^{\zeta-1}(1+k^{\zeta-1})}\left[\frac{\varrho\sum_{i=1}^{\infty}\alpha_i g(\eta_i,\upsilon)}{\Delta\Gamma(\zeta)}+\frac{1}{\Delta_1}\int_1^{+\infty}h(\sigma)\mathscr{Q}_1(\sigma,s)dB(\sigma)\right]\\ \geq&\frac{1}{k^{\zeta-1}(1+k^{\zeta-1})}\frac{1+k^{1-\zeta}}{k^{\zeta-1}}\cdot\sup_{t\in J}\frac{(\ln\sigma)^{\zeta-1}}{1+(\ln\sigma)^{\zeta-1}}\\ &\times\left[\frac{\varrho\sum_{i=1}^{\infty}\alpha_i g(\eta_i,\upsilon)}{\Delta\Gamma(\zeta)}+\frac{1}{\Delta_1}\int_1^{+\infty}h(\sigma)\mathscr{Q}_1(\sigma,\upsilon)dB(t)\right]\\ \geq&\frac{1}{k^{2\zeta-2}(1+k^{\zeta-1})}\sup_{\sigma\in J}\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}},\end{aligned}$$

hence,

$$\begin{aligned}\min_{\sigma\in[e^{\frac{1}{k}},e^{k}]}\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln t)^{\zeta-1}}\geq&\frac{1}{4k^{2}(1+k^{\zeta-1})}\frac{1}{4k^{2}(1+k^{\zeta-1})}\sup_{\sigma\in J}\frac{g(\sigma,s)}{1+(\ln\sigma)^{\zeta-1}}\\&+\frac{1}{k^{2\zeta-2}(1+k^{\zeta-1})}\sup_{\sigma\in J}\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}\\\geq&\mu(k)\sup_{\sigma\in J}\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}.\end{aligned}$$

Hence the proof is concluded. □

Motivated by Lemma 2.4, we construct the operator A as

$$A\varrho(\sigma)=\int_{1}^{\infty}\mathscr{Q}(\sigma,\upsilon)\Psi_{q}(g(\upsilon))\Psi_{q}(f(\upsilon,\varrho(\upsilon),{}^{H}D_{1^{+}}^{\zeta-2}\varrho(\upsilon),{}^{H}D_{1^{+}}^{\zeta-1}\varrho(\upsilon)))\frac{d\upsilon}{\upsilon}. \tag{2.13}$$

The BVP (1.1, 1.2) has a solution precisely when a fixed point exists for the operator equation $Ax=x$.

3. Main result

The analysis presented herein rests on the following conditions:

(H_1) For any $r>0$, there exists a function $\varphi_r(\sigma)\in L[1,+\infty)$ and $\varphi_r(\sigma)\geq 0$, with $\int_1^{+\infty}\varphi_r(\upsilon)\Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon}<+\infty$, for $\max\{\|u\|,\|v\|,\|w\|\}\leq r$ and a.e. $\sigma\in[1,+\infty)$, such that

$$\Psi_q\left(f(\sigma,\ (1+(\ln\sigma)^{\zeta-1})u,(1+(\ln\sigma)v,w)\right)\leq\varphi_r(\sigma).$$

(H_2) $g:[1,+\infty)\to[1,+\infty)$, $g(\sigma)$ do not vanish on any subset of $[1,+\infty)$ and $\int_1^{+\infty}\Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon}<\infty$.

(H_3) $h(\sigma)\in L[1,+\infty)$, for $\sigma\in[1,+\infty)$ and

$$\int_{1}^{\infty}h(\sigma)(\ln\sigma)^{\zeta-1}dB(\sigma)+\varrho\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\zeta-1}<\Gamma(\zeta).$$

Lemma 3.1. *Let $V\subset E$ be a bounded set. Then V is relatively compact in E if the following conditions hold: (i) For any $\varrho(\sigma)\in V$, $\frac{\varrho(\sigma)}{1+(\ln\sigma)^{\zeta-1}}$, $\frac{{}^{H}D_{1^{+}}^{\zeta-2}\varrho(\sigma))}{1+(\ln\sigma)}$ and ${}^{H}D_{1^{+}}^{\zeta-1}\varrho(t)$ are equicontinuous on any compact interval of $[1,\infty)$.*

(ii) For any $\varepsilon>0$, there exists a constant $T=T(\varepsilon)>1$ such that

$$\left|\frac{\varrho(\sigma_1)}{1+(\ln\sigma_1)^{\zeta-1}}-\frac{\varrho(\sigma_2)}{1+(\ln\sigma_2)^{\zeta-1}}\right|<\varepsilon,\left|\frac{{}^{H}D_{1^{+}}^{\zeta-2}\varrho(\sigma_1)}{1+\ln\sigma_1}-\frac{{}^{H}D_{1^{+}}^{\zeta-2}\varrho(\sigma_2)}{1+\ln\sigma_2}\right|<\varepsilon, \tag{3.1}$$

and $|{}^{H}D_{1^{+}}^{\zeta-1}\varrho(\sigma_2)-{}^{H}D_{1^{+}}^{\zeta-1}\varrho(\sigma_2)|<\varepsilon,$

for any $\sigma_1, \sigma_2 \geq T$ *and* $\varrho \in V$.

Proof. The proof is similar with [21], we omit the detail here. □

Lemma 3.2. *If* (H_1)-(H_3) *are satisfied, then* A *maps* P *to* P *as a completely continuous operator.*

Proof. Initially, we establish the well-definedness of $A : P \to P$. Let Ω be bounded. For every $\varrho \in \Omega$, there exists $l_0 \in \mathbb{R}^+$ satisfying

$$\|u\|, \|v\|, \|w\| < l_0.$$

From Remark 3.1, it follows that

$$\begin{aligned}
\|A\varrho\| =& \sup_{0\leq\sigma<+\infty} \frac{1}{1+(\ln\sigma)^{\zeta-1}} \left| \int_1^\infty \mathscr{Q}(\sigma,\upsilon)\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho(s)))\frac{d\upsilon}{\upsilon} \right| \\
\leq& \Upsilon \int_1^\infty g(\upsilon)\varphi_{l_0}(\upsilon))\frac{d\upsilon}{\upsilon} \\
<& +\infty, \\
\|{}^H D_{1^+}^{\zeta-2} A\varrho\| =& \sup_{1\leq\sigma<+\infty} \frac{1}{1+(\ln\sigma)} \\
& \times \left| \int_1^\infty \mathscr{Q}_0(\sigma,\upsilon)\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho(\upsilon)))\frac{d\upsilon}{\upsilon} \right| \\
\leq& \Gamma(\zeta)\Upsilon \int_1^\infty \Psi_q(g(\upsilon))\varphi_{l_0}(\upsilon))\frac{d\upsilon}{\upsilon} \\
<& +\infty, \\
\|{}^H D_{1^+}^{\zeta-1} A\varrho\| =& \sup_{0\leq\sigma<+\infty} \left| \int_1^\infty \mathscr{Q}_0(\sigma,\upsilon)\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho(\upsilon)))\frac{d\upsilon}{\upsilon} \right| \\
\leq& \Gamma(\zeta)\Upsilon \int_1^\infty \Psi_q(g(\upsilon))\varphi_{l_0}(\upsilon))\frac{d\upsilon}{\upsilon} \\
<& +\infty.
\end{aligned}$$

So $A\Omega$ is bounded. The equicontinuity of $A\Omega$ over any compact interval within $[1,\infty)$ will be demonstrated. To formalize this, fix a compact interval $\varpi = [1,\sigma] \subset [1,\infty)$. For any $\varrho \in \Omega, \sigma_1, \sigma_2 \in \varpi$, we may assume that $\sigma_2 > \sigma_1$, then

$$\begin{aligned}
& \left| \frac{(A\varrho)(\sigma_1)}{1+(\ln\sigma_1)^{\zeta-1}} - \frac{(A\varrho)(\sigma_2)}{1+(\ln\sigma_2)^{\zeta-1}} \right| \\
\leq& \int_1^\infty \left| \left(\frac{\mathscr{Q}(\sigma_2,s)}{1+(\ln\sigma)^{\zeta-1}} - \frac{\mathscr{Q}(\sigma_1,s)}{1+(\ln\sigma)^{\zeta-1}}\right) \right| \Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D^{\zeta-2}\varrho(\upsilon),{}^H D^{\zeta-1}\varrho(\upsilon)))\frac{d\upsilon}{\upsilon} \\
\leq& \int_1^{\sigma_2} \left| \frac{\mathscr{Q}(t_1,\upsilon)}{1+(\ln\sigma_1)^{\zeta-1}} - \frac{\mathscr{Q}(\sigma_2,s)}{1+(\ln\sigma_2)^{\zeta-1}} \right| \Psi_q(g(\upsilon))\varphi_\sigma(\upsilon))\frac{d\upsilon}{\upsilon} \\
& + \int_{\sigma_2}^{+\infty} \left| \frac{\mathscr{Q}(\sigma_1,\upsilon)}{1+(\ln\sigma_1)^{\zeta-1}} - \frac{\mathscr{Q}(\sigma_2,\upsilon)}{1+(\ln\sigma_2)^{\zeta-1}} \right| \Psi_q(g(\upsilon))\varphi_\sigma(\upsilon))\frac{d\upsilon}{\upsilon} \\
\leq& \int_1^{\sigma_2} \left| \frac{\mathscr{Q}(\sigma_1,\upsilon)}{1+(\ln\sigma_1)^{\zeta-1}} - \frac{\mathscr{Q}(\sigma_2,\upsilon)}{1+(\ln\sigma_1)^{\zeta-1}} \right| \Psi_q(g(\upsilon))\varphi_\sigma(\upsilon))\frac{d\upsilon}{\upsilon} \\
& + \Upsilon \int_{\sigma_2}^{\infty} \left| \frac{(\ln\sigma_1)^{\zeta-1}}{1+(\ln\sigma_1)^{\zeta-1}} - \frac{(\ln\sigma_2)^{\zeta-1}}{1+(\ln\sigma_2)^{\zeta-1}} \right| \Psi_q(g(\upsilon))\varphi_\sigma(\upsilon))\frac{d\upsilon}{\upsilon}.
\end{aligned}$$

Given the uniform continuity of the functions $\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}, \frac{(\ln\sigma)^{\zeta-1}}{1+(\ln\sigma)^{\zeta-1}}$ on $[1,\sigma]\times[1,\sigma]$ and $[1,\sigma]$ respectively, we deduce that

$$\left|\frac{A\varrho(\sigma_1)}{1+(\ln\sigma_1)^{\zeta-1}} - \frac{A\varrho(\sigma_2)}{1+(\ln\sigma_2)^{\zeta-1}}\right| \to 0, \text{as } \sigma_1 \to \sigma_2.$$

Hence, $\frac{A\varrho(\sigma)}{1+(\ln\sigma)^{\zeta-1}}$ is equicontinuous on $[1,\sigma]$.

Now we investigate

$$^{H}D_{1^+}^{\zeta-2}A\varrho(\sigma) = \int_1^{+\infty} \mathscr{Q}_0(\sigma,s)\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),^{H}D_{1^+}^{\zeta-2}\varrho(\upsilon),^{H}D_{1^+}^{\zeta-1}\varrho(\upsilon)))\frac{d\upsilon}{\upsilon}.$$

Then

$$\begin{aligned}
&\left|\frac{(^{H}D_{1^+}^{\alpha-2}A\varrho)(\sigma_1)}{1+\ln\sigma_1} - \frac{(^{H}D_{1^+}^{\zeta-2}A\varrho)(\sigma_2)}{1+\ln\sigma_2}\right| \\
&\leq \int_1^{\infty} \left|(\frac{\mathscr{Q}(\sigma_2,\upsilon)}{1+\ln\sigma_1} - \frac{\mathscr{Q}(\sigma_1,\upsilon)}{1+\ln\sigma_2})\right| \Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),^{H}D_{1^+}^{\zeta-2}\varrho(\upsilon),^{H}D_{1^+}^{\zeta-1}\varrho(\upsilon)))\frac{d\upsilon}{\upsilon} \\
&\leq \int_1^{\sigma_2} \left|\frac{\mathscr{Q}(\sigma_1,\upsilon)}{1+\ln\sigma_\sigma} - \frac{\mathscr{Q}(t_2,s)}{1+\ln\sigma_2}\right| \Psi_q(g(\upsilon))\varphi_\sigma(\upsilon))\frac{d\upsilon}{\upsilon} \\
&\quad + \int_{\sigma_2}^{+\infty} \left|\frac{\mathscr{Q}(\sigma_1,\upsilon)}{1+\ln\sigma_1} - \frac{\mathscr{Q}(\sigma_2,\upsilon)}{1+\ln\sigma_2}\right| \Psi_q(g(\upsilon))\varphi_\sigma(\upsilon))\frac{d\upsilon}{\upsilon} \\
&\leq \int_1^{\sigma_2} \left|\frac{\mathscr{Q}(\sigma_1,\upsilon)}{1+(\ln\sigma_1)^{\zeta-1}} - \frac{\mathscr{Q}(\sigma_2,\upsilon)}{1+\ln\sigma_1}\right| g(\upsilon)\varphi_\sigma(\upsilon))\frac{d\upsilon}{\upsilon} \\
&\quad + \Upsilon\int_{\sigma_2}^{\infty} \left|\frac{\ln\sigma_1}{1+(\ln\sigma_1)^{\zeta-1}} - \frac{\ln\sigma_2}{1+\ln\sigma_2}\right| g(\upsilon)\varphi_\sigma(\upsilon))\frac{d\upsilon}{\upsilon}.
\end{aligned}$$

Since $\frac{\mathscr{Q}_0(\sigma,s)}{1+(\ln\sigma)}, \frac{(\ln\sigma)^{\zeta-1}}{1+(\ln\sigma)}$ are uniformly continuous on $[1,\sigma]\times[1,\sigma]$ and $[1,\sigma]$, respectively, we have

$$\left|\frac{^{H}D_{1^+}^{\zeta-2}A\varrho(\sigma_1)}{1+(\ln\sigma_1)} - \frac{^{H}D_{1^+}^{\zeta-2}A\varrho(\sigma_2)}{1+(\ln\sigma_2)}\right| \to 0, \text{as } \sigma_1 \to \sigma_2.$$

Hence, $\frac{^{H}D_{1^+}^{\zeta-2}A\varrho(\sigma)}{1+\ln\sigma}$ is equicontinuous on $[1,\sigma]$. Now we consider

$$^{H}D_{1^+}^{\zeta-1}A\varrho(\sigma) = \int_1^{+\infty} \overline{\mathscr{Q}}(\sigma,\upsilon)\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),^{H}D_{1^+}^{\zeta-2}\varrho(\upsilon),^{H}D_{1^+}^{\zeta-1}\varrho(\upsilon)))\frac{d\upsilon}{\upsilon}.$$

Similarly, $^{H}D_{1^+}^{\zeta-1}A\varrho(\sigma)$ is equicontinuous on ϖ.

The equiconvergence of $A\varrho(\sigma)$ at infinity is established for all $\varrho(\sigma)\in\Omega$. By applying condition (H_1) to an arbitrary $\varrho\in\Omega$, it follows that

$$\begin{aligned}
&\lim_{\sigma\to\infty}\left|\frac{A\varrho(\sigma)}{1+(\ln\sigma)^{\zeta-1}}\right| \\
&=\lim_{t\to\infty}\int_1^{\infty}\frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}\left|\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),^{H}D_{1^+}^{\zeta-2}\varrho(\upsilon),^{H}D_{1^+}^{\zeta-1}\varrho(\upsilon)))\right|\frac{d\upsilon}{\upsilon} \\
&\leq\Upsilon\lim_{\sigma\to\infty}\int_1^{\infty}\Psi_q(g(\upsilon))\varphi_r(\upsilon)\frac{d\upsilon}{\upsilon}<\infty.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\lim_{\sigma\to\infty}\left|\frac{{}^H D_{1^+}^{\zeta-2}A\varrho(\sigma)}{1+(\ln\sigma)}\right| \\
=&\lim_{\sigma\to\infty}\int_1^\infty \frac{\mathscr{Q}_0(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}\left|\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho(\upsilon)))\right|\frac{d\upsilon}{\upsilon} \\
\le&\Gamma(\zeta)\Upsilon\lim_{\sigma\to\infty}\int_1^\infty \Psi_q(g(\upsilon))\varphi_r(\upsilon)\frac{d\upsilon}{\upsilon} \\
<&\infty,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\sigma\to\infty}\left|\frac{{}^H D_{1^+}^{\zeta-1}A\varrho(\sigma)}{1+(\ln\sigma)}\right| =&\lim_{\sigma\to\infty}\int_1^\infty \overline{\mathscr{Q}}(\sigma,\upsilon)\left|\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho(\upsilon)))\right|\frac{d\upsilon}{\upsilon} \\
\le&\Gamma(\zeta)\Upsilon\lim_{\sigma\to\infty}\int_1^\infty \Psi_q(g(\upsilon))\varphi_r(\upsilon)\frac{d\upsilon}{\upsilon} \\
<&\infty.
\end{aligned}$$

Hence, $A, {}^H D_{1^+}^{\zeta-2}A$ and ${}^H D_{1^+}^{\zeta-1}A$ are equiconvergent at ∞. By Lemma 3.1, we get $A: P\to P$ is completely continuous. The proof is completed.

The continuity of A is finally verified. Suppose $\varrho_n\to\varrho$ in P as $n\to\infty$. Then $\|\varrho_n\|<\hbar$, hence $\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho(\upsilon)))\le\varphi_\hbar(\upsilon)$. So

$$\int_1^{+\infty}\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho(\upsilon)))\frac{d\upsilon}{\upsilon}<\int_1^{+\infty}\Psi_q(g(\upsilon))\varphi_\hbar(\upsilon)\frac{d\upsilon}{\upsilon}<+\infty.$$

Combining the continuity of f with an application of the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\int_1^{+\infty}\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho_n(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho_n(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho_n(\upsilon)))\frac{d\upsilon}{\upsilon} \\
\to&\int_1^{+\infty}\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho(\upsilon)))\frac{d\upsilon}{\upsilon},\quad \text{as } n\to+\infty.
\end{aligned}$$

Hence, by Lemma 2.6, we get

$$\begin{aligned}
\|A\varrho_n-A\varrho\|\le\Upsilon\int_1^{+\infty}&\Psi_q(g(\upsilon))\Big|\Psi_q(f(\upsilon,\varrho_n(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho_n(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho_n(\upsilon))) \\
&-\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D_{1^+}^{\zeta-2}\varrho(\upsilon),{}^H D_{1^+}^{\zeta-1}\varrho(\upsilon))))\Big|\frac{d\upsilon}{\upsilon} \\
\to&\, 0, n\to+\infty.
\end{aligned}$$

So A is continuous. In conclusion, we have determined that operator $A:P\to P$ is complete continuity. □

The following notation is introduced for convenience

$$M=\max\{1,\Gamma(\zeta)\}\Upsilon\int_1^{+\infty}p(\upsilon)\Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon},\qquad m=\frac{\mu(k)}{k^{\zeta-1}}\int_{\frac{1}{k}}^{k}\Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon},$$

where $p(\upsilon)$ is given in Theorem 3.1.

Theorem 3.1. *Assume* (H_1)-(H_3) *holds, and for parameters* $0 < a < b < d \leq c$, *the functions* $f : [1, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^+$ *and* $p(t) \in L[1, +\infty)$ *such that*

(H_4) $\Psi_q(f(\upsilon, (1 + (\ln \upsilon)^{\zeta-1})u, (1 + (\ln \upsilon))v, w)) < \frac{c}{M}p(\upsilon)$, $1 \leq \upsilon < +\infty, 0 \leq u, v, w \leq c$;

(H_5) $\Psi_q(f(\upsilon, (1 + (\ln \upsilon)^{\zeta-1})u, (1 + (\ln \upsilon))v, w)) > \frac{b}{m}$, $\frac{1}{k} \leq \upsilon \leq k, b \leq u, v, w \leq c$;

(H_6) $\Psi_q(f(\upsilon, (1 + (\ln \upsilon)^{\zeta-1})u, (1 + (\ln \upsilon))v, w)) < \frac{a}{M}p(\upsilon)$, $1 \leq \upsilon < +\infty, 0 \leq u, v, w \leq a$.

Then fractional boundary value problem (1.1,1.2) exists at least three positive solutions ϱ_1, ϱ_2 *and* ϱ_3 *satisfying* $\|\varrho_1\| < a$,$\gamma(\varrho_2) > b, \|\varrho_3\| > a$ and $\gamma(\varrho_3) < b$.

Proof. Let us begin by defining the functional $\gamma(\varrho)$ over E through

$$\gamma(\varrho) = \min_{\sigma \in [\frac{1}{k}, k]} \frac{\varrho(\sigma)}{1 + (\ln \sigma)^{\zeta-1}},$$

and proceed to confirm the fulfillment the conditions of Theorem 2.1. □

For any elementϱ contained in $\overline{P_c}$, the inequality $\|\varrho\| \leq c$ is satisfied, yielding

$$0 \leq \frac{\varrho(\sigma)}{1 + (\ln \sigma)^{\zeta-1}}, \frac{{}^H D_{1^+}^{\zeta-2}\varrho(\sigma)}{1 + (\ln \sigma)}, {}^H D_{1^+}^{\zeta-1}\varrho(\sigma) \leq c \text{ for } 1 \leq \sigma < +\infty.$$

From (H_4), one obtains

$$\Psi_q(f(\sigma, u, v, w)) = \Psi_q\left(f(\sigma, (1 + (\ln \sigma)^{\zeta-1})\frac{u}{(1 + (\ln \sigma)^{\zeta-1})}, (1 + \ln \sigma)\frac{v}{(1 + \ln \sigma)}, w)\right) \leq \frac{c}{M}.$$

From (H_4), one has

$$\begin{aligned}
&\sup_{1\leq\sigma<+\infty} \frac{|A\varrho(\sigma)|}{1 + (\ln \sigma)^{\zeta-1}} \\
\leq &\sup_{1\leq\sigma<+\infty} \frac{1}{1 + (\ln \sigma)^{\zeta-1}} \left|\int_1^\infty \mathscr{Q}(\sigma, \upsilon)\Psi_q(g(\upsilon))\Psi_q(f(\upsilon, \varrho(\upsilon), {}^H D_{1^+}^{\zeta-2}\varrho(\upsilon), {}^H D_{1^+}^{\zeta-1}\varrho(\upsilon))))\frac{d\upsilon}{\upsilon}\right| \\
< &\frac{\Upsilon c}{M}\int_1^\infty p(\upsilon)\Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon} \\
\leq &c, \\
&\sup_{1\leq\sigma<+\infty} \frac{|{}^H D_{1^+}^{\zeta-2}A\varrho(\sigma)|}{1 + (\ln \sigma)} \\
\leq &\sup_{1\leq\sigma<+\infty} \frac{1}{1 + \ln \sigma} \left|\int_1^\infty \mathscr{Q}_0(\sigma, \upsilon)\Psi_q(g(\upsilon))\Psi_q(f(\upsilon, \varrho(\upsilon), {}^H D_{1^+}^{\zeta-2}\varrho(\upsilon), {}^H D_{1^+}^{\zeta-1}\varrho(\upsilon))))\frac{d\upsilon}{\upsilon}\right| \\
< &\frac{\Gamma(\zeta)\Upsilon c}{M}\int_1^\infty p(\upsilon)\Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon} \\
\leq &c, \\
&\sup_{1\leq\sigma<+\infty} |{}^H D_{1^+}^{\zeta-1}A\varrho(\sigma)| \\
\leq &\sup_{1\leq\sigma<+\infty} \left|\int_1^\infty \overline{\mathscr{Q}}(\sigma, \upsilon)\Psi_q(g(\upsilon))\Psi_q(f(\upsilon, \varrho(\upsilon), {}^H D_{1^+}^{\zeta-2}\varrho(\upsilon), {}^H D_{1^+}^{\zeta-1}\varrho(\upsilon))))\frac{d\upsilon}{\upsilon}\right|
\end{aligned}$$

$$\leq \frac{\Gamma(\zeta)\Upsilon c}{M}\int_1^{\infty} p(\upsilon))\Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon}$$
$$\leq c.$$

Therefore, $\|A\varrho\| \leq c$, hence, $A : P_c \to P_c$ and by Lemma 3.2, we have that A is completely continuous. Similarly, from condition (H_6) we get that if $\varrho \in \overline{P_a}$ then $\|A\varrho\| < a$. Then condition A_2 of Lemma 2.4 holds.

Let

$$\varrho^{\star}(\sigma) = \frac{b+c}{2}(1+(\ln\sigma)^{\zeta-1}),\ 1 \leq t < +\infty.$$

Evidently, $\varrho^{\star} \in P$ and $\|\varrho^{\star}\| = \frac{b+c}{2} < c$. According to the definition of $\gamma(\varrho)$, we have

$$\gamma(\varrho^{\star}) = \frac{b+c}{2} > b.$$

Hence, one has

$$\varrho^{\star} \in \{\varrho \in P(\gamma, b, d) : \gamma(\varrho) > b\} \neq \emptyset.$$

For functions ϱ belonging to $P(\gamma, b, d)$, it follows that

$$b \leq \min_{\frac{1}{k}\leq\sigma\leq k} \frac{\varrho(\sigma)}{1+(\ln\sigma)^{\zeta-1}} \leq c.$$

Then assumption (H_5) implies

$$\Psi_q f(\sigma,u,v,w)) = \Psi_q\left((f(\sigma,(1+(\ln\sigma)^{\zeta-1})\frac{u}{(1+(\ln\sigma)^{\zeta-1})},(1+\ln\sigma)\frac{v}{(1+\ln\sigma)},w)\right) \geq \frac{b}{m}$$
$$\text{for all } \frac{1}{k} \leq \sigma \leq k, b \leq \|u\|, \|v\|, \|w\| \leq c.$$

Consequently, from (H_5) of Theorem 3.1, we get

$$\begin{aligned}\gamma(A\varrho) &= \min_{e^{\frac{1}{k}}<\sigma\leq e^k} \frac{A\varrho(\sigma)}{1+(\ln\sigma)^{\zeta-1}}\\ &= \min_{e^{\frac{1}{k}}<\sigma\leq e^k} \frac{\int_1^{\infty} \mathscr{Q}(\sigma,\upsilon)\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D^{\zeta-2}\varrho(\upsilon),{}^H D^{\zeta-1}\varrho(\upsilon))))\frac{d\upsilon}{\upsilon}}{1+(\ln\sigma)^{\zeta-1}}\\ &\geq \int_1^{\infty} \min_{e^{\frac{1}{k}}<\sigma\leq e^k} \frac{\mathscr{Q}(\sigma,\upsilon)}{1+(\ln\sigma)^{\zeta-1}}\Psi_q(g(\upsilon))\Psi_q(f(\upsilon,\varrho(\upsilon),{}^H D^{\zeta-2}\varrho(\upsilon),{}^H D^{\zeta-1}\varrho(\upsilon))))\frac{d\upsilon}{\upsilon}\\ &> \frac{b\mu(k)}{mk^{\zeta-1}}\int_{e^{\frac{1}{k}}}^{e^k} \Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon}\\ &= b,\end{aligned}$$

which means $\gamma(A\varrho) > b$, for all $\varrho \in P(\gamma, b, d)$. Condition (A_1) of Theorem 2.1 is fulfilled.

Suppose $\varrho \in P(\gamma, b, c)$ with $\|A\varrho\| > d$. From the definitions:

$$\|\varrho\| \leq c \quad \text{and} \quad b \leq \frac{\varrho(\sigma)}{1+(\ln\sigma)^{\zeta-1}} \leq c \quad (\sigma \geq 1).$$

Invoking (H_5), we derive $\gamma(A\varrho) > b$. Consequently, the condition of Theorem 2.1 (A_3) holds.

Hence, A has at least three fixed points $\varrho_1, \varrho_2, \varrho_3$ such that $\|\varrho_1\| < a$,$\gamma(\varrho_2) > b$, $\|\varrho_3\| > a$ and $\gamma(\varrho_3) < b$. It is thus demonstrated that equations (1.1) and (1.2) possess a minimum of three positive solutions, and the argument is thereby finalized. □

4. An example

In the section, we analyze the following fractional-order boundary value problem posed on an unbounded domain, featuring integral and infinite point boundary constraints

$$\begin{cases} {}^{H}D_{1^+}^{\frac{5}{2}}\varrho(\sigma)+\dfrac{1}{t^2}\Psi_3(f(\sigma,\varrho(\sigma),{}^{H}D^{\frac{1}{2}}\varrho(\sigma),{}^{H}D^{\frac{3}{2}}\varrho(\sigma))=0, \\ \varrho(1)=\varrho'(1)=0,{}^{H}D_{1^+}^{\frac{3}{2}}\varrho(+\infty)=\displaystyle\int_1^{\infty}\frac{e^{-\sigma}}{\sigma^{\frac{3}{2}}}\varrho(\sigma)d\sigma+\sum_{i=1}^{\infty}\frac{1}{2j^2}{}^{H}\varrho(\frac{1}{e^{j^{\frac{4}{3}}}}),1<\sigma<\infty, \end{cases} \tag{4.1}$$

where

$$\Psi_3(f(\sigma,u,v,w))=\begin{cases} \dfrac{1}{1000}|\sin\sigma|+10(\dfrac{u}{1+(\ln\sigma)^{\frac{3}{2}}})^9+5(\dfrac{v}{1+(\ln\sigma)})^6+3w,\|u\|,\|v\|,\|w\|\leq\dfrac{1}{2}, \\ \dfrac{1}{1000}|\sin\sigma|+10(\dfrac{u}{1+\sigma^{\frac{3}{2}}})^9+5(\dfrac{v}{1+(\ln\sigma)})^6+3w+10^5(u-\dfrac{1}{2}), \\ \dfrac{1}{2}\leq\|u\|,\|v\|,\|w\|\leq 1, \\ \dfrac{1}{1000}|\sin\sigma|+10(\dfrac{u}{1+\sigma^{\frac{3}{2}}})^9 \\ +5(\dfrac{v}{1+(\ln\sigma)})^6+3w+5\times10^5,\|u\|,\|v\|,\|w\|\geq 1. \end{cases}$$

In this case, $\zeta=\frac{5}{2},g(\sigma)=\frac{1}{t^2},h(\sigma)=\frac{e^{-\sigma}}{\sigma^{\frac{3}{2}}},\alpha_i=\frac{1}{2j^2},\eta_j=\frac{1}{e^{j^{\frac{4}{3}}}},b=\frac{1}{2}$, and

$$B(\sigma)=\begin{cases} 0, & \sigma\in[0,1), \\ 4, & \sigma\in[1,e), \\ 1, & \sigma\in[e,\infty). \end{cases} \tag{4.2}$$

Select $a=\frac{1}{2},b=1,c=10^5,\ k=4,\ p(t)=\frac{t^3}{1+t^2}$. Then according to direct calculation we get that $\mu(4)=\frac{1}{576}$, and

$$\begin{aligned} &\int_1^{\infty}\Psi_q(g(\sigma))\frac{d\sigma}{\sigma}=\int_1^{\infty}\sigma^{-3}d\sigma=\frac{1}{2}, \\ &\int_1^{+\infty}p(\upsilon)\Psi_q(g(\sigma))\frac{d\upsilon}{\upsilon}=\int_1^{+\infty}\frac{\sigma^3}{1+\sigma^2}\frac{1}{\sigma^2}\frac{d\sigma}{\sigma}=\frac{\pi}{4}, \\ &G_1(\sigma,\upsilon)=g(\sigma,\upsilon)+\frac{\frac{1}{2}\sum_{i=1}^{\infty}\alpha_i g(\eta_i,\upsilon)(\ln\sigma)^{\frac{3}{2}}}{\Delta\Gamma(\frac{5}{2})}, \end{aligned} \tag{4.3}$$

and

$$g(\sigma,\upsilon)=\begin{cases} \dfrac{(\ln\sigma)^{\frac{3}{2}}-(\ln\frac{\sigma}{\upsilon})^{\frac{3}{2}}}{\Gamma(\frac{5}{2})},1\leq\upsilon\leq\sigma<+\infty, \\ \dfrac{(\ln\sigma)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})},1\leq\sigma\leq\upsilon<+\infty, \end{cases}$$

$$\Delta=\Gamma(\zeta)-b\sum_{i=1}^{\infty}\alpha_i(\ln\eta_i)^{\zeta-1}=\frac{3}{4}\sqrt{\pi}-\frac{1}{4}\sum_{j=1}^{\infty}\frac{1}{j^4}\approx 1.3293-0.7028-0.2706=1.0587,$$

$$\Delta_1 = \Delta - \int_1^{\infty} h(\sigma)(\ln\sigma)^{\zeta-1} dB(\sigma) = 1.0587 - 0.7028 = 0.3559,$$

$$\begin{aligned}
\Upsilon =& \frac{1}{\Gamma(\alpha)} + \frac{b\sum_{i=1}^{\infty}\alpha_i}{\Delta\Gamma(\alpha)} g(\eta_i, \upsilon) + \frac{1}{\Delta_1}\int_1^{\infty} h(\sigma) G_1(\sigma, s) dB(t) \\
=& \frac{1}{\Gamma(\frac{5}{2})} + \frac{\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{2j^2}}{1.0587\Gamma(\frac{5}{2}) g(\eta_i, s)} + \frac{1}{0.3559}\int_1^{\infty}\frac{e^{-t}}{t^{\frac{3}{2}}} G_1(t, s) dB(t) \\
\approx& 20.50,
\end{aligned}$$

$$M = \max\{1, \Gamma(\zeta)\}\Upsilon\int_1^{+\infty} p(\upsilon)\Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon} = \max\{1, \Gamma(\frac{5}{2})\}\frac{\pi\Upsilon}{4} = \frac{3\sqrt{\pi}}{4}\frac{3.14\times 20.50}{4} = 21.29,$$

$$m = \frac{\mu(k)}{k^{\alpha-1}}\int_{\frac{1}{k}}^{k}\Psi_q(g(\upsilon))\frac{d\upsilon}{\upsilon} = \frac{\frac{1}{576}}{4^{\frac{3}{2}}}\int_{\frac{1}{4}}^{4}\frac{1}{\upsilon^2}\frac{d\upsilon}{\upsilon} = \frac{1}{576\times 8}\times\frac{1}{2}(4^4 - 4^{-4}) \approx 0.0278.$$

So the nonlinear term f satisfies

$$\begin{aligned}
&\Psi_3(f(\upsilon, (1+(\ln\upsilon)^{\zeta-1})u, (1+(\ln\upsilon))v, w)) \\
<& 0.002 + 5\times(\frac{1}{2})^8 + 5\times(\frac{1}{2})^6 + 3\times\frac{1}{2} < 2349 \\
\leq& \frac{c}{M}p(\upsilon), (\upsilon, u, v, w)\in[0, +\infty)\times[0, 10^5]^3, \\
&\Psi_3(f(\upsilon, (1+(\ln\upsilon)^{\zeta-1})u, (1+(\ln s))v, w)) > 2349 > 35.97 \\
\approx& \frac{b}{m}, (\upsilon, u, v, w)\in[\frac{1}{4}, 4]\times[1, 10^5]^3, \\
&\Psi_3(f(\upsilon, (1+(\ln\upsilon)^{\zeta-1})u, (1+(\ln s))v, w)) < 0.001 + 10(\frac{1}{2})^9 < 0.40783 \\
\approx& \frac{a}{M}p(\upsilon), (\upsilon, u, v, w)\in[0, +\infty)\times[0, \frac{1}{2}].
\end{aligned}$$

Application of Theorem 4.1 guarantees** that (4.1) possesses at least three positive solutions $\{\varrho_i\}_{i=1}^{3}$ with the property that

$$\begin{aligned}
&\sup_{0\leq\sigma<+\infty}\frac{|\varrho_1(\sigma)|}{1+(\ln\sigma)^{\zeta-1}} < \frac{1}{2}, 1 < \min_{\frac{1}{4}\leq\sigma\leq 4}\frac{\varrho_2(\sigma)}{1+(\ln\sigma)^{\zeta-1}}, \\
&\frac{1}{2} < \sup_{0\leq\sigma<+\infty}\frac{|\varrho_3(\sigma)|}{1+(\ln\sigma)^{\zeta-1}}\ with\ \min_{\frac{1}{4}\leq\sigma\leq 4}\frac{\varrho_3(\sigma)}{1+(\ln\sigma)^{\zeta-1}} < 1.
\end{aligned}$$

5. Conclusions

In this study, we have conducted a theoretical investigation into a class of boundary value problems defined on an infinite interval, which involve Hadamard fractional derivatives coupled with mixed integral and infinite-point boundary conditions. Such problems are instrumental in modeling complex processes in science and engineering. The core of our analysis lay in the detailed derivation of the properties of the associated Green function. Based on these properties, we successfully constructed an appropriate cone within a carefully selected functional space. By applying fixed point theorems on this cone, we established rigorous criteria for the existence of multiple positive solutions to the proposed problem. The theoretical findings were further substantiated through a computational case study, which confirmed the practical applicability and effectiveness of the developed framework. This work thus provides a solid theoretical foundation and a verifiable method for analyzing nonlinear fractional systems subject to complex constraints over unbounded domains.

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