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AN ALTERNATIVE RIEMANN-HILBERT APPROACH AND ITS APPLICATION TO THE GLOBAL ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS

Jie Yan[1,2,3], Huili Han[1,†], Min Ku[4] and Xin Wei[5]

Abstract This work develops a direct solution method for a class of matrix Riemann-Hilbert (MRH) problems, avoiding the conventional reduction to scalar formulations. By leveraging the nonlinear steepest descent framework introduced by Deift and Zhou, we construct three invertible transformations that systematically reduce the MRH problem for orthogonal polynomials to an explicitly solvable model. Through analytic continuation and a rigorous treatment of boundary conditions, we obtain precise asymptotic expansions in both oscillatory and exponentially decaying regimes. Furthermore, our approach yields global asymptotics for orthogonal polynomials on extended complex domains, surpassing existing results confined to the unit circle or the real axis. These findings demonstrate the efficiency of our alternative Riemann-Hilbert approach and significantly broaden the utility of such techniques in the asymptotic analysis of orthogonal polynomials.

Keywords Alternative Riemann-Hilbert approach, global asymptotic analysis, orthogonal polynomials with analytic weight.

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1. Introduction

Orthogonal polynomials have been extensively studied with respect to their classical properties, such as recurrence relations, zero distributions, and asymptotic behavior [6, 24]. Nevertheless, high-precision asymptotic analysis, particularly for polynomials of large degree as $n \to \infty$, remains a significant challenge in mathematical research [3]. The asymptotic theory of orthogonal polynomials has traditionally focused on two principal limiting regimes: The large-degree limit $n \to \infty$, and the behavior of the variable x within specific regions (such as inside or outside the oscillatory domain) [1, 4, 12, 17, 25, 28]. Although these methods are effective within certain parameter ranges, they often fail to yield uniform asymptotic approximations over the entire domain. More importantly, deriving globally valid asymptotic expressions remains a substantial theoretical challenge within classical analytical frameworks [5].

[†]The corresponding author.
[1]School of Mathematics and Statistics, Ningxia University, Yinchuan 750021, China
[2]School of Mathematics and Computer Science, Ningxia Normal University, Guyuan 756099, China
[3]Ningxia Key Laboratory of Artificial Intelligence and Basic Education, Ningxia Normal University, Guyuan 756099, China
[4]Institute for Computing and Information Sciences, University of Radboud, the Netherlands
[5]School of Science, Xi'an Shiyou University, Xi'an 710065, China
Email: 18795390879@163.com(J. Yan), nxhan@126.com(H. Han), kumin0844@163.com(M. Ku), xwei@xsyu.edu.cn(X. Wei)

A transformative breakthrough occurred in the 1990s with the development of the Riemann-Hilbert approach, pioneered by Deift, Zhou, and their collaborators [8]. In 1992, Fokas, Its, and Kitaev [13] first applied this approach to orthogonal polynomials, establishing a powerful framework for deriving strong asymptotic behavior. Their foundational work spurred numerous subsequent advances [7, 8, 10, 15, 20, 27]. The Riemann-Hilbert approach reformulates the asymptotic analysis of orthogonal polynomials as a matrix Riemann-Hilbert (MRH) problem in the complex plane, as described below

Problem 1.1. *Let $\Gamma \subset \mathbb{C}$ be a contour such that 0 is contained in the bounded domain, denoted by Ω^+, which is enclosed by Γ. Define $\Omega^- = \mathbb{C} \setminus \overline{\Omega^+}$. Given an $m \times m$ matrix function $\mathbf{J}(t)$ defined on Γ that is Hölder continuous, we determine matrix functions $\mathbf{\Psi}^{\pm}(z)$ satisfying*

(1) $\mathbf{\Psi}^+(z)$ is analytic in Ω^+ and continuous up to Γ.

(2) $\mathbf{\Psi}^-(z)$ is analytic in Ω^- and continuous up to Γ, with $\mathbf{\Psi}^-(\infty) = \mathbf{I}_{m\times m}$.

(3) For all $t \in \Gamma$, the boundary condition holds

$$\mathbf{\Psi}^+(t) = \mathbf{\Psi}^-(t)\mathbf{J}(t). \tag{1.1}$$

When $m = 1$, Problem 1.1 reduces to the scalar case, for which the solvability conditions and solution structures are well-established [21, 23]. However, for $m > 1$, Problem 1.1 becomes a challenging matrix boundary value problem for general $\mathbf{J}(t)$. The main difficulty, compared to the scalar case, arises because the involved matrix functions do not commute. Consequently, the order of operations becomes crucial, and explicit solutions are exceedingly rare [18]. The MRH problem typically requires reformulation into an $m \times m$ system of scalar Riemann-Hilbert problems [9, 10, 15, 16, 27]. Although this approach yields solvability conditions and solution constructions [26], it involves intricate mathematical manipulations, especially as m increases. Given these challenges, developing a direct solution method for a class of MRH problems, without reduction to scalar form, is theoretically and practically significant.

In this paper, we propose a direct solution method for the MRH Problem 2.1 that preserves the matrix structure, circumventing the limitations of scalar-based approaches especially as the dimensionality increases. Our approach offers an explicit solution representation, supported by rigorous theoretical foundations developed in Section 2.1. Several illustrative examples in Section 2.2 demonstrate the computational efficiency and practical utility of the method. In Section 3, we characterize Problem 2.1 through the distinctive properties of orthogonal polynomials. Building upon this theoretical framework, Section 4 employs the Deift-Zhou nonlinear steepest descent method to construct three invertible transformations. These transformations systematically reduce Problem 2.1 to an explicitly solvable model, thereby establishing a complete analytical framework for its solution. In Section 5, we rigorously characterize the asymptotic behavior of orthogonal polynomials in both oscillatory and exponentially decaying regimes through analytic continuation and a detailed analysis of boundary conditions derived from an alternative Riemann-Hilbert approach. Finally, in Section 6, we establish strong asymptotic results for orthogonal polynomials on extended complex domains. These results not only generalize existing theory but also validate the efficacy of our alternative Riemann-Hilbert approach. Unlike classical Riemann-Hilbert approach, which often restrict asymptotic analysis to the unit circle or the real axis, our framework provides complete global asymptotic behavior, encompassing interior, exterior, and their boundaries, along with a full characterization of zero distributions. This significantly broadens the scope of applicability and resolves key limitations in prior work.

2. Direct solution to the MRH problem

With $0 < \rho < 1$, we consider the strictly positive weights ω on the unit circle $\mathbb{T}_1$ that can be extended to an open set $A_\rho = \{\rho < |z| < \frac{1}{\rho}\}$ containing $\mathbb{T}_1$ as a non-vanishing holomorphic function, which can be expressed as a Laurent series with the following representation

$$\omega(z) = \sum_{j=-\infty}^{\infty} e_j z^j, \ z \in A_\rho. \tag{2.1}$$

The coefficients e_j are given by

$$e_j = \frac{1}{2\pi}\int_{\mathbb{T}_1} \omega(t)t^{-j}\frac{\mathrm{d}t}{it}, \tag{2.2}$$

and they satisfy the symmetry condition $e_{-j} = \bar{e}_j$ for all $j \in \mathbb{Z}$. To decompose $\omega(z)$, we need setting

$$\omega^+(z) = \sum_{j=0}^{\infty} e_j z^j, \ \omega^-(z) = -\sum_{j=-\infty}^{-1} e_j z^j, \tag{2.3}$$

where $\omega^+(z)$ and $\omega^-(z)$ are holomorphic in $\mathbb{D}_\rho^+ = \{|z| < \frac{1}{\rho}\}$ and $\mathbb{D}_\rho^- = \{|z| > \rho\}$, respectively, with the equality $\omega(z) = \omega^+(z) - \omega^-(z)$. Additionally, the symmetry property is clear that

$$\omega(z) = \overline{\omega}(1/z), \ z \in A_\rho, \tag{2.4}$$

where $\overline{\omega}(z) = \overline{\omega(\overline{z})}$ denotes the complex conjugate of $\omega(z)$.

In reference [10], a 2×2 MRH problem emerges from the investigation of orthogonal polynomials defined on the unit circle $\mathbb{T}_1$ with respect to a weight function $\omega(t)$. The problem formulation requires the construction of a sectionally holomorphic matrix function $\mathbf{\Psi} : \mathbb{C} \setminus \mathbb{T}_1 \to \mathbb{C}^{2\times 2}$ that satisfies

Problem 2.1.

(1) $\mathbf{\Psi}$ is analytic in $\mathbb{C} \setminus \mathbb{T}_1$.

(2) For $|t| = 1$, the boundary values satisfy

$$\mathbf{\Psi}^+(t) = \mathbf{\Psi}^-(t)\begin{pmatrix} 1 & t^{-n}\omega(t) \\ 0 & 1 \end{pmatrix}, \tag{2.5}$$

where n is a positive integer, and the weight function $\omega(t) \in L^1(\mathbb{T}_1)$, such that a key constant

$$e_0 = \frac{1}{2\pi}\int_{\mathbb{T}_1} \omega(t)\frac{\mathrm{d}t}{it} > 0. \tag{2.6}$$

(3) $\mathbf{\Psi}^+(0) = \mathbf{0}$.

(4) As $z \to \infty$ in $\mathbb{C}$, the asymptotic behavior

$$\mathbf{\Psi}^-(z)\begin{pmatrix} z^{-n-1} & 0 \\ 0 & z^{n-1} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.7}$$

where $\mathbf{\Psi}^+(z)$ and $\mathbf{\Psi}^-(z)$ are unknown analytic 2×2 matrix functions defined on $\mathbb{D}^+ = \{|z| < 1\}$ and $\mathbb{D}^- = \{|z| > 1\}$, respectively.

Set

$$\mathbf{R}(z) = \begin{pmatrix} R_{1,1}(z) \; R_{1,2}(z) \\ R_{2,1}(z) \; R_{2,2}(z) \end{pmatrix}, \; z \in \Omega, \tag{2.8}$$

where each entry $R_{j,k}(z),\ j,k = 1,2$ is defined on an open domain $\Omega \subseteq \mathbb{C}$ and shares the same analytic or continuous properties as $\mathbf{R}(z)$. The max norm of $\mathbf{R}(z)$ is defined by

$$\|\mathbf{R}(z)\| = \max\left\{|R_{1,1}(z)|,\ |R_{1,2}(z)|,\ |R_{2,1}(z)|,\ |R_{2,2}(z)|\right\},\ z \in \Omega, \tag{2.9}$$

and its supremum norm over Ω is given by

$$||\mathbf{R}(z)||_\Omega = \sup_{z\in\Omega} ||\mathbf{R}(z)||. \tag{2.10}$$

If $\mathbf{R}(z)$ is continuous on the closure $\overline{\Omega} = \Omega \cup \partial\Omega$, then by the maximum principle for norms, $\|\mathbf{R}(z)\|$ attains its supremum on the boundary $\partial\Omega$.

Subsequently, we present a direct solution method for solving the MRH problem 2.1. Our approach yields an explicit solution, which we rigorously derive and analyze.

2.1. Direct solution method to the MRH Problem 2.1

Let

$$\mathbf{\Psi}(z) = \begin{pmatrix} \Psi_{1,1}(z) \; \Psi_{1,2}(z) \\ \Psi_{2,1}(z) \; \Psi_{2,2}(z) \end{pmatrix}, \; z \in \mathbb{C} \setminus \mathbb{T}_1. \tag{2.11}$$

Firstly, we say that $\omega^+(z)$ and $\omega^-(z)$ be analytic functions defined on the domains $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively. These functions satisfy the boundary condition

$$\omega(t) = \omega^+(t) - \omega^-(t), \quad |t| = 1. \tag{2.12}$$

This leads to the MRH Problem 2.1, which admits the equivalent formulation

Problem 2.2.

(1) $\Psi_{1,1}(z),\ \Psi_{1,2}(z),\ \Psi_{2,1}(z)$, and $\Psi_{2,2}(z)$ are analytic in $\mathbb{C} \setminus \mathbb{T}_1$.

(2) For $|t| = 1$, the boundary values satisfy

$$\begin{pmatrix} \Psi^+_{1,1}(t) \; \Psi^+_{1,2}(t) - t^{-n}\Psi^+_{1,1}(t)\omega^+(t) \\ \Psi^+_{2,1}(t) \; \Psi^+_{2,2}(t) - t^{-n}\Psi^+_{2,1}(t)\omega^+(t) \end{pmatrix} = \begin{pmatrix} \Psi^-_{1,1}(t) \; \Psi^-_{1,2}(t) - t^{-n}\Psi^-_{1,1}(t)\omega^-(t) \\ \Psi^-_{2,1}(t) \; \Psi^-_{2,2}(t) - t^{-n}\Psi^-_{2,1}(t)\omega^-(t) \end{pmatrix}. \tag{2.13}$$

(3) $\Psi^+_{1,1}(0) = 0,\ \Psi^+_{1,2}(0) = 0,\ \Psi^+_{2,1}(0) = 0,\ \Psi^+_{2,2}(0) = 0$.

(4) As $z \to \infty$, we have the following asymptotic expansions

$$\begin{aligned} &\Psi^-_{1,1}(z) = z^{n+1} + \mathcal{O}(z^n),\ \Psi^-_{1,2}(z) = \mathcal{O}(z^{-n}), \\ &\Psi^-_{2,1}(z) = \mathcal{O}(z^n),\ \Psi^-_{2,2}(z) = z^{-n+1} + \mathcal{O}(z^{-n}), \end{aligned} \tag{2.14}$$

where $\Psi^+_{j,k}(z)$ and $\Psi^-_{j,k}(z)$ for $j,k = 1,2$ are analytic functions on $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively.

From the $(1,1)$ entry of the MRH Problem 2.2, we obtain the boundary condition

$$\Psi_{1,1}^{+}(t) = \Psi_{1,1}^{-}(t),\ |t| = 1. \tag{2.15}$$

The functions $\Psi_{1,1}^{\pm}(z)$ are analytic in $\mathbb{D}^{\pm}$ and continuous up to $\mathbb{T}_1$, respectively. By the Painlevé theorem, $\Psi_{1,1}(z)$ admits an analytic continuation to an entire function. From the growth condition

$$\Psi_{1,1}^{-}(z) = z^{n+1} + \mathcal{O}(z^{n}),\ z \to \infty, \tag{2.16}$$

and $\Psi_{1,1}^{+}(0) = 0$, we get

$$\Psi_{1,1}(z) = z^{n+1} + a_1 z^{n} + \cdots + a_n z = z^{n+1} + A(z),\ z \in \mathbb{C}, \tag{2.17}$$

where $A(z) = \sum\limits_{j=1}^{n} a_j z^{n-j+1}$. Similarly, $\Psi_{2,1}(z)$ has the form

$$\Psi_{2,1}(z) = c_1 z^{n} + \cdots + c_n z,\ z \in \mathbb{C}. \tag{2.18}$$

Given a formal Laurent series

$$f(z) = \sum_{j=-\infty}^{\infty} f_j z^{j},\ f_j \in \mathbb{C}, \tag{2.19}$$

we define the extraction operator $\mho_n$ as

$$\mho_n(f(z)) = \sum_{j=-n+1}^{0} f_j z^{j}, \tag{2.20}$$

which selects the part of $f(z)$ with terms ranging from z^{-n+1} to z^{0}, while the complementary operator G_n removes these terms, given by

$$G_n(f(z)) = f(z) - \mho_n(f(z)). \tag{2.21}$$

Moreover, by the $(1,2)$ entry of the MRH Problem 2.2, we know

$$\Psi_{1,2}^{+}(t) - t^{-n}\Psi_{1,1}(t)\omega^{+}(t) = \Psi_{1,2}^{-}(t) - t^{-n}\Psi_{1,1}(t)\omega^{-}(t),\ |t| = 1. \tag{2.22}$$

It is noted that $\Psi_{1,2}^{\pm}(z) - z^{-n}\Psi_{1,1}(z)\omega^{\pm}(z)$ admits an analytic continuation to an entire function, denoted by $W(z)$. As $z \to \infty$, then using the asymptotic behavior

$$W(z) = \Psi_{1,2}^{-}(z) - z^{-n}\Psi_{1,1}(z)\omega^{-}(z) = \mathcal{O}(z^{-n}) - z^{-n} \cdot z^{n+1} \cdot \mathcal{O}(z^{-1}), \tag{2.23}$$

where $\mathcal{O}(z^{-n}) \to 0$ as $z \to \infty$, and the term $z \cdot \mathcal{O}(z^{-1})$ remains bounded. Hence, $W(z)$ is bounded and entire, and by Liouville's theorem, it must be

$$W(z) = C, \tag{2.24}$$

for some constant C. Consequently, we have

$$\Psi_{1,2}^{\pm}(z) = z^{-n}\Psi_{1,1}(z)\omega^{\pm}(z) + C,\ z \in \mathbb{D}^{\pm}. \tag{2.25}$$

Since the Laurent series of both $\Psi^{\pm}_{1,2}(z)$ contain no terms of the form z^{-j} for $0 \leq j \leq n-1$, we then obtain

$$\begin{aligned}\mho_n\big(\Psi^{+}_{1,2}(z)\big) &= \mho_n\big(z^{-n}\Psi_{1,1}(z)\omega^{+}(z) + C\big) = 0,\\ \mho_n\big(\Psi^{-}_{1,2}(z)\big) &= \mho_n\big(z^{-n}\Psi_{1,1}(z)\omega^{-}(z) + C\big) = 0,\end{aligned} \tag{2.26}$$

combining these two conditions, we deduce that

$$\mho_n\big(z^{-n}\Psi_{1,1}(z)\omega(z)\big) = 0,\ z \in A_\rho. \tag{2.27}$$

Therefore, the action of G_n on $\Psi_{1,2}(z)$ simplifies to

$$z^{-n}\Psi_{1,1}(z)\omega(z) = G_n\big(z^{-n}\Psi_{1,1}(z)\omega(z)\big),\ z \in A_\rho, \tag{2.28}$$

substituting (2.28) into (2.25), one can immediately have

$$\begin{aligned}\Psi^{+}_{1,2}(z) &= G_n\big(z^{-n}\Psi_{1,1}(z)\omega^{+}(z)\big),\ z \in \mathbb{D}^{+},\\ \Psi^{-}_{1,2}(z) &= G_n\big(z^{-n}\Psi_{1,1}(z)\omega^{-}(z)\big),\ z \in \mathbb{D}^{-}.\end{aligned} \tag{2.29}$$

Following a similar argument as in the derivation of (2.25), we obtain

$$\Psi^{\pm}_{2,2}(z) = z^{-n}\Psi_{2,1}(z)\omega^{\pm}(z),\ z \in \mathbb{D}^{\pm}. \tag{2.30}$$

Applying the extraction operator $\mho_n$ to the power series expansions of $\Psi^{\pm}_{1,2}(z)$ yields

$$\begin{aligned}\mho_n\big(\Psi^{+}_{2,2}(z)\big) &= \mho_n\big(z^{-n}\Psi_{2,1}(z)\omega^{+}(z)\big) = 0,\\ \mho_n\big(\Psi^{-}_{2,2}(z)\big) &= \mho_n\big(z^{-n}\Psi_{2,1}(z)\omega^{-}(z)\big) = z^{-n+1},\end{aligned} \tag{2.31}$$

combining these results gives

$$\mho_n\big(z^{-n}\Psi_{2,1}(z)\omega(z)\big) = -z^{-n+1},\ z \in A_\rho. \tag{2.32}$$

Consequently, we obtain the following expressions

$$\begin{aligned}\Psi^{+}_{2,2}(z) &= G_n\big(z^{-n}\Psi_{2,1}(z)\omega^{+}(z)\big),\ z \in \mathbb{D}^{+},\\ \Psi^{-}_{2,2}(z) &= G_n\big(z^{-n}\Psi_{2,1}(z)\omega^{-}(z)\big) + z^{-n+1},\ z \in \mathbb{D}^{-}.\end{aligned} \tag{2.33}$$

To summarize the foregoing discussion, the solution to the MRH Problem 2.2 takes the form

$$\boldsymbol{\Psi}^{\pm}(z) = \begin{pmatrix}\Psi_{1,1}(z) & G_n\big(z^{-n}\Psi_{1,1}(z)\omega^{\pm}(z)\big)\\ \Psi_{2,1}(z) & G_n\big(z^{-n}\Psi_{2,1}(z)\omega^{\pm}(z)\big) + \gamma^{\pm}(z)\end{pmatrix},\ z \in \mathbb{D}^{\pm}, \tag{2.34}$$

where $\gamma^{+}(z) = 0$, $\gamma^{-}(z) = z^{-n+1}$, which serves as a boundary value representation.

From (2.32) we obtain

$$z^{-n}\Psi_{2,1}(z)\omega(z) = G_n\big(z^{-n}\Psi_{2,1}(z)\big) - z^{-n+1}, \tag{2.35}$$

multiplying both sides by z^{n-2} gives

$$\begin{aligned}z^{-2}\Psi_{2,1}(z)\omega(z) &= z^{n-2}G_n\big(z^{-n}\Psi_{2,1}(z)\omega(z)\big) - z^{-1}\\ &= z^{n-2}G_n\big(z^{-n}\Psi_{2,1}(z)\omega^{+}(z)\big) - \big(z^{n-2}G_n\big(z^{-n}\Psi_{2,1}(z)\omega^{-}(z)\big) + z^{-1}\big)\\ &= \Lambda_1(z) - \Lambda_2(z),\ z \in \mathbb{D}^{\pm},\end{aligned} \tag{2.36}$$

here, $\Lambda_1(z)$ consists of terms with non-negative powers of z (i.e., terms z^j for $j \geq 0$), while $\Lambda_2(z)$ contains the negative powers of z (i.e., terms z^j for $j < 0$). To decompose the left hand side of (2.36) into these two parts, we apply the Cauchy integral formula yields

$$\frac{1}{2\pi i}\int_{\mathbb{T}_1}\frac{\Psi_{2,1}(t)\omega(t)\,\mathrm{d}t}{t^2(t-z)} = z^{n-2}G_n\big(z^{-n}\Psi_{2,1}(z)\omega^{\pm}(z)\big) + z^{n-2}\gamma^{\pm}(z),\ z\in\mathbb{D}^{\pm}, \tag{2.37}$$

or equivalently,

$$\frac{z^{-n+2}}{2\pi i}\int_{\mathbb{T}_1}\frac{\Psi_{2,1}(t)\omega(t)\,\mathrm{d}t}{t^2(t-z)} = G_n\big(z^{-n}\Psi_{2,1}(z)\omega^{\pm}(z)\big) + \gamma^{\pm}(z),\ z\in\mathbb{D}^{\pm}. \tag{2.38}$$

Similarly, (2.28) can be expressed as

$$\frac{z^{-n+1}}{2\pi i}\int_{\mathbb{T}_1}\frac{\Psi_{1,1}(t)\omega(t)\,\mathrm{d}t}{t(t-z)} = G_n\big(z^{-n}\Psi_{1,1}(z)\omega^{\pm}(z)\big),\ z\in\mathbb{D}^{\pm}. \tag{2.39}$$

Substituting the left sides of the last two equations into (2.34), it is readily seen the integral representation as

$$\mathbf{\Psi}(z) = \begin{pmatrix} \Psi_{1,1}(z) & \dfrac{z^{-n+1}}{2\pi i}\displaystyle\int_{\mathbb{T}_1}\frac{\Psi_{1,1}(t)\omega(t)\,\mathrm{d}t}{t(t-z)} \\ \Psi_{2,1}(z) & \dfrac{z^{-n+2}}{2\pi i}\displaystyle\int_{\mathbb{T}_1}\frac{\Psi_{2,1}(t)\omega(t)\,\mathrm{d}t}{t^2(t-z)} \end{pmatrix},\ z\in\mathbb{C}\setminus\mathbb{T}_1. \tag{2.40}$$

Now, (2.34) or (2.40) provides the solution to the MRH problem 2.2, provided that the functions $\Psi_{1,1}(z)$ and $\Psi_{2,1}(z)$ can be determined. From (2.17) and (2.27), we observe that

$$\mho_n\big(z^{-n}A(z)\omega(z)\big) = -\mho_n\big(z\omega(z)\big), \tag{2.41}$$

which is

$$\mho_n\big(z^{-n}A(z)\omega(z)\big) = -\sum_{j=0}^{n-1}\bar{e}_{j+1}z^{-j}, \tag{2.42}$$

we calculate the left side to be

$$\mho_n\Big(z\sum_{k=1}^{n}a_kz^{-k}\sum_{j=-\infty}^{\infty}e_jz^j\Big) = \sum_{j=0}^{n-1}\Big(\sum_{k=1}^{n}a_ke_{k-j-1}\Big)z^{-j}, \tag{2.43}$$

by equating the last two expressions, we obtain

$$\sum_{k=1}^{n}a_ke_{k-j-1} = -\bar{e}_{j+1},\ j=0,1,\cdots,n-1. \tag{2.44}$$

Therefore,

$$\mathbf{E}_n\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix} = -\begin{pmatrix}\bar{e}_1\\\bar{e}_2\\\vdots\\\bar{e}_n\end{pmatrix}, \tag{2.45}$$

where $\mathbf{E}_n$ is an $n \times n$ Toeplitz matrix whose elements are Laurent coefficients of ω, given by

$$\mathbf{E}_n = \begin{pmatrix} e_0 & e_1 & \cdots & e_{n-1} \\ e_{-1} & e_0 & \cdots & e_{n-2} \\ \vdots & \vdots & & \vdots \\ e_{-n+1} & e_{-n+2} & \cdots & e_0 \end{pmatrix}, \tag{2.46}$$

since $e_{-j} = \bar{e}_j$, $\mathbf{E}_n$ is Hermitian. Note that (2.32) is equivalent to

$$\mathbf{E}_n \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}. \tag{2.47}$$

Consequently, we have the following result.

Theorem 2.1. *Let $\omega \in L^1(\mathbb{T}_1)$ be a real valued function. If $\omega > 0$ almost everywhere on $\mathbb{T}_1$ and e_0, defined by* (2.6)*, is the main diagonal elements of the matrix $\mathbf{E}_n$ given in* (2.46)*, then $\mathbf{E}_n$ is positive definite for every positive integer n.*

Proof. For any non-zero column vector $(u_0, \cdots, u_{n-1})^T$, the quadratic form

$$\begin{aligned} (\bar{u}_0 \ \cdots \ \bar{u}_{n-1})\mathbf{E}_n \begin{pmatrix} u_0 \\ \vdots \\ u_{n-1} \end{pmatrix} &= \sum_{j,k=0}^{n-1} \bar{u}_j e_{k-j} u_k \\ &= \sum_{j,k=0}^{n-1} \bar{u}_j \cdot \frac{1}{2\pi} \int_{\mathbb{T}_1} t^j w(t) t^{-k} \frac{\mathrm{d}t}{it} \cdot u_k \\ &= \frac{1}{2\pi} \int_{\mathbb{T}_1} \omega(t) \big|\bar{u}_0 + \bar{u}_1 t + \cdots + \bar{u}_{n-1} t^{n-1}\big|^2 \frac{\mathrm{d}t}{it}. \end{aligned} \tag{2.48}$$

Obviously, the last expression must be positive, because a non-zero polynomial cannot be identically zero on the set $\{t \in \mathbb{T}_1 \mid \omega(t) > 0\}$, which has positive (one-dimensional Lebesgue) measure. Thus, $\mathbf{E}_n$ is positive definite. The proof is complete. $\square$

The matrix $\mathbf{E}_n$ is positive definite and hence non-singular. We therefore obtain

$$\Psi_{1,1}(z) = z^{n+1} + \sum_{j=1}^{n} a_j z^{n-j+1}, \ \Psi_{2,1}(z) = \sum_{j=1}^{n} c_j z^{n-j+1}, \tag{2.49}$$

with

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = -\mathbf{E}_n^{-1} \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \vdots \\ \bar{e}_n \end{pmatrix}, \quad \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{E}_n^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \tag{2.50}$$

and $\mathbf{E}_n$ is defined in (2.46).

To summarize the foregoing discussion, we now state the main theorem.

Theorem 2.2. *Let n be a positive integer, and let ω be a weight function satisfying* (2.6)*. Then, the MRH Problem 2.1 admits a unique solution, explicitly expressed by the formulas* (2.49) *and* (2.34)*, or equivalent, by* (2.49) *and* (2.40)*.*

2.2. Examples

For an $m \times p$ matrix $\mathbf{Q} = (q_{jk})$, define the $m \times p$ flipped conjugate matrix $\mathbf{Q}^s$ by

$$\mathbf{Q}^s = (q^s_{jk}),\ q^s_{jk} = q_{m-j+1,\ p-k+1},\ j = 1, \cdots, m,\ k = 1, \cdots, p, \tag{2.51}$$

this operation reverses both rows and columns. It is straightforward to verify that $\mathbf{Q}^s\mathbf{R}^s = (\mathbf{QR})^s$, if $\mathbf{QR}$ is well-defined. Let $\bar{\mathbf{Q}}^s$ denote the complex conjugate of $\mathbf{Q}^s$, and then $(\overline{\mathbf{QR}})^s = \bar{\mathbf{Q}}^s\bar{\mathbf{R}}^s$.

We demonstrate the result through the following examples.

Example 2.1. For the case $n = 2$ with the weight function $\omega(t) = 1 + \frac{1}{3}t + \frac{1}{3}t^{-1}$, we consider the MRH Problem 2.1.

A key observation now is that

$$\omega^+(t) = 1 + \frac{1}{3}t,\ \omega^-(t) = -\frac{1}{3}t^{-1}. \tag{2.52}$$

By (2.45) and (2.47), we have, for (2.46) with (2.2),

$$\begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \tag{2.53}$$

solving for a_1, a_2, c_1, c_2, in terms of (2.49), we obtain

$$\Psi_{1,1}(z) = z^3 - \frac{3}{8}z^2 + \frac{1}{8}z,\ \Psi_{2,1}(z) = \frac{3}{8}z^2 - \frac{9}{8}z. \tag{2.54}$$

Thus,

$$\mathbf{\Psi}^+(z) = \begin{pmatrix} z^3 - \frac{3}{8}z^2 + \frac{1}{8}z & \frac{1}{3}z^2 + \frac{7}{8}z \\ \frac{3}{8}z^2 - \frac{9}{8}z & \frac{1}{8}z \end{pmatrix},\ \mathbf{\Psi}^-(z) = \begin{pmatrix} z^3 - \frac{3}{8}z^2 + \frac{1}{8}z & -\frac{1}{24}z^{-2} \\ \frac{3}{8}z^2 - \frac{9}{8}z & z^{-1} + \frac{3}{8}z^{-2} \end{pmatrix}, \tag{2.55}$$

constitutes the unique solution satisfying all boundary conditions and analyticity requirements of the MRH problem in Example 2.1.

Example 2.2. Let n be a positive integer, and let b be a fixed complex number with $|b| < \frac{1}{2}$ and $\omega(t) = 1 + bt + \bar{b}t^{-1}$, where $\bar{b}$ denotes the complex conjugate of b. Provide a direct solution to the MRH Problem 2.1.

By straightforward calculations, we have $e_0 = 1, e_1 = b, \bar{e}_1 = \bar{b}$, and $e_j = \bar{e}_j = 0$ for $2 \leq j \leq n$. The $n \times n$ matrix $\mathbf{E}_n$ is defined by

$$\mathbf{E}_n = \begin{pmatrix} 1 & b & 0 & \cdots & 0 & 0 \\ \bar{b} & 1 & b & \cdots & 0 & 0 \\ 0 & \bar{b} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{b} & 1 \end{pmatrix}, \tag{2.56}$$

from (2.51), $\mathbf{E}_n^s$ can be represented as

$$\mathbf{E}_n^s = \begin{pmatrix} 1 & \bar{b} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \bar{b} & 0 \\ 0 & 0 & \cdots & b & 1 & \bar{b} \\ 0 & 0 & \cdots & 0 & b & 1 \end{pmatrix} = \begin{pmatrix} 1 & \bar{b} & 0 & \cdots & 0 & 0 \\ b & 1 & \bar{b} & \cdots & 0 & 0 \\ 0 & b & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b & 1 \end{pmatrix}, \tag{2.57}$$

one easily gets $\overline{\mathbf{E}_n^s} = \mathbf{E}_n$. Let $\mathbf{E}_n^{-1} = (\mathbf{L}_1, \cdots, \mathbf{L}_n)$, and $\mathbf{L}_1 = (\tau_1, \cdots, \tau_n)^T$ be the first column of $\mathbf{E}_n^{-1}$. To see that

$$\overline{(\mathbf{E}_n \mathbf{E}_n^{-1})^s} = \overline{\mathbf{E}_n^s}\,\overline{(\mathbf{E}_n^{-1})^s} = \mathbf{I}, \tag{2.58}$$

with $\mathbf{I}$ being the n dimensional identity matrix, from which we deduce

$$(\overline{\mathbf{E}_n^s})^{-1} = \overline{(\mathbf{E}_n^{-1})^s} = \overline{(\mathbf{L}_1, \cdots, \mathbf{L}_n)^s} = (\overline{\mathbf{L}_n^s}, \cdots, \overline{\mathbf{L}_1^s}), \tag{2.59}$$

and

$$\mathbf{E}_n^{-1} = \overline{(\mathbf{E}_n^{-1})^s} = (\mathbf{L}_1, \cdots, \mathbf{L}_n), \tag{2.60}$$

which implies that $\overline{\mathbf{L}_j^s} = \mathbf{L}_{n-j+1}$ for $j = 1, \cdots, n$. In particular, $\mathbf{L}_n = \overline{\mathbf{L}_1^s} = (\bar{\tau}_n, \cdots, \bar{\tau}_1)^T$ is the last column of $\mathbf{E}_n^{-1}$.

By setting

$$X_0 = 1,\ X_j = \det \mathbf{E}_j,\ j \geq 1, \tag{2.61}$$

consequently, the sequence $\{X_j\}$ satisfies the recurrence relation

$$\begin{aligned} &X_0 = X_1 = 1, \\ &X_j = X_{j-1} - |b|^2 X_{j-2},\ j \geq 2, \end{aligned} \tag{2.62}$$

for all $j = 0, 1, 2, \cdots$, we immediately obtain

$$X_j = \frac{\lambda^{j+1} - \mu^{j+1}}{\sqrt{1 - 4|b|^2}},\ \lambda = \frac{1 + \sqrt{1 - 4|b|^2}}{2},\ \mu = \frac{1 - \sqrt{1 - 4|b|^2}}{2}. \tag{2.63}$$

Through straightforward calculations, we derive the following result

$$\mathbf{L}_1 = \frac{1}{X_n}\big(X_{n-1}, -\bar{b}X_{n-2}, \bar{b}^2 X_{n-3}, \cdots, (-1)^{n-1}\bar{b}^{n-1}\big), \tag{2.64}$$

which yields, for each $j = 1, 2, \cdots, n$, the relation

$$\tau_j = \frac{(-\bar{b})^{j-1} X_{n-j}}{X_n}. \tag{2.65}$$

From (2.50), it follows that

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = -\bar{b} \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_n \end{pmatrix}, \quad \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = - \begin{pmatrix} \bar{\tau}_n \\ \vdots \\ \bar{\tau}_1 \end{pmatrix}, \tag{2.66}$$

and $a_j = -\bar{b}\tau_j, c_j = -\bar{\tau}_{n-j+1}$, we combine (2.49), (2.63) and (2.65) to derive the expression

$$\begin{aligned} \Psi_{1,1}(z) &= z^{n+1} - \frac{\bar{b}}{\lambda^{n+1} - \mu^{n+1}} \left[\frac{\lambda z[(\lambda z)^n - (-\bar{b})^n]}{\lambda z + \bar{b}} - \frac{\mu z[(\mu z)^n - (-\bar{b})^n]}{\mu z + \bar{b}} \right], \\ \Psi_{2,1}(z) &= -\frac{1}{\lambda^{n+1} - \mu^{n+1}} \left[\frac{\lambda z[\lambda^n - (-bz)^n]}{\lambda + bz} - \frac{\mu z[\mu^n - (-bz)^n]}{\mu + bz} \right]. \end{aligned} \tag{2.67}$$

Furthermore, from (2.34), we obtain

$$\begin{aligned} \mathbf{\Psi}^+(z) &= \begin{pmatrix} z^{n+1} - \frac{\bar{b}}{\lambda^{n+1} - \mu^{n+1}} \left[\frac{\lambda z[(\lambda z)^n - (-\bar{b})^n]}{\lambda z + \bar{b}} - \frac{\mu z[(\mu z)^n - (-\bar{b})^n]}{\mu z + \bar{b}} \right] & bz^2 + z - \frac{|b|^2(\lambda^n - \mu^n) z}{\lambda^{n+1} - \mu^{n+1}} \\ -\frac{1}{\lambda^{n+1} - \mu^{n+1}} \left[\frac{\lambda z[\lambda^n - (-bz)^n]}{\lambda + bz} - \frac{\mu z[\mu^n - (-bz)^n]}{\mu + bz} \right] & \frac{(-b)^n(\lambda - \mu) z}{\lambda^{n+1} - \mu^{n+1}} \end{pmatrix}, \\ \mathbf{\Psi}^-(z) &= \begin{pmatrix} z^{n+1} - \frac{\bar{b}}{\lambda^{n+1} - \mu^{n+1}} \left[\frac{\lambda z[(\lambda z)^n - (-\bar{b})^n]}{\lambda z + \bar{b}} - \frac{\mu z[(\mu z)^n - (-\bar{b})^n]}{\mu z + \bar{b}} \right] & \frac{(-\bar{b})^{n+1}(\lambda - \mu) z^{-n}}{\lambda^{n+1} - \mu^{n+1}} \\ -\frac{1}{\lambda^{n+1} - \mu^{n+1}} \left[\frac{\lambda z[\lambda^n - (-bz)^n]}{\lambda + bz} - \frac{\mu z[\mu^n - (-bz)^n]}{\mu + bz} \right] & \frac{\bar{b}(\lambda^n - \mu^n) z^{-n}}{\lambda^{n+1} - \mu^{n+1}} + z^{-n+1} \end{pmatrix}, \end{aligned} \tag{2.68}$$

where the parameters λ and μ are given in (2.63). This result indicates that the MRH problem, as described in Example 2.2, admits only this distinct solution.

The foregoing derivations naturally lead us to establish the subsequent principal finding.

Theorem 2.3. *Let n be a positive integer and $b \in \mathbb{C}$ satisfy $|b| < \frac{1}{2}$. Then the MRH Problem 2.1 with weight function $\omega(t) = 1 + bt + \bar{b}t^{-1}$ admits a unique solution, which can be explicitly represented by* (2.68).

In the special case $n = 2$, (2.68) is reduced to clearer and more explicit expressions.

Theorem 2.4. *For $n = 2$, $b \in \mathbb{C}$ with $|b| < \frac{1}{2}$, and $\omega(t) = 1 + bt + \bar{b}t^{-1}$, the unique solution to the MRH Problem 2.1 simplifies to the following explicit form*

$$\mathbf{\Psi}^+(z) = \begin{pmatrix} z^3 - \frac{\bar{b}z^2 - \bar{b}^2 z}{1 - |b|^2} & bz^2 + z - \frac{|b|^2 z}{1 - |b|^2} \\ \frac{bz^2 - z}{1 - |b|^2} & \frac{b^2 z}{1 - |b|^2} \end{pmatrix}, \quad \mathbf{\Psi}^-(z) = \begin{pmatrix} z^3 - \frac{\bar{b}z^2 - \bar{b}^2 z}{1 - |b|^2} & -\frac{\bar{b}^3 z^{-2}}{1 - |b|^2} \\ \frac{bz^2 - z}{1 - |b|^2} & \frac{\bar{b}z^{-2}}{1 - |b|^2} + z^{-1} \end{pmatrix}. \tag{2.69}$$

When $b = \frac{1}{3}$, (2.69) and (2.55) become identical.

Example 2.3. Let n be a positive integer, and define the weight function $\omega(t) = 1 + bt + \bar{b}t^{-1}$, where $b \in \mathbb{C}$ satisfies $|b| = \frac{1}{2}$ and $\bar{b}$ denotes its complex conjugate. We consider the MRH Problem 2.1.

Consistent with the discussion in Example 2.2, the same methodology can be employed to derive the expression for $\mathbf{E}_n$ presented in (2.56). Furthermore, for $j \geq 0$, X_j is defined as in (2.61). Under these conditions, it suffices to show

$$\begin{aligned} &X_0 = X_1 = 1, \\ &X_j = X_{j-1} - \frac{1}{4}X_{j-2},\ j \geq 2, \end{aligned} \tag{2.70}$$

we derive $X_j = \frac{j+1}{2^j}$ for $j = 1, 2, \cdots$. By direct computation, the coefficient

$$\tau_j = \frac{2(-2\bar{b})^{j-1}(n-j+1)}{n+1},\ j = 1, 2, \cdots, n, \tag{2.71}$$

we thus establish the following expressions

$$\begin{aligned} \Psi_{1,1}(z) &= \frac{(n+1)z^{n+3} + 2(n+2)\bar{b}z^{n+2} + (-2\bar{b})^{n+2}z}{(n+1)(z+2\bar{b})^2}, \\ \Psi_{2,1}(z) &= \frac{(-2)^{n+2}b^{n+1}z^{n+2} - 4(n+1)bz^2 - 2nz}{(n+1)(1+2bz)^2}. \end{aligned} \tag{2.72}$$

Substituting $\Psi_{1,1}(z)$ and $\Psi_{2,1}(z)$ into (2.34) yields the desired result

$$\begin{aligned} \mathbf{\Psi}^+(z) &= \begin{pmatrix} \frac{(n+1)z^{n+3} + 2(n+2)\bar{b}z^{n+2} + (-2\bar{b})^{n+2}z}{(n+1)(z+2\bar{b})^2} & bz^2 + \frac{(n+2)z}{2(n+1)} \\ \frac{(-2)^{n+2}b^{n+1}z^{n+2} - 4(n+1)bz^2 - 2nz}{(n+1)(1+2bz)^2} & \frac{(-2b)^n z}{n+1} \end{pmatrix}, \\ \mathbf{\Psi}^-(z) &= \begin{pmatrix} \frac{(n+1)z^{n+3} + 2(n+2)\bar{b}z^{n+2} + (-2\bar{b})^{n+2}z}{(n+1)(z+2\bar{b})^2} & -\frac{\bar{b}(-2\bar{b})^n z^{-n}}{n+1} \\ \frac{(-2)^{n+2}b^{n+1}z^{n+2} - 4(n+1)bz^2 - 2nz}{(n+1)(1+2bz)^2} & z^{-n+1} + \frac{2n\bar{b}z^{-n}}{n+1} \end{pmatrix}, \end{aligned} \tag{2.73}$$

which represents a uniquely determined solution to the MRH problem as formulated in Example 2.3.

To conclude, the preceding analysis leads to the fundamental theorem below.

Theorem 2.5. *Let n be a positive integer, $b \in \mathbb{C}$ with $|b| = \frac{1}{2}$ and $\omega(t) = 1 + bt + \bar{b}t^{-1}$. Then the MRH Problem 2.1 is shown to possess a unique solution, which is provided in* (2.73).

In particular, if $b = \frac{1}{2}$, the MRH problem in Example 2.3 admits a uniquely determined solution

$$\mathbf{\Psi}^+(z) = \begin{pmatrix} \frac{(n+1)z^{n+3} + (n+2)z^{n+2} + (-1)^n z}{(n+1)(z+1)^2} & \frac{1}{2}z^2 + \frac{(n+2)z}{2(n+1)}) \\ \frac{2(-z)^{n+2} - 2(n+1)z^2 - 2nz}{(n+1)(z+1)^2} & \frac{(-1)^n z}{n+1} \end{pmatrix},$$

$$\mathbf{\Psi}^{-}(z)=\begin{pmatrix}\dfrac{(n+1)z^{n+3}+(n+2)z^{n+2}+(-1)^n z}{(n+1)(z+1)^2} & \dfrac{(-1)^{n+1}z^{-n}}{2(n+1)}\\ \dfrac{2(-z)^{n+2}-2(n+1)z^2-2nz}{(n+1)(z+1)^2} & z^{-n+1}+\dfrac{nz^{-n}}{n+1}\end{pmatrix}. \tag{2.74}$$

3. Characterization of the MRH problem for orthogonal polynomials

Let $p(t),q(t)\in\text{span}\{\cdots,t^{-2},t^{-1},1,t,t^2,\cdots\}$, the vector space of Laurent polynomials. We consider a weight function $\omega(t)$ on $\mathbb{T}_1$ satisfying (2.6). The matrices $\mathbf{E}_n$ are defined as in (2.46). We define an inner product by

$$\langle p,q\rangle=\frac{1}{2\pi}\int_{\mathbb{T}_1}\omega(t)p(t)\bar{q}(t)\frac{dt}{it}. \tag{3.1}$$

Beginning with the standard monomial basis $\{1,t,t^2,\cdots\}$, we construct a sequence of monic orthogonal polynomials through the Gram-Schmidt orthogonalization procedure. The recursive construction yields

$$\begin{aligned}&p_0(t)=1,\\&p_n(t)=t^n-\sum_{j=0}^{n-1}\frac{\langle t^n,p_j(t)\rangle}{\langle p_j(t),p_j(t)\rangle}p_j(t),\ n=1,2,\cdots,\end{aligned} \tag{3.2}$$

the generated polynomial sequence $\{p_n(t)\}_{n=0}^{\infty}$ maintains the fundamental property that

$$\text{span}\{p_0(t),p_1(t),\cdots,p_n(t)\}\cong\text{span}\{1,t,\cdots,t^n\}, \tag{3.3}$$

and each polynomial $p_n(t)$ has the specific form

$$p_n(t)=t^n+\sum_{j=1}^{n}a_{n,j}t^{n-j}, \tag{3.4}$$

where it is monic (i.e., the leading coefficient is 1) and the remaining coefficients $a_{n,j}$ are determined by the orthogonalization process described in (3.2). The polynomials satisfy the orthogonality condition

$$\langle p_j(t),p_k(t)\rangle=h_k\delta_{jk},\ \text{ for }\ k=0,1,2,\cdots,n, \tag{3.5}$$

where δ_{jk} denotes the Kronecker delta, and the normalization constants h_k are given by

$$h_k=\frac{1}{2\pi}\int_{\mathbb{T}_1}\omega(t)|p_k(t)|^2\frac{dt}{it}>0. \tag{3.6}$$

In consequence, the normalized polynomials $\left\{h_n^{-\frac{1}{2}}p_n(t)\right\}_{n=0}^{\infty}$ constitute a complete orthonormal system in the polynomial space $\text{span}\{1,t,t^2,\cdots\}$. This ensures that each $p_n(t)$ is perpendicular to all lower degree polynomials.

For a positive integer n, we could rewrite the monic orthogonal polynomials as

$$p_n(t)=\sum_{j=0}^{n}a_{n,j}t^{n-j},\ a_{n,0}=1, \tag{3.7}$$

an $(n+1)$-dimensional square matrix $\mathbf{S}_{n+1}$ is constructed to be upper triangular, taking the form

$$\mathbf{S}_{n+1} = \begin{pmatrix} 1 & a_{1,1} & a_{2,2} & \cdots & a_{n,n} \\ 0 & 1 & a_{2,1} & \cdots & a_{n,n-1} \\ 0 & 0 & 1 & \cdots & a_{n,n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{3.8}$$

From its expression, we obtain

$$\bar{\mathbf{S}}_{n+1}^T \bar{\mathbf{E}}_{n+1} \mathbf{S}_{n+1} = \begin{pmatrix} h_0 & 0 & \cdots & 0 \\ 0 & h_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & h_n \end{pmatrix}, \tag{3.9}$$

where $\bar{\mathbf{E}}_{n+1}$ is the conjugate of matrix $\mathbf{E}_{n+1}$ defined as (2.46), and $\bar{\mathbf{S}}_{n+1}^T$ denotes the transpose of $\bar{\mathbf{S}}_{n+1}$. This, together with the multiplicative property of the determinant and (2.61), implies

$$\det\left(\bar{\mathbf{S}}_{n+1}^T \bar{\mathbf{E}}_{n+1} \mathbf{S}_{n+1}\right) = \det \bar{\mathbf{S}}_{n+1}^T \det \bar{\mathbf{E}}_{n+1} \det \mathbf{S}_{n+1} = X_{n+1} = h_0 h_1 \cdots h_n, \tag{3.10}$$

we note that

$$h_n = \frac{X_{n+1}}{X_n}. \tag{3.11}$$

Let $\mathbf{H}_{n+1} = \bar{\mathbf{S}}_{n+1}^T \bar{\mathbf{E}}_{n+1} \mathbf{S}_{n+1}$, and then

$$\bar{\mathbf{E}}_{n+1} \mathbf{S}_{n+1} = (\bar{\mathbf{S}}_{n+1}^T)^{-1} \mathbf{H}_{n+1}, \tag{3.12}$$

where the right hand side is a lower triangular matrix with h_n as its lower right corner. Considering the $(n+1)$-th column of both sides of (3.12), we get

$$\bar{\mathbf{E}}_{n+1} \begin{pmatrix} a_{n,n} \\ \vdots \\ a_{n,1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h_n \end{pmatrix}, \tag{3.13}$$

taking the complex conjugate of both sides and replacing n with $n-1$ yields

$$-h_{n-1}^{-1} \mathbf{E}_n \begin{pmatrix} \bar{a}_{n-1,n-1} \\ \vdots \\ \bar{a}_{n-1,1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}. \tag{3.14}$$

Combining the above formula with (2.50), we deduce the relation $c_j = -h_{n-1}^{-1}\bar{a}_{n-1,n-j}$ for $j = 1, \cdots, n$. It is readily seen from (2.49) that

$$\Psi_{2,1}(z) = -h_{n-1}^{-1} z^n \sum_{j=1}^{n} \bar{a}_{n-1,n-j} z^{1-j} = -h_{n-1}^{-1} z \tilde{p}_{n-1}(z), \tag{3.15}$$

where

$$\tilde{p}_{n-1}(z) = z^{n-1}\bar{p}_{n-1}(z^{-1}) = \sum_{j=0}^{n-1} \bar{a}_{n-1,j} z^j, \tag{3.16}$$

is the reversed monic orthogonal polynomial of degree $n-1$. Following the definition of the flipped conjugate operator in (2.51), we apply it to both sides of (3.12) and utilizing the identity $\bar{\mathbf{E}}^s = \mathbf{E}$, we derive the matrix relation

$$\mathbf{E}_{n+1}\mathbf{S}_{n+1}^s = \left((\bar{\mathbf{S}}_{n+1}^T)^{-1}\mathbf{H}_{n+1}\right)^s, \tag{3.17}$$

considering the $(n+1)$-th column of both sides in the above expression, we have

$$\mathbf{E}_{n+1}\begin{pmatrix} 1 \\ a_{n,1} \\ \vdots \\ a_{n,n} \end{pmatrix} = \begin{pmatrix} h_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.18}$$

which implies that rows 2 to $n+1$ can be expressed as

$$\mathbf{E}_{n}\begin{pmatrix} a_{n,1} \\ \vdots \\ a_{n,n} \end{pmatrix} = -\begin{pmatrix} \bar{e}_1 \\ \vdots \\ \bar{e}_n \end{pmatrix}. \tag{3.19}$$

A combination of the above formula with (2.50), we establish the relationship $a_j = a_{n,j}$ for $j = 1, \cdots, n$. We know the function $\Psi_{1,1}(z)$ in (2.49), that then

$$\Psi_{1,1}(z) = z\big(z^n + \sum_{j=1}^{n} a_{n,j} z^{n-j}\big) = z p_n(z). \tag{3.20}$$

Moreover, we have the subsequent theorem.

Theorem 3.1. *For the weight function ω on $\mathbb{T}_1$ satisfying (2.6), $\{p_n(z)\}_{n=0}^{\infty}$ denote the sequence of monic orthogonal polynomials. Then, the MRH Problem 2.1 admits a unique solution, which can be expressed in either of the following forms.*

(1) Sectional representation

$$\mathbf{\Psi}^{\pm}(z) = \begin{pmatrix} z p_n(z) & G_n\big(z^{1-n} p_n(z)\omega^{\pm}(z)\big) \\ -h_{n-1}^{-1} z\tilde{p}_{n-1}(z) & -h_{n-1}^{-1} G_n\big(z^{1-n}\tilde{p}_{n-1}(z)\omega^{\pm}(z)\big) + \gamma^{\pm}(z) \end{pmatrix}, \ z \in \mathbb{D}^{\pm}, \tag{3.21}$$

where G_n is the complementary operator defined in (2.21), *h_n is the constant from* (3.6), *$\gamma^+(z) = 0$ and $\gamma^-(z) = z^{1-n}$.*

(2) Integral representation

$$\mathbf{\Psi}(z) = \begin{pmatrix} zp_n(z) & \dfrac{z^{-n+1}}{2\pi i}\displaystyle\int_{\mathbb{T}_1}\dfrac{p_n(t)\omega(t)\,\mathrm{d}t}{t-z} \\ -h_{n-1}^{-1}z\tilde{p}_{n-1}(z) & -\dfrac{z^{-n+2}}{2\pi h_{n-1}i}\displaystyle\int_{\mathbb{T}_1}\dfrac{\tilde{p}_{n-1}(t)\omega(t)\,\mathrm{d}t}{t(t-z)} \end{pmatrix}, \ z \in \mathbb{C}\setminus\mathbb{T}_1, \tag{3.22}$$

where $\tilde{p}_n(z)$ denotes the reversed monic orthogonal polynomial of $p_n(z)$.

Proof. By substituting the expressions for $\Psi_{1,1}(z)$ and $\Psi_{2,1}(z)$ from (3.20) and (3.15) into (2.34) and (2.40), respectively, we systematically derive the solutions given in (3.21) and (3.22). □

Let $\mathbf{E}_{n+1}$ be an $(n+1)$-th dimensional square matrix constructed according to (2.46), and

$$\mathbf{E}_{n+1}^* = \begin{pmatrix} E_{00} & E_{01} & \cdots & E_{0n} \\ E_{10} & E_{11} & \cdots & E_{1n} \\ \vdots & \vdots & & \vdots \\ E_{n0} & E_{n1} & \cdots & E_{nn} \end{pmatrix}, \tag{3.23}$$

is the adjoint matrix of $\mathbf{E}_{n+1}$. Then, the inverse matrix $\mathbf{E}_{n+1}^{-1} = X_{n+1}^{-1}\mathbf{E}_{n+1}^*$, where X_{n+1} is given in (2.61). From (3.18), a simple calculation leads to

$$p_n(z) = (z^n, z^{n-1}, \cdots, 1)\mathbf{E}_{n+1}^{-1}\begin{pmatrix} h_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{h_n}{X_{n+1}}(z^n, z^{n-1}, \cdots, 1)\begin{pmatrix} E_{00} \\ E_{10} \\ \vdots \\ E_{n0} \end{pmatrix}, \tag{3.24}$$

inserting (3.11) into the right hand side of the preceding formula, we can deduce that

$$p_n(z) = \frac{1}{X_n}(z^n E_{00} + z^{n-1}E_{10} + \cdots + E_{n0}), \tag{3.25}$$

or equivalently,

$$p_n(z) = \frac{1}{X_n}\begin{vmatrix} z^n & z^{n-1} & \cdots & 1 \\ e_{-1} & e_0 & \cdots & e_{n-1} \\ \vdots & \vdots & & \vdots \\ e_{-n} & e_{-n+1} & \cdots & e_0 \end{vmatrix}. \tag{3.26}$$

Thereby, the result for the reversed orthogonal polynomial is

$$\tilde{p}_n(z) = z^n\bar{p}_n(z^{-1}) = 1 + \bar{a}_{n,1}z + \cdots + \bar{a}_{n,n}z^n, \tag{3.27}$$

similarly, one can check that the above expression can be reformulated as

$$\tilde{p}_n(z) = \frac{1}{X_n} \begin{vmatrix} 1 & z & \cdots & z^n \\ e_1 & e_0 & \cdots & e_{-n+1} \\ \vdots & \vdots & & \vdots \\ e_n & e_{n-1} & \cdots & e_0 \end{vmatrix}. \tag{3.28}$$

In summary, the above discussion leads to the following conclusion.

Theorem 3.2. *Suppose $\omega(t) = 1 + bt + \bar{b}t^{-1}$ be defined on $\mathbb{T}_1$, where $b \in \mathbb{C}$, $\bar{b}$ denotes the complex conjugate of b, and $|b| < \frac{1}{2}$. Let $\{p_n(z)\}_{n=0}^{\infty}$ represent the sequence of monic orthogonal polynomials. Then, the solution to the MRH Problem 2.1 is uniquely determined by either* (3.21) *or* (3.22)*, with*

$$p_n(z) = \sum_{j=0}^{n} \frac{(-\bar{b})^j(\lambda^{n-j+1} - \mu^{n-j+1})}{\lambda^{n+1} - \mu^{n+1}} z^{n-j},\ h_n = \frac{\lambda^{n+2} - \mu^{n+2}}{\lambda^{n+1} - \mu^{n+1}}, \tag{3.29}$$

where λ and μ are given in (2.63).

Proof. The matrix $\mathbf{E}_{n+1}$ is given as (2.56). For each $j = 0, 1, \cdots, n$, the entry E_{j0} in (3.23) equals $(-1)^j$ multiplied by the determinant of the submatrix resulting from deleting 0-th row and j-th column of $\mathbf{E}_{n+1}$, we note that it is straightforward to obtain

$$E_{j0} = (-\bar{b})^j X_{n-j},\ j = 0, 1, \cdots, n. \tag{3.30}$$

Hence, inserting the above formula into (3.25), we also write

$$p_n(z) = \frac{1}{X_n} \sum_{j=0}^{n} (-\bar{b})^j X_{n-j} z^{n-j}. \tag{3.31}$$

Using mathematical induction, we can prove (2.63). Substituting this result into (3.31) and (3.11), we conclude that (3.29) holds. □

Theorem 3.3. *For $\omega(t) = 1 + bt + \bar{b}t^{-1}$, where $b \in \mathbb{C}$ and $|b| = \frac{1}{2}$. Let $\{p_n(t)\}_{n=0}^{\infty}$ be the sequence of monic orthogonal polynomials. Then* (3.21) *or* (3.22) *provide the unique solution to the MRH Problem 2.1, with*

$$p_n(z) = \sum_{j=0}^{n} \frac{(-2\bar{b})^j(n - j + 1)}{n + 1} z^{n-j},\ h_n = \frac{n + 2}{2n + 2}. \tag{3.32}$$

Proof. Following the methodology of Theorem 3.2, we observe that $X_j = \frac{j+1}{2^j}$, substituting this result into (3.31) and (3.11), it is readily seen (3.32). □

4. Steepest descent analysis of the MRH problem

For a weight function $\omega(t)$ defined on $\mathbb{T}_1$ satisfying (2.6), the Szegö constant η is define as

$$\eta = \exp\left(\frac{1}{2\pi} \int_{\mathbb{T}_1} \log \omega(t) \frac{\mathrm{d}t}{it}\right). \tag{4.1}$$

If $\log\omega \in L^1(\mathbb{T}_1)$, then $\eta > 0$. If $\log\omega \notin L^1(\mathbb{T}_1)$, then $\eta = 0$. A fundamental result in spectral theory, Szegö's theorem, states that

$$\eta = \lim_{n\to\infty} h_n, \tag{4.2}$$

where h_n is defined in (3.6). This theorem was first proved in 1914 by Gábor Szegö, while he was still a 19-year-old undergraduate.

We start with strictly positive analytic weight functions. Suppose ω is such a function, there exists a number ρ with $0 < \rho < 1$ such that $\omega(z)$ is holomorphic and non-zero on the annulus domain $A_\rho = \{\rho < |z| < 1/\rho\}$. Given that ω is strictly positive on $\mathbb{T}_1$, its index with respect to $\mathbb{T}_1$ vanishes, i.e., $I(\omega, \mathbb{T}_1) = 0$. Consequently, $I(\omega, \chi) = 0$ for each closed continuous curve χ in the annulus A_ρ. It follows that a holomorphic logarithm $\log\omega(z)$ can be defined on A_ρ so that it takes real values on $\mathbb{T}_1$.

Let $r \in (0, 1)$ be a fixed positive constant with $\rho < r$, we define holomorphic functions $\Phi^\pm(z)$ on the regions $\{|z| < \frac{1}{\rho}\}$ and $\{|z| > \rho\}$, respectively, as follows

$$\begin{aligned}\Phi^+(z) &= \exp\left(\frac{1}{2\pi i}\int_{\mathbb{T}_{\frac{1}{r}}} \frac{\log\omega(t)\,\mathrm{d}t}{t-z}\right),\ |z| < \frac{1}{r} < \frac{1}{\rho},\\ \Phi^-(z) &= \exp\left(\frac{1}{2\pi i}\int_{\mathbb{T}_r} \frac{\log\omega(t)\,\mathrm{d}t}{t-z}\right),\ \rho < r < |z|.\end{aligned} \tag{4.3}$$

Cauchy's theorem tells us that the above two integrals are independent of r as long as $|z| < \frac{1}{r} < \frac{1}{\rho}$ for the first integral, or $\rho < r < |z|$ for the second. Then Cauchy's integral formula provides

$$\begin{aligned}\frac{\Phi^+(z)}{\Phi^-(z)} &= \exp\left(\frac{1}{2\pi i}\int_{\mathbb{T}_{\frac{1}{r}}} \frac{\log\omega(t)\,\mathrm{d}t}{t-z} - \frac{1}{2\pi i}\int_{\mathbb{T}_r} \frac{\log\omega(t)\,\mathrm{d}t}{t-z}\right)\\ &= \exp(\log\omega(z)) = \omega(z),\end{aligned} \tag{4.4}$$

where $\rho < r < |z| < 1/r < 1/\rho$. This shows that the pair $\left(\Phi^+(z), \Phi^-(z)\right)$ uniquely solves the scalar Riemann-Hilbert problem below.

Problem 4.1.

(1) $\Phi^+(z)$ is holomorphic and nonzero in $\{|z| < 1/\rho\}$.

(2) $\Phi^-(z)$ is holomorphic and nonzero in $\{|z| > \rho\}$.

(3) For $\rho < |z| < \frac{1}{\rho}$, we have $\Phi^+(z) = \Phi^-(z)\omega(z)$.

(4) $\Phi^+(0) = \eta$, where η is the Szegö constant given in (4.1).

(5) As $z \to \infty$, we have $\Phi^-(z) \to 1$.

From (2.4), we deduce the following symmetry relations

$$\omega^+(z) = -\overline{\omega^-}(z^{-1}) + e_0,\ \omega^-(z) = -\overline{\omega^+}(z^{-1}) + e_0. \tag{4.5}$$

Through examination of the scalar Riemann-Hilbert Problem 4.1, we note that the pair $\left(\frac{\eta}{\overline{\Phi^-}(z^{-1})}, \frac{\eta}{\overline{\Phi^+}(z^{-1})}\right)$ similarly satisfies the problem conditions, so the uniqueness implies that

$$\Phi^+(z) = \frac{\eta}{\overline{\Phi^-}(z^{-1})},\ \Phi^-(z) = \frac{\eta}{\overline{\Phi^+}(z^{-1})}. \tag{4.6}$$

We analyze the characterization using the steepest descent method introduced by Deift-Zhou in [8] which typically involves a series of explicit transformations, structured as follows

$$\mathbf{\Psi}(z) \mapsto \mathbf{T}(z) \mapsto \mathbf{S}(z) \mapsto \mathbf{U}(z),$$

where $\mathbf{\Psi}(z)$ is the same as that in (3.22). Each transformation step brings us closer to our claim.

First transformation $\mathbf{\Psi}(z) \mapsto \mathbf{T}(z)$. The purpose is explicit, that is, to normalize the new MRH problem at ∞. Let

$$\mathbf{H}(z) = \begin{cases} \begin{pmatrix} z^{-n-1} & 0 \\ 0 & z^{n-1} \end{pmatrix}, & |z| > 1, \\ \mathbf{I}, & |z| < 1, \end{cases} \tag{4.7}$$

where $\mathbf{I}$ denote the identity matrix, the first transformation is established by introducing

$$\mathbf{T}(z) = \mathbf{\Psi}(z)\mathbf{H}(z),\ z \in \mathbb{C} \setminus \mathbb{T}_1. \tag{4.8}$$

Therefore, it is straightforward to check that $\mathbf{T}(z)$ satisfies the MRH problem stated below.

Problem 4.2.

(1) $\mathbf{T}(z)$ *is analytic in* $\mathbb{C} \setminus \mathbb{T}_1$.

(2) For $t \in \mathbb{T}_1$, *we get*

$$\mathbf{T}^{+}(t) = \mathbf{T}^{-}(t) \begin{pmatrix} t^{n+1} & t\omega(t) \\ 0 & t^{-n+1} \end{pmatrix}. \tag{4.9}$$

(3) $\mathbf{T}^{+}(0) = \mathbf{0}$.

(4) As $z \to \infty$, *we have* $\mathbf{T}^{-}(z) \to \mathbf{I}$.

Remark 4.1. The solution to the MRH Problem 4.2 exists uniquely, since it being equivalent to the MRH Problem 2.1 through the transform (4.8).

Second transformation $\mathbf{T}(z) \mapsto \mathbf{S}(z)$. This crucial step is derived by factorizing the jump matrix in the boundary condition (4.9). In fact, the matrix decomposition takes the form

$$\begin{pmatrix} t^{n+1} & t\omega(t) \\ 0 & t^{-n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{t^n\omega(t)} & 1 \end{pmatrix} \begin{pmatrix} 0 & t\omega(t) \\ -\frac{t}{\omega(t)} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{t^n}{\omega(t)} & 1 \end{pmatrix}. \tag{4.10}$$

With a fixed $r \in (0,1)$, we introduce the contours $\mathbb{T}_r = \left\{z \in \mathbb{C} \middle| |z| = r\right\}, \mathbb{T}_{\frac{1}{r}} = \left\{z \in \mathbb{C} \middle| |z| = \frac{1}{r}\right\}$, and $\Gamma = \mathbb{T}_r \cup \mathbb{T}_1 \cup \mathbb{T}_{\frac{1}{r}}$, where $\mathbb{T}_r$ and $\mathbb{T}_{\frac{1}{r}}$ are oriented counterclockwise. The second transformation is defined as

$$\mathbf{S}(z) = \mathbf{T}(z)\mathbf{K}(z),\ z \in \mathbb{C} \setminus \Gamma, \tag{4.11}$$

where

$$\mathbf{K}(z)=\begin{cases}\mathbf{I}, & |z|<r,\\ \begin{pmatrix}1 & 0\\ \dfrac{z^n}{\omega(z)} & 1\end{pmatrix}^{-1}, & r<|z|<1,\\ \begin{pmatrix}1 & 0\\ \dfrac{1}{z^n\omega(z)} & 1\end{pmatrix}, & 1<|z|<\dfrac{1}{r},\\ \mathbf{I}, & |z|>\dfrac{1}{r},\end{cases} \tag{4.12}$$

Obviously, the asymptotic behavior of $\mathbf{S}(z)$ at both infinity and the origin aligns with that of $\mathbf{T}(z)$. It can be easily verified that $\mathbf{S}(z)$ satisfy the subsequent MRH problem.

Problem 4.3.

(1) $\mathbf{S}(z)$ *is analytic in* $\mathbb{C}\setminus\Gamma$.

(2) For $t\in\Gamma$, *we have*

$$\mathbf{S}^+(t)=\mathbf{S}^-(t)\mathbf{J}_S(t), \tag{4.13}$$

where

$$\mathbf{J}_S(t)=\begin{cases}\begin{pmatrix}1 & 0\\ \dfrac{t^n}{\omega(t)} & 1\end{pmatrix}, & t\in\mathbb{T}_r,\\ \begin{pmatrix}0 & t\omega(t)\\ -\dfrac{t}{\omega(t)} & 0\end{pmatrix}, & t\in\mathbb{T}_1,\\ \begin{pmatrix}1 & 0\\ \dfrac{1}{t^n\omega(t)} & 1\end{pmatrix}, & t\in\mathbb{T}_{\frac{1}{r}}.\end{cases} \tag{4.14}$$

(3) $\mathbf{S}^+(0)=\mathbf{0}$.

(4) As $z\to\infty$, *we have* $\mathbf{S}^-(z)\to\mathbf{I}$.

Remark 4.2. The uniqueness of the solution to the MRH Problem 4.3 follows from its equivalence to the MRH Problem 4.2 under transformation (4.11).

Third transformation $\mathbf{S}(z)\mapsto\mathbf{U}(z)$. To guarantee the analyticity of $\mathbf{U}(z)$ on $\mathbb{T}_1$, it is necessary to take out the discontinuity on $\mathbb{T}_1$. Now, setting

$$\mathbf{M}(z)=\begin{cases}\begin{pmatrix}\Phi^-(z) & 0\\ 0 & \dfrac{1}{\Phi^-(z)}\end{pmatrix}, & |z|>1,\\ \begin{pmatrix}0 & z\Phi^+(z)\\ -\dfrac{z}{\Phi^+(z)} & 0\end{pmatrix}, & |z|<1,\end{cases} \tag{4.15}$$

here, the functions $\Phi^+(z)$ and $\Phi^-(z)$ are given in (4.3), one easily gets

$$\begin{pmatrix} 0 & t\omega(t) \\ -\dfrac{t}{\omega(t)} & 0 \end{pmatrix} = [\mathbf{M}^-(t)]^{-1}\mathbf{M}^+(t),\ t \in \mathbb{T}_1. \tag{4.16}$$

Building upon the matrix decomposition in (4.16), we establish the third transformation, which is formulated as

$$\mathbf{U}(z) = \mathbf{S}(z)\mathbf{M}^{-1}(z),\ z \in \mathbb{C} \setminus \Gamma_*, \tag{4.17}$$

where $\mathbf{M}^{-1}(z)$ denotes the matrix inverse of $\mathbf{M}(z)$ defined in (4.15), with $\Gamma_* = \mathbb{T}_r \cup \mathbb{T}_{\frac{1}{r}}$. Notably, this transformation (4.17) does not change the asymptotic behavior both at infinity and in the neighborhood of the origin. These considerations motivate the analysis of the subsequent MRH problem.

Problem 4.4.

(1) $\mathbf{U}(z)$ *is analytic in* $\mathbb{C} \setminus \Gamma_*$.

(2) For $t \in \Gamma_*$, *we have*

$$\mathbf{U}^+(t) = \mathbf{U}^-(t)\mathbf{J}_U(t), \tag{4.18}$$

where the factor matrix

$$\mathbf{J}_U(t) = \mathbf{M}^-(t)\mathbf{J}_S(t)[\mathbf{M}^+(t)]^{-1} = \begin{cases} \begin{pmatrix} 1 & -t^n\Phi^+(t)\Phi^-(t) \\ 0 & 1 \end{pmatrix}, & t \in \mathbb{T}_r, \\ \begin{pmatrix} 1 & 0 \\ \dfrac{1}{t^n\Phi^+(t)\Phi^-(t)} & 1 \end{pmatrix}, & t \in \mathbb{T}_{\frac{1}{r}}. \end{cases} \tag{4.19}$$

(3) $\mathbf{U}^+(0) = \mathbf{0}$.

(4) As $z \to \infty$, *we have* $\mathbf{U}^-(z) \to \mathbf{I}$.

Remark 4.3. The MRH Problem 4.4 can be converted to the MRH Problem 4.3 through the transform (4.17). Thus, it also admits a unique solution.

Remark 4.4. From expression (4.19), it can be observed that uniformly for all $n \to \infty$,

$$\mathbf{J}_U(t) = \mathbf{I} + O(r^n),\ t \in \Gamma_*. \tag{4.20}$$

This indicates that as $n \to \infty$, the jump matrix $\mathbf{J}_U$ uniformly approaches the identity matrix $\mathbf{I}$.

5. Asymptotic expressions of orthogonal polynomials

Tracing back the steps $\mathbf{\Psi}(z) \mapsto \mathbf{T}(z) \mapsto \mathbf{S}(z) \mapsto \mathbf{U}(z)$ given in (4.8), (4.11), and (4.17), unravelling these transformations we have the essential technical theorem.

Theorem 5.1. *The solution to the MRH Problem 4.4 is uniquely determined as*

$$\mathbf{U}(z) = \mathbf{\Psi}(z)\mathbf{H}(z)\mathbf{K}(z)\mathbf{M}^{-1}(z),\ z \in \mathbb{C} \setminus \Gamma_*, \tag{5.1}$$

where the matrix functions $\mathbf{H}(z), \mathbf{K}(z)$, *and* $\mathbf{M}(z)$ *defined respectively by* (4.7), (4.12), *and* (4.15). *Here,* $\mathbf{\Psi}(z)$ *being the preconstructed solution to the MRH problem 2.1 given in* (3.22).

Proof. Since each transformation in (4.8), (4.11) and (4.17) is reversible. □

The desired expression (4.20) can be obtained as discussion above.

Theorem 5.2. *The solution* $\mathbf{U}(z)$ *to the MRH Problem 4.4 is controlled by the jump matrix* $\mathbf{J}_U(t)$. *More precisely, there exist positive constants* η *and* δ *satisfying the following norm estimates*

$$\big\|\mathbf{U}-\mathbf{I}\big\|_{\mathbb{C}\setminus\Gamma_*} < \eta\big\|\mathbf{J_U}-\mathbf{I}\big\|_{\Omega}, \ \textit{while} \ \big\|\mathbf{J_U}-\mathbf{I}\big\|_{\Omega} < \delta, \tag{5.2}$$

where $\mathbf{J}_U(t)$ *is holomorphic in an open neighborhood* Ω *of the contour* Γ_*, *and* $\det \mathbf{J}_U(t) \neq 0$ *for any* $t \in \Gamma_*$.

Proof. These results were established in [2, 19], and their proofs remain directly applicable. □

In order to discuss the asymptotic behavior for $\mathbf{\Psi}(z)$, it necessitates separate consideration of all regions.

Case 1. If $|z| > \frac{1}{r}$. Considering (4.7), (4.12) and (4.15) together with Theorem 5.1, we can directly obtain

$$\begin{aligned}\mathbf{U}(z) &= \begin{pmatrix} \Psi_{1,1}(z) & \Psi_{1,2}(z) \\ \Psi_{2,1}(z) & \Psi_{2,2}(z)\end{pmatrix}\begin{pmatrix} z^{-n-1} & 0 \\ 0 & z^{n-1}\end{pmatrix}\mathbf{I}\begin{pmatrix} \dfrac{1}{\Phi^-(z)} & 0 \\ 0 & \Phi^-(z)\end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1}{z^{n+1}\Phi^-(z)}\Psi_{1,1}(z) & z^{n-1}\Phi^-(z)\Psi_{1,2}(z) \\ \dfrac{1}{z^{n+1}\Phi^-(z)}\Psi_{2,1}(z) & z^{n-1}\Phi^-(z)\Psi_{2,2}(z)\end{pmatrix},\end{aligned} \tag{5.3}$$

by combining Theorem 5.2 and Remark 4.4 with (5.3), one gets

$$\begin{aligned}&\Psi_{1,1}(z) = z^{n+1}\Phi^-(z)\big(1+\mathcal{O}(z^{-1})\big), \ \Psi_{1,2}(z) = \mathcal{O}\big(z^{-n}[\Phi^-(z)]^{-1}\big), \\ &\Psi_{2,1}(z) = \mathcal{O}\big(z^n\Phi^-(z)\big), \ \Psi_{2,2}(z) = \frac{1}{z^{n-1}\Phi^-(z)}\big(1+\mathcal{O}(z^{-1})\big).\end{aligned} \tag{5.4}$$

As seen from (3.22), it is evident that

$$p_n(z) = z^n\Phi^-(z)\big(1+\mathcal{O}(z^{-1})\big), \ \tilde{p}_{n-1}(z) = -h_{n-1}\mathcal{O}\big(z^{n-1}\Phi^-(z)\big). \tag{5.5}$$

Case 2. If $1 < |z| < \frac{1}{r}$. Similarly, a combination of (4.7), (4.12) and (4.15) with Theorem 5.1, we derive

$$\begin{aligned}\mathbf{U}(z) &= \begin{pmatrix} \Psi_{1,1}(z) & \Psi_{1,2}(z) \\ \Psi_{2,1}(z) & \Psi_{2,2}(z)\end{pmatrix}\begin{pmatrix} z^{-n-1} & 0 \\ 0 & z^{n-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{1}{z^n\omega(z)} & 1\end{pmatrix}\begin{pmatrix} \dfrac{1}{\Phi^-(z)} & 0 \\ 0 & \Phi^-(z)\end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1}{z^{n+1}\Phi^-(z)}\Psi_{1,1}(z) + \dfrac{1}{z\Phi^-(z)\omega(z)}\Psi_{1,2}(z) & z^{n-1}\Phi^-(z)\Psi_{1,2}(z) \\ \dfrac{1}{z^{n+1}\Phi^-(z)}\Psi_{2,1}(z) + \dfrac{1}{z\Phi^-(z)\omega(z)}\Psi_{2,2}(z) & z^{n-1}\Phi^-(z)\Psi_{2,2}(z)\end{pmatrix},\end{aligned} \tag{5.6}$$

one needs the asymptotic formula

$$\begin{aligned}&\Psi_{1,1}(z) = z^{n+1}\Phi^-(z)\big(1+\mathcal{O}(z^{-1})\big), \ \Psi_{1,2}(z) = \mathcal{O}\big(z^{-n}[\Phi^-(z)]^{-1}\big), \\ &\Psi_{2,1}(z) = -\frac{z}{\Phi^-(z)\omega(z)}\big(1+\mathcal{O}(z^{-1})\big), \ \Psi_{2,2}(z) = \frac{1}{z^{n-1}\Phi^-(z)}\big(1+\mathcal{O}(z^{-1})\big).\end{aligned} \tag{5.7}$$

An immediate consequence of (3.22) is that

$$p_n(z) = z^n \Phi^-(z)\big(1+\mathcal{O}(z^{-1})\big),\ \tilde{p}_{n-1}(z) = \frac{h_{n-1}}{\Phi^-(z)\omega(z)}\big(1+\mathcal{O}(z^{-1})\big). \tag{5.8}$$

Case 3. If $r<|z|<1$. In view of (4.7), (4.12), (4.15) and Theorem 5.1, it can be inferred that

$$\begin{aligned}\mathbf{U}(z) &= \begin{pmatrix}\Psi_{1,1}(z) & \Psi_{1,2}(z)\\ \Psi_{2,1}(z) & \Psi_{2,2}(z)\end{pmatrix}\mathbf{I}\begin{pmatrix}1 & 0\\ -\dfrac{z^n}{\omega(z)} & 1\end{pmatrix}\begin{pmatrix}0 & -\dfrac{\Phi^+(z)}{z}\\ \dfrac{1}{z\Phi^+(z)} & 0\end{pmatrix}\\ &= \begin{pmatrix}\dfrac{1}{z\Phi^+(z)}\Psi_{1,2}(z) & -\dfrac{\Phi^+(z)}{z}\Psi_{1,1}(z)+\dfrac{z^{n-1}\Phi^+(z)}{\omega(z)}\Psi_{1,2}(z)\\ \dfrac{1}{z\Phi^+(z)}\Psi_{2,2}(z) & -\dfrac{\Phi^+(z)}{z}\Psi_{2,1}(z)+\dfrac{z^{n-1}\Phi^+(z)}{\omega(z)}\Psi_{2,2}(z)\end{pmatrix},\end{aligned} \tag{5.9}$$

the asymptotic expansions of $\boldsymbol{\Psi}$ take the form

$$\begin{aligned}&\Psi_{1,1}(z) = \frac{z^{n+1}\Phi^+(z)}{\omega(z)}\big(1+\mathcal{O}(z)\big),\ \Psi_{1,2}(z) = z\Phi^+(z)\big(1+\mathcal{O}(z)\big),\\ &\Psi_{2,1}(z) = -\frac{z}{\Phi^+(z)}\big(1+\mathcal{O}(z)\big),\ \Psi_{2,2}(z) = \mathcal{O}\big(z^2\Phi^+(z)\big).\end{aligned} \tag{5.10}$$

As shown in (3.22),

$$p_n(z) = \frac{z^n\Phi^+(z)}{\omega(z)}\big(1+\mathcal{O}(z)\big),\ \tilde{p}_{n-1}(z) = \frac{h_{n-1}}{\Phi^+(z)}\big(1+\mathcal{O}(z)\big). \tag{5.11}$$

Case 4. If $|z|<r$. From (4.7), (4.12), (4.15) and Theorem 5.1, we obtain

$$\begin{aligned}\mathbf{U}(z) &= \begin{pmatrix}\Psi_{1,1}(z) & \Psi_{1,2}(z)\\ \Psi_{2,1}(z) & \Psi_{2,2}(z)\end{pmatrix}\mathbf{I}\begin{pmatrix}0 & -\dfrac{\Phi^+(z)}{z}\\ \dfrac{1}{z\Phi^+(z)} & 0\end{pmatrix}\\ &= \begin{pmatrix}\dfrac{1}{z\Phi^+(z)}\Psi_{1,2}(z) & -\dfrac{\Phi^+(z)}{z}\Psi_{1,1}(z)\\ \dfrac{1}{z\Phi^+(z)}\Psi_{2,2}(z) & -\dfrac{\Phi^+(z)}{z}\Psi_{2,1}(z)\end{pmatrix},\end{aligned} \tag{5.12}$$

substituting Theorem 5.2 and Remark 4.4 into (5.12), then find

$$\begin{aligned}&\Psi_{1,1}(z) = \mathcal{O}\big(z^2[\Phi^+(z)]^{-1}\big),\ \Psi_{1,2}(z) = z\Phi^+(z)\big(1+\mathcal{O}(z)\big),\\ &\Psi_{2,1}(z) = -\frac{z}{\Phi^+(z)}\big(1+\mathcal{O}(z)\big),\ \Psi_{2,2}(z) = \mathcal{O}\big(z^2\Phi^+(z)\big).\end{aligned} \tag{5.13}$$

(3.22) leads to the result that

$$p_n(z) = \mathcal{O}\big(z[\Phi^+(z)]^{-1}\big),\ \tilde{p}_{n-1}(z) = \frac{h_{n-1}}{\Phi^+(z)}\big(1+\mathcal{O}(z)\big). \tag{5.14}$$

To sum up the discussion above, we have

$$
\begin{aligned}
p_n(z) &= \begin{cases} z^n\Phi^-(z)\big(1+\mathcal{O}(z^{-1})\big), & |z|>1,\\ \dfrac{z^n\Phi^+(z)}{\omega(z)}\big(1+\mathcal{O}(z)\big), & r<|z|<1,\\ \mathcal{O}\big(z[\Phi^+(z)]^{-1}\big), & |z|<r,\end{cases}\\
\tilde{p}_{n-1}(z) &= \begin{cases} -h_{n-1}\mathcal{O}\big(z^{n-1}\Phi^-(z)\big), & |z|>\dfrac{1}{r},\\ \dfrac{h_{n-1}}{\Phi^-(z)\omega(z)}\big(1+\mathcal{O}(z^{-1})\big), & 1<|z|<\dfrac{1}{r},\\ \dfrac{h_{n-1}}{\Phi^+(z)}\big(1+\mathcal{O}(z)\big), & |z|<1.\end{cases}
\end{aligned}
\tag{5.15}
$$

6. Asymptotic analysis for orthogonal polynomials on extended domains

For a function $g \in L^2(\mathbb{T}_1)$, define its norm by

$$\|g\| = \Big(\frac{1}{2\pi}\int_{\mathbb{T}_1}|g(t)|^2\frac{dt}{it}\Big)^{\frac{1}{2}}. \tag{6.1}$$

The asymptotic zero set $Z_{\text{asym}}^{\{p_n\}}$ associated with a sequence of orthogonal polynomials $\{p_n(z)\}_{n=1}^{\infty}$ is characterized as the collection of all points $z \in \mathbb{C}$ for which there exist a subsequence $\{p_{n_j}(z)\}_{j=1}^{\infty}$ and a corresponding sequence of zeros $\{z_j\}_{j=1}^{\infty}$ (with $p_{n_j}(z_j) = 0$ for all j) such that $\lim\limits_{j\to\infty} z_j = z$.

We introduce the functions

$$\varphi_n(z) = z^{-n}p_n(z)[\Phi^-(z)]^{-1} - 1, \tag{6.2}$$

where $\Phi^-(z)$ is defined in (4.3), η is the constant given in (4.1), and $\tilde{p}_n(z)$ represents the monic reversal of $p_n(z)$. As a consequence, we establish the following technical lemma.

Lemma 6.1. *The following limits hold*

$$\lim_{n\to\infty}\|\varphi_n(z)\| = 0. \tag{6.3}$$

Proof. Define the auxiliary function $g_n(z) = \varphi_n(z)+1$, by Parseval's identity, we immediately obtain

$$\|g_n\|^2 = \|\varphi_n\|^2 + 1. \tag{6.4}$$

For $t \in \mathbb{T}_1$, using (4.6), the following equalities hold

$$|g_n(t)|^2 = \frac{|p_n(t)|^2}{(|t|^2)^n|\Phi^-(t)|^2} = \frac{|p_n(t)|^2\Phi^+(t)}{\eta\Phi^-(t)} = \frac{|p_n(t)|^2\omega(t)}{\eta}, \tag{6.5}$$

from which it is readily seen that

$$\|g_n\|^2 = \frac{h_n}{\eta}. \tag{6.6}$$

Substituting Szegö's theorem (4.2) and (6.6) into (6.4), a straightforward computation shows that $\lim\limits_{n\to\infty}\|\varphi_n(z)\| = 0$. $\square$

Lemma 6.2. *Consider a strictly positive analytic weight function ω defined on $\mathbb{T}_1$. Suppose ω is holomorphic and non vanishing in an annulus region $\rho < |z| < \frac{1}{\rho}$ for some $\rho \in (0,1)$. Then, the functions $\varphi_n(z)$ defined in* (6.2) *satisfy the asymptotic estimate*

$$\lim_{n\to\infty} \sup \|\varphi_n\|^{\frac{1}{n}} \leq \rho. \tag{6.7}$$

Proof. By Theorem 1 in [22], the asymptotic bound $\limsup |p_n(0)|^{\frac{1}{n}} \leq \rho$ holds as $n \to \infty$. Furthermore, Formula (8.6) in [14, p. 156] yields the recurrence relation

$$h_n = h_0 \prod_{j=1}^{n} \big(1 - |p_j(0)|^2\big),\ n \geq 1, \tag{6.8}$$

which establishes a connection between the constants h_n given in (3.6) and the values of the orthogonal polynomials $p_n(z)$ at $z = 0$. By combining these findings, we derive

$$\frac{h_n}{\eta} = \prod_{j=n+1}^{\infty} \frac{1}{1 - |p_j(0)|^2}, \tag{6.9}$$

implies that the desired conclusion as

$$\limsup_{n\to\infty} \big(\frac{h_n}{\eta} - 1\big)^{\frac{1}{n}} \leq \rho^2. \tag{6.10}$$

Consequently, (6.7) is established, and the proof is concluded. □

In general, we have successfully obtained the key conclusions.

Theorem 6.1. *Under the same assumptions as in Lemma 6.2, and let $0 < \rho < r < 1$. Suppose $p_n(z)$ represent the sequence of orthogonal polynomials. Then the following results hold*

(1)

$$\lim_{n\to\infty} z^{-n} p_n(z) = \Phi^-(z),\ |z| > \rho, \tag{6.11}$$

uniformly on $|z| \geq r$.

(2)

$$\lim_{n\to\infty} \big(p_n(z) - z^n \Phi^-(z)\big) = 0,\ \rho < |z| < \frac{1}{\rho}, \tag{6.12}$$

uniformly on $r \leq |z| \leq \frac{1}{r}$.

(3) The asymptotic zero set satisfies $Z^{\{p_n\}}_{asym} \subset \{|z| \leq \rho\}$ for sufficiently large n, or equivalently, $Z^{\{p_n\}}_{asym} \subseteq \{|z| \leq r\}$.

Proof. From (6.2), we obtain the norm estimate

$$\|t^{-n} p_n(t) - \Phi^-(t)\| = \|\Phi^-(t)\varphi_n(t)\| \leq C_1 \|\varphi_n(t)\|,\ t \in \mathbb{T}_1, \tag{6.13}$$

where $C_1 = \max\limits_{|t|=1} |\Phi^-(t)|$. According to Lemma 6.2, choose $\sigma \in (\rho, r)$ such that $\|\varphi_n\| \leq C_2 \sigma^n$ for all n, where $C_2 > 0$. Consider the expansion

$$z^{-n} p_n(z) - \Phi^-(z) = \sum_{j=1}^{\infty} \beta_j z^{-j}. \tag{6.14}$$

For $1 \leq j \leq n$, we have $|\beta_j| \leq C_1 C_2 \sigma^n$, while for $j > n$, it follows from the expansion of $\Phi^-(z)$ that $|\beta_j| \leq C_3 \sigma^j$ for some $C_3 > 0$. On $t \in \mathbb{T}_r$, the bound

$$\begin{aligned} |t^{-n} p_n(t) - \Phi^-(t)| &\leq C_1 C_2 \sigma^n \sum_{j=1}^{n} \frac{1}{r^j} + C_3 \sum_{j=n+1}^{\infty} \left(\frac{\sigma}{r}\right)^j \\ &\leq \frac{C_1 C_2 (\frac{\sigma}{r})^n}{1-r} + \frac{C_3 (\frac{\sigma}{r})^{n+1}}{1-\frac{\sigma}{r}} \\ &\leq C_4 \left(\frac{\sigma}{r}\right)^n, \ C_4 = 2\max\left\{\frac{C_1 C_2}{1-r}, \frac{C_3 \sigma}{r-\sigma}\right\}. \end{aligned} \tag{6.15}$$

Applying the maximum principle for $|z| \geq r$ combined with the boundary estimate (6.15), we deduce that

$$|z^{-n} p_n(z) - \Phi^-(z)| \leq C_4 \left(\frac{\sigma}{r}\right)^n, \ |z| \geq r, \tag{6.16}$$

since $\sigma < r$, the bound decays exponentially as $n \to \infty$, which establishes (6.11).

To verify (6.12), consider $t \in \mathbb{T}_{\frac{1}{r}}$, then

$$|p_n(t) - t^n \Phi^-(t)| = \left| t^n \sum_{j=1}^{\infty} \beta_j t^{-j} \right| \leq C_1 C_2 \left(\frac{\sigma}{r}\right)^n \cdot \frac{r}{1-r} \leq C_5 \left(\frac{\sigma}{r}\right)^n, \tag{6.17}$$

with $C_5 = \frac{C_1 C_2 r}{1-r}$. Additionally, from (6.15) we conclude

$$|p_n(t) - t^n \Phi^-(t)| \leq C_4 \sigma^n, \ t \in \mathbb{T}_r. \tag{6.18}$$

Applying the maximum principle over the annular $r \leq |z| \leq \frac{1}{r}$ and combining (6.17) and (6.18), we get

$$|p_n(z) - z^n \Phi^-(z)| \leq C_4 \sigma^n + C_5 \left(\frac{\sigma}{r}\right)^n, \ r \leq |z| \leq \frac{1}{r}, \tag{6.19}$$

given $\sigma < r < 1$, both terms vanish exponentially as $n \to \infty$, thereby proving (6.12).

Conclusion (3) is a consequence of (6.11) and Rouché's Theorem. This completes the proof of Theorem 6.1. □

Analogously to Theorem 6.1, the following result holds. The proof follows a similar argument, and we omit the details here.

Theorem 6.2. *Let ω, ρ and r be defined as in Theorem 6.1, and let $\tilde{p}_{n-1}(z)$ denote the reversed polynomial of $p_{n-1}(z)$. Then, we have*

(1)

$$\lim_{n\to\infty} \widetilde{p}_{n-1}(z) = \frac{\eta}{\Phi^+(z)}, \ |z| < \frac{1}{\rho}, \tag{6.20}$$

with uniform convergence on $|z| \leq \frac{1}{r}$.

(2) The asymptotic zero distribution satisfies $Z_{asym}^{\{\widetilde{p}_{n-1}\}} \subset \{|z| \geq \frac{1}{\rho}\}$ for all sufficiently large n, or $Z_{asym}^{\{\widetilde{p}_{n-1}\}} \subseteq \{|z| \geq \frac{1}{r}\}$.

Section 6 extends the asymptotic properties of orthogonal polynomials to a broader domain, while also verifying the scientific validity and effectiveness of our alternative Riemann-Hilbert approach for solving asymptotic analysis problems in orthogonal polynomials.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[2] A. I. Aptekarev, *Sharp constants for rational approximations of analytic functions*, Sb. Math., 2002, 193(1), 1–72.

[3] A. I. Aptekarev, S. Yu. Dobrokhotov, D. N. Tulyakov and A. V. Tsvetkova, *Plancherel-Rotach type asymptotic formulae for multiple orthogonal Hermite polynomials and recurrence relations*, Izv. Math., 2022, 86(1), 36–97.

[4] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*, Springer, New York, 1999.

[5] P. Bleher and K. Liechty, *Uniform asymptotics for discrete orthogonal polynomials with respect to varying exponential weights on a regular infinite lattice*, Int. Math. Res. Not., 2011, 2011(2), 342–386.

[6] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Dover Publications, New York, 2011.

[7] P. Deift, A. R. Its and X. Zhou, *A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics*, Ann. Math., 1997, 146, 26–47.

[8] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems, asymptotics for the MKdV equation*, Ann. Math., 1993, 137, 295–368.

[9] X. C. Ding and T. Trogdon, *A Riemann-Hilbert approach to the perturbation theory for orthogonal polynomials: Applications to numerical linear algebra and random matrix theory*, Int. Math. Res. Not., 2024, 2024(5), 3975–4061.

[10] Z. H. Du and J. Y. Du, *Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on the unit circle*, Chin. Ann. Math. Ser. A, 2006, 27(5), 701–718.

[11] A. Martinez-Finkelshtein, K. T.-R. McLaughlin and E. B. Saff, *Szegö orthogonal polynomials with respect to an analytic weight: Canonical representation and strong asymptotics*, Constr. Approx., 2006, 24, 319–363.

[12] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[13] A. S. Fokas, A. R. Its and A. V. Kitaev, *The isomonodromy approach to matic models in 2D quantum gravity*, Comm. Math. Phys., 1992, 147, 395–430.

[14] Y. L. Geronimus, *Orthogonal Polynomials: Estimates, Asymptotic Formulas, and Series of Polynomials Orthogonal on the Unit Circle and on an Interval*, Authorized Translation from the Russian, Consultants Bureau, New York, 1961.

[15] H. L. Han, H. Liu and Y. F. Wang, *The strong asymptotic analysis of the first kind orthogonal trigonometric polynomial*, preprint, 2021. arXiv:2103.03601v1 [math.CV].

[16] H. Hedenmalm, *Soft Riemann-Hilbert problems and planar orthogonal polynomials*, Comm. Pure Appl. Math., 2024, 77, 2413–2451.

[17] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, Cambridge, 2005.

[18] I. Karpenko, D. Shepelsky and G. Teschl, *A Riemann-Hilbert approach to the modified Camassa-Holm equation with step-like boundary conditions*, Monatsh. Math., 2023, 201, 127–172.

[19] A. B. J. Kuijlaars, *Riemann-Hilbert analysis for orthogonal polynomials*, in: E. Koelink, W. Van Assche, eds., Orthogonal Polynomials and Special Functions, Lecture Notes in Mathematics, Vol. 1817, Springer, Berlin, Heidelberg, 2003, 167–210.

[20] M. D. Liu and B. Li, *Riemann-Hilbert approach for the complex Sharma-Tasso-Olver equation with high-order poles*, Qual. Theory Dyn. Syst., 2024, 23, 225.

[21] J. K. Lu, *Boundary Value Problems for Analytic Functions*, World Scientific, Singapore, 1993.

[22] P. Nevai and V. Totik, *Orthogonal polynomials and their zeros*, Acta Sci. Math., 1989, 53, 99–104.

[23] P. L. Shabalin, *The Riemann boundary value problem for generalized analytical functions with supersingular points on the contour of the boundary condition*, Lobachevskii J. Math., 2024, 45, 5244–5253.

[24] P. K. Suetin, *Orthogonal Polynomials in Two Variables*, Translated by E. V. Pankratiev, Routledge, London, 2022.

[25] F. G. Tricomi, *Sul comportamento asintotico dei polinomi di Laguerre*, Ann. Mat. Pura Appl., 1949, 28(1), 263–289.

[26] N. P. Vekua, *Systems of Singular Integral Equations and Some Boundary Problems*, Noordhoff, Groningen, 1967.

[27] Y. F. Wang, Y. F. Lu and J. Y. Du, *Strong asymptotic analysis of OLPs on the unit circle by Riemann- Hilbert approach*, in: S. Rogosin and A. Çelebi, Analysis as a Life: Dedicated to Heinrich Begehr on the Occasion of his 80th Birthday, Birkhauser, Cham, 2019, 139–169.

[28] R. Wong, *Asymptotic Approximations of Integrals*, Society for Industrial and Applied Mathematics, Philadelphia, 2001.

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