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POSITIVE RADIAL SOLUTIONS OF A FOURTH-ORDER ELLIPTIC BOUNDARY VALUE PROBLEM ON UNIT BALL WITH NONLINEAR GRADIENT TERM*

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Abstract In this paper, we discuss the existence of positive radial solution of the fourth-order elliptic equation $\Delta^2 u = f(|x|, u, |\nabla u|, \Delta u)$ on the unit ball Ω of $\mathbb{R}^N$ with Navier boundary condition $u|_{\partial\Omega} = 0$ and $\Delta u|_{\partial\Omega} = 0$, where $N \geq 2$, $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}$ is a continuous function. Under certain local inequality conditions of f, an existence result of positive radial solution is obtained. The inequality conditions relate to the principal eigenvalue of Laplacian $-\Delta$ on $u|_{\partial\Omega} = 0$. The discussion is based on the method of lower and upper solutions and truncating function technique.

Keywords Fourth-order elliptic boundary value problem, positive radial solution, lower and upper solutions, truncating function technique.

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1. Introduction

In this paper, we discuss the existence of positive radial solution for the fourth-order elliptic boundary value problem (BVP)

$$\begin{cases} \Delta^2 u = f(|x|, u, |\nabla u|, \Delta u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball in $\mathbb{R}^N$, $N \geq 2$, $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}$ is a continuous function.

The fourth order elliptic equation boundary values arise in the study of traveling waves in suspension bridges and the study of the static deflection of an elastic plate in a fluid, see [5, 12, 17, 27]. BVP (1.1) is a general fourth order elliptic boundary value problem with Navier boundary condition, and some of its special situations have been studied by many researchers, see [1–3, 6–11, 13, 16, 24–26, 32]. Many researches are focused on simple fourth-order boundary value problem

$$\begin{cases} \Delta^2 u = f(x, u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

see [1, 2, 6–11, 13, 16, 25, 26]. Dalmasso [6] obtained uniqueness and positivity results of radial solutions when $f(x, u) = |u|^p$ on a ball. In [1,2,10,11,16,25,26], the authors applied the mountain

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pass lemma and critical point theory to obtain the existence and multiplicity results of solutions of BVP (1.2). In [7–9], the authors using the fixed point theorem on cones obtained the existence results of positive solutions of BVP (1.2). Recently, Ghoudi and Lahrach [13] obtained existence of signchanging solution by finite-dimensional reduction procedure for perturbed variational functionals.

There are some researchers discussed the case of the right side of BVP (1.2) is added a linear term of Δu

$$\begin{cases} \Delta^2 u = c\,\Delta u + f(x,\, u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{1.3}$$

see [3, 4, 18, 24, 28, 32]. The authors of these references mainly applied the variational method and critical theory to discuss the existence of nontrivial solutions of BVP (1.3).

For the case of the nonlinearity f contains Δu,

$$\begin{cases} \Delta^2 u = f(x,\, u,\, \Delta u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{1.4}$$

has also been studied by several researchers, see [20, 22, 29, 31]. In [29], Pao introduced the concept of a pair of coupled upper and lower solutions, and he showed that if BVP (1.4) has a pair of coupled upper and lower solutions $\tilde{u}$ and $\hat{u}$, then BVP (1.1) has a solution between $\tilde{u}$ and $\hat{u}$ under f satisfies proper regularity condition. However, it is not easy to find a pair of coupled upper and lower solutions in the applications. In [31], Wang used a pair of uncoupled upper and lower solutions $\tilde{u}$ and $\hat{u}$ with $\tilde{u} \geq \hat{u}$ and $\Delta\tilde{u} \leq \Delta\hat{u}$, to replace the pair of Pao's coupled upper and lower solutions, under the nonlinearity $f(x,\, \xi,\, \eta)$ satisfies an order condition obtained the existence of a solution between $\tilde{u}$ and $\hat{u}$ and established a monotonic iterative program for seeking the solutions. See [31, Theorem 2.1]. In [20], the authors improved Wang's theorem of upper and lower solutions and delete wang's order condition. See [20, Theorem 3.3]. Then them used the new theorem of lower and upper solutions to obtain existence result. In [22], the authors obtained existence result of a positive solution by applying fixed point index theory in cones under $f(x,\, \xi,\, \eta)$ is nonnegative and satisfies linear growth condition on $(\xi,\, \eta)$ at infinity.

The purpose of this paper is to study existence of positive radial solution for the general BVP (1.1) with nonlinear gradient term $|\nabla u|$. When Ω is an annular domain, the existence of radial solutions has been discussed in [21, 23]. In [21], the authors obtained existence and uniqueness results of radial symmetric solutions under certain inequality conditions that allow $f(r,\, \xi,\, \zeta,\, \eta)$ is downward superlinear growth on $\xi,\, \zeta,\, \eta$ as $|(\xi,\, \zeta,\, \eta)| \to \infty$ by Leary-Schauder fixed point theorem and technique of prior estimates. In [23], the authors obtained existence results of positive radial solutions when f is nonnegative and satisfies certain superlinear or sublinear growth conditions by the fixed point index theory in cones. However, the discussion methods in [21, 23] depend on some special estimations of the solution of the corresponding linear elliptic boundary value problem on annular domain, and these estimations don't hold for ball. Hence, the discussion methods in [21, 23] are not applicable to BVP (1.1) on the unit ball. In this paper, we use the lower and upper solutions method and truncating function technique to discussed the existence of positive radial solutions for BVP (1.1) on the unit ball.

Next, we always assume that Ω is the unit ball in R^N, $I = [0,\, 1]$, and λ_1 is the principal eigenvalue of the Laplace operator $-\Delta$ on the boundary condition $u|_{\partial\Omega} = 0$. Our main result is as follows:

Theorem 1.1. *Let $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}$ be continuous and satisfy the following conditions*

(F1) *there exist constants a, b, $c \geq 0$ with restriction $\frac{a}{{\lambda_1}^2} + \frac{b}{{\lambda_1}^{3/2}} + \frac{c}{\lambda_1} < 1$ and $H > 0$, such that*

$$f(r, \xi, \zeta, \eta) \leq a\xi + b\zeta - c\eta, \quad r \in I, \ \ \xi, \ \ \zeta \geq 0, \ \ \eta \leq 0, \ \ |(\xi, \zeta, \eta)| > H;$$

(F2) *there exist constants a_0, $c_0 \geq 0$ with restriction $\frac{a_0}{{\lambda_1}^2} + \frac{c_0}{\lambda_1} \geq 1$ and $\delta > 0$, such that*

$$f(r, \xi, \zeta, \eta) \geq a_0\xi - c_0\eta, \quad r \in I, \ \ \xi, \ \ \zeta \geq 0, \ \ \eta \leq 0, \ \ |(\xi, \zeta, \eta)| < \delta;$$

(F3) *for every $r \in I$ and $\eta \leq 0$, $f(r, \xi, \zeta, \eta)$ is increasing on ξ and ζ in $\mathbb{R}^+$,*

then BVP (1.1) has at least one positive radial solution.

In Theorem 1.1, (F1) and (F2) are local inequality conditions of nonlinearity $f(r, \xi, \zeta, \eta)$ on (ξ, ζ, η) at ∞ and O respectively, and they allow $f(r, \xi, \zeta, \eta)$ to be superlinear growth on η. Consider the following fourth-order elliptic boundary value problem

$$\begin{cases} \Delta^2 u = \alpha\sqrt{u} + \beta\sqrt{|\nabla u|} + \gamma(\Delta u)^3, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{1.5}$$

where α, β, $\gamma > 0$ are constants. It is easy to verify that the corresponding nonlinearity

$$f(r, \xi, \zeta, \eta) := \alpha\sqrt{\xi} + \beta\sqrt{\zeta} + \gamma\,\eta^3, \quad r \in [0, 1], \ \ \xi, \zeta \geq, \ \ \eta \leq 0 \tag{1.6}$$

satisfies the conditions (F1)-(F3) of Theorem 1.1, and it is cubic growth on η. Obviously, $u \equiv 0$ is a trivial solution of BVP (1.5). Additionally, by Theorem 1.1, BVP (1.5) has also a nontrivial positive radial solution. This example shows that Theorem 1.1 has certain applicability.

Taking the proof of Theorem 1.1 is based on the lower and upper solutions method and truncating function technique, which will be given in Section 3. Some preliminaries to prove Theorem 1.1 are presented in Section 2.

2. Preliminaries

Let $\Omega = \{x \in \mathbb{R}^N \mid |x| < 1\}$, $I = [0, 1]$. Let $W^{n,p}(\Omega)$ be the usual Sobolev space on Ω, where $1 < p < +\infty$, $n \in \mathbb{N}$, and $H^n(\Omega) := W^{n,2}(\Omega)$. Let $C^\mu(\overline{\Omega})$ be the μ-order Hölder continuous function space on $\overline{\Omega}$ with exponent $\mu \in (0, 1)$, and $C^{n+\mu}(\overline{\Omega}) := \{u \in C^n(\overline{\Omega}) \mid D^\alpha u \in C^\mu(\overline{\Omega}),\ |\alpha| = n\}$. We use $W_r^{n,p}(\Omega)$, $H_r^n(\Omega)$, $C_r^\mu(\overline{\Omega})$, $L_r^p(\Omega)$ to denote the radially symmetric subspaces of $W^{n,p}(\Omega)$, $H^n(\Omega)$, $C^\mu(\overline{\Omega})$, $L^p(\Omega)$ respectively, etc. Obviously, these subspaces are closed subspaces of their mother space. A radially symmetric function on $\overline{\Omega}$ is equivalent to a function on I. Let $C(I)$ be the Banach space of all continuous function $u(r)$ on I with norm $\|u\|_C = \max_{r\in I}|u(r)|$. Generally, we use $C^n(I)$ to denote the Banach space of all nth-order continuous differentiable function on I with the norm $\|u\|_{C^n} = \max\{\|u\|_C, \|u'\|_C, \cdots, \|u^{(n)}\|_C\}$. Let $C^+(I)$ be the cone of nonnegative functions in $C(I)$.

To discuss BVP (1.1), we review some conclusions on the linear elliptic boundary value problem (LBVP)

$$\begin{cases} -\Delta u = h(x)\,, & x \in \Omega\,, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $h \in L^p(\Omega)(1 < p < \infty)$. By the L^p-theory of the linear elliptic equations [19], for every $h \in L^p(\Omega)$, LBVP (2.1) has a unique L^p-solution $u := Th \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, and the solution operator $T : L^p(\Omega) \to W^{2,p}(\Omega)$ is a linear bounded operator. When $h \in C^\mu(\overline{\Omega})$, by the Schauder theory of the linear elliptic equations [19], $u = Th \in C^{2+\mu}(\overline{\Omega})$ is a classical solution of LBVP (2.1). When $p > N/2$, by the compactness of the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, $T : L^p(\Omega) \to C(\overline{\Omega})$ is a compact linear operator. Especially, the restriction of T on $C(\overline{\Omega})$, $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact. When $h \geq 0$, by the maximum principle of elliptic operators [15], $u = Th \geq 0$. Hence, $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ is a positive compact linear operator.

It is well-known that the Laplace operator $-\Delta$ with the boundary condition $u|_{\partial\Omega} = 0$ has a minimum positive real eigenvalue eigenvalue λ_1, which given by

$$\lambda_1 = \inf\left\{ \frac{\|\nabla u\|_2}{\|u\|_2} \;\middle|\; u \in H^2(\Omega) \cap H_0^1(\Omega),\ \|u\|_2 \neq 0 \right\}. \tag{2.2}$$

Moreover, λ_1 has a positive unit eigenfunction, that is, there exists a function $\phi_1 \in C^2(\overline{\Omega}) \cap C^+(\overline{\Omega})$ with $\|\phi_1\|_{C(\overline{\Omega})} = 1$ satisfies the equation

$$\begin{cases} -\Delta\phi_1 = \lambda_1\,\phi_1, & x \in \Omega, \\ \phi_1|_{\partial\Omega} = 0. \end{cases} \tag{2.3}$$

Lemma 2.1. *For every $h \in L^2(\Omega)$, the L^2-solution $u = Th$ of BVP (2.1) satisfies*

$$\|Th\|_2 \leq \frac{1}{\lambda_1}\,\|h\|_2, \qquad \|\nabla(Th)\|_2 \leq \frac{1}{\sqrt{\lambda_1}}\|h\|_2. \tag{2.4}$$

Proof. For every $h \in L^2(\Omega)$, since $u = Sh \in H^2(\Omega) \cap H_0^1(\Omega)$ is a L^2-solution of LBVP (2.1), by (2.2) and the Schwartz inequality, we have

$$\lambda_1\|u\|_2{}^2 \leq \|\nabla u\|_2{}^2 = (-\Delta u,\, u) \leq \|-\Delta u\|_2\|u\|_2 = \|h\|_2\|u\|_2.$$

So we have

$$\|u\|_2 \leq \frac{1}{\lambda_1}\,\|h\|_2, \qquad \|\nabla u\|_2{}^2 \leq \|h\|_2\|u\|_2 \leq \frac{1}{\sqrt{\lambda_1}}\|h\|_2\|\nabla u\|_2.$$

Hence, (2.4) holds. □

Let $h = h(|x|)$ be radially symmetric. Consider the radial symmetry of the solution of LBVP (2.1). Let $u(|x|)$ be a radial solution of LBVP (2.1). Setting $r = |x|$, we have

$$\begin{aligned} &\frac{\partial u}{\partial x_i} = u'(r)\,\frac{x_i}{r}, \quad i = 1, 2, \cdots, N, \\ &\nabla u = \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_N}\Big) = u'(r)\,\frac{x}{r}, \qquad |\nabla u| = |u'(r)|, \\ &\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = u''(r) + \frac{N-1}{r}u'(r) = \frac{1}{r^{N-1}}\big(r^{N-1}u'(r)\big)'. \end{aligned} \tag{2.5}$$

Hence, $u(r)$ satisfies the ordinary differential equation

$$\begin{cases} -\dfrac{1}{r^{N-1}}\big(r^{N-1}u'(r)\big)' = h(r), & r \in I, \\ u'(0) = 0, \quad u(1) = 0. \end{cases} \tag{2.6}$$

Conversely, if $u(r)$ is a solution of BVP (2.6), by (2.5) $u(|x|)$ is a radially symmetric solution of BVP (2.1).

Lemma 2.2. *When $h \in L^2_r(\Omega)$, the L^2-solution $u = Th$ of LBVP (2.1) is radially symmetric, and $u = Th \in H^2_r(\Omega) \cap C^1(I)$. Moreover,*

1) *when $h \in C(I)$, $u = Th \in C^2(I)$ is a classical radial solution of LBVP (2.1);*
2) *when $h \in C^+(I)$, $u = Th$ satisfies: $u \geq 0$, $u' \leq 0$.*

Proof. Let $h \in L^2_r(\Omega)$ and set $u = Th$. By the transformation of spherical coordinates, we have

$$\|h\|_2{}^2 = \int_\Omega h^2(|x|)dx = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^1 r^{N-1} h^2(r)dr.$$

Hence, $r^{N-1}h^2(r)$ is Lebesgue integrable on I. By this and the Hölder inequality, we easily verify that the function

$$u(r) = \int_r^1 \frac{1}{t^{N-1}}\, dt \int_0^t s^{N-1} h(s)ds \tag{2.7}$$

belongs to $C^1(I)$ and

$$u'(r) = -\frac{1}{r^{N-1}} \int_0^r s^{N-1} h(s)ds, \quad r \in (0,\, 1]; \qquad u'(0) = 0. \tag{2.8}$$

By (2.7), (2.8) and (2.5), we easily verify that $u(|x|)$ belongs to $H^2_r(\Omega) \cap H^1_{0\,r}(\Omega)$ and is L^2-solution of LBVP (2.1). By the uniqueness of the solution of LBVP (2.1), $u = Th$. Hence, Th is radially symmetric, and $Th \in H^2_r(\Omega) \cap C^1(I)$.

When $h \in C(I)$, by (2.7) and (2.8), $u = Th \in C^2(I)$ and it satisfies the quation (2.6). By (2.5), $u = Th$ is a classical radial solution of LBVP (2.1). Hence, the conclusion 1) holds. When $h \in C^+(I)$, by (2.7) $u \geq 0$, and by (2.8) $u' \leq 0$. Hence, the conclusion 2) holds. □

Lemma 2.3. *Let $f : I \times \mathbb{R}^3 \to \mathbb{R}$ be continuous. If there exist constants a, b, $c \geq 0$ satisfying $\frac{a}{\lambda_1^2} + \frac{b}{\lambda_1^{3/2}} + \frac{c}{\lambda_1} < 1$ and a constant $d > 0$ such that*

$$|f(r,\, \xi,\, \zeta,\, \eta)| \leq a|\xi| + b|\zeta| + c|\eta| + d, \qquad r \in I, \quad \xi,\ \zeta,\ \eta \in \mathbb{R}, \tag{2.9}$$

then BVP (1.1) has a classical radial solution $u \in C^4(I)$.

Proof. Let $T : L^2(\Omega) \to H^2(\Omega)$ be the solution operator of LBVP (2.1). Then it is a bounded linear operator. By the compactness of the Sobolev embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$, $T : L^2(\Omega) \to L^2(\Omega)$ is a compact linear operator. By Lemma 2.2, $T(L^2_r(\Omega)) \subset L^2_r(\Omega)$. Hence, the restriction of T on $L^2_r(\Omega)$, $T : L^2_r(\Omega) \to L^2_r(\Omega)$ is compact linear operator.

For BVP (1.1), set $v = -\Delta u$, then it becomes second order elliptic boundary value problem

$$\begin{cases} -\Delta v = f(|x|,\, Tv,\, |\nabla(Tv)|, -v), & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.10}$$

We show that BVP (2.10) has a radial solution $v_0 \in C^2(I)$.

Define a mapping F on $L^2_r(\Omega) \to L^2_r(\Omega)$ by

$$F(v)(|x|) = f(|x|,\, (Tv)(|x|),\, |\nabla(Tv)(|x|)|, -v(|x|)), \qquad v \in L^2_r(\Omega),\ \ x \in \Omega. \tag{2.11}$$

By (2.9), $F : L_r^2(\Omega) \to L_r^2(\Omega)$ is continuous. For $F(v)$ taking the norm of $L^2(\Omega)$, by (2.9) and Lemma 2.1, we have

$$\begin{aligned} \|F(v)\|_2 &\le a\|Tv\|_2 + b\|\nabla(Tv)\|_2 + c\|v\|_2| + d|\Omega|^{1/2} \\ &\le \Big(\frac{a}{\lambda_1} + \frac{b}{\lambda_1{}^{1/2}} + c\Big)\|v\|_2 + d|\Omega|^{1/2}, \quad v \in L_r^2(\Omega). \end{aligned} \tag{2.12}$$

Hence, the composite mapping of F and T

$$A = T \circ F : L_r^2(\Omega) \to L_r^2(\Omega) \tag{2.13}$$

is completely continuous. We use the Schauder fixed point theorem to show that A has a fixed point. Choose

$$R_0 = \frac{c|\Omega|^{1/2}}{\lambda_1\big(1 - \big(\frac{a}{\lambda_1^2} + \frac{b}{\lambda_1^{3/2}} + \frac{c}{\lambda_1}\big)\big)}, \qquad D_r = \{v \in L_r^2(\Omega) \mid \|v\|_2 \le R_0\}.$$

Then D_r is a bounded closed convex set in $L_r^2(\Omega)$. For every $v \in D_r$, by (2.4) and (2.12), we have

$$\begin{aligned} \|Av\|_2 &= \|T(F(v))\|_2 \\ &\le \frac{1}{\lambda_1}\|F(v)\|_2 \\ &\le \Big(\frac{a}{\lambda_1^2} + \frac{b}{\lambda_1^{3/2}} + \frac{c}{\lambda_1}\Big)\|v\|_2 + \frac{c|\Omega|^{1/2}}{\lambda_1} \\ &\le \Big(\frac{a}{\lambda_1^2} + \frac{b}{\lambda_1^{3/2}} + \frac{c}{\lambda_1}\Big)R_0 + \frac{c|\Omega|^{1/2}}{\lambda_1} \\ &= R_0. \end{aligned}$$

Hence, $Av \in D_r$. This means that $A(D_r) \subset D_r$. By the Schauder fixed point theorem, A has a fixed point $v_0 \in D_r$. Since $h := F(v_0) \in L_r^2(\Omega)$, by Lemma 2.2, $v_0 = Th \in C^1(I)$. By the definition (2.11) of F and the continuity of f, $h = F(v_0) \in C(I)$. Hence, by Lemma 2.2 1), $v_0 = Th \in C^2(I)$ is a classical radial solution of LBVP (2.1) for $h = F(v_0)$. By the definition of F, $v_0 \in C^2(I)$ is a classical radial solution of BVP (2.10). Hence, $u_0 = Tv_0 \in C^4(I)$ is a classical radial solution of BVP (1.1). □

Lemma 2.4. *Let $f : I \times \mathbb{R}^3 \to \mathbb{R}$ be continuous. If there exist constant a, b, $c \ge 0$ satisfying $\frac{a}{\lambda_1^2} + \frac{b}{\lambda_1^{3/2}} + \frac{c}{\lambda_1} < 1$, such that*

$$\begin{aligned} |f(r, \xi_2, \zeta_2, \eta_2) - f(r, \xi_1, \zeta_1, \eta_2)| \le & a|\xi_2 - \xi_1| + |\zeta_2 - \zeta_1| + c|\eta_2 - \eta_1|, \\ & \textit{for} \quad r \in I, \;\; (\xi_1, \zeta_1, \eta_1), \, (\xi_2, \zeta_2, \eta_2) \in \mathbb{R}^3, \end{aligned} \tag{2.14}$$

then BVP (1.1) has a unique classical radial solution.

Proof. From (2.14) we easily see that f satisfies the condition (2.9) of Lemma 2.3. By Lemma 2.3, BVP (1.1) has at least one radial solution in $C^4(I)$. Let $u_1, u_2 \in C^4(I)$ be two radial solutions of BVP (1.1). Set $v_1 = -\Delta u_1$, $v_2 = -\Delta u_2$. Then v_1, v_2 are the radial solutions of BVP (2.10). By the definition of T and F,

$$v_1 = T(F(v_1)), \qquad v_2 = T(F(v_2)). \tag{2.15}$$

By (2.11) and (2.14), we have

$$\begin{aligned}
&|F(v_2)(|x|) - F(v_1)(|x|)| \\
=&|\, f(|x|,\, Tv_2(|x|),\, |\nabla(Tv_2)(|x|)|,\, -v_2(|x|)) \\
&- f(|x|,\, Tv_1(|x|),\, |\nabla(Tv_1)(|x|)|,\, -v_1(|x|))\,| \\
\le& a|Tv_2(|x|) - Tv_1(|x|)| + b|\,|\nabla(Tv_2)(|x|)| - |\nabla(Tv_1)(|x|)|\,| + c|v_2(|x|) - v_1(|x|)| \\
\le& a|T(v_2 - v_1)(|x|)| + b|\nabla(T(v_2 - v_1)(|x|)| + c|(v_2 - v_1)(|x|)|, \quad x \in \Omega.
\end{aligned}$$

From this inequality and (2.4) it follows that

$$\begin{aligned}
\|F(v_2) - F(v_1)\|_2 &\le a\|T(v_2 - v_1)\|_2 + b\|\nabla(T(v_2 - v_1)\|_2 + c\|v_2 - v_1\|_2 \\
&\le \Big(\frac{a}{\lambda_1} + \frac{b}{{\lambda_1}^{1/2}} + c\Big)\|v_2 - v_1\|_2.
\end{aligned}$$

Hence, by (2.15) and (2.4), we have

$$\begin{aligned}
\|v_2 - v_1\|_2 &= \|T(F(v_2)) - T(F(v_1))\|_2 \\
&= \|T(F(v_2) - F(v_1))\|_2 \\
&\le \frac{1}{\lambda_1}\|F(v_2) - f(v_1)\|_2 \\
&\le \Big(\frac{a}{{\lambda_1}^2} + \frac{b}{{\lambda_1}^{3/2}} + \frac{c}{\lambda_1}\Big)\|v_2 - v_1\|_2.
\end{aligned}$$

Since $\frac{a}{{\lambda_1}^2} + \frac{b}{{\lambda_1}^{3/2}} + \frac{c}{\lambda_1} < 1$, from this inequality it follows that $\|v_2 - v_1\|_2 = 0$. Hence, $v_1 = v_2$, so we have $u_1 = Tv_1 = Tv_2 = u_2$. This means that BVP (1.1) has a unique radial solution. □

Lemma 2.5. *Let a, b, $c \ge 0$ be constants and $\frac{a}{{\lambda_1}^2} + \frac{b}{{\lambda_1}^{3/2}} + \frac{c}{\lambda_1} < 1$, $h \in C^+(I)$. Then the fourth order elliptic boundary value problem*

$$\begin{cases} \Delta^2 u = au + b|\nabla u| - c\Delta u + h(|x|), & x \in \Omega\,, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0 \end{cases} \tag{2.16}$$

has a unique classical radial solution $\tilde{u} \in C^4(I)$, moreover $\tilde{u} \ge 0$, $\tilde{u}' \le 0$ and $\Delta\tilde{u} \le 0$.

Proof. It is easy to verify that corresponding nonlinearity of BVP (2.16)

$$f(r,\, \xi,\, \zeta,\, \eta) := a\,\xi + b\,|\zeta| - c\,\eta + h(r), \qquad r \in I,\ \ \xi,\ \zeta,\ \eta \in \mathbb{R}$$

satisfies the condition (2.14) of Lemma 2.4. By Lemma 2.4, BVP (2.16) has a unique classical radial solution $\tilde{u} \in C^4(I)$.

On the other hand, consider the fourth order elliptic boundary value

$$\begin{cases} \Delta^2 u = a\,|u| + b\,|\nabla u| + c\,|\Delta u| + h(|x|), & x \in \Omega\,, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0. \end{cases} \tag{2.17}$$

It is easy to verify that the corresponding nonlinearity of BVP (2.17)

$$f(r,\,\xi,\,\zeta,\,\eta) := a\,|\xi| + b\,|\zeta| + c\,|\eta| + h(r), \qquad r \in I,\ \ \xi,\ \zeta,\ \eta \in \mathbb{R}$$

also satisfies the condition (2.14). By Theorem 2.4, BVP (2.17) has a unique radial solution $u_0 \in C^4(I)$. Set

$$v_0 = -\Delta u_0, \qquad h_0 = a|(Tv_0)| + b|\nabla v_0| + c|v_0| + h(r). \tag{2.18}$$

Then by the definition of T, $u_0 = Tv_0$, $v_0 = Th_0$. By (2.18), $h_0 \ge 0$. By the positivity of T and Lemma 2.2 2), $v_0 = Th_0 \ge 0$, $u_0 = Tv_0 \ge 0$ and $u_0{}' \le 0$. Hence, $u_0 \ge 0$, $\Delta u_0 = -v_0 \le 0$. Hence by the equation (2.17), u_0 is also a radial solution of BVP (2.16). By the uniqueness of radial solution of BVP (2.16), $\tilde{u} = u_0$. Hence, $\tilde{u} \ge 0$, $\tilde{u}' \le 0$ and $\Delta\tilde{u} \le 0$. □

3. Proof of the main result

In this section, we use the upper and lower solutions method and truncating function technique to prove Theorem 1.1. If a function $w \in C^4(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta^2 w \ge f(|x|,\, w,\, |\nabla w|,\, \Delta w), & x \in \Omega, \\ w|_{\partial\Omega} \ge 0, \quad \Delta w|_{\partial\Omega} \le 0, \end{cases} \tag{3.1}$$

we call it an upper solution of BVP (1.1), and if a function $v \in C^4(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta^2 v \le f(|x|,\, v,\, |\nabla v|,\, \Delta v), & x \in \Omega, \\ v|_{\partial\Omega} \le 0, \quad \Delta v|_{\partial\Omega} \ge 0, \end{cases} \tag{3.2}$$

we call it a lower solution of BVP (1.1).

Proof of Theorem 1.1. Let a, b, c, H be the constant in Condition (F1). Choose a positive constant by $C_0 = \max\{|f(r,\xi,\zeta,\eta) - (a\xi + b\zeta - c\eta)| \mid r \in I,\ \ \xi,\ \zeta \ge 0,\ \eta \le 0,\ |(\xi,\zeta,\eta)| \le H\} + 1$. Then by Condition (F1), we have

$$f(r,\,\xi,\,\zeta,\,\eta) \le a\xi + b\zeta - c\eta + C_0, \qquad r \in I, \quad \xi,\ \zeta \ge 0, \quad \eta \le 0. \tag{3.3}$$

By Lemma 2.5, the fourth order elliptic boundary value problem

$$\begin{cases} \Delta^2 u = au + b|\nabla u| - c\Delta u + C_0, & x \in \Omega\,, \\ u = \Delta u = 0, & x \in \partial\Omega \end{cases} \tag{3.4}$$

has a unique radial solution $w_0 \in C^4(I)$, and $w_0 \ge 0$, $w_0' \le 0$, $\Delta w_0 \le 0$. By this equation and inequality (3.3), we have

$$\Delta^2 w_0(r) = aw_0(r) + b|\nabla w_0(r)| - c\Delta w_0(r) + C_0$$

$$\geq f(r,\, w_0(r),\, |\nabla w_0(r)|,\, \Delta w_0(r)), \qquad r \in I.$$

Hence, w_0 is a radial upper solution of BVP (1.1).

Let ϕ_1 be the positive unit eigenfunction of the first eigenvalue λ_1 of the Laplace operator $-\Delta$ on the boundary condition $u|_{\partial\Omega} = 0$, namely $\phi_1 \in C^2(\overline{\Omega}) \cap C^+(\overline{\Omega})$, $\|\phi_1\|_C = 1$ and it satisfies (2.3). By the well-known symmetry result on the elliptic boundary value problem [14, Theorem 1], ϕ_1 is radially symmetric. By (2.3) and the definition of T, $-\Delta\phi_1 = \lambda_1\phi_1$, and $\phi = T(\lambda_1\phi)$. By (2.5) and (2.8), we have

$$|\nabla\phi_1(r)| = |\phi_1'(r)| = \Big|\frac{\lambda_1}{r^{N-1}} \int_0^r s^{N-1}\phi_1(s)ds\Big| \leq \frac{\lambda_1}{N}. \tag{3.5}$$

Let $\delta > 0$ be the constant in condition (F1). Set $\delta_0 = \min\{\delta/(1+\lambda_1/N+\lambda_1),\, C_0/{\lambda_1}^2\}$. Choose $\sigma \in (0,\, \delta_0)$ and $v_0 = \sigma\phi_1$. We show that v_0 is a lower solution of BVP (1.1). For every $r \in I$, since $\xi := (v_0)(r) = \sigma\phi_1(r) \geq 0$, $\zeta := |\nabla v_0(r)| = \sigma|\phi_1'(r)| \geq 0$, $\eta := \Delta v_0(r) = -\sigma\lambda_1\phi_1(r) \leq 0$, and

$$\begin{aligned}
|(\xi,\, \zeta,\, \eta)| &= \big|\big(v_0(r),\, |\nabla v_0(r)|,\, \Delta v_0(r)\big)\big| \\
&= \sigma\big|\big(\phi_1(r),\, |\phi_1'(r)|,\, -\lambda_1\phi_r\big)\big| \\
&\leq \sigma\big(|\phi_1(r)| + |\phi_1'(r)| + |\lambda_1\phi_r)|\big) \\
&\leq \sigma\big(1 + \frac{\lambda_1}{N} + \lambda_1\big) \\
&< \delta,
\end{aligned}$$

by condition (F1) and (2.3), we have

$$\begin{aligned}
f\big(v_0(r),\, |\nabla v_0(r)|,\, \Delta v_0(r)\big) &\geq a_0 v_0(r) - c_0\Delta v_0(r) \\
&= (a_0 + c_0\lambda_1)\sigma\phi_1(r) \\
&= \big(\frac{a_0}{\lambda_1^2} + \frac{b_0}{\lambda_1}\big)\sigma\Delta^2\phi_1(r) \\
&\geq \Delta^2 v_0(r), \qquad r \in I.
\end{aligned}$$

Hence, v_0 is a radial lower solution of BVP (1.1). We show that

$$0 \leq v_0 \leq w_0, \quad w_0' \leq v_0' \leq 0, \quad \Delta w_0 \leq \Delta v_0 \leq 0. \tag{3.6}$$

Consider $u := w_0 - v_0$. Since w_0 satisfies (3.4) and $\Delta^2 v_0 = \lambda_1^2\sigma\phi_1$, it follows that

$$\begin{aligned}
\Delta^2 u &= \Delta^2 w_0 - \Delta^2 v_0 \\
&= a w_0 + b|\nabla w_0| - c\Delta w_0 + C_0 - \lambda_1^2\sigma\phi_1 \\
&\geq +C_0 - \lambda_1^2\sigma \\
&\geq 0, \qquad x \in \Omega.
\end{aligned}$$

Hence, $h = \Delta^2 u \in C^+(I)$, and u is a radial solution of the fourth order elliptic boundary value problem

$$\begin{cases} \Delta^2 u = h(|x|), & x \in \Omega\,, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0. \end{cases}$$

By Lemma 2.5, $u \geq 0$ and $u' \leq 0$ and $\Delta u \leq 0$. Hence (3.6) holds. By (2.5), we have

$$|\nabla w_0(r)| = |w_0'(r)| = -w_0'(r), \quad |\nabla v_0(r)| = |v_0'(r)| = -v_0'(r), \qquad r \in I. \tag{3.7}$$

Define three functions $\theta_1, \theta_2, \theta_3 : I \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \theta_1(r, \xi) &= \min\{\max\{v_0(r), \xi\}, \ w_0(r)\}, \\ \theta_2(r, \zeta) &= \min\{\max\{-v_0'(r), \zeta\}, \ -w_0'(r)\}, \\ \theta_3(r, \eta) &= \min\{\max\{\Delta w_0(r), \eta\}, \ \Delta v_0(r)\}. \end{aligned} \tag{3.8}$$

Then $\theta_1, \theta_2, \theta_3 : I \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy

$$\begin{aligned} & v_0(r) \leq \theta_1(r, \xi) \leq w_0(r), \quad (r, \xi) \in I \times \mathbb{R}, \\ & -v_0'(r) \leq \theta_2(r, \zeta) \leq -w_0'(r), \quad (r, \zeta) \in I \times \mathbb{R}, \\ & \Delta w_0(r) \leq \theta_3(r, \eta) \leq \Delta v_0(r), \quad (r, \eta) \in I \times \mathbb{R}. \end{aligned} \tag{3.9}$$

Choose a constant $c \in (0, \lambda_1)$ and make a truncating function f^* of f by

$$f^*(r, \xi, \zeta, \eta) = f(r, \theta_1(r, \xi), \theta_2(r, \zeta), \theta_3(r, \eta)) + c(\eta - \theta_3(r, \eta)). \tag{3.10}$$

Then by the continuity of f, $f^* : I \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, and by (3.9),

$$|f^*(r, \xi, \zeta, \eta)| \leq c|\eta| + C_1, \qquad r \in I, \quad \xi, \zeta, \eta \in \mathbb{R},$$

where $C_1 = \sup_{r \in I,\, \xi, \zeta, \eta \in \mathbb{R}} |f(r, \theta_1(r, \xi), \theta_2(r, \zeta), \theta_3(r, \eta))| + c\|\Delta w_0\|_C$. Hence, $f^* : I \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the condition (2.9) of Lemma 2.3. By Lemma 2.9, the fourth order elliptic boundary value problem

$$\begin{cases} \Delta^2 u = f^*(|x|, u, |\nabla u|, \Delta u), \qquad x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0 \end{cases} \tag{3.11}$$

has a radial solution $u_0 \in C^4(I)$. We show that

$$\Delta w_0 \leq \Delta u_0 \leq \Delta v_0. \tag{3.12}$$

On the contrary, suppose that $\Delta w_0 \not\leq \Delta u_0$. We consider the function $\omega(x) = u_0(|x|) - w_0(|x|)$. By the assumption, $\min_{x \in \overline{\Omega}} \omega(x) < 0$. Since $\omega|_{\partial\Omega} = 0$, by the minimum value theorem of continuous functions, there exists $x_0 \in \Omega$ such that $\omega(x_0) = \min_{x \in \overline{\Omega}} \omega(x) < 0$. By the properties of the minimum point of a C^2-function, $\nabla\omega(x_0) = 0$ and $\Delta\omega(x_0) \geq 0$. Hence, by $\omega(x_0) = \Delta u_0(|x_0|) - \Delta u_0(|x_0|) < 0$ and $\Delta\omega(x_0) = \Delta^2 u_0(|x_0|) - \Delta^2 w_0(|x_0|) \geq 0$, we have

$$\Delta u_0(|x_0|) < \Delta w_0(|x_0|), \qquad \Delta^2 u_0(|x_0|) \geq \Delta^2 w_0(|x_0|). \tag{3.13}$$

Hence, by the definitions (3.8), $\theta_3(|x_0|, \Delta u_0(|x_0|)) = \Delta w_0(|x_0|)$. By the equation (3.11), the definition of f^*, the first inequality of (3.13), (3.7), (3.9), Condition (F3) and the definition of upper solution w_0, we have

$$\Delta^2 u_0(|x_0|) = f^*(|x_0|, u_0(|x_0|), |\nabla u_0(|x_0|)|, \Delta u_0(|x_0|))$$

$$
\begin{aligned}
&= f\big(|x_0|, \theta_1(|x_0|, u_0(|x_0|)), \theta_2(|x_0|, |\nabla u_0(|x_0|)|), \theta_3(|x_0|, \Delta u_0(|x_0|))\big) \\
&\quad + c(\Delta u_0(|x_0|) - \theta_3(|x_0|, \Delta u_0(|x_0|))) \\
&= f\big(|x_0|, \theta_1(|x_0|, u_0(|x_0|)), \theta_2(|x_0|, |\nabla u_0(|x_0|)|), \Delta w_0(|x_0|)\big) \\
&\quad + c(\Delta u_0(|x_0|) - \Delta w_0(|x_0|)) \\
&< f\big(|x_0|, \theta_1(|x_0|, u_0(|x_0|)), \theta_2(|x_0|, |\nabla u_0(|x_0|)|), \Delta w_0(|x_0|)\big) \\
&\leq f\big(|x_0|, w_0(|x_0|), |\nabla w_0(|x_0|)|, \Delta w_0(|x_0|)\big) \\
&\leq \Delta^2 w_0(|x_0|),
\end{aligned}
$$

namely, $\Delta^2 u_0(|x_0|) < \Delta^2 w_0(|x_0|)$. This contradicts to the second inequality of (3.13). Hence, $\Delta w_0 \leq \Delta u_0$.

With a similarly method, we can show that $\Delta u_0 \leq \Delta v_0$. Hence, (3.12) holds. Set

$$h_1 := -\Delta(u_0 - v_0), \qquad h_2 := -\Delta(w_0 - u_0).$$

Then by (3.12), $h_1, h_2 \in C^+(I)$. By the definition of T, $u_0 - v_0 = Th_1$, $w_0 - u_0 = Th_2$. By Lemma 2.2 2), $u_0 - v_0 \geq 0$, $(u_0 - v_0)' \leq 0$; $w_0 - u_0 \geq 0$, $(w_0 - v_0)' \leq 0$. Hence,

$$v_0 \leq u_0 \leq w_0, \qquad -v_0' \leq -u_0' \leq -w_0'. \tag{3.14}$$

Now, by (3.12), (3.14), (3.7) and the definitions (3.8) of θ_1, θ_2 and θ_3, we have

$$\theta_1(r, u_0(r)) = u_0(r), \;\; \theta_2(r, |\Delta u_0(r)|) = |\Delta u_0(r)|, \;\; \theta_3(r, \Delta u_0(r)) = \Delta u_0(r), \;\; r \in I.$$

Hence by the definition (3.10) of f^*, we have

$$f^*(|x|, u_0(|x|), |\nabla u_0(|x|)|, \Delta u_0(|x|)) = f(|x|, u_0(|x|), |\nabla u_0(|x|)|, \Delta u_0(|x|)), \quad x \in \overline{\Omega}.$$

By the equation (3.11), u_0 is also a radial solution of BVP (1.1). Since $u_0 \geq v_0 = \sigma\phi_1$, so u_0 is a positive radial solution of BVP (1.1). □

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