Journal of Applied Analysis and Computation
Volume 17, Number 1, February 2027, 1–13
Website:http://www.jaac-online.com
DOI:10.11948/20250391

GROUND STATE SOLUTIONS FOR A CLASS OF HALF-WAVE CHOQUARD EQUATION WITH ASYMMETRIC PERTURBATION FUNCTION*

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Abstract In this paper, we investigate a class of half-wave Choquard equation with asymmetric perturbation function. By establishing a generalized Lieb type compactness theorem and employing the Nehari manifold method, we prove that the aforementioned equation admits a ground state solution.

Keywords Half-wave Choquard equation, ground state solution, generalized version of Lieb's compactness theorem, Nehari manifold method.

MSC(2010) 35J20, 35J60.

1. Introduction and main result

In this paper, we consider the following half-wave Choquard equation:

$$\sqrt{-\Delta}u + u = K(x)\left(I_\alpha * |u|^p\right)|u|^{p-2}u, \tag{$\mathcal{C}$}$$

where $N \geq 3$, $0 < \alpha < N$ and $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-1}$. Here, $\sqrt{-\Delta}$ denotes the half-wave operator, and I_α is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}}.$$

The function $K(x)$ is an asymmetric perturbation satisfying the following condition:

(K) $1 \not\equiv K(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$\lim_{|x|\to+\infty} K(x) = \inf_{x\in\mathbb{R}^N} K(x) = 1.$$

The half-wave operator $\sqrt{-\Delta}$ holds dual significance in mathematics and physics. On the physical side, it describes systems with nonlocal interactions or fractional dispersion, with clear application backgrounds in cutting-edge condensation [10], nonlinear optics [5], and fractional quantum mechanics [14]. On the mathematical point of view, it can be rigorously defined via the

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Fourier spectral representation or as a singular integral operator [8,10]. For more details of the operator $\sqrt{-\Delta}$, we refer the readers to [16]. Equation ($\mathcal{C}$) describes a nonlocal problem arising in various research fields, including Bose-Einstein condensation [19], multi-particle systems [21], and single-component plasmas [30]. In particular, when $K(x) = 1$, $N = 3$, $\alpha = 2$, $p = 2$ and the half-wave operator $\sqrt{-\Delta}$ is replaced by the classical Laplacian $-\Delta$, equation ($\mathcal{C}$) reduces to the well-known Choquard equation, which reads as follows:

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u, \quad x \in \mathbb{R}^3. \tag{1.1}$$

In [23], Pekar gave an equivalent characterization of equation (1.1), regarding it as a prototype of the steady-state polaron model in quantum theory. Choquard [17] reinterpreted this equation as a single-electron model and applied it to the theory of single-component plasmas. Later, Penrose [24] treated equation (1.1) as a model of self-gravitating matter, where quantum state reduction is interpreted as a gravitational effect. For more physical applications, we refer to [9,13,20,26,27,31].

Due to its strong physical significance, the existence and properties of solutions to equation ($\mathcal{C}$) have attracted considerable attention. We briefly review some known results. Lenzmann [14] considered the Cauchy problem for the following semi-relativistic Hartree equation:

$$\begin{cases} i\dfrac{\partial u}{\partial t} = \sqrt{-\Delta + m^2}\,u + \left(\dfrac{\lambda e^{-\mu|x|}}{|x|} * |u|^2\right) u, & x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & u : [0,T) \times \mathbb{R}^3 \to \mathbb{C}, \end{cases} \tag{1.2}$$

where $m \geq 0$, $\lambda \in \mathbb{R}$ and $\mu \geq 0$ are given parameters. By virtue of the Kato's inequality, the author established the global well-posedness and higher regularity of solutions to equation (1.2). Furthermore, Lenzmann [14] further extended his results to a large class of external potentials and obtained some regularity results in more regular spaces.

In [15], Lenzmann proved the uniqueness of ground state solution for the following pseudo-relativistic Hartree equation:

$$\sqrt{-\Delta + m^2}\,u - (|x|^{-1} * |u|^2)u = \mu u, \quad x \in \mathbb{R}^3, \tag{1.3}$$

where $m \geq 0$ and $\mu \in \mathbb{R}$ stands for certain Lagrange multipliers. Combining the variational arguments with a nonrelativistic limit, the author investigated the existence and properties of solutions to equation (1.3).

Recently, Bellazzini et al. [2,3] studied a class of Cauchy problem for a nonlinear half-wave equation with focusing power-type nonlinearity as follows:

$$\begin{cases} i\dfrac{\partial u}{\partial t} = \sqrt{-\Delta}\,u - |u|^{p-1}u, \quad x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \end{cases} \tag{1.4}$$

where $1 < p < \frac{N+1}{N-1}$. The authors proved the non-existence of nontrivial traveling solitary waves for equation (1.4) with speeds grater than unity ($|v| \geq 1$). Moreover, they also proved that equation (1.4) with $p < \frac{N+1}{N-1}$ for $N \geq 2$ and $p = \infty$ for $N = 1$ admits no small data scattering in the energy space, using the properties of traveling solitary waves with speeds $|v| < 1$. In the energy-critical case $p = \frac{N+1}{N-1}$ with $N \geq 2$, Bellazzini et al. [2,3] established the failure of small data scattering in the energy space. For further developments concerning the half-wave equation and the related nonlocal problems, we refer to [1,5–7,11,12,22,25,29] and the references therein.

Motivated by the above works, it is natural to raise a question. **Can we obtain some existence results of the half-wave Choquard equation with an asymmetric perturbation function?** To the best of our knowledge, there is no result concerning the existence of ground state solutions of $(\mathcal{C})$. To this end, in this paper, we focus our attention on this problem and aim to provide an affirmative answer.

Before stating our main result, we first give some notations. $L^t(\mathbb{R}^N)$ denotes the Lebesgue space for $1 \leq t \leq \infty$, equipped with the norm $\|u\|_{L^t} = \left(\int_{\mathbb{R}^N} |u|^t \, dx\right)^{\frac{1}{t}}$. The homogeneous Sobolev space $D^{\frac{1}{2},2}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the following norm:

$$||u||^2_{D^{\frac{1}{2},2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+1}} dxdy.$$

The standard Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^N)$ is defined by

$$H^{\frac{1}{2}}(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+1}} dxdy < +\infty \right\}$$

endowed with the norm

$$\|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+1}} dxdy + \int_{\mathbb{R}^N} |u|^2 dx.$$

It is standard to prove that the weak solutions of equation $(\mathcal{C})$ is the critical points of the following energy functional:

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+1}} \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \mathrm{d}x.$$

The main result of this paper is as follows:

Theorem 1.1. *Assume that $N \geq 3$, $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-1}$ and (K), equation $(\mathcal{C})$ possesses a ground state solution.*

Remark 1.1. Compared to [2,3,15], the lack of translation invariance induced by the asymmetric perturbation $K(x)$ introduces essential difficulties in restoring compactness for the corresponding energy functional of equation $(\mathcal{C})$. To address these issues, some are required to recover compactness in the variational setting and to establish sharper estimates for equation $(\mathcal{C})$.

We now outline the main ideas of the proof for Theorem 1.1. Our strategy consists of three key steps. First, we establish a generalized Lieb type compactness theorem utilizing the refined Sobolev inequality (see Lemma 2.1). Second, we apply this compactness theorem (see Lemma 2.2) to rule out the vanishing of the (PS) sequence for the energy functional I. Finally, we obtain the existence of ground state solutions to equation $(\mathcal{C})$ via the Nehari manifold method.

The paper is organized as follows. In Section 2, we establish the generalized Lieb type compactness theorem. In Section 3, we devoted to prove Theorem 1.1.

2. The generalized Lieb type compactness theorem

In this section, we prove the generalized Lieb type compactness theorem, which plays a key role in compactness analysis.

Lemma 2.1. *Assume that $t \in (2, 2_s^*)$, where $2_s^* = \frac{2N}{N-1}$, we have that*

$$\int_{\mathbb{R}^N} |u|^t dx \leq C \left(\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u|^s dx \right)^{\frac{t-2}{t}} \|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)}, \quad \forall u \in H^{\frac{1}{2}}(\mathbb{R}^N).$$

Proof. Let $t \in (2, 2_s^*)$ and suppose $u \in D^{\frac{1}{2},2}(\mathbb{R}^N)$. From the Sobolev inequality, we have that

$$\int_{B_1(y)} |u|^t dx \leq \left(\int_{B_1(y)} \int_{B_1(y)} \frac{|u(x) - u(y)|^2}{|x-y|^{N+1}} dxdy + \int_{B_1(y)} |u|^2 dx \right)^{\frac{t}{2}}.$$

Then

$$\begin{aligned} \int_{B_1(y)} |u|^t dx &= \left(\int_{B_1(y)} |u|^t dx \right)^{\frac{2}{t}} \left(\int_{B_1(y)} |u|^t dx \right)^{\frac{t-2}{t}} \\ &\leq \left(\int_{B_1(y)} \int_{B_1(y)} \frac{|u(x) - u(y)|^2}{|x-y|^{N+1}} dxdy + \int_{B_1(y)} |u|^2 dx \right) \left(\int_{B_1(y)} |u|^t dx \right)^{\frac{t-2}{t}}. \end{aligned}$$

We cover $\mathbb{R}^N$ with unit-radius balls such that every point in $\mathbb{R}^N$ is contained in at most $N+1$ balls. Therefore

$$\int_{\mathbb{R}^N} |u|^t dx \leq C \left(\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u|^t dx \right)^{\frac{t-2}{t}} \|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)}.$$

The proof is complete. □

Next, we present the generalized Lieb type compactness theorem.

Lemma 2.2. (Generalized Lieb type compactness theorem) *Suppose that $t \in (2, 2_s^*)$ and $\{u_n\}$ is a bounded sequence in $H^{\frac{1}{2}}(\mathbb{R}^N)$ such that $\lim\limits_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^t dx > 0$. Then there exists a sequence*

$$\bar{u}_n := u_n(x + y_n) \to \bar{u} \not\equiv 0, \quad \bar{u} \in L^t_{loc}(\mathbb{R}^N),$$

such that if $\{y_n\} \subset \mathbb{R}^N$ is bounded, we have that

$$\bar{u}_n \to u \not\equiv 0, \quad u \in L^t_{loc}(\mathbb{R}^N).$$

Proof. Let $\{u_n\}$ be a bounded sequence in $H^{\frac{1}{2}}(\mathbb{R}^N)$. Up to a subsequence, we assume

$$u_n \rightharpoonup u \in H^{\frac{1}{2}}(\mathbb{R}^N) \text{ and } u_n \to u \in L^t_{loc}(\mathbb{R}^N), \quad \forall t \in (2, 2_s^*).$$

By Lemma 2.1, there exists a constant $C_1 > 0$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^t dx \geq C_1.$$

Since $\{u_n\}$ is a bounded sequence in $H^{\frac{1}{2}}(\mathbb{R}^N)$ and by the Sobolev embedding theorem, we have

$$\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^t dx \le \int_{\mathbb{R}^N}|u_n|^t dx \le C.$$

It follows that $C_1, C_2 > 0$ such that

$$C_1 \le \sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^t dx \le C_2.$$

From the above inequality, we know there exists $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n)}|u_n|^t dx \ge \sup_{y\in\mathbb{R}^N}\int_{B_1(y_n)}|u_n|^t dx - \frac{C}{2n} \ge C_1.$$

Let $\bar{u}_n := u_n(x+y_n)$. Then

$$\|\bar{u}_n\|_{H^{\frac{1}{2}}(\mathbb{R}^N)} = \|u_n\|_{H^{\frac{1}{2}}(\mathbb{R}^N)},$$

and $\int_{B_1(0)}|\bar{u}_n|^t dx \ge C_1$. Thus, up to a subsequence, there exists $\bar{u} \in H^{\frac{1}{2}}(\mathbb{R}^N)$ such that $\bar{u}_n \rightharpoonup \bar{u}$. Since the embedding $H^{\frac{1}{2}}(\mathbb{R}^N) \hookrightarrow L^t_{Loc}(\mathbb{R}^N)$ is compact, we can conclude that $\bar{u} \not\equiv 0$.

If $\{y_n\}$ is bounded, then there exists $C > 0$ such that $B_1(y_n) \subset B_C(0)$ and

$$\int_{B_C(0)}|u_n|^t dx \ge \int_{B_1(y_n)}|u_n|^t dx \ge C_1.$$

This implies $u \not\equiv 0$. The proof is complete. □

Lemma 2.3. ([18], Hardy-Littlewood-Sobolev inequality) Let $s, r > 1$ and $0 < \mu < N$ satisfy

$$\frac{1}{s} + \frac{1}{r} = 1 + \frac{\mu}{N}.$$

Then there exists a sharp constant $C(N, \mu, s, r) > 0$ such that

$$\left|\iint_{\mathbb{R}^{2N}} \frac{u(x)v(y)}{|x-y|^{N-\mu}} dxdy\right| \le C(N,\mu,s,r)\|u\|_{L^s(\mathbb{R}^N)}\|v\|_{L^r(\mathbb{R}^N)},$$

holds for all $u \in L^s(\mathbb{R}^N)$ and $v \in L^r(\mathbb{R}^N)$. Moreover, when $s = r = \frac{2N}{N+\mu}$, the optimal constant is given explicitly by

$$C(N,\mu) = \pi^{\frac{N-\mu}{2}} \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{N+\mu}{2})} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right)^{-\frac{\mu}{N}}.$$

Lemma 2.4. ([4], Brézis-Lieb lemma) Let Ω be an open subset of $\mathbb{R}^N$ and $\{u_n\}$ be a sequence in $L^p(\Omega)$ with $1 \le p < +\infty$. Assume that

(i) $\{u_n\}$ is bounded in $L^p(\Omega)$;

(ii) $u_n \to u$ almost everywhere in Ω.

Then

$$\lim_{n\to\infty}\left(\int_\Omega |u_n|^p\, dx - \int_\Omega |u_n - u|^p\, dx\right) = \int_\Omega |u|^p\, dx.$$

3. Proof of Theorem 1.1

In this section, we investigate the existence of ground state solutions to equation $(\mathcal{C})$ and present the proof of Theorem 1.1. To this end, we first define the following mountain pass level and path set [28]:

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)) \quad \text{and} \quad \Gamma := \{\gamma \in C([0,1], H^{\frac{1}{2}}(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

We define the Nehari manifold associated with equation $(\mathcal{C})$, the minimum energy level on the Nehari manifold, and the minimax energy level of the functional I as follows:

$$\mathcal{N} := \{u \in H^{\frac{1}{2}}(\mathbb{R}^N) \setminus \{0\} : \langle I'(u), u\rangle = 0\}, \bar{c} = \inf_{u\in\mathcal{N}} I(u) \quad \text{and} \quad \bar{\bar{c}} = \inf_{u\in H^{\frac{1}{2}}(\mathbb{R}^N)\setminus\{0\}} \sup_{t\geq 0} I(tu).$$

Lemma 3.1. *Under the assumptions of Theorem 1.1, we have*

(1) the functional I satisfies the mountain pass geometry;

(2) for any $u \in H^{\frac{1}{2}}(\mathbb{R}^N)\setminus\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ and $I(t_u u) = \max_{t>0} I(t_u)$;

(3) $\bar{c} > 0$;

(4) $c = \bar{c} = \bar{\bar{c}}$;

(5) if $u \in \mathcal{N}$, then we have

$$\langle \varphi'(u), u\rangle = 2\|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - p\int_{\mathbb{R}^N} K(x)|u|^p dx. \tag{3.1}$$

Moreover, if $u \in \mathcal{N}$ and $I(u) = c$, then u is a ground state solution of equation $(\mathcal{C})$.

Proof. (1) By condition (K) and the Sobolev embedding theorem, we have that

$$I(u) \geq \frac{1}{2}\|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - C\|K\|_{L^\infty(\mathbb{R}^N)}\|u\|^{2p}_{H^{\frac{1}{2}}(\mathbb{R}^N)}.$$

To ensure that the energy functional $I(u)$ is well-defined for all u in $H^{\frac{1}{2}}(\mathbb{R}^N)$ and that the non-local term can be controlled, it is necessary to place the integrability index $\frac{2Np}{N+\alpha}$ required by the Hardy-Littlewood-Sobolev inequality within the allowable range of the Sobolev embedding, that is:

$$2 \leq \frac{2Np}{N+\alpha} \leq \frac{2N}{N-1}.$$

Thereby it concludes that

$$\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-1}.$$

Since $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-1}$, there exists a sufficiently small constant $\rho > 0$ such that

$$T := \inf_{\|u\|_{H^{\frac{1}{2}}(\mathbb{R}^N)}=\rho} I(u) > I(0) = 0.$$

It follows that

$$I(tu) = \frac{t^2}{2}||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx.$$

As $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-1}$, it is straightforward to verify that $I(tu) < 0$ for sufficiently large t. Therefore, there exists a constant $t_u > 0$ depending on u, such that $I(t_u u) < 0$ for $t > t_u$ and $||t_u u||_{H^{\frac{1}{2}}(\mathbb{R}^N)} > \rho$.

(2) Let

$$f(t) = I(tu) = \frac{t^2}{2}||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx,$$

for any $u \in H^{\frac{1}{2}}(\mathbb{R}^N) \setminus \{0\}$ and $t \in (0, \infty)$. Then we have

$$f'(t) = t||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - t^{2p-1}\int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx.$$

From the above equation, we know that $f'(t) = 0$ if and only if

$$||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} = t^{2p-2}\int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx.$$

Set

$$h(t) = t^{2p-2}\int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx.$$

Then $\lim\limits_{t\to+\infty} h(t) = +\infty$ and $\lim\limits_{t\to 0} h(t) = 0$. By the intermediate value theorem and the monotonicity of $h(t)$, there exists a unique $t_u \in (0, +\infty)$ such that

$$h(t_u) = ||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)}.$$

Thus, $t_u u \in \mathcal{N}$.

Next, we show the uniqueness of t. By contradiction, we suppose there exists $0 < t_1 < t_2$ such that

$$f'(t_1)t_1 = \langle I'(t_1 u), t_1 u\rangle = t_1^2||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - t_1^{2p}\int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx = 0$$

and

$$f'(t_2)t_2 = \langle I'(t_2 u), t_2 u\rangle = t_2^2||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - t_2^{2p}\int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx = 0.$$

This implies

$$\frac{f'(t_1)}{t_1} - \frac{f'(t_2)}{t_2} = (t_2^{2p-2} - t_1^{2p-2})\int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx > 0,$$

which contradicts with $t_1 < t_2$. Thus, we have that t_u is unique.

(3) By $\langle I'(u), u\rangle = 0$, according to condition (K) and the Sobolev embedding theorem, we can get

$$0 = \langle I'(u), u\rangle \geq \frac{1}{2}||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - C||K||_{L^\infty(\mathbb{R}^N)}||u||^{2p}_{H^{\frac{1}{2}}(\mathbb{R}^N)}.$$

It follows that $C||K||_{L^\infty(\mathbb{R}^N)}||u||^{2p-2}_{H^{\frac{1}{2}}(\mathbb{R}^N)} \geq 1$ and $||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} \geq C$.

We have

$$I(u) = I(u) - \frac{1}{2p}\langle I'(u), u\rangle = \left(\frac{1}{2} - \frac{1}{2p}\right)||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} \geq C.$$

From the above relations, we know that I has a lower bound on $\mathcal{N}$.

(4) From (2), we have $\bar{c} = \bar{\bar{c}}$. For any $u \in H^{\frac{1}{2}}(\mathbb{R}^N)\setminus\{0\}$, choosing $\bar{t} > 0$ sufficiently large such that $I(\bar{t}u) < 0$. We can construct a path $\gamma : [0,1] \to H^{\frac{1}{2}}(\mathbb{R}^N)$, which satisfies $\gamma(t) = t\bar{t}u$. Then, it is not difficult to see that $\gamma \in \Gamma$ and $c \geq \bar{\bar{c}}$. For any $\gamma \in \Gamma$, we define $h(t) := \langle I'(\gamma(t)), \gamma(t)\rangle$. When $t > 0$ small enough, we have $h(t) > 0$ and

$$I(\gamma(1)) - \frac{1}{2p}\langle I'(\gamma(1)), \gamma(1)\rangle \geq \left(\frac{1}{2} - \frac{1}{2p}\right) \|\gamma(1)\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} \geq 0.$$

This implies

$$h(1) = \langle I'(\gamma(1)), \gamma(1)\rangle \leq 2pI(\gamma(1)) = 2pI(\bar{t}u) < 0.$$

It follows that there exists $\bar{\bar{t}} \in (0,1)$ such that $h(\bar{\bar{t}}) = 0$ as $\gamma(\bar{\bar{t}}) \in \mathcal{N}$ and $c \geq \bar{c}$. In conclusion, we can get $c = \bar{c} = \bar{\bar{c}}$.

(5) From (3.1), for any $u \in \mathcal{N}$, we know that

$$\varphi(u) = \|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} K(x)\big(I_\alpha * |u|^p\big)|u|^p dx$$

and

$$\langle \varphi'(u), u\rangle = \langle \varphi'(u), u\rangle - 2p\varphi(u) = (2-2p)\|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} < 0.$$

Therefore, we have that $\varphi'(u) \neq 0$. For any $u \in \mathcal{N}$, suppose that $u \in \mathcal{N}$ and $I(u) = \bar{c}$, we know that exists a $\lambda \in \mathbb{R}$ such that $I'(u) = \lambda\varphi'(u)$. From the Lagrangian multiplier principle, it follows that

$$\lambda\langle \varphi'(u), u\rangle = \langle I'(u), u\rangle = 0.$$

This shows the $\lambda = 0$ and $I'(u) = 0$. The proof is complete. □

Lemma 3.2. *Assume that $\{u_n\} \subset H^{\frac{1}{2}}(\mathbb{R}^N)$ is a $(PS)_c$ sequence of I, then $\{u_n\}$ is bounded on $H^{\frac{1}{2}}(\mathbb{R}^N)$.*

Proof. Clearly, we have that

$$c + 1 \geq I(u_n) - \frac{1}{2p}\langle I'(u_n), u_n\rangle \geq \left(\frac{1}{2} - \frac{1}{2p}\right) \|u_n\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)},$$

which gets our desired result. The proof is complete. □

Lemma 3.3. *Assume that $\{u_n\} \subset H^{\frac{1}{2}}(\mathbb{R}^N)$ is a $(PS)_c$ sequence of I, then I satisfies the $(PS)_c$ condition.*

Proof. We divide the proof into four steps.

Step 1. We will prove that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx > 0.$$

In fact, if it is not true, then we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx = 0.$$

It follows that
$$c = \lim_{n\to+\infty} \frac{1}{2}\|u_n\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)}$$
and
$$0 = \lim_{n\to+\infty} \|u_n\|_{H^{\frac{1}{2}}(\mathbb{R}^N)}.$$
So, it is easy to obtain that $c = 0$. This leads to a contradiction with Lemma 3.1. Hence,
$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)(I_\alpha * |u|^p)|u|^p dx > 0.$$
Moreover, from condition (K), one can deduce
$$\begin{aligned} 0 &< \lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)(I_\alpha * |u_n|^p)|u_n|^p dx \\ &\le \|K\|_{L^\infty(\mathbb{R}^N)} \lim_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p dx \\ &\le \|K\|_{L^\infty(\mathbb{R}^N)} \lim_{n\to+\infty} \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N+\alpha}} dx \right)^{\frac{N+\alpha}{N}}. \end{aligned}$$
Thus, one sees that
$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N+\alpha}} dx > 0.$$

Step 2. From Lemma 2.2, we can find a sequence $\{y_n\} \subset \mathbb{R}^N$ such that
$$\bar{u}_n := u_n(x + y_n) \to \bar{u} \not\equiv 0, \bar{u} \in L^t_{\text{loc}}(\mathbb{R}^N), \quad \forall t \in (2, 2^*_s),$$
and
$$\int_{B_1(0)} |\bar{u}_n|^{\frac{2Np}{N+\alpha}} dx = \int_{B_1(y_n)} |u_n|^{\frac{2Np}{N+\alpha}} dx \ge C > 0.$$
The presence of $K(x)$ implies that the functional I is not translation-invariant. We proceed to prove that
$$\lim_{n\to+\infty} \bar{I}(\bar{u}_n) = \lim_{n\to+\infty} I(u_n) = c, \lim_{n\to+\infty} I'(u_n) = 0,$$
where
$$\bar{I}(\bar{u}_n) := \frac{1}{2}\|\bar{u}_n\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - \frac{1}{p}\int_{\mathbb{R}^N} K(x + y_n)(I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^p dx > 0.$$
Then, for any $H^{\frac{1}{2}}(\mathbb{R}^N)$, we have
$$|\langle \bar{I}'(\bar{u}_n), \varphi\rangle| = |\langle I'(u_n), \varphi_n\rangle| \le \|I'(u_n)\|_{(H^{\frac{1}{2}}(\mathbb{R}^N))^{-1}} \|\varphi_n\|_{H^{\frac{1}{2}}(\mathbb{R}^N)} = o_n(1),$$
where $\varphi_n = \varphi(x-y_n)$ and $\|\varphi_n\|_{H^{\frac{1}{2}}(\mathbb{R}^N)} = \|\varphi\|_{H^{\frac{1}{2}}(\mathbb{R}^N)}$. This implies that $\bar{I}'(u_n) \to 0$, as $n \to +\infty$. From this fact, we obtain that $\{\bar{u}_n\}$ is a $(PS)_c$ sequence of I.

Step 3. We claim that $\{y_n\}$ is a bounded sequence in $\mathbb{R}^N$. Otherwise, then as $n \to +\infty$, we have $|y_n| \to +\infty$. Taking $n \to +\infty$, one has
$$\int_{\mathbb{R}^N} K(x + y_n)(I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^{p-2}\bar{u}_n\varphi dx = \int_{\mathbb{R}^N} (I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^{p-2}\bar{u}_n\varphi dx. \tag{3.2}$$

It follows from condition (K), the Hölder inequality, Lemma 2.3 and $|y_n| \to +\infty$, we have that

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\mathbb{R}^N}(K(x+y_n)-1)(I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^{p-1}\varphi dx\\
\leq &\lim_{n\to+\infty}(\|K\|_{L^\infty(\mathbb{R}^N)}-1)\int_{\mathbb{R}^N}(I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^{p-1}\varphi dx\\
\leq &\lim_{n\to+\infty}(\|K\|_{L^\infty(\mathbb{R}^N)}-1)\left(\int_{\mathbb{R}^N}|\bar{u}_n|^{\frac{2Np}{N+\alpha}}dx\right)^{\frac{N+\alpha}{2N}}\left(\int_{\mathbb{R}^N}|\bar{u}_n|^{\frac{2N(p-1)}{N+\alpha}}\varphi^{\frac{2N}{N+\alpha}}dx\right)^{\frac{N+\alpha}{2N}}\\
\leq &\lim_{n\to+\infty}(\|K\|_{L^\infty(\mathbb{R}^N)}-1)\left(\int_{\mathbb{R}^N}|\bar{u}_n|^{\frac{2Np}{N+\alpha}}dx\right)^{\frac{N+\alpha}{2N}\left(2-\frac{1}{p}\right)}\left(\int_{\mathbb{R}^N}|\varphi|^{\frac{2Np}{N+\alpha}}dx\right)^{\frac{N+\alpha}{2Np}}\\
\leq &C\lim_{n\to+\infty}(\|K\|_{L^\infty(\mathbb{R}^N)}-1)\left(\int_{\mathbb{R}^N}|\varphi|^{\frac{2N}{N-\alpha}}dx\right)^{\frac{N+\alpha}{2N\alpha}}\\
= &0.
\end{aligned}$$

Arguing as above, we can conclude that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}K(x+y_n)(I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^{p-2}\bar{u}_n\varphi dx = \lim_{n\to+\infty}\int_{\mathbb{R}^N}(I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^{p-2}\bar{u}_n\varphi dx.$$

Then we can conclude from (3.2) and $\bar{u}_n \rightharpoonup \bar{u} \in H^{\frac{1}{2}}(\mathbb{R}^N)$ that

$$\lim_{n\to+\infty}\langle \bar{I}'(\bar{u}_n), \varphi\rangle = \langle I'_\infty(\bar{u}), \varphi\rangle = 0,$$

which implies that $\bar{u}$ is a weak solution of equation

$$\sqrt{-\Delta u} + u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{$\mathcal{C}_\infty$}$$

The corresponding energy functional for the equation $(\mathcal{C}_\infty)$ is

$$I_\infty(u) = \frac{1}{2}||u||^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - \frac{1}{p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx.$$

Let c_∞ be the ground state energy of the functional I_∞. Then, following the similar proof of Lemma 3.1, we can deduce that $c < c_\infty$.

According to condition (K), Lemma 2.4 and $\langle I'_\infty(\bar{u}), \varphi\rangle = 0$, we have that

$$\begin{aligned}
c_\infty &> c\\
&= \lim_{n\to+\infty}\left[\bar{I}(\bar{u}_n) - \frac{1}{2}\langle \bar{I}'(\bar{u}_n), \bar{u}_n\rangle\right]\\
&= \left(\frac{1}{2}-\frac{1}{2p}\right)\lim_{n\to+\infty}\int_{\mathbb{R}^N}K(x+y_n)(I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^p dx\\
&\geq \left(\frac{1}{2}-\frac{1}{2p}\right)\lim_{n\to+\infty}\int_{\mathbb{R}^N}(I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^p dx\\
&\geq \left(\frac{1}{2}-\frac{1}{2p}\right)\lim_{n\to+\infty}\int_{\mathbb{R}^N}(I_\alpha * |\bar{u}_n|^p)|\bar{u}_n|^p dx\\
&= \lim_{n\to+\infty}\left[I_\infty(\bar{u}) - \frac{1}{2}\langle I'_\infty(\bar{u}), \bar{u}\rangle\right]\\
&= I_\infty(\bar{u})\\
&\geq c_\infty,
\end{aligned}$$

which leads to a contradiction. Thus, $\{y_n\}$ is a bounded sequence in $\mathbb{R}^N$. By Lemma 2.2, we obtain that

$$u_n \to u \not\equiv 0, u \in L^t_{loc}(\mathbb{R}^N), \quad \forall t \in (2, 2^*_s).$$

At this point, we have demonstrated the boundedness of $\{y_n\}$, and thereby obtained that the original sequence $\{u_n\}$ converges to a non-zero function u. Next, in order to prove that u is the desired ground state solution, we need to verify $I(u) = c$.

Step 4. We demonstrate that $I(u) = c$. For $u_n \rightharpoonup u \in H^{\frac{1}{2}}(\mathbb{R}^N)$, we have that

$$\lim_{n\to+\infty} \langle I'(u_n), \varphi\rangle = \langle I'(u), \varphi\rangle = 0, \quad \forall \varphi \in H^{\frac{1}{2}}\left(\mathbb{R}^N\right).$$

Moreover, $u \in \mathcal{N}$. By Lemma 2.4, it is not difficult to show that

$$\begin{aligned} c &\le I(u) \\ &= I(u) - \frac{1}{2p}\langle I'(u), u\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right) \|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} \\ &\le \left(\frac{1}{2} - \frac{1}{2p}\right) \lim_{n\to+\infty} \|u_n\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} \\ &= \lim_{n\to+\infty} \left[I(u_n) - \frac{1}{2p}\langle I'(u_n), u_n\rangle\right] \\ &= c, \end{aligned}$$

which implies that $I(u) = c$ and

$$\lim_{n\to+\infty} \|u_n\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} = \|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)}.$$

This completes the proof. □

Proof of Theorem 1.1. From Lemmas 3.1–3.3, we can obtain our desired result in Theorem 1.1. The proof is complete. □

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Received December 2025; Accepted June 2026; Available online July 2026.