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DYNAMICS OF A DISCRETE COMMENSALISM MODEL WITH NON-SELECTIVE HARVESTING: DEPENDENCE ON THE HARVESTABLE PROPORTION M

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Abstract This paper proposes and studies a discrete commensalism model with non-selective harvesting and partial population closure. By applying difference inequality theory and standard computational techniques, sufficient conditions are derived for the permanence and extinction of the system, respectively. Furthermore, by constructing a suitable Lyapunov function, sufficient conditions for the global attractivity of the system are obtained. Finally, numerical simulations are provided to illustrate the validity of the theoretical results. The results reveal that increasing the closure fraction promotes species persistence, while excessive harvesting may lead to extinction.

Keywords Commensalism, harvesting, permanence, extinction, global attractivity.

MSC(2010) 39A30, 39A22, 39A60, 92D25.

1. Introduction

Commensalism is a symbiotic interaction between two populations in which one population derives benefit while the other is neither harmed nor benefited from the interaction [15]. There are many real-life examples of commensalism, such as the relationship between a squirrel and an oak, or between a clownfish and a sea anemone, etc.

During the last decade, research on the dynamic behaviors of the commensalism model has been widely carried out [5, 9, 11, 14–20, 22, 24]. Such topics as the existence of the equilibrium point [19], global stability [5, 24], Hopf-bifurcation [9, 14], global attractivity [22], and positive periodic solutions [11] have been extensively studied. Recently, Sun et al. [18] proposed the following commensalism model:

$$\begin{cases} \dfrac{dx}{dt} = r_1 x \left(1 - \dfrac{x}{K_1} + \alpha \dfrac{y}{K_1}\right), \\ \dfrac{dy}{dt} = r_2 y \left(1 - \dfrac{y}{K_2}\right), \end{cases} \tag{1.1}$$

where $\alpha, r_i, K_i (i = 1, 2)$ are all positive constants. They investigated the stability properties of the equilibria of system (1.1).

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On the other hand, as was pointed out by Chakraborty et al. [1], the study of resource management, including fisheries, forestry, and wildlife management, is very important. They argued that it is necessary to harvest the population, but harvesting should be regulated so that both ecological sustainability and conservation of the species can be implemented in the long run. Deng [6] proposed the flowing discrete commensalism model:

$$\begin{cases} \dfrac{dx}{dt} = r_1 x\left(1 - \dfrac{x}{K_1} + \alpha\dfrac{y}{K_1}\right) - q_1 Emx, \\ \dfrac{dy}{dt} = r_2 y\left(1 - \dfrac{y}{K_2}\right) - q_2 Emy. \end{cases} \tag{1.2}$$

They investigated the dynamic behaviors of the system (1.2) and determined the influence of harvesting and stock fraction.

As we all know, though most dynamic behaviors of population models are based on the continuous models governed by differential equations, discrete-time models governed by difference equation are more appropriate than the continuous ones when the population size is relatively small or the population has non-overlapping generations. It has been found that the dynamic behaviors of discrete systems are rather complex and contain rich dynamics than the continuous ones. Recently, more and more scholars had paid attention to studying discrete population models (see [2, 4, 7, 8, 10–13, 15, 21–23, 25] and the references cited therein).

However, to the best of our knowledge, few scholars have proposed and investigated the influence of harvesting on the discrete commensalism model. This motivates us to propose and study a discrete version of system (1.2). The aim of this paper is to investigate the permanence, extinction, and global attractivity of the following system:

$$\begin{cases} x_1(n+1) = x_1(n)\exp\left\{r_1(n)\left(1 - \dfrac{x_1(n)}{p_1(n)} + \dfrac{\alpha(n)}{p_1(n)}x_2(n)\right) - q_1(n)Em\right\}, \\ x_2(n+1) = x_2(n)\exp\left\{r_2(n)\left(1 - \dfrac{x_2(n)}{p_2(n)}\right) - q_2(n)Em\right\}, \end{cases} \tag{1.3}$$

where $x_1(n), x_2(n)$ denote the population densities of the two species at any time n. r_i and p_i represent the intrinsic growth rate and environmental carrying capacity of the $i-th$ species, respectively, and q_i are the catchability coefficients of the two species. The parameter α is the interaction parameter between $x_1(n)$ and $x_2(n)$. E is the combined fishing effort used for harvesting, and $m(0 < m < 1)$ is the fraction of the stock available for harvesting. One may refer to [6, 16, 18] for more background and the formulation of system (1.3). Throughout this paper, we assume that $\{\alpha(n)\}, \{r_i(n)\}, \{p_i(n)\}$, and $\{q_i(n)\}$ are bounded non-negative almost periodic sequences such that

$$\begin{aligned} &0 < \alpha^l \le \alpha(n) \le \alpha^u, 0 < r_i^l \le r_i(n) \le r_i^u, \\ &0 < p_i^l \le p_i(n) \le p_i^u, 0 < q_i^l \le q_i(n) \le q_i^u, \quad i = 1, 2. \end{aligned} \tag{1.4}$$

Here, for any bounded sequence $\{a(n)\}$, $a^u = \sup_{n\in N}\{a(n)\}, a^l = \inf_{n\in N}\{a(n)\}$.

From a biological perspective, we assumed that $x_i(0) > 0$ for $i = 1, 2$. Then it is easy to see that the solutions to (1.3) with the above initial conditions remain positive for all $n \in N^+ = \{0, 1, 2...\}$.

The organization of this paper is as follows. In section 2, we present some useful lemmas. Sufficient conditions for the permanence and extinction of system (1.3) are established in Sections 3 and 4, respectively. In section 5, we derive sufficient conditions for the global attractivity of (1.3). Some examples, together with their numeric simulations, are presented in section 6. A brief discussion concludes the paper.

2. Preliminaries

In this section, we introduce several useful lemmas.

Lemma 2.1 (Lemma 2.1, [3]). *Assume that* $\{x(k)\}$ *is a positive sequence satisfying*

$$x(k+1) \leq x(k)\exp\{a(k) - b(k)x(k)\}$$

for $k \in N$*, where* $a(k)$ *and* $b(k)$ *are sequences bounded above and below by positive constants. Then*

$$\limsup_{k\to+\infty} x(k) \leq \frac{1}{b^l}\exp(a^u - 1).$$

Lemma 2.2 (Lemma 2.2, [3]). *Assume that* $\{x(k)\}$ *satisfies*

$$x(k+1) \geq x(k)\exp\{a(k) - b(k)x(k)\}, \quad k \geq N_0,$$

with $\limsup_{k\to+\infty} x(k) \leq x^*$ *and* $x(N_0) > 0$*, where* $a(k)$ *and* $b(k)$ *are positive sequences bounded above and below by positive constants, and* $N_0 \in N$*. Then*

$$\liminf_{k\to+\infty} x(k) \geq \min\left\{\frac{a^l}{b^u}\exp(a^l - b^u x^*), \frac{a^l}{b^u}\right\}.$$

3. Permanence

Theorem 3.1. *Assume that*

$$m < \min\left\{\frac{r_1^l}{q_1^u E}, \frac{r_2^l}{q_2^u E}\right\}. \tag{3.1}$$

Then system (1.3) *is permanent.*

Proof. From the second equation of system (1.3), it follows that

$$x_2(n+1) \leq x_2(n)\exp\left\{r_2(n) - \frac{r_2(n)x_2(n)}{p_2(n)}\right\}.$$

It follows from Lemma 2.1 that

$$\limsup_{n\to+\infty} x_2(n) \leq \frac{p_2^u}{r_2^l}\exp(r_2^u - 1) \triangleq M_2.$$

For any sufficiently small $\varepsilon > 0$, there exists an integer $n_1 > 0$ such that for all $n > n_1$,

$$x_2(n) \leq M_2 + \varepsilon.$$

Then, for $n > n_1$, we have

$$x_1(n+1) \le x_1(n)\exp\left\{r_1(n) + \frac{\alpha(n)r_1(n)}{p_1(n)}(M_2+\varepsilon) - \frac{r_1(n)}{p_1(n)}x_1(n)\right\}.$$

Letting $\varepsilon \to 0$, it follows from Lemma 2.1 that

$$\limsup_{n\to+\infty} x_1(n) \le \frac{p_1^u}{r_1^l}\exp\left\{r_1^u + \frac{r_1^u\alpha^u}{p_1^l}M_2 - 1\right\} \triangleq M_1.$$

From the equations of system (1.3), it follows that

$$x_i(n+1) \ge x_i(n)\exp\left\{r_i^l - \frac{r_i^u}{p_i^l}x_i(n) - q_i^u Em\right\}, \quad (i=1,2).$$

By (3.1), we have

$$r_1^l - q_1^u Em > 0, \qquad r_2^l - q_2^u Em > 0.$$

It follows from Lemma 2.2 that

$$\liminf_{n\to+\infty} x_i(n) \ge m_i, \quad (i=1,2),$$

where

$$m_i = \min\left\{\frac{r_i^l p_i^l - p_i^l q_i^u Em}{r_i^u}\exp\left\{r_i^l - q_i^u Em - \frac{r_i^u M_i}{p_i^l}\right\}, \frac{r_i^l p_i^l - p_i^l q_i^u Em}{r_i^u}\right\}.$$

So the proof of Theorem 3.1 is completed. □

4. Extinction

Theorem 4.1. *Assume that*

$$m > \max\left\{\frac{r_1^u(p_1^l + \alpha^u M_2)}{p_1^l q_1^l E}, \frac{r_2^u}{q_2^l E}\right\}, \tag{4.1}$$

let $(x_1(n), x_2(n))^T$ *be any positive solution of system* (1.3)*, then*

$$\lim_{n\to\infty} x_i(n) = 0, \quad i = 1,2.$$

Proof. For the above $\varepsilon > 0$, there exists $n_2 > n_1$ such that for all $n > n_2$, we have

$$m > \max\left\{\frac{r_1^u(p_1^l + \alpha^u(M_2+\varepsilon))}{p_1^l q_1^l E}, \frac{r_2^u}{q_2^l E}\right\}, \tag{4.2}$$

$$x_1(n+1) \le x_1(n)\exp\left\{r_1^u + \frac{r_1^u\alpha^u}{p_1^l}(M_2+\varepsilon) - q_1^l Em\right\}, \tag{4.3}$$

$$x_2(n+1) \le x_2(n)\exp\left\{r_2^u - q_2^l Em\right\}. \tag{4.4}$$

By using (4.3) and (4.4), we get

$$\prod_{p=0}^{n-1} x_1(p+1) \le \prod_{p=0}^{n-1} x_1(p) \exp\left\{ r_1^u + \frac{r_1^u \alpha^u}{p_1^l}(M_2+\varepsilon) - q_1^l Em \right\},$$
$$\prod_{p=0}^{n-1} x_2(p+1) \le \prod_{p=0}^{n-1} x_2(p) \exp\left\{ r_2^u - q_2^l Em \right\}.$$

That is

$$x_1(n) \le x_1(0) \exp\left\{ n\left(r_1^u + \frac{r_1^u \alpha^u}{p_1^l}(M_2+\varepsilon) - q_1^l Em \right) \right\},$$
$$x_2(n) \le x_2(0) \exp\left\{ n\left(r_2^u - q_2^l Em \right) \right\}.$$

For $x_1(0) > 0$, $x_2(0) > 0$, it is easy to see that $x_1(n) > 0$, $x_2(n) > 0$. Since (4.2) holds, we have

$$\lim_{n\to\infty} x_i(n) = 0, \quad (i = 1, 2).$$

The proof of Theorem 4.1 is completed. □

Theorem 4.2. *Assume that*

$$\frac{r_1^u - r_2^l + \frac{r_1^u \alpha^u}{p_1^l} M_2 + \frac{r_2^u}{p_2^l} M_2}{q_1^l E - q_2^u E} < m < \frac{r_2^l}{q_2^u E} \tag{4.5}$$

holds, and let $(x_1(n), x_2(n))^T$ *be any positive solution of system* (1.3). *Then species* x_2 *is permanent while species* x_1 *is driven to extinction.*

Proof. By (4.5), we have

$$r_2^l - q_2^u Em > 0, \qquad r_2^l - q_2^u Em - \frac{r_2^u}{p_2^l} M_2 > r_1^u - q_1^l Em + \frac{r_1^u \alpha^u}{p_1^l} M_2.$$

It follows from Theorem 3.1 that species x_2 is permanent. Thus, for the above $\varepsilon > 0$ and $n_1 > 0$, for all $n > n_1$,

$$x_2(n) \le M_2 + \varepsilon. \tag{4.6}$$

Then, we can choose a positive constant δ such that

$$\left\{ r_2^l - q_2^u Em - \frac{r_2^u}{p_2^l}(M_2+\varepsilon) \right\} - \left\{ r_1^u - q_1^l Em + \frac{r_1^u \alpha^u}{p_1^l}(M_2+\varepsilon) \right\} > \delta. \tag{4.7}$$

Let $(x_1(n), x_2(n))^T$ be any positive solution of system (1.3). For any $k > n_1$, we have

$$\begin{aligned} \ln \frac{x_1(k+1)}{x_1(k)} &= r_1(k)\left(1 - \frac{x_1(k)}{p_1(k)} + \frac{\alpha(k)}{p_1(k)} x_2(k) \right) - q_1(k) Em \\ &\le r_1^u - \frac{r_1^l}{p_1^u} x_1(k) + \frac{r_1^u \alpha^u}{p_1^l} x_2(k) - q_1^l Em, \end{aligned} \tag{4.8}$$

$$\begin{aligned}\ln\frac{x_2(k+1)}{x_2(k)} &= r_2(k)\left(1-\frac{x_2(k)}{p_2(k)}\right)-q_2(k)Em\\ &\geq r_2^l-\frac{r_2^u}{p_2^l}x_2(k)-q_2^uEm. \end{aligned}\tag{4.9}$$

Then, inequalities (4.6) - (4.9) lead to

$$\begin{aligned}\ln\frac{x_1(k+1)}{x_1(k)}-\ln\frac{x_2(k+1)}{x_2(k)} \leq&(r_1^u-q_1^lEm-r_2^l+q_2^uEm)+\left(\frac{r_1^u\alpha^u}{p_1^l}+\frac{r_2^u}{p_2^l}\right)x_2(k)-\frac{r_1^l}{p_1^u}x_1(k)\\ \leq& r_1^u-q_1^lEm+\frac{r_1^u\alpha^u}{p_1^l}(M_2+\varepsilon)-\left(r_2^l-q_2^uEm-\frac{r_2^u}{p_2^l}(M_2+\varepsilon)\right)\\ <&-\delta\\ <&0. \end{aligned}\tag{4.10}$$

Summing both sides of (4.10) from 0 to $n-1$, we obtain

$$\ln\frac{x_1(n)}{x_1(0)}-\ln\frac{x_2(n)}{x_2(0)}<-n\delta. \tag{4.11}$$

Then

$$x_1(n)<x_1(0)\frac{x_2(n)}{x_2(0)}\exp\{-n\delta\}. \tag{4.12}$$

Theorem 3.1 implies that $x_2(n)$ is eventually bounded. Then inequality (4.12) shows that $\lim\limits_{n\to\infty}x_1(n)=0$. The proof of Theorem 4.2 is completed. □

Similarly, we can obtain the following theorem.

Theorem 4.3. *Assume that*

$$\frac{r_2^u-r_1^l+\frac{r_1^u}{p_1^l}M_1}{q_2^lE-q_1^uE}<m<\frac{r_1^l}{q_1^uE} \tag{4.13}$$

holds, and let $(x_1(n),x_2(n))^T$ be any positive solution of system (1.3). Then species x_1 is permanent while species x_2 is driven to extinction.

5. Globally attractive

Theorem 5.1. *Assume that*

$$m>\frac{r_2^u}{q_2^lE} \tag{5.1}$$

and there exists a positive constant $\eta>0$ such that

$$\min\left\{\frac{r_1^l}{p_1^u},\frac{2}{M_1}-\frac{r_1^u}{p_1^l}\right\}>\eta \tag{5.2}$$

holds. Then species x_1 is globally attractive while species x_2 is driven to extinction.

Proof. Suppose that $(x_1(n), x_2(n))^T$ and $(x_1^*(n), x_2^*(n))^T$ are any two positive solutions of system (1.3). Under assumption (5.1), it follows from Theorem 4.1 that $\lim\limits_{n\to+\infty} x_2(n) = 0$. Since $\limsup\limits_{n\to\infty} x_1(n) < M_1$, for sufficiently small $\varepsilon > 0$, there exists an integer $N_0 > 0$ such that for all $n > N_0$, we have

$$x_1(n) < M_1 + \varepsilon, \qquad x_2(n) < \varepsilon, \qquad \min\left\{\frac{r_1^l}{p_1^u}, \frac{2}{M_1+\varepsilon} - \frac{r_1^u}{p_1^l}\right\} > \eta.$$

To complete the proof of Theorem 5.1, it suffices to show that $\lim\limits_{n\to+\infty}(x_1(n) - x_1^*(n)) = 0$.

Let $V(n) = |\ln x_1(n) - \ln x_1^*(n)|$. From (1.3), we have

$$\begin{aligned}
|\ln x_1(n+1) - \ln x_1^*(n+1)| &= \bigg|\ln x_1(n) - \ln x_1^*(n) - \frac{r_1(n)}{p_1(n)}(x_1(n) - x_1^*(n)) \\
&\quad + \frac{\alpha(n) r_1(n)}{p_1(n)}(x_2(n) - x_2^*(n))\bigg| \\
&\le \left|\ln x_1(n) - \ln x_1^*(n) - \frac{r_1(n)}{p_1(n)}(x_1(n) - x_1^*(n))\right| \\
&\quad + \frac{\alpha(n) r_1(n)}{p_1(n)}|x_2(n) - x_2^*(n)|.
\end{aligned}$$

Since $\ln x_1(n) - \ln x_1^*(n) = \frac{1}{\xi_1(n)}(x_1(n) - x_1^*(n))$, where $\min\{x_1(n), x_1^*(n)\} \le \xi_1(n) \le \max\{x_1(n), x_1^*(n)\} \le M_1 + \varepsilon$, we have

$$\begin{aligned}
\Delta V(n) &\le -\left(\frac{1}{\xi_1(n)} - \left|\frac{1}{\xi_1(n)} - \frac{r_1(n)}{p_1(n)}\right|\right)|x_1(n) - x_1^*(n)| + \frac{\alpha^u r_1^u}{p_1^l}|x_2(n) - x_2^*(n)| \\
&\le -\min\left\{\frac{r_1^l}{p_1^u}, \frac{2}{M_1+\varepsilon} - \frac{r_1^u}{p_1^l}\right\}|x_1(n) - x_1^*(n)| + 2\frac{\alpha^u r_1^u}{p_1^l}\varepsilon \\
&\le -\eta|x_1(n) - x_1^*(n)| + 2\frac{\alpha^u r_1^u}{p_1^l}\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, it follows that

$$\Delta V(n) \le -\eta|x_1(n) - x_1^*(n)|.$$

Then

$$\sum_{p=N_0}^{n}(V(p+1) - V(p)) \le -\eta\sum_{p=N_0}^{n}|x_1(p) - x_1^*(p)|,$$

that is

$$V(n+1) - V(N_0) \le -\eta\sum_{p=N_0}^{n}|x_1(p) - x_1^*(p)|.$$

Therefore,

$$\sum_{p=N_0}^{n}|x_1(p) - x_1^*(p)| \le \frac{V(N_0)}{\eta} < +\infty.$$

Hence, $\lim\limits_{n\to+\infty}(x_1(n) - x_1^*(n)) = 0$. The proof of Theorem 5.1 is completed. □

Similarly, we can obtain the following theorem.

Theorem 5.2. *Assume that there exists a positive constant $\gamma > 0$ such that*

$$\min\left\{\frac{r_2^l}{p_2^u}, \frac{2}{M_2} - \frac{r_2^u}{p_2^l}\right\} > \gamma$$

holds. Then species x_2 is globally attractive.

6. Examples and numerical simulations

The following examples lend credence to the plausibility of the main results.

Example 6.1. Corresponding to system (1.3), we assume that

$$\begin{aligned} &r_1(n) = 0.5\sin n + 1, \quad r_2(n) = 1 - 0.8\cos n, \quad m = 0.1, \\ &p_1(n) = 0.1(\sin n + 2), \quad p_2(n) = 0.2(\cos n + 1.5), \quad E = 1, \\ &q_1(n) = 0.2\cos n + 0.8, \quad q_2(n) = 0.2(\cos n + 1.5), \quad \alpha(n) = 0.5\cos n + 0.6. \end{aligned} \tag{6.1}$$

It is easy to see that $\frac{r_1^l}{q_1^u E} = 0.5$, $\frac{r_2^l}{q_2^u E} = 0.4$, $m = 0.1 < \min\{0.5, 0.4\}$. Then the conditions of Theorem 3.1 are satisfied (see Figure 1).

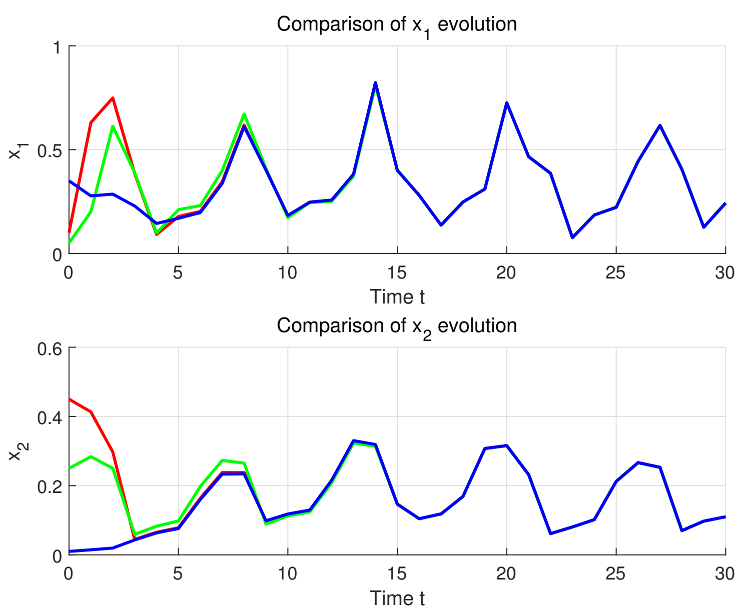

Figure 1. Dynamic behaviors of system (1.3) with initial conditions $(x_1(0), x_2(0))^T = (0.1, 0.45)^T, (0.05, 0.25)^T$, and $(0.35, 0.01)^T$.

Example 6.2. Corresponding to system (1.3), we assume that

$$\begin{aligned} &r_1(n) = 0.5 + 0.001\sin n, \quad r_2(n) = 2 + 0.001\cos(\sqrt{2}n), \\ &p_1(n) = 1 + 0.001\cos(\sqrt{2}n), \quad p_2(n) = 0.1 + 0.001\sin n, \\ &q_1(n) = 100 + 0.1\sin n, \quad q_2(n) = 10 + 0.001\cos(\sqrt{2}n), \end{aligned} \tag{6.2}$$

$$\alpha(n) = 0.005\left(1 + \frac{1+\sin n}{2}\right), \quad m = 0.3, \quad E = 1.$$

It is easy to see that $M_2 \approx 0.1374$, $\frac{r_1^u(p_1^l+\alpha^u M_2)}{p_1^l q_1^l E} \approx 0.005022$, $\frac{r_2^u}{q_2^l E} \approx 0.20012$, $m = 0.3 > \max\{0.005022, 0.20012\}$. Then the conditions of Theorem 4.1 are satisfied (see Figure 2).

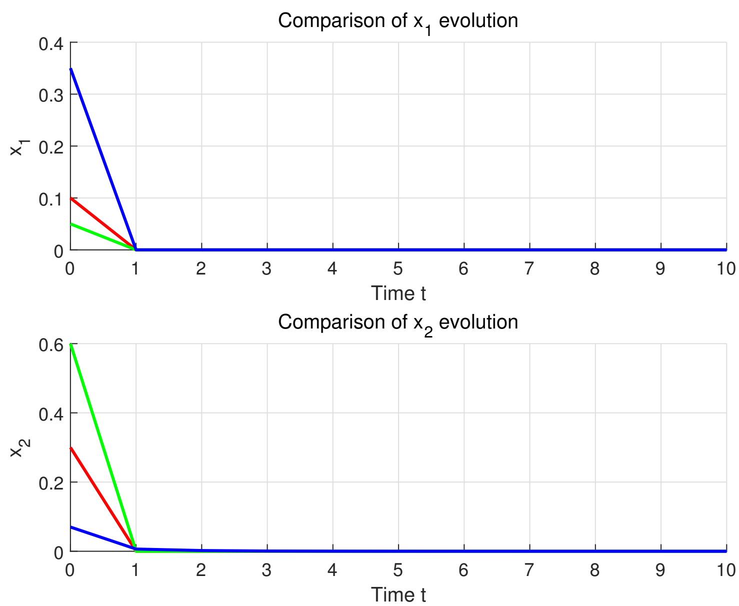

Figure 2. Dynamic behaviors of system (1.3) with initial conditions $(x_1(0), x_2(0))^T = (0.1, 0.3)^T, (0.05, 0.6)^T$, and $(0.35, 0.07)^T$.

Example 6.3. Corresponding to system (1.3), we assume that

$$\begin{aligned}
&r_1(n) = 1 + 0.001\sin n, \quad r_2(n) = 0.6 + 0.001\cos(\sqrt{2}n),\\
&p_1(n) = 1 + 0.001\cos(\sqrt{2}n), \quad p_2(n) = 0.1 + 0.001\sin n,\\
&q_1(n) = 1000 + 0.1\sin n, \quad q_2(n) = 1 + 0.001\cos(\sqrt{2}n),\\
&\alpha(n) = 0.005\left(1 + \frac{1+\sin n}{2}\right), \quad m = 0.1, \quad E = 1.
\end{aligned} \tag{6.3}$$

It is easy to see that $M_2 = \frac{p_2^u}{r_2^l}\exp(r_2^u - 1) \approx 0.1132$, $\frac{r_1^u - r_2^l + \frac{r_1^u\alpha^u}{p_1^l}M_2 + \frac{r_2^u}{p_2^l}M_2}{q_1^l E - q_2^u E} \approx 0.001092 < m < \frac{r_2^l}{q_2^u E} \approx 0.5984$. Then the conditions of Theorem 4.2 are satisfied (see Figure 3).

Example 6.4. Corresponding to system (1.3), we assume that

$$\begin{aligned}
&r_1(n) = 1 + 0.01\sin n, \quad r_2(n) = 2 + 0.01\cos(\sqrt{2}n),\\
&p_1(n) = 5 + 0.01\cos(\sqrt{2}n), \quad p_2(n) = 0.2 + 0.01\sin n,\\
&q_1(n) = 0.1 + 0.001\sin n, \quad q_2(n) = 20 + 0.001\cos(\sqrt{2}n),
\end{aligned} \tag{6.4}$$

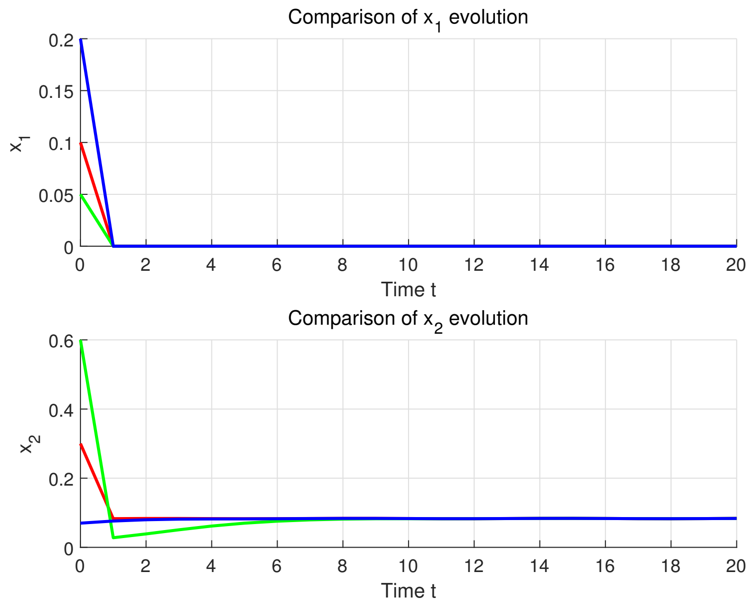

Figure 3. Dynamic behaviors of system (1.3) with initial conditions $(x_1(0), x_2(0))^T = (0.1, 0.3)^T, (0.05, 0.6)^T$, and $(0.2, 0.07)^T$.

$$\alpha(n) = 0.001\left(1 + \frac{1+\sin n}{2}\right), \quad m = 0.15, \quad E = 1.$$

It is easy to see that $M_2 = \frac{p_2^u}{r_2^l}\exp(r_2^u - 1) \approx 0.2897$, $M_1 = \frac{p_1^u}{r_1^l}\exp\left\{r_1^u + \frac{r_1^u\alpha^u}{p_1^l}M_2 - 1\right\} \approx 5.112$, $\frac{r_2^u - r_1^l + \frac{r_1^u}{p_1^l}M_1}{q_2^l E - q_1^u E} \approx 0.1033 < m < \frac{r_1^l}{q_1^u E} \approx 9.802$. Then the conditions of Theorem 4.3 are satisfied (see Figure 4).

Example 6.5. Corresponding to system (1.3), we assume that

$$\begin{aligned}
&r_1(n) = 1 + 0.1\sin n, \quad r_2(n) = 1 + 0.1\cos(\sqrt{2}n),\\
&p_1(n) = 2 + 0.1\sin(\sqrt{2}n), \quad p_2(n) = 1 + 0.1\sin n,\\
&q_1(n) = 0.1 + 0.001\sin n, \quad q_2(n) = 4 + 0.1\cos(\sqrt{2}n),\\
&\alpha(n) = 0.1 + 0.05\cos n, \quad m = 0.5, \quad E = 1.
\end{aligned} \tag{6.5}$$

It is easy to see that $M_2 = \frac{p_2^u}{r_2^l}\exp(r_2^u - 1) \approx 1.3509$, $M_1 = \frac{p_1^u}{r_1^l}\exp\left\{r_1^u + \frac{r_1^u\alpha^u}{p_1^l}M_2 - 1\right\} \approx 2.8994$, $m = 0.5 > \frac{r_2^u}{q_2^l E} \approx 0.28205$, $\min\left\{\frac{r_1^l}{p_1^u}, \frac{2}{M_1} - \frac{r_1^u}{p_1^l}\right\} \approx \min\{0.42857, 0.11085\} > 0$. Then the conditions of Theorem 5.1 are satisfied (see Figure 5).

Example 6.6. Corresponding to system (1.3), we assume that

$$\begin{aligned}
&r_1(n) = 1 + 0.2\cos(2\pi n), \quad r_2(n) = 1 + 0.05\cos(\sqrt{2}n),\\
&p_1(n) = 1 + 0.2\sin(2\pi n), \quad p_2(n) = 0.9 + 0.05\sin n,\\
&q_1(n) = 0.2 + 0.1\sin(\sqrt{2}n), \quad q_2(n) = 0.2 + 0.1\cos(\sqrt{2}n),
\end{aligned} \tag{6.6}$$

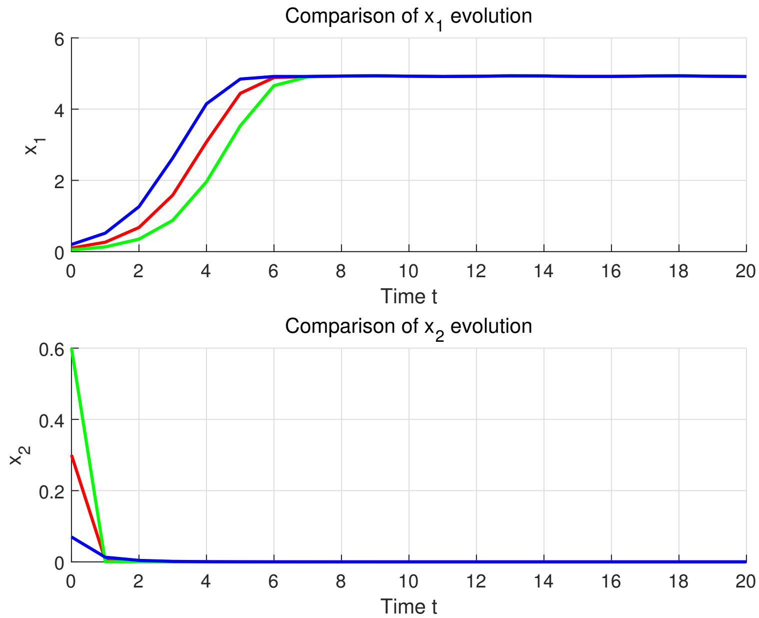

Figure 4. Dynamic behaviors of system (1.3) with initial conditions $(x_1(0), x_2(0))^T = (0.1, 0.3)^T, (0.05, 0.6)^T$, and $(0.2, 0.07)^T$.

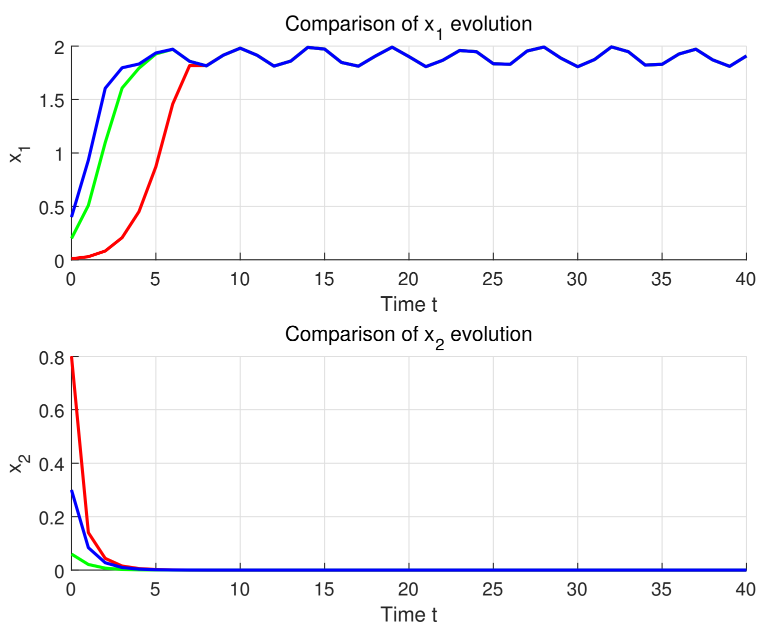

Figure 5. Dynamic behaviors of system (1.3) with initial conditions $(x_1(0), x_2(0))^T = (0.01, 0.8)^T, (0.2, 0.06)^T$, and $(0.4, 0.3)^T$.

$$\alpha(n) = 0.5 + 0.4\cos(2\pi n), \quad m = 0.15, \quad E = 10.$$

It is easy to see that $M_2 = \frac{p_2^u}{r_2^l}\exp(r_2^u - 1) \approx 1.05$, $\min\left\{\frac{r_2^l}{p_2^u}, \frac{2}{M_2} - \frac{r_2^u}{p_2^l}\right\} \approx \min\{1, 0.6667\} > 0$.

Then the conditions of Theorem 5.2 are satisfied (see Figure 6).

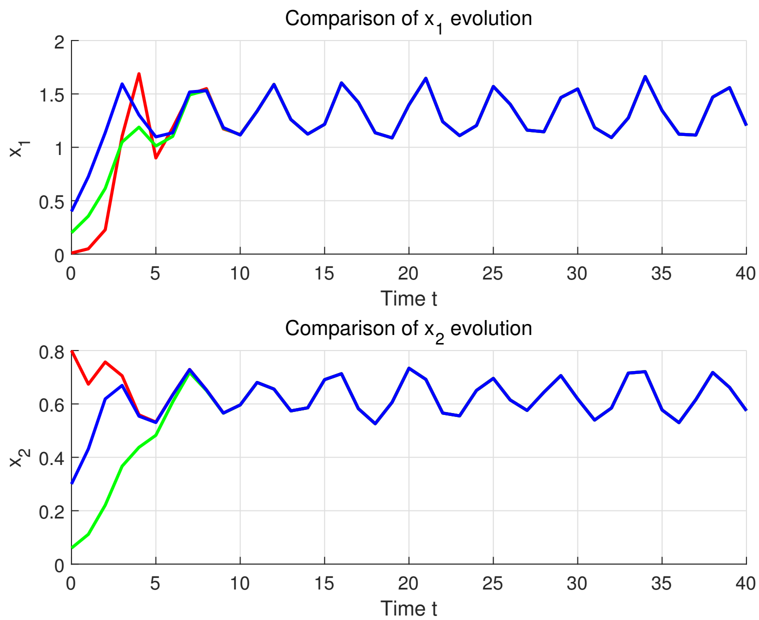

Figure 6. Dynamic behaviors of system (1.3) with initial conditions $(x_1(0), x_2(0))^T = (0.01, 0.8)^T, (0.2, 0.06)^T$, and $(0.4, 0.3)^T$.

7. Conclusion

Aiming at long-term ecological sustainability and species conservation, this paper investigates the dynamic behaviors of a non-selective harvesting discrete commensalism model. Section 3 shows that when the harvesting fraction m is sufficiently small, the system is permanent. This implies that low harvesting intensity allows both species to coexist, highlighting the importance of controlled exploitation for biodiversity maintenance.

Section 4 reveals threshold effects of harvesting intensity. Theorem 4.1 indicates that if m is too large, the entire system goes extinct, warning that overharvesting may cause ecological collapse. Theorem 4.2 shows that when m is below a certain value and (4.5) holds, the beneficiary species x_2 persists while the host x_1 goes extinct. This suggests that under moderate harvesting, the benefiting species may outlast the host, so protecting the host is key to system stability. Theorem 4.3 presents the opposite scenario: Within a specific range of m, x_1 persists while x_2 goes extinct, indicating that harvesting can selectively drive a commensal species to extinction.

Section 5 provides conditions for global attractivity. Theorem 5.1 shows that when m exceeds a threshold and (5.1) holds, x_1 is globally attractive while x_2 goes extinct, meaning that high harvesting can stabilize one species at the cost of the other. Theorem 5.2 shows that m has no impact on the global attractivity of x_2, suggesting that the long-term dynamics of x_2 are governed by other intrinsic factors.

Overall, these results offer practical insights for designing nature reserves: Harvesting thresholds must be carefully set to avoid species loss, and the asymmetric dependence in commensal relationships should be fully considered. Sustainable harvesting is not merely about reducing catch but about respecting ecological thresholds that maintain interspecific interactions.

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