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DYNAMIC ANALYSIS FOR COURNOT COMPETITION MODEL WITH EXPONENTIAL DEMAND FUNCTION AND RELATIVE PROFIT MAXIMIZATION

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Abstract This paper introduces a nonlinear system of difference equations which describes a qualitative study of Cournot oligopoly games with two boundedly rational players. With nonlinear demand function of exponential form, the model is built by following the traditional role in which players have limited information. The model describes a competition between two firms whose main interest is to maximize their relative profits. Several local and global analysis about the equilibrium points of the model are analyzed.

Keywords Cournot model, relative profit, stability, global asymptotic behavior.

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1. Introduction

Market economy is fundamentally a dynamic system, which can usually be described mathematically by difference equations. In the dynamic study of economics, a couple of models represented by difference equations are investigated, such as the classical cobweb model describing the variation of the supply and demand, the Cournot models of oligopoly, an so on [1,3,5,18,21]. Many researchers have made landmark contributions to the development of Cournot game and have developed various interesting results, as shown in [2,4,6–9,22,23]. While difference equations of exponential form can be used in many fields to describe realistic problem and has been attracting the interest of many scholars [11,14,16,20]. In the study of oligopoly game, the demand function of exponential form can describe the bounded rationality duopoly game more realistic [19] and at the same time the dynamic of exponential system will be more complicated [10,13,15,17,24]. Those cited works have given a lot of interest results on the complex dynamic characteristics on oligopolistic competition models. And most of them focused on the maximization of absolute profits. Along these studies we introduce a certain type of Cournot models where the demand function follows exponential growth and the relative profit maximization is presented.

Assume in this paper the demand function

$$p = f(Q) = a\mathrm{e}^{-(q_1+q_2)}, \tag{1.1}$$

where a is a parameter of maximum price in the market and q_i, $i = 1, 2$ denotes the quantity supplied by firm i with $i = 1$, 2. Assume the cost function

$$C_i(Q) = c_i q_i, \quad i = 1,\ 2, \tag{1.2}$$

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where c_i is the marginal cost of the $i-$th firm. Then the absolute profit resulting from the above Cournot duopoly game is given by

$$\pi_i(q_1, q_2) = f(Q(t))q_i(t) - c_i q_i(t) = a\mathrm{e}^{-(q_1(t)+q_2(t))}q_i(t) - c_i q_i(t). \tag{1.3}$$

Now each firm calculates its relative profit as the difference between its absolute profit and the average profit of its opponents. It means that the relative profit of each firm is

$$\begin{aligned}\Pi_1 &= \pi_1(q_1, q_2) - \pi_2(q_1, q_2) = a\mathrm{e}^{-(q_1+q_2)}(q_1 - q_2) - c_1 q_1 + c_2 q_2,\\ \Pi_2 &= \pi_2(q_1, q_2) - \pi_1(q_1, q_2) = a\mathrm{e}^{-(q_1+q_2)}(q_2 - q_1) + c_1 q_1 - c_2 q_2.\end{aligned}$$

The classical oligopoly games, and the associated notion of Nash equilibrium, are based on quite demanding notion of rationality. However, since the available information in the oligopoly market is incomplete, the rational players make their price decisions on a local estimate of the expected marginal relative profit $\frac{\partial \Pi_i}{\partial q_i}$. Hence, the dynamical equation of the bounded rationality player i has the form

$$q_i(t+1) = q_i(t) + \alpha_i q_i(t)\frac{\partial \Pi_i}{\partial q_i}, \quad i = 1,\, 2, \tag{1.4}$$

where $\alpha_i,\ i = 1,\, 2$ is a positive parameter which represents the relative speed of the quantity adjustment by producer i. Therefore, using (1.3), the discrete dynamical system becomes an iterated two-dimensional mapping which has the form

$$\begin{cases} q_1(t+1) = (1-\alpha_1 c_1)q_1(t) + a\alpha_1 q_1(t)(1 - q_1(t) + q_2(t))\mathrm{e}^{-(q_1(t)+q_2(t))},\\ q_2(t+1) = (1-\alpha_2 c_2)q_2(t) + a\alpha_2 q_2(t)(1 - q_2(t) + q_1(t))\mathrm{e}^{-(q_1(t)+q_2(t))}.\end{cases} \tag{1.5}$$

We can rewrite this system in the new form

$$\begin{cases} u(t+1) = (1-a_1)u(t) + b_1 u(t)(1 - u(t) + v(t))\mathrm{e}^{-(u(t)+v(t))},\\ v(t+1) = (1-a_2)v(t) + b_2 v(t)(1 - v(t) + u(t))\mathrm{e}^{-(u(t)+v(t))},\end{cases} \tag{1.6}$$

where $a_i = \alpha_i c_i,\ b_i = a\alpha_i,\ i = 1, 2$. In this paper we study the boundedness and the global asymptotic behavior of the positive solutions of the system (1.6). The following theorem will be used later.

Theorem 1.1 (Theorem 3, [12])**.** *Suppose $T = (f, g)$ be a monotone map on a closed and bounded rectangular region $S \subset \mathbb{R}^2$. If T has a unique fixed point $E = (\bar{x}, \bar{y})$ in S, then E is a global attractor of T on S.Please state your theorem here.*

2. Equilibria points of (1.6)

In this section, the existence of non-negative equilibrium points of (1.6) are examined. Observe that the equilibria points $(\bar{u}, \bar{v})$ of (1.6) are given by the equations

$$\begin{cases} \bar{u} = (1-a_1)\bar{u} + b_1\bar{u}(1 - \bar{u} + \bar{v})\mathrm{e}^{-(\bar{u}+\bar{v})}, & (2.1)\\ \bar{v} = (1-a_2)\bar{v} + b_2\bar{v}(1 - \bar{v} + \bar{u})\mathrm{e}^{-(\bar{u}+\bar{v})}, & (2.2)\end{cases}$$

the following results can be obtained.

Proposition 2.1. *(i) When $a_i \geq b_i,\ i = 1, 2$, (1.6) has a unique equilibrium point $E_0 = (0, 0)$;*

(ii) When $a_i < b_i,\ i = 1, 2$, (1.6) has two equilibrium points $E_1 = (u^*, 0)$ and $E_2 = (0,\ v^*)$.

(iii) (1.6) has a unique positive equilibrium point $(\bar{u}, \bar{v})$ when

$$\frac{2}{\mathrm{e}^2} < \frac{a_1}{b_1} + \frac{a_2}{b_2} < 2 \tag{2.3}$$

and

$$\max\{\frac{a_1b_2 + a_2b_1}{2b_1b_2}, \frac{2b_1b_2}{\mathrm{e}^2(a_1b_2 + a_2b_1)}\} < \mathrm{e}^r < \min\{\frac{2b_1b_2}{a_1b_2 + a_2b_1}, \frac{\mathrm{e}^2(a_1b_2 + a_2b_1)}{2b_1b_2}\}, \tag{2.4}$$

where $r = \frac{a_1b_2 - a_2b_1}{a_1b_2 + a_2b_1}$.

Proof. From (2.1), we have $\bar{u} = 0$ or $a_1 = b_1(1 - \bar{u} + \bar{v})\mathrm{e}^{-(\bar{u}+\bar{v})}$. Similarly, $\bar{v} = 0$ or $a_2 = b_2(1 - \bar{v} + \bar{u})\mathrm{e}^{-(\bar{u}+\bar{v})}$ from (2.2). Set $f(x) = a_1 - b_1(1 - x)\mathrm{e}^{-x}$. Then

$$f(0) = a_1 - b_1,\ f(1) = a_1,\ \lim_{x\to+\infty} f(x) = a_1,\ f'(x) = b_1(2 - x)\mathrm{e}^{-x}.$$

Obviously, f has a unique critical point $x = 2$ in $(0, +\infty)$, which means $f(2)$ is the absolute maximum of f in $(0, +\infty)$.

(i) If $a_1 \geq b_1$, then $f(0) \geq 0$, $f(1) > 0$, $f'(x) > 0$ in $(0, 2)$, $f'(x) < 0$ in $(2, +\infty)$ and $\lim_{x\to+\infty} f(x) = a_1 > 0$. Thus f and has no positive root in $(-\infty, +\infty)$. Similar analysis can be applied to (2.2) if we set $g(y) = a_2 - b_2(1 - y)\mathrm{e}^{-y}$. Thus (1.6) has a unique equilibrium point $E_0 = (0, 0)$ provided $a_i \geq b_i,\ i = 1, 2$.

(ii) If $a_1 < b_1$, then $f(0) < 0$ and since $f(1) > 0$, $f'(x) > 0$ in $(0, 2)$, f has a unique root $u^* \in (0, 1)$ in $(0, +\infty)$. In addition, from $f'(x) < 0$ in $(2, +\infty)$ and $\lim_{x\to+\infty} f(x) = a_1 > 0$, the positive root u^* is the unique root of f. Hence $(u^*, 0)$ is a nonzero root of (2.1). Similar analysis can be applied to (2.2) if $a_2 < b_2$. Thus, (1.6) has two equilibrium points $E_1 = (u^*, 0)$ and $E_2 = (0,\ v^*)$, where u^* and v^* satisfy $a_1 = b_1(1 - u^*)\mathrm{e}^{-u^*}$ and $a_2 = b_2(1 - v^*)\mathrm{e}^{-v^*}$, respectively.

(iii) Assume that $(\bar{u}, \bar{v})$ is a solution of

$$\begin{cases} a_1 = b_1(1 - \bar{u} + \bar{v})\mathrm{e}^{-(\bar{u}+\bar{v})}, \\ a_2 = b_2(1 - \bar{v} + \bar{u})\mathrm{e}^{-(\bar{u}+\bar{v})}, \end{cases}$$

with $\bar{u} > 0$ and $\bar{v} > 0$. It follows that

$$\mathrm{e}^{\bar{u}+\bar{v}} = \frac{b_1}{a_1}(1 - \bar{u} + \bar{v}), \quad \mathrm{e}^{\bar{u}+\bar{v}} = \frac{b_2}{a_2}(1 - \bar{v} + \bar{u}).$$

Obviously, $|\bar{u} - \bar{v}| < 1$ guarantees that $1 - \bar{u} + \bar{v} > 0$ and $1 - \bar{v} + \bar{u} > 0$. Notice that $\frac{b_1}{a_1}(1 - \bar{u} + \bar{v}) = \frac{b_2}{a_2}(1 - \bar{v} + \bar{u})$, we have $\bar{v} = \frac{a_1b_2 - a_2b_1}{a_1b_2 + a_2b_1} + \bar{u}$. Consequently,

$$\bar{u} + \bar{v} = \frac{a_1b_2 - a_2b_1}{a_1b_2 + a_2b_1} + 2\bar{u} \quad \text{or} \quad \bar{u} + \bar{v} = 2\bar{v} - \frac{a_1b_2 - a_2b_1}{a_1b_2 + a_2b_1},$$

and

$$\bar{v} - \bar{u} = \frac{a_1b_2 - a_2b_1}{a_1b_2 + a_2b_1}.$$

Now set $g(x) = a_1 - b_1 e^{-r}(1+r)e^{-2x}$, where $r = \frac{a_1b_2 - a_2b_1}{a_1b_2 + a_2b_1}$. We have $1 + r = \frac{2a_1b_2}{a_1b_2+a_2b_1}$ and $1 - r = \frac{2a_2b_1}{a_1b_2+a_2b_1}$. If $1 + r = 0$, g has no zero point, while if $1 + r \neq 0$,

$$g(-\infty) = \begin{cases} -\infty, & 1+r>0, \\ +\infty, & 1+r<0, \end{cases} \qquad g(+\infty) = a_1,$$

$$g'(x) = 2b_1 e^{-r}(1+r)e^{-2x} \begin{cases} >0, & 1+r>0, \\ <0, & 1+r<0, \end{cases}$$

$$g''(x) = -4b_1 e^{-r}(1+r)e^{-2x} \begin{cases} <0, & 1+r>0, \\ >0, & 1+r<0. \end{cases}$$

Notice that $g(0) = a_1 - b_1 e^{-r}(1+r)$ and $g(1) = a_1 - b_1 e^{-r}(1+r)e^{-2}$, it follows from $\frac{2b_1b_2}{a_1b_2+a_2b_1}e^{-2} < e^r < \frac{2b_1b_2}{a_1b_2+a_2b_1}$ that $g(0) < 0$ and $g(1) > 0$. Since g is a continuous in $(-\infty, +\infty)$, we have $g(x) = 0$ has a unique positive root in $(0, 1)$. Similarly, set $h(y) = a_2 - b_2 e^r(1-r)e^{-2y}$. If $\frac{a_1b_2+a_2b_1}{2b_1b_2} < e^r < \frac{a_1b_2+a_2b_1}{2b_1b_2}e^2$, $h(y) = 0$ has a unique positive root in $(0, 1)$. Therefore, (1.6) has the unique equilibrium $(\bar{u}, \bar{v})$ satisfying $\bar{v} = \bar{u} + r$. □

3. Local stability of the equilibrium points

In the following, we deal with the local stability of the above four equilibrium points of (1.6). Observe that the equilibria points of (1.6) are given by the equations

$$\begin{cases} \bar{u} = (1-a_1)\bar{u} + b_1\bar{u}(1-\bar{u}+\bar{v})e^{-(\bar{u}+\bar{v})}, \\ \bar{v} = (1-a_2)\bar{v} + b_2\bar{v}(1-\bar{v}+\bar{u})e^{-(\bar{u}+\bar{v})}. \end{cases}$$

Set $F(u, v) = (1-a_1)u + b_1u(1-u+v)e^{-(u+v)}, G(u, v) = (1-a_2)v + b_2v(1-v+u)e^{-(u+v)}$, where F and G are continuous functions. Then we obtain

$$\begin{aligned} &\frac{\partial F}{\partial u} = 1 - a_1 + b_1[u^2 - (v+3)u + 1 + v]e^{-(u+v)}, \frac{\partial F}{\partial v} = b_1u(u-v)e^{-(u+v)}, \\ &\frac{\partial G}{\partial u} = b_2v(v-u)e^{-(u+v)}, \frac{\partial G}{\partial v} == 1 - a_2 + b_2[v^2 - (u+3)v + 1 + u]e^{-(u+v)}. \end{aligned}$$

Proposition 3.1. *The equilibrium point $E_0 = (0, 0)$ of (1.6) is locally asymptotically stable if $b_i < a_i < 2 + b_i$, $i = 1, 2$.*

Proof. The Jacobian matrix of (1.6) about the equilibrium point E_0 is

$$J(E_0) = \begin{pmatrix} 1 - a_1 + b_1 & 0 \\ 0 & 1 - a_2 + b_2 \end{pmatrix}. \tag{3.1}$$

The eigenvalues of $J(E_0)$ are given by $\lambda_1 = 1 - a_1 + b_1$ and $\lambda_2 = 1 - a_2 + b_2$. Since $b_i < a_i < 2 + b_i$, $i = 1, 2$, we have $|\lambda_i| < 1$, $i = 1, 2$ which imply the equilibrium point E_0 of (1.6) is locally asymptotically stable. □

Proposition 3.2. *The equilibrium points E_1 and E_2 of (1.6) are saddle points.*

Proof. The Jacobian matrix of (1.6) about the equilibrium point $E_1 = (u^*, 0)$ is

$$J(E_1) = \begin{pmatrix} 1 - a_1 + b_1(u^{*2} - 3u^* + 1)\mathrm{e}^{-u^*} & b_1 u^{*2}\mathrm{e}^{-u^*} \\ 0 & 1 - a_2 + b_2(1 + u^*)\mathrm{e}^{-u^*} \end{pmatrix}. \tag{3.2}$$

The eigenvalues of $J(E_1)$ are given by $\lambda_1 = 1 - a_1 + b_1(u^{*2} - 3u^* + 1)\mathrm{e}^{-u^*}$ and $\lambda_2 = 1 - a_2 + b_2(1 + u^*)\mathrm{e}^{-u^*}$. Note that in this case $a_1 = b_1(1 - u^*)\mathrm{e}^{-u^*}$, we obtain

$$\begin{aligned} \lambda_1 &= 1 - a_1 + b_1(u^{*2} - 3u^* + 1)\mathrm{e}^{-u^*} \\ &= 1 - b_1(1 - u^*)\mathrm{e}^{-u^*} + b_1(u^{*2} - 3u^* + 1)\mathrm{e}^{-u^*} \\ &= 1 - b_1 u^*(2 - u^*)\mathrm{e}^{-u^*} \\ &< 1, \end{aligned} \tag{3.3}$$

$$\lambda_2 = 1 - a_2 + b_2(1 + u^*)\mathrm{e}^{-u^*} = 1 - a_2 + a_2\frac{1 + u^*}{1 - u^*} = 1 + a_2\frac{2u^*}{1 - u^*} > 1, \tag{3.4}$$

which imply the equilibrium points E_1 of system (1.6) is a saddle point. We can similarly prove that the equilibrium point E_2 of system (1.6) is also a saddle point. □

Proposition 3.3. *The Nash equilibrium point E_3 of (1.6) is asymptotically stable if*

$$\begin{aligned} a_1 a_2 \frac{4\bar{u}\bar{v}}{(1 - \bar{u} + \bar{v})(1 - \bar{v} + \bar{u})} &< a_1\frac{(2 + \bar{v} - \bar{u})\bar{u}}{1 - \bar{u} + \bar{v}} + a_2\frac{(2 + \bar{u} - \bar{v})\bar{v}}{1 - \bar{v} + \bar{u}} \\ &< 2 + a_1 a_2\frac{2\bar{u}\bar{v}}{(1 - \bar{u} + \bar{v})(1 - \bar{v} + \bar{u})}. \end{aligned} \tag{3.5}$$

Proof. The Jacobian matrix of (1.6) about the equilibrium point $E_3 = (\bar{u}, \bar{v})$ is

$$\begin{aligned} &J(E_3) \\ &= \begin{pmatrix} 1 - a_1 + b_1[\bar{u}^2 - (3 + \bar{v})\bar{u} + 1 + \bar{v}]\mathrm{e}^{-(\bar{u}+\bar{v})} & b_1\bar{u}(\bar{u} - \bar{v})\mathrm{e}^{-(\bar{u}+\bar{v})} \\ b_2\bar{v}(\bar{v} - \bar{u})\mathrm{e}^{-(\bar{u}+\bar{v})} & 1 - a_2 + b_2[\bar{v}^2 - (3 + \bar{u})\bar{v} + 1 + \bar{u}]\mathrm{e}^{-(\bar{u}+\bar{v})} \end{pmatrix}. \end{aligned}$$

Note that in this case $a_1 = b_1(1 - \bar{u} + \bar{v})\mathrm{e}^{-(\bar{u}+\bar{v})}$, $a_2 = b_2(1 - \bar{v} + \bar{u})\mathrm{e}^{-(\bar{u}+\bar{v})}$, we have

$$\begin{aligned} J(E_3) &= \begin{pmatrix} 1 - a_1 + \dfrac{a_1[\bar{u}^2 - (3 + \bar{v})\bar{u} + 1 + \bar{v}]}{1 - \bar{u} + \bar{v}} & \dfrac{a_1\bar{u}(\bar{u} - \bar{v})}{1 - \bar{u} + \bar{v}} \\ \dfrac{a_2\bar{v}(\bar{v} - \bar{u})}{1 - \bar{v} + \bar{u}} & 1 - a_2 + \dfrac{a_2[\bar{v}^2 - (3 + \bar{u})\bar{v} + 1 + \bar{u}]}{1 - \bar{v} + \bar{u}} \end{pmatrix} \\ &= \begin{pmatrix} 1 - a_1\dfrac{(2 + \bar{v} - \bar{u})\bar{u}}{1 - \bar{u} + \bar{v}} & \dfrac{a_1\bar{u}(\bar{u} - \bar{v})}{1 - \bar{u} + \bar{v}} \\ \dfrac{a_2\bar{v}(\bar{v} - \bar{u})}{1 - \bar{v} + \bar{u}} & 1 - a_2\dfrac{(2 + \bar{u} - \bar{v})\bar{v}}{1 - \bar{v} + \bar{u}} \end{pmatrix}. \end{aligned} \tag{3.6}$$

Consequently,

$$Tr(J(E_3)) = 2 - a_1\frac{(2 + \bar{v} - \bar{u})\bar{u}}{1 - \bar{u} + \bar{v}} - a_2\frac{(2 + \bar{u} - \bar{v})\bar{v}}{1 - \bar{v} + \bar{u}} \tag{3.7}$$

and

$$\begin{aligned}Det(J(E_3)) =&1 - a_1\frac{(2+\bar{v}-\bar{u})\bar{u}}{1-\bar{u}+\bar{v}} - a_2\frac{(2+\bar{u}-\bar{v})\bar{v}}{1-\bar{v}+\bar{u}}\\ &+ a_1a_2[\frac{(2+\bar{v}-\bar{u})\bar{u}}{1-\bar{u}+\bar{v}}\cdot\frac{(2+\bar{u}-\bar{v})\bar{v}}{1-\bar{v}+\bar{u}} - \frac{\bar{u}(\bar{u}-\bar{v})}{1-\bar{u}+\bar{v}}\cdot\frac{\bar{v}(\bar{v}-\bar{u})}{1-\bar{v}+\bar{u}}]\\ =&1 - a_1\frac{(2+\bar{v}-\bar{u})\bar{u}}{1-\bar{u}+\bar{v}} - a_2\frac{(2+\bar{u}-\bar{v})\bar{v}}{1-\bar{v}+\bar{u}}\\ &+ a_1a_2\frac{4\bar{u}\bar{v}}{(1-\bar{u}+\bar{v})(1-\bar{v}+\bar{u})}.\end{aligned} \tag{3.8}$$

From (3.5), $|Det(J(E_3))| < 1$ and $|Tr(J(E_3))| < 1 + Det(J(E_3))$ which guarantee the Nash equilibrium point E_3 is asymptotically stable. The proof completes. □

4. Boundedness and invariant

Theorem 4.1. *Assume $a_i + b_i\mathrm{e}^{-2} < 1$. Then every solution of (1.6) with $u(0) > 0, v(0) > 0$ satisfies $u(t) > 0, v(t) > 0$ for all $t > 0$.*

Proof. Let

$$F_i(x,y) = 1 - a_i + b_i(1-x+y)\mathrm{e}^{-(x+y)}, \quad i = 1, 2. \tag{4.1}$$

Then (1.6) can be rewritten in the form

$$\begin{aligned}u(t+1) &= u(t)F_1(u(t),v(t)),\\ v(t+1) &= v(t)F_2(v(t),u(t)).\end{aligned}$$

Assume $\{(u(t),v(t))\}_{t=0}^{\infty}$ is a solution of (1.6) with $u(0) > 0, v(0) > 0$. Observe that

$$\frac{\partial F_i(x,y)}{\partial x} = -b_i(2-x+y)\mathrm{e}^{-(x+y)}, \quad \frac{\partial F_i(x,y)}{\partial y} = b_i(x-y)\mathrm{e}^{-(x+y)}, \quad i = 1, 2.$$

It is obvious that F_i has no positive critical points. Consider the domain

$$D = \{(x,y) : 0 \le x \le a, 0 \le y \le b\},$$

where a, b are arbitrary positive real numbers. Then

$$\begin{aligned}F_i(0,y) &= 1 - a_i + b_i(1+y)\mathrm{e}^{-y}, \quad 0 \le y \le b,\\ F_i(a,y) &= 1 - a_i + b_i(1-a+y)\mathrm{e}^{-(a+y)}, \quad 0 \le y \le b,\\ F_i(x,0) &= 1 - a_i + b_i(1-x)\mathrm{e}^{-x}, \quad 0 \le x \le a,\\ F_i(x,b) &= 1 - a_i + b_i(1-x+b)\mathrm{e}^{-(x+b)}, \quad 0 \le x \le a.\end{aligned}$$

The minimum of the above four functions is $1 - a_i - b_i\mathrm{e}^{-2}$. Therefore, for all $(x,y) \in D$,

$$F_i(x,y) \ge 1 - a_i - b_i\mathrm{e}^{-2} > 0. \tag{4.2}$$

Since a and b are arbitrary, it can be obtained $F_i(x,y) > 0$ for all $x, y > 0$, $i = 1, 2$. □

Theorem 4.2. *Let $(u(t),v(t))_{t=0}^{\infty}$ be a solution of (1.6) with $\{(u(t_0),v(t_0))\} \in (0,1]^2$ for some $t_0 \ge 0$. If one of the following statements is held,* *(i)* $b_i \leq \mathrm{e}\min\{a_i, 1-a_i);$

(ii) $(1-a_i)\mathrm{e} \leq b_i \leq \frac{\mathrm{e}}{2};$

(iii) $\frac{(2b_i+1-a_i)^2}{4b_i} \leq 1,$

then $(u(t), v(t)) \in (0,1]^2$ *for all* $t \geq t_0$.

Proof. Let $t_0 \geq 0$ such that $\{(u(t_0), v(t_0))\} \in (0,1]^2$. From (1.6)

$$\begin{aligned} u(t_0+1) &= (1-a_1)u(t_0) + b_1 u(t_0)(1-u(t_0)+v(t_0))\mathrm{e}^{-(u(t_0)+v(t_0))} \\ &\leq (1-a_1)u(t_0) + b_1 u(t_0)(2-u(t_0))\mathrm{e}^{-u(t_0)}. \end{aligned} \tag{4.3}$$

Set $r(x) = (1-a_1)x + b_1 x(2-x)\mathrm{e}^{-x},\ x \in (0,1]$. Then $r'(x) = (1-a_1) + b_1(x^2-4x+2)\mathrm{e}^{-x}$ and $r''(x) = -b_1(x^2-6x+6)\mathrm{e}^{-x}$. For $x \in (0,1]$, $r''(x) \leq 0$ and consequently by (i), $r'(x) > r'(1) = 1-a_1-b_1\mathrm{e}^{-1} > 0$. So r is increasing on $(0,1]$ and then

$$u(t_0+1) \leq r(u(t_0)) \leq r(1) = (1-a_1) + b_1\mathrm{e}^{-1} \leq 1.$$

If $a_1 + b_1\mathrm{e}^{-1} > 1$, from (4.3), we can see by (ii)

$$\begin{aligned} u(t_0+1) &\leq (1-a_1)u(t_0) + b_1\mathrm{e}^{-1}(2-u(t_0)) \\ &= (1-a_1-b_1\mathrm{e}^{-1})u(t_0) + 2b_1\mathrm{e}^{-1} \\ &\leq 1. \end{aligned}$$

On the other hand, by(4.3),

$$u(t_0+1) \leq (1-a_1)u(t_0) + b_1 u(t_0)(2-u(t_0)).$$

Set $s(x) = (1-a_1)x + b_1 x(2-x),\ x \in (0,1]$. It is easy to show that the maximum value of $s(x)$ on (0,1] is no more than $\frac{(2b_1+1-a_1)^2}{4b_1}$. By (iii), $u(t_0+1) \leq \frac{(2b_1+1-a_1)^2}{4b_1} \leq 1$. For all cases, we conclude that $u(t_0+1) \leq 1$ whenever $u(t_0) \leq 1$. Similar results can be obtained for $v(t)$. The proof completes. □

Theorem 4.3. *For every solution* $(u(t), v(t))_{t=0}^{\infty}$ *of (1.6),*

(i) $u(t) \leq u(t_0)(1-a_1+\frac{b_1}{\mathrm{e}})^{t-t_0} + \frac{b_1}{a_1\mathrm{e}-b_1}[1-(1-a_1+\frac{b_1}{\mathrm{e}})^{t-t_0}];$

(ii) $v(t) \leq v(t_0)(1-a_2+\frac{b_2}{\mathrm{e}})^{t-t_0} + \frac{b_2}{a_2\mathrm{e}-b_2}[1-(1-a_2+\frac{b_2}{\mathrm{e}})^{t-t_0}];$

hold for $t \geq t_0 \geq 0$.

Proof.

$$\begin{aligned} u(t+1) &= (1-a_1)u(t) + b_1 u(t)(1-u(t)+v(t))\mathrm{e}^{-(u(t)+v(t))} \\ &\leq (1-a_1)u(t) + b_1 u(t)\mathrm{e}^{-u(t)} + b_1 u(t)v(t)\mathrm{e}^{-v(t)} \\ &\leq (1-a_1)u(t) + \frac{b_1}{\mathrm{e}} + \frac{b_1}{\mathrm{e}}u(t) \\ &= (1-a_1+\frac{b_1}{\mathrm{e}})u(t) + \frac{b_1}{\mathrm{e}}. \end{aligned}$$

By induction, for $t \geq t_0$,

$$u(t) \leq u(t_0)(1-a_1+\frac{b_1}{\mathrm{e}})^{t-t_0} + \frac{b_1}{a_1\mathrm{e}-b_1}[1-(1-a_1+\frac{b_1}{\mathrm{e}})^{t-t_0}]. \tag{4.4}$$

Thus (i) holds. (ii) could be obtained by similar analysis. □

Corollary 4.1. *Assume* $0 < a_i\mathrm{e} - b_i < 2\mathrm{e},\ i = 1, 2$. *Then for every positive solution* $(u(t), v(t))$ *with* $(u(t_0), v(t_0)) \in (0, \frac{b_1}{a_1\mathrm{e}-b_1}] \times (0, \frac{b_2}{a_2\mathrm{e}-b_2}]$ *for some* $t_0 \geq 0$, $(u(t), v(t)) \in (0, \frac{b_1}{a_1\mathrm{e}-b_1}] \times (0, \frac{b_2}{a_2\mathrm{e}-b_2}]$ *for all* $t \geq t_0$.

Corollary 4.2. *Assume* $0 < a_i\mathrm{e} - b_i < 2\mathrm{e},\ i = 1, 2$. *Then every positive solution* $(u(t), v(t))$ *is bounded. In addition,*

$$\limsup_{t\to\infty} u(t) \leq \frac{b_1}{a_1\mathrm{e} - b_1}$$

and

$$\limsup_{t\to\infty} v(t) \leq \frac{b_2}{a_2\mathrm{e} - b_2}.$$

Theorem 4.4. *Assume* $(u(t), v(t))_{t=0}^{\infty}$ *is a positive solution of (1.6). If one of the following condition holds,*

(i) $b_i \leq \frac{a_i}{2}\mathrm{e}$;

(ii) $2+\tau_i^2+\frac{(1+\mathrm{e}^{-1})^2}{4}(\mathrm{e}^{2\tau_i}-\mathrm{e}^{2\tau_j})+2\mathrm{e}^{\tau_i}(1+\mathrm{e}^{-1}) \geq \tau_i[4+\mathrm{e}^{\tau_i}(1+\mathrm{e}^{-1})]$, $1-a_i+b_i\tau_i\mathrm{e}^{-\tau_i}[\tau_i^2\mathrm{e}^{-\tau_j} - \tau_i(2\mathrm{e}^{-\tau_j}+\mathrm{e}^{-1}+1)+\mathrm{e}^{-1}+1] \geq 0$ *and* $(1-a_i)\tau_i + b_i(1+\mathrm{e}^{-1})\tau_i\mathrm{e}^{-\tau_i} - b_i\mathrm{e}^{-(\tau_i+\tau_j)}\tau_i^2 \leq 1$,

where $\tau_i = \frac{b_i}{a_i\mathrm{e}-b_i}$, $i, j = 1, 2$ *and* $i \neq j$, *there exists* $t_0 \geq 0$ *such that* $(u(t), v(t)) \in (0, 1]^2$ *for all* $t \geq t_0$.

Proof. By Corollary 4.2, (i) guarantees the result. Moreover, Corollary 4.2 implies that $u(t) \leq \tau_1 + \varepsilon_1$ and $v(t) \leq \tau_2 + \varepsilon_2$ when $t \geq t_0$ for some $t_0 \geq 0$, where ε_1 and ε_2 are sufficiently small positive number. Set $\gamma_i = \tau_i + \varepsilon_i$, $\delta_i = \mathrm{e}^{-\gamma_i}$, $i = 1, 2$. Then $\delta_i \to \mathrm{e}^{-\tau_i}$ when $\varepsilon_i \to 0$, $i = 1, 2$. Notice that $\mathrm{e}^{-v(t)} \geq \delta_2$, we have

$$\begin{aligned}
u(t+1) &= (1-a_1)u(t) + b_1u(t)(1-u(t)+v(t))\mathrm{e}^{-(u(t)+v(t))}\\
&= (1-a_1)u(t) + b_1u(t)\mathrm{e}^{-(u(t)+v(t))} - b_1u^2(t)\mathrm{e}^{-(u(t)+v(t))} + b_1u(t)v(t)\mathrm{e}^{-(u(t)+v(t))}\\
&\leq (1-a_1)u(t) + b_1u(t)\mathrm{e}^{-u(t)} + \frac{b_1}{\mathrm{e}}u(t)\mathrm{e}^{-u(t)} - b_1u^2(t)\mathrm{e}^{-(u(t)+v(t))}\\
&\leq -b_1\delta_2\mathrm{e}^{-u(t)}u^2(t) + b_1(1+\mathrm{e}^{-1})\mathrm{e}^{-u(t)}u(t) + (1-a_1)u(t).
\end{aligned} \tag{4.5}$$

Set $H(x) = -b_1\delta_2\mathrm{e}^{-x}x^2 + b_1(1+\mathrm{e}^{-1})\mathrm{e}^{-x}x + (1-a_1)x,\ x \leq \gamma_1$. Then

$$\begin{aligned}
H'(x) &= b_1\mathrm{e}^{-x}[\delta_2x^2 - (2\delta_2+1+\mathrm{e}^{-1})x + (1+\mathrm{e}^{-1})] + (1-a_1),\\
H''(x) &= -b_1\mathrm{e}^{-x}[\delta_2x^2 - (4\delta_2+1+\mathrm{e}^{-1})x + 2(\delta_2+1+\mathrm{e}^{-1})].
\end{aligned}$$

(ii) imply that $\gamma_1^2+2+\frac{(1+\mathrm{e}^{-1})^2}{4}(\frac{1}{\delta_1^2}-\frac{1}{\delta_1^2})+\frac{2(1+\mathrm{e}^{-1})}{\delta_1} \geq (4+\frac{(1+\mathrm{e}^{-1})}{\delta_1})\gamma_1$, which guarantees the roots of $\delta_2x^2 - (4\delta_2+1+\mathrm{e}^{-1})x + 2(\delta_2+1+\mathrm{e}^{-1}) > 0$ for $x \leq \gamma_1$. Thus $H''(x) < 0$ for all $x \leq \gamma_1$. Therefore $H'(x) > H'(\gamma_1)$ for $x < \gamma_1$. Notice that (ii) yields $H'(\gamma_1) \geq 0$. Then H is increasing in $(0, \gamma_1)$. (ii) also implies $H(\gamma_1) \leq 1$. Therefore $H(x) \leq 1$ for $x \leq \gamma_1$. By $u(t) \leq \gamma_1$, it follows that $u(t+1) \leq H(u(t)) \leq H(\gamma_1) \leq 1$ for $t \geq t_0$. Similar proof could be applied to v. The proof completes. □

Theorem 4.5. *Assume* $\{(u(t), v(t))\}_{t=0}^{\infty}$ *is a positive solution of (1.6). If one of the following condition holds,*

(i) $[b_i(1+\mathrm{e}^{-1}) + (1-a_i)]^2 < 4b_i\mathrm{e}^{-(\tau_1+\tau_2)}$;

(ii) $\frac{b_i}{4}\mathrm{e}^{\tau_j}(1+\mathrm{e}^{-1})+(1-a_i)\tau_i<1$, *$i,j=1,2$ and $i\neq j$;*

where τ_i, $i=1,2$ are defined as in Theorem 4.3, there exists $t_0\geq 0$ such that $(u(t),v(t))\in(0,1]^2$ for all $t\geq t_0$.

Proof. Let H be defined as in Theorem 4.3. When $u(t)\leq\gamma_1$,

$$\begin{aligned}H(u(t))&=-b_1\delta_2\mathrm{e}^{-u(t)}u^2(t)+b_1(1+\mathrm{e}^{-1})\mathrm{e}^{-u(t)}u(t)+(1-a_1)u(t)\\&\leq -b_1\delta_1\delta_2u^2(t)+b_1(1+\mathrm{e}^{-1})u(t)+(1-a_1)u(t)\\&=-b_1\delta_1\delta_2u^2(t)+[b_1(1+\mathrm{e}^{-1})+(1-a_1)]u(t)\\&:=\bar{H}(u(t)).\end{aligned}$$

Then $\bar{H}(x)$ has its max value at $x_0=\frac{b_1(1+\mathrm{e}^{-1})+(1-a_1)}{2b_1\delta_1\delta_2}$ and

$$\bar{H}(x)\leq\bar{H}(x_0)=\frac{[b_1(1+\mathrm{e}^{-1})+(1-a_1)]^2}{4b_1\delta_1\delta_2}. \tag{4.6}$$

On the other hand, for $t\geq t_0$,

$$\begin{aligned}H(u(t))&=-b_1\delta_2\mathrm{e}^{-u(t)}[u^2(t)-\frac{1+\mathrm{e}^{-1}}{\delta_2}u(t)]+(1-a_1)u(t)\\&=-b_1\delta_2\mathrm{e}^{-u(t)}[u(t)-\frac{1+\mathrm{e}^{-1}}{2\delta_2}]^2+\frac{(1+\mathrm{e}^{-1})^2}{4\delta_2^2}b_1\delta_2(1+\mathrm{e}^{-1})\mathrm{e}^{-u(t)}+(1-a_1)u(t)\\&\leq\frac{b_1}{4\delta_2}(1+\mathrm{e}^{-1})+(1-a_1)\gamma_1.\end{aligned} \tag{4.7}$$

One can choose ε_1 and ε_2 so small that the assumptions imply that for $t\geq t_0$, $\frac{[b_1(1+\mathrm{e}^{-1})+(1-a_1)]^2}{4b_1\delta_1\delta_2}\leq 1$ and $\frac{b_1}{4\delta_2}(1+\mathrm{e}^{-1})+(1-a_1)\gamma_1\leq 1$. Therefore,

$$\begin{aligned}u(t+1)&\leq\bar{H}(u(t))\leq\frac{[b_1(1+\mathrm{e}^{-1})+(1-a_1)]^2}{4b_1\delta_1\delta_2}\leq 1,\\u(t+1)&\leq H(u(t))\leq\frac{b_1}{4\delta_2}(1+\mathrm{e}^{-1})+(1-a_1)\gamma_1\leq 1.\end{aligned}$$

Similar results can be obtained for $v(t)$. □

5. Global analysis

Theorem 5.1. *If $b_i(1+\mathrm{e}^{-1})<a_i<1+b_i(1+\mathrm{e}^{-1})$, $i=1,2$, Then E_0 is a global attractor of all positive solution of (1.6).*

Proof. Let $\{(u(t),v(t))\}_{t=0}^{\infty}$ be a positive solution of (1.6). Then

$$\begin{aligned}u(t+1)&=(1-a_1)u(t)+b_1u(t)(1-u(t)+v(t))\mathrm{e}^{-(u(t)+v(t))}\\&\leq(1-a_1)u(t)+b_1u(t)[1+v(t)]\mathrm{e}^{-(u(t)+v(t))}\\&\leq(1-a_1+b_1)u(t)+b_1u(t)v(t)\mathrm{e}^{-v(t)}\\&\leq[1-a_1+b_1(1+\mathrm{e}^{-1})]u(t)\\&<u(t)\end{aligned} \tag{5.1}$$

and

$$v(t+1) \leq [1 - a_2 + b_2(1 + \mathrm{e}^{-1})]v(t) < v(t). \tag{5.2}$$

Therefore, there exist $u_0, v_0 \geq 0$ such that $\lim_{t\to\infty} u(t) = u_0$ and $\lim_{t\to\infty} v(t) = v_0$. Since the possible value of (u_0, v_0) is E_0, it can be obtained $\lim_{t\to\infty} u(t) = 0$ and $\lim_{t\to\infty} v(t) = 0$. The proof completes. □

Theorem 5.2. *Assume $a_i + b_i\mathrm{e}^{-2} < 1$ and $0 < a_i\mathrm{e} - b_i < 2\mathrm{e}$, $i = 1, 2$. Then the unique positive equilibrium point $(\bar{u}, \bar{v})$ of (1.6) is a global attractor of all positive solutions of (1.6).*

Proof. Let $\{(u(t), v(t))\}_{t=0}^{\infty}$ be a positive solution of (1.6). Set $h(u) = a_1 - b_1(1-u+v)\mathrm{e}^{-(u+v)}$. In the first case, if $u(0) \leq \bar{u}$, noticing that $\bar{u} < 1$, $h'(u) = b_1(2 - u + v)\mathrm{e}^{-(u+v)} > 0$ leads h is increasing on $(0, \bar{u}]$ in u. Then $h(u(0)) \leq h(\bar{u})$ for any $v > 0$, which yields that $h(u(0)) \leq a_1 - b_1(1 - \bar{u} + \bar{v})\mathrm{e}^{-(\bar{u}+\bar{v})} = 0$. Consequently,

$$\begin{aligned} u(1) &= (1 - a_1)u(0) + b_1 u(0)(1 - u(0) + v(0))\mathrm{e}^{-(u(0)+v(0))} \\ &= u(0) - u(0)[a_1 - b_1(1 - u(0) + v(0))\mathrm{e}^{-(u(0)+v(0))}] \\ &= u(0) - u(0)h(u(0)) \\ &\geq u(0). \end{aligned} \tag{5.3}$$

Then the sequence $\{u(t)\}_{t=0}^{\infty}$ is increasing and since it was shown in corollary 4.2 that it is bounded above, then it converges to the unique positive equilibrium point $\bar{u}$. The second case is to assume $u(0) > \bar{u}$. With similar arguments, we have $u(1) < u(0)$. If $u(1) \leq \bar{u}$, the results holds as in the first case. If $u(1) > \bar{u}$, then $u(2) < u(1)$. By induction, $\{u(t)\}_{t=0}^{\infty}$ converges to the unique positive equilibrium point $\bar{u}$. Similar analysis can be applied to $\{v(t)\}_{t=0}^{\infty}$. The proof completes. □

As a special case, when $a_1 = a_2 = a$ and $b_1 = b_2 = b$, now contemplate the system

$$\begin{cases} u(t+1) = (1 - a)u(t) + bu(t)(1 - u(t) + v(t))\mathrm{e}^{-(u(t)+v(t))}, \\ v(t+1) = (1 - a)v(t) + bv(t)(1 - v(t) + u(t))\mathrm{e}^{-(u(t)+v(t))}, \end{cases} \tag{5.4}$$

we have

Theorem 5.3. *Assume the condition in Theorem 4.4 or Theorem 4.5 holds. If $3b < 1 - a$, the unique positive equilibrium point $(\bar{u}, \bar{u})$ of (1.6) is a global attractor of all positive solutions of (5.4).*

Proof. Rewrite (5.4) as

$$\begin{cases} u(t+1) = F(u(t), v(t)), \\ v(t+1) = G(u(t), v(t)), \end{cases}$$

where $F(u, v) = (1 - a)u + bu(1 - u + v)\mathrm{e}^{-(u+v)}, G(u, v) = (1 - a)v + bv(1 - v + u)\mathrm{e}^{-(u+v)}$ are continuous functions.

Now consider

$$\begin{cases} m_1 = F(m_1, M_2), & M_1 = F(M_1, m_2), \\ m_2 = G(M_1, m_2), & M_2 = G(m_1, M_2). \end{cases}$$

That is,

$$\begin{cases} m_1 = (1-a)m_1 + bm_1(1-m_1+M_2)\mathrm{e}^{-(m_1+M_2)}, \\ M_1 = (1-a)M_1 + bM_1(1-M_1+m_2)\mathrm{e}^{-(M_1+m_2)}, \\ m_2 = (1-a)M_1 + bM_1(1-M_1+m_2)\mathrm{e}^{-(M_1+m_2)}, \\ M_2 = (1-a)M_2 + bM_2(1-M_2+m_1)\mathrm{e}^{-(m_1+M_2)}. \end{cases}$$

Solving the above equations directly can obtain

$$m_1 = M_1 = m_2 = M_2 = \frac{1}{2}\ln\frac{b}{a}.$$

So (5.4) has unique fixed point $(\bar{u}, \bar{v}) = (\frac{1}{2}\ln\frac{b}{a}, \frac{1}{2}\ln\frac{b}{a})$.

On the other hand, we can see

$$\begin{aligned} \frac{\partial F}{\partial u} &= 1 - a + b[u^2 - (v+3)u + 1 + v]\mathrm{e}^{-(u+v)} \\ &\geq 1 - a + b(u^2 - 4u + 1)\mathrm{e}^{-(u+v)}. \end{aligned}$$

If $3b < 1 - a$, notice that

$$4u - u^2 - 1 = u(4-u) - 1 \leq 3,$$

so we have

$$b(4u - u^2 - 1)\mathrm{e}^{-(u+v)} \leq 3b\mathrm{e}^{-(u+v)} \leq 3b < 1 - a,$$

which means $\frac{\partial F}{\partial u} > 0$. For $\frac{\partial G}{\partial u} = bv(v-u)\mathrm{e}^{-(u+v)}$,

$$\frac{\partial^2 G}{\partial u^2} = bv(1 - v + u)\mathrm{e}^{-(u+v)} > 0,$$

which imply that $\frac{\partial G}{\partial u}$ is increasing. While $\frac{\partial G}{\partial u}|_{u=0} > 0$, $\frac{\partial G}{\partial u} > 0$ for all $u, v > 0$. Thus, by Theorem 1.1, it follows that the unique positive equilibrium point $(\bar{u}, \bar{u})$ is a global attractor of all positive solutions of system (5.4). □

Example 5.1. Set $a_1 = 0.75, a_2 = 0.25$, $b_1 = 5, b_2 = 1$, Then

$$\frac{2}{\mathrm{e}^2} < \frac{a_1}{b_1} + \frac{a_2}{b_2} = 0.4 < 2.$$

In addition, $a_1b_2 + a_2b_1 = 2$, $a_1b_2 - a_2b_1 = -0.5$, $2b_1b_2 = 10$, $\mathrm{e}^r = \mathrm{e}^{-0.25} \approx 0.7788$,

$$\frac{2b_1b_2}{a_1b_2 + a_2b_1} = 5, \quad \frac{\mathrm{e}^2(a_1b_2 + a_2b_1)}{2b_1b_2} = \frac{\mathrm{e}^2}{5}.$$

The conditions in Proposition 3 (iii) are satisfied. Figure 1 shows the stability of equilibrium of (1.6). Notice that $\bar{u} \approx 0.93, \bar{v} \approx 0.68$, (3.5) in Proposition 3.3 holds.

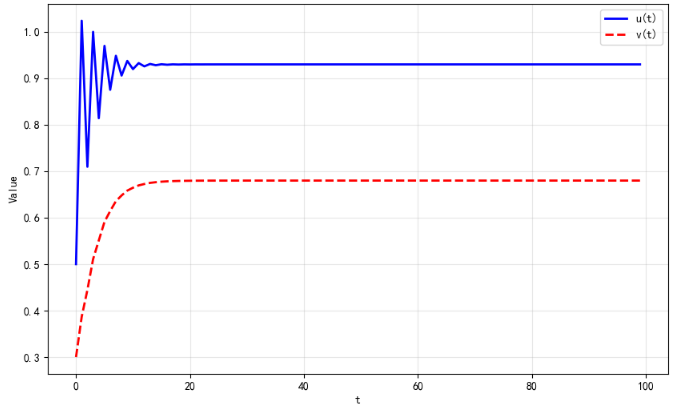

Figure 1. The stability of equilibrium of (1.6).

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