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PARAMETRIC MARCINKIEWICZ INTEGRAL OPERATORS AND COMMUTATORS ON HERZ-HARDY SPACES WITH VARIABLE EXPONENTS*

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Abstract In this paper, the authors obtain the boundedness of parametric Marcinkiewicz integral operator with variable kernel and its commutators on Herz spaces with variable exponents $\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ when the parameters $\alpha(\cdot), p(\cdot)$ and $q(\cdot)$ satisfies some conditions. Meanwhile, the corresponding estimates on Herz-Hardy spaces with variable exponents $H\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ are also established.

Keywords Marcinkiewicz integral operators, Herz space, Herz-Hardy spaces, variable exponent, commutators.

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1. Introduction and main results

Denote by S^{n-1} the unit sphere of $\mathbb{R}^n (n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Suppose that $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$,$(1 \leq r \leq \infty)$ satisfies

$$\| \Omega \|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})}= \sup_{x\in\mathbb{R}^n} (\int_{S^{n-1}} | \Omega(x, z') |^r \, d\sigma(z'))^{\frac{1}{r}} < \infty, \tag{1.1}$$

where $z' = \frac{z}{|z|}, \forall z \in \mathbb{R}^n \setminus \{0\}$, $\Omega(x, \lambda z) = \Omega(x, z)$, $\forall x, z \in \mathbb{R}^n, \forall \lambda > 0$.

$$\int_{S^{n-1}} \Omega(x, z')dz' = 0, \forall z \in \mathbb{R}^n. \tag{1.2}$$

In 1958, Stein [24] introduced the Marcinkiewicz integral of higher dimensions μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

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He also proved that μ_Ω is of weak type $(1,1)$ and type (p,p) for $(1 < p \leq 2)$, when Ω is a continuous function and satisfies Lip_α-condition $(1 < \alpha \leq 2)$.

In 1960, Hömander [11] first defined the parametric Marcinkiewicz integrals μ_Ω^ρ as

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty |F_{\Omega,t}^\rho(f)(x)|^2 \frac{dt}{t^{2\rho+1}} \right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}^\rho(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy.$$

They obtained the following result:

Theorem 1.1 ([11]). *Let $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})(0 < \alpha \leq 1)$, $\rho > 0$ and $1 < p < \infty$, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, there exists a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega^\rho(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$

The boundedness of the Marcinkiewicz integral operator μ_Ω has been extensively studied in [5, 7, 20, 25, 26].

On the other hand, Calderón and Zygmund [2] pioneered the study of the L^p boundedness of singular integral operators with variable kernels, establishing their close connection to problems concerning second-order linear elliptic equations with variable coefficients. Building on this foundational work, the boundedness properties of such operators have attracted significant attention in recent years. Notable advances include the work of Ding, Lin and Shao [6], who obtained the L^p boundedness of the Marcinkiewicz integral operator with variable kernels. Furthermore, Shao, Lu and Wang [19] established the boundedness for parametric Marcinkiewicz integral operator with variable kernel and their commutators on variable exponent weak λ-central Morrey spaces. For additional related results specifically on the Marcinkiewicz integral operator with variable kernels, see references [21–23].

The Herz spaces with variable exponents $\mathcal{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ and Herz-Hardy spaces with variable exponents $H\mathcal{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ were first introduced by Izuki and Noi in [14]. Drihem and Seghiri [9] obtained a new equivalent norm of these function spaces. In reference [1], the author proved the boundedness properties of singular integral operators on variable weak Herz spaces and variable weak Herz-type Hardy spaces. For recent results of these function spaces, see [8] et al.

Motivated by the above research, in this paper, we will consider the boundedness of the parametric Marcinkiewicz integral operators with variable kernels and their commutators on the variable Herz spaces and Herz-Hardy spaces, where the smoothness condition on Ω has been removed.

The parametric Marcinkiewicz integrals with variable kernels $\mu_\Omega^\theta(\theta > 0)$ difined as

$$\mu_\Omega^\theta(f)(x) = \left(\int_0^\infty |F_{\Omega,t}^\theta(f)(x)|^2 \frac{dt}{t^{2\theta+1}} \right)^{\frac{1}{2}}, \tag{1.3}$$

where

$$F_{\Omega,t}^\theta(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-\theta}} f(y)dy.$$

Given a local integrable function b, the corresponding commutator is defined by

$$\mu^{\theta}_{\Omega,b}(f)(x) := b(x)\mu^{\theta}_{\Omega}(f)(x) - \mu^{\theta}_{\Omega}(bf)(x) = \left(\int_0^{\infty} |F^{\theta}_{\Omega,b,t}(f)(x)|^2 \frac{dt}{t^{2\theta+1}}\right)^{\frac{1}{2}}, \tag{1.4}$$

where

$$F^{\theta}_{\Omega,b,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-\theta}}(b(x)-b(y))^m f(y)dy.$$

Before stating the main results of this article, we first recall some necessary definitions and notations.

For every $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{A_k}$ be the characteristic function. $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$. We say $x \sim y$ if there exist constants $C_1, C_2 > 0$ such that $C_1 y \le x \le C_2 y$.

BMO$(\mathbb{R}^n)$ space is defined as

$$\mathrm{BMO}(\mathbb{R}^n) = \left\{ b \in L^1_{loc}(\mathbb{R}^n) : \|b\|_* = \sup_Q \frac{1}{|Q|}\int_Q |b(x) - b_Q|dx < \infty \right\},$$

where the supreme is taken over all cubes $Q \subset \mathbb{R}^n$ and $b_Q = \frac{1}{|Q|}\int_Q b(y)dy$.

Define $\mathcal{P}(\mathbb{R}^n)$ to be the set of $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ such that

$$1 < p_- := \text{ess inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ := \text{ess sup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all Lebesgue measurable function f satisfying

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{ \eta > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx \le 1 \right\} < \infty.$$

$L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach function space when equipped with the Luxemburg-Nakano norm above.

Let $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximum operator is defined by

$$\mathrm{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy.$$

By $\mathcal{B}(\mathbb{R}^n)$ we denotes the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ which satisfies the following conditions

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}, \quad |x-y| \le \frac{1}{2}, \tag{1.5}$$

and

$$|p(x) - p(\infty)| \le \frac{C}{\log(|x|+e)}, \quad x \in \mathbb{R}^n, \tag{1.6}$$

where $p(\infty) = \lim_{|x|\to\infty} p(x)$. It is proved that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ as $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ in [3].

Let $S(\mathbb{R}^n)$ be the Schwartz space on $\mathbb{R}^n$, and $S'(\mathbb{R}^n)$ is the duality space of $S(\mathbb{R}^n)$. Let $G_N f(x)$ be the Grand maximal function of f defined by

$$G_N f(x) = \sup_{\Phi \in \mathcal{A}_N} |\varphi^{\star}_{\triangledown}(f)(x)|,$$

where $\mathcal{A}_N = \{\varphi \in S(\mathbb{R}^n) : \sup_{|\alpha|\le N, |\beta|\le N} |x^{\alpha} D^{\beta}\varphi(x)| \le 1\}$ and $N > n+1$. $\varphi^{\star}_{\triangledown}$ is nontangential maximum operator defined by $\varphi^{\star}_{\triangledown}(f)(x) = \sup_{|y-x|<t} |\varphi_t * (f)(y)|$, with $\varphi_t(x) = t^{-n}\varphi(x/t)$.

Definition 1.1 ([8]). Given measurable functions $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the mixed Lebesgue-sequence space $l^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequence of $L^{p(\cdot)}-$functions by the modular $L^{q(\cdot)}(\mathbb{R}^n)$ are defined as follows:

$$\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}((f_\nu)_\nu) = \sum_\nu \inf\left\{\lambda_\nu > 0 : \varrho_{L^{p(\cdot)}}\left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\right) \leq 1\right\},$$

where

$$\varrho_{L^{p(\cdot)}}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx.$$

The quasi-norm is defined from this as usual

$$\|(f_\nu)_\nu\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf\{\eta > 0 : \varrho_{l^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\eta}(f_\nu)_\nu\right) \leq 1\}.$$

Since $q_+ < \infty$, then we can replace by the simpler expression

$$\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}((f_\nu)_\nu) = \sum_\nu \||f_\nu|^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Definition 1.2 ([14]). Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \left\|\left(2^{k\alpha(\cdot)} f\chi_k\right)_{k\in\mathbb{Z}}\right\|_{l^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The inhomogeneous Herz space with variable exponent $\mathcal{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\mathcal{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n) : \|f\|_{\mathcal{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\mathcal{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \|f\chi_{B_0}\|_{L^{p(\cdot)}} + \left\|\left(2^{k\alpha(\cdot)} f\chi_k\right)_{k\geq 1}\right\|_{l^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Proposition 1.1 ([9]). *Let $\alpha \in L^\infty(\mathbb{R}^n)$, p and q satisfy (1.5). If α and q satisfy (1.6), then*

$$\mathcal{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \mathcal{K}^{\alpha_\infty,q_\infty}_{p(\cdot)}(\mathbb{R}^n).$$

Additionally, α and q satisfy (1.5), then

$$\|f\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{-1} \|2^{k\alpha(0)} f\chi_k\|^{q(0)}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{\infty} \|2^{k\alpha_\infty} f\chi_k\|^{q_\infty}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q_\infty}}.$$

Definition 1.3 ([10]). Let $p(\cdot), q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous Herz-Hardy space with variable exponent $H\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : G_N f(x) \in \dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)\},$$

the norm of f in $H\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined as follows

$$\|f\|_{H\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.$$

One recognizes immediately that if $\alpha(\cdot), q(\cdot)$ are constants, then the spaces $H\dot{\mathcal{K}}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ were just studied in [18], $\alpha(\cdot), q(\cdot)$ and $p(\cdot)$ are all constants, then the spaces $H\dot{\mathcal{K}}^{\alpha,q}_{p}(\mathbb{R}^n)$ are the usual Herz-Hardy spaces see [16, 17].

Definition 1.4 ([10]). Let $\alpha \in L^\infty(\mathbb{R}^n), p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and non-negative integer m. A function a on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))-$atom, if it satisfies

(1) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leq r\}, r > 0$.

(2) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq |\overline{B(0,r)}|^{-\alpha(0)/n}, 0 < r < 1$.

(3) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq |\overline{B(0,r)}|^{-\alpha_\infty/n}, r \geq 1$.

(4) $\int_{\mathbb{R}^n} a(x) x^\beta dx = 0, \quad |\beta| \leq m$.

The function a on $\mathbb{R}^n$ is said to be central $(\alpha(\cdot), p(\cdot))-$atom of restricted type, if it satisfies the above conditions (3) and (4), and for some $r \geq 1$, $\operatorname{supp} a \subset B(0, r)$.

The main results of this paper are stated as follows.

Theorem 1.2. *Suppose that $p(\cdot), q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\alpha \in L^\infty(\mathbb{R}^n)$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^s(S^{n-1})$, $s > p'_-$ satisfies (1.1) and (1.2). If $\alpha(\cdot), p(\cdot), q(\cdot)$ satisfies*

$$-\frac{n}{p(0)} < \alpha(0) < n - \frac{n}{p(0)} \quad \text{and} \quad -\frac{n}{p_\infty} < \alpha_\infty < n - \frac{n}{p_\infty},$$

then there exists a constant $C > 0$ independent of f such that

$$\|\mu^\theta_\Omega(f)\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$$

and

$$\|\mu^\theta_\Omega(f)\|_{\mathcal{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.3. *Suppose that $b \in \mathrm{BMO}(\mathbb{R}^n)$, under the hypotheses of Theorem 1.2, then there exists a constant $C > 0$ independent of f such that*

$$\|\mu^\theta_{\Omega,b}(f)\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}$$

and

$$\|\mu^\theta_{\Omega,b}(f)\|_{\mathcal{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.4. *Suppose that $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $(p_1)_+ < 2n$ and $\frac{1}{p_1(\cdot)} - \frac{n}{p_2(\cdot)} = \frac{1}{2n}$, $\alpha \in L^\infty(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ $\Omega \in L^\infty(\mathbb{R}^n) \times L^s(S^{n-1})$, $s > (p'_1)_-$ satisfies (1.1) and (1.2). If $\alpha(\cdot), q_1(\cdot), q_2(\cdot)$ satisfies*

$$n\left(1 - \frac{1}{(p_1)_-}\right) < \alpha(\cdot), q_1(0) < q_2(0) \quad \text{and} \quad (q_1)_\infty < (q_2)_\infty,$$

then there exists a constant $C > 0$ independent of f such that

$$\|\mu_\Omega^\theta(f)\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{H\dot{\mathcal{K}}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.5. *Suppose that $b \in \mathrm{BMO}(\mathbb{R}^n)$, under the hypotheses of Theorem 1.4, then there exists a constant $C > 0$ independent of f such that*

$$\|\mu_{\Omega,b}^\theta(f)\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{H\dot{\mathcal{K}}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

2. Preliminaries lemmas

We first give eight lemmas which will be used in the proof of Theorem.

Lemma 2.1 ([15]). *(Generalized Hölder inequality) Let $q(\cdot) : \mathbb{R}^n \to [1,\infty)$. If $f \in L^{q(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q'(\cdot)}(\mathbb{R}^n)$, then f and fg are integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(y)|dx \leq \ r_q\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)},$$

where $r_q = 1 + \frac{1}{q_-} - \frac{1}{q_+}$, $\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1$.

Lemma 2.2 ([12]). *Suppose that $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then exists a constant $C > 0$, such that for all balls B in $\mathbb{R}^n$,*

$$C^{-1} \leq \frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.3 ([8]). *Let $0 < a < 1$ and $0 < q \leq \infty$, $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\|_{l^q} < \infty.$$

Then the sequences $\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\}_{k\in\mathbb{Z}}$ and $\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\}_{k\in\mathbb{Z}}$ belong to l^q, and

$$\|\{\delta_k\}_{k\in\mathbb{Z}}\|_{l^q} + \|\{\eta_k\}_{k\in\mathbb{Z}}\|_{l^q} \leq C\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\|_{l^q}$$

with $C > 0$ only depending on a and q.

Lemma 2.4. *If Ω satisfy (1.1) and (1.2), Suppose $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, for all $f \in L^{q(\cdot)}(\mathbb{R}^n)$, then exists a constant C, such that*

(1) $\|\mu_\Omega^\theta(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C_p\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}$,

(2) $\|\mu_{\Omega,b}^\theta(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}$.

Using a similar argument as in the proofs of Lemmas 2.4 and 2.6 in [20], it is easy to draw the above conclusion, the details are omitted here.

Lemma 2.5 ([4]). *Define a variable exponent $\tilde{q}(\cdot)$ by $1/p(\cdot) = 1/\tilde{q}(\cdot) + 1/s$. Then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\|g\|_{L^s(\mathbb{R}^n)},$$

for all measurable functions f and g.

Lemma 2.6 ([4]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.5). Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{1/p(x)} & \text{if } |Q| \leq 2^n \text{ and } x \in Q, \\ |Q|^{1/p_\infty} & \text{if } |Q| \geq 1, \end{cases}$$

for every cube (or ball) $Q \subset \mathbb{R}^n$.

Lemma 2.7 ([13]). *Suppose that $b \in \mathrm{BMO}, m \in \mathbb{N}, i, j \in \mathbb{Z}$ and $i < j$, then*

$$C^{-1}\|b\|_*^m \leq \sup_B \frac{\|(b-b_B)^m \chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C\|b\|_*^m,$$
$$\|(b-b_{B_i})^m \chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C(j-i)^m \|b\|_*^m \|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

where $B = (x, r), B_i = (x, 2^i r)$.

Lemma 2.8 ([9]). *Let $\alpha(\cdot), p(\cdot), q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $1 < p_- \leq p_+ < \infty$, for any $f \in H\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ we have*

$$f = \sum_{k=-\infty}^{+\infty} \lambda_k a_k$$

in the sense of $S'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot), p(\cdot))-$atom with support contained in B_k and

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \leq C\|f\|_{H\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Conversely, if $\alpha(\cdot) \geq n(1-\frac{1}{p_-})$ and $s \geq [\alpha_+ + n(\frac{1}{p_-}-1)]$, then $f \in H\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{H\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \approx \inf\left\{\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all above decompositions of f.

3. Proofs of Theorems 1.2-1.5

The proofs of Theorem 1.2 and Theorem 1.4 can be proved by referring to [10]. Their details are not repeated here. Below, we only prove the case of the commutators, that is, Theorem 1.3 and Theorem 1.5.

Proof of Theorem 1.3. We only prove the homogeneous case, the inhomogeneous case can be proved in the same way.

Let $f \in \dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$. For each $j \in \mathbb{Z}$ denote $f_j = f\chi_j$, we have

$$f(x) = \sum_{j=-\infty}^{\infty} f\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

In view of Proposition 1.1, we have that

$$\|\mu_{\Omega,b}^{\theta}(f)\|_{\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \approx \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \|\mu_{\Omega,b}^{\theta}(f)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q(0)} \right\}^{1/q(0)} + \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \|\mu_{\Omega,b}^{\theta}(f)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_\infty} \right\}^{1/q_\infty}.$$

Then

$$\begin{aligned}
\|\mu_{\Omega,b}^{\theta}(f)\|_{\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq & \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=-\infty}^{k-2} \|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{q(0)} \right\}^{1/q(0)} \\
& + \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=k-2}^{k+1} \|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{q(0)} \right\}^{1/q(0)} \\
& + \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=k+2}^{\infty} \|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{q(0)} \right\}^{1/q(0)} \\
& + \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=-\infty}^{k-2} \|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{q_\infty} \right\}^{1/q_\infty} \\
& + \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=k-2}^{k+1} \|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{q_\infty} \right\}^{1/q_\infty} \\
& + \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=k+2}^{\infty} \|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{q_\infty} \right\}^{1/q_\infty} \\
=: & \, \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3 + \mathrm{E}_4 + \mathrm{E}_5 + \mathrm{E}_6.
\end{aligned}$$

We first estimate $\mathrm{E}_2 + \mathrm{E}_5$. By applying (2) in Lemma 2.4, we can get

$$\begin{aligned}
\mathrm{E}_2 + \mathrm{E}_5 & \leq C \left\{ \left(\sum_{k=-\infty}^{-1} \|2^{k\alpha(0)} f\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q(0)} \right)^{\frac{1}{q(0)}} + \left(\sum_{k=0}^{\infty} \|2^{k\alpha_\infty} f\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_\infty} \right)^{\frac{1}{q_\infty}} \right\} \\
& \leq C\|f\|_{\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Now estimate E_1 and E_4. We consider

$$\begin{aligned}
|\mu_{\Omega,b}^{\theta}(f_j)(x)| \leq & \left(\int_0^{|x|} \left| \int_{|x-y|\leq t} \frac{\Omega(x,x-y)}{|x-y|^{n-\theta}} (b(x)-b(y))^m f_j(y)dy \right|^2 \frac{dt}{t^{2\theta+1}} \right)^{\frac{1}{2}} \\
& + \left(\int_{|x|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x,x-y)}{|x-y|^{n-\theta}} (b(x)-b(y))^m f_j(y)dy \right|^2 \frac{dt}{t^{2\theta+1}} \right)^{\frac{1}{2}}
\end{aligned}$$

$$=: \mathrm{E}_{11} + \mathrm{E}_{12}.$$

Note that $j \le k-2, x \in A_k, y \in A_j$. So $|x-y| \sim |x|$, we have

$$\left| \frac{1}{|x-y|^{2\theta}} - \frac{1}{|x|^{2\theta}} \right| \le \frac{C|y|}{|x-y|^{2\theta+1}}.$$

Via the Minkowski inequality and the Lemma 2.1, we have

$$\begin{aligned}
\mathrm{E}_{11} &\le C \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\theta}} |b(x)-b(y)|^m |f_j(y)| \left(\int_{|x-y|}^{|x|} \frac{dt}{t^{2\theta+1}} \right)^{\frac{1}{2}} dy \\
&\le C \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\theta}} |b(x)-b(y)|^m |f_j(y)| \left| \frac{1}{|x-y|^{2\theta}} - \frac{1}{|x|^{2\theta}} \right|^{\frac{1}{2}} dy \\
&\le C \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\theta}} |b(x)-b(y)|^m |f_j(y)| \frac{|y|^{1/2}}{|x-y|^{\theta+1/2}} dy \\
&\le C \frac{2^{j/2}}{|x|^{n+1/2}} \int_{A_j} |\Omega(x,x-y)| |b(x)-b(y)|^m |f_j(y)| dy \\
&\le C 2^{\frac{j-k}{2}} 2^{-kn} \Bigg\{ |b(x)-b_{B_j}|^m \int_{A_j} |\Omega(x,x-y)| |f_j(y)| dy \\
&\quad + \int_{A_j} |\Omega(x,x-y)| |b_{B_j} - b(y)|^m |f_j(y)| dy \Bigg\} \\
&\le C 2^{\frac{j-k}{2}} 2^{-kn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Bigg\{ |b(x)-b_{B_j}|^m \|\Omega(x,x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad + \|\Omega(x,x-\cdot)(b_{B_j}-b(\cdot))^m \chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \Bigg\}.
\end{aligned}$$

For E_{12}. Note that $|x-y| \sim |x|$, by applying the Minkowski inequality and Lemma 2.1, we obtain

$$\begin{aligned}
\mathrm{E}_{12} &\le C \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\theta}} |b(x)-b(y)|^m |f_j(y)| \left(\int_{|x|}^{|\infty|} \frac{dt}{t^{2\theta+1}} \right)^{\frac{1}{2}} dy \\
&\le C \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n}} |b(x)-b(y)|^m |f_j(y)| dy \\
&\le C 2^{-kn} \Bigg\{ |b(x)-b_{B_j}|^m \int_{A_j} |\Omega(x,x-y)| |f_j(y)| dy \\
&\quad + \int_{A_j} |\Omega(x,x-y)| |b_{B_j} - b(y)|^m |f_j(y)| dy \Bigg\} \\
&\le C 2^{-kn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Bigg\{ |b(x)-b_{B_j}|^m \|\Omega(x,x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$+\|\Omega(x,x-\cdot)(b_{B_j}-b(\cdot))^m\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\Bigg\}.$$

So, we can get

$$|\mu^{\theta}_{\Omega,b}(f_j)(x)|\le C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg\{|b(x)-b_{B_j}|^m\|\Omega(x,x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$
$$+\|\Omega(x,x-\cdot)(b_{B_j}-b(\cdot))^m\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\Bigg\}.$$

For $x\in B$ and $y\in B_j$, $x-y\in B_{j+1}$. Noting that $s>p'_-$, denote $\tilde{q}'(\cdot)>1$ and $1/p'(x)=1/\tilde{q}'(x)+1/s$. By Lemma 2.5 we have

$$\begin{aligned}\|\Omega(x,x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}&\le\|\Omega(x,x-\cdot)\chi_j\|_{L^{s}(S^{n-1})}\|\chi_j\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\&\le C\left(\int_{A_j}|\Omega(x,x-y)|^s dy\right)^{1/s}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\&\le C\left(\int_{B_{j+1}}|\Omega(x,z)|^s dz\right)^{1/s}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\&\le C\left(\int_0^{2^{j+1}}\int_{S^{n-1}}|\Omega(x,z')|^s dz' r^{n-1}dr\right)^{1/s}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\&\le C2^{jn/s}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})}\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

When $|B_j|\le 2^n$ and $x\in B_j$, by Lemma 2.6 we have

$$\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\approx|B_j|^{1/\tilde{q}'(x)}\approx\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}.$$

When $|B_j|\ge 1$ we have

$$\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\approx|B_j|^{1/(\tilde{q})'_\infty}\approx\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_j|^{-1/s}.$$

So, we obtain

$$\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\approx\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}|B_j|^{1/s}.$$

Then

$$\begin{aligned}\|\Omega(x,x-\cdot)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\&\le C2^{jn}\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})}\|\chi_{B_j}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

According to Lemma 2.7, we have

$$|b(x)-b_{B_j}|^m=\frac{1}{|B_j|}\int_{B_j}|b(x)-b_{B_j}|^m dy$$

$$\begin{aligned}&\leq \frac{1}{|B_j|}\|(b(x)-b_{B_j})^m\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\&\leq C\|b\|_*^m\frac{1}{|B_j|}\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\&\leq C\|b\|_*^m.\end{aligned}$$

Similarly, it follows from Lemma 2.5 and Lemma 2.7 that

$$\begin{aligned}&\|\Omega(x,x-\cdot)(b_{B_j}-b(\cdot))^m\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\&\leq \|\Omega(x,x-\cdot)\chi_j\|_{L^s(\mathbb{R}^n)}\|(b_{B_j}-b(\cdot))^m\chi_j\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\&\leq C2^{jn/s}\|b\|_*^m\|\chi_{B_j}\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\|\Omega(x,x-\cdot)\chi_j\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})}\\&\leq C2^{jn/s}\|b\|_*^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})}\frac{|B_j|^{1-\frac{1}{s}}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\\&\leq C2^{jn}\|b\|_*^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})}\|\chi_{B_j}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned}\|\mu^\theta_{\Omega,b}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}&\leq C2^{(j-k)n}\|b\|_*^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})(\mathbb{R}^n)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\\&\leq C2^{(j-k)n}\|b\|_*^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})(\mathbb{R}^n)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\\&\leq C2^{(j-k)(n-\frac{n}{p(0)})}\|b\|_*^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})(\mathbb{R}^n)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Which gives

$$\begin{aligned}\mathrm{E}_1&\leq C\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n-\frac{n}{p(0)})q(0)}\|f_j\|^{q(0)}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)\right\}^{1/q(0)}\\&\leq C\left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n-\alpha(0)-\frac{n}{p(0)})q(0)}(2^{j\alpha(0)q(0)}\|f_j\|^{q(0)}_{L^{p(\cdot)}(\mathbb{R}^n)})\right)\right\}^{1/q(0)}.\end{aligned}$$

Since $n-\alpha(0)-\frac{n}{p(0)}>0$, then by Lemma 2.3 and Proposition 1.1,we get

$$\mathrm{E}_1\leq C\left\{\sum_{j=-\infty}^{-1}2^{j\alpha(0)q(0)}\|f_j\|^{q(0)}_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q(0)}=C\|f\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.$$

Now let us estimate E_4. then by Lemma 2.3 we have

$$\mathrm{E}_4\ \leq\left\{\sum_{k=0}^{\infty}2^{k\alpha_\infty q_\infty}\left(\sum_{j=-\infty}^{-1}\|\mu^\theta_{\Omega,b}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q_\infty}\right\}^{1/q_\infty}$$

$$+\left\{\sum_{k=0}^{\infty}2^{k\alpha_\infty q_\infty}\left(\sum_{j=0}^{k-2}\|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q_\infty}\right\}^{1/q_\infty}$$

$$=: \mathrm{E}_{41}+\mathrm{E}_{42}.$$

For E_{41}, in view of $j<0\le k$, by Lemma 2.6, we can obtain

$$\|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C2^{k(\frac{n}{p_\infty}-n)}2^{j(n-\frac{n}{p(0)})}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Therefore

$$\mathrm{E}_{41}\le C\left\{\sum_{k=0}^{\infty}2^{k(\alpha_\infty+\frac{n}{p_\infty}-n)q_\infty}\left(\sum_{j=-\infty}^{-1}2^{j(n-\frac{n}{p(0)}-\alpha(0))}2^{j\alpha(0)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q_\infty}\right\}^{1/q_\infty}.$$

Since $\alpha_\infty+\frac{n}{p_\infty}-n<0<n-\frac{n}{p(0)}-\alpha(0)$, by embedding $l^{q(0)}\hookrightarrow l^\infty$, we have

$$\mathrm{E}_{41}\le C\left\{\sum_{j=-\infty}^{-1}2^{j\alpha(0)q(0)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q(0)}\right\}^{1/q(0)}=C\|f\|_{\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Similar to the estimate of E_1, we replace $\alpha(0),p(0)$ and $q(0)$ by α_∞,p_∞ and q_∞ respectively, then

$$\begin{aligned}\mathrm{E}_{42}&\le C\left\{\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k-2}2^{(k-j)(\alpha_\infty+\frac{n}{p_\infty}-n)}(2^{j\alpha_\infty}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)})\right)^{q_\infty}\right\}^{1/q_\infty}\\&\le C\left\{\sum_{j=0}^{\infty}2^{j\alpha_\infty q_\infty}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_\infty}\right\}^{1/q_\infty}\\&\le C\|f\|_{\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Next we estimate E_3, we split

$$\begin{aligned}\mathrm{E}_3\le&\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\left(\sum_{j=k+2}^{-1}\|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(0)}\right\}^{1/q(0)}\\&+\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\left(\sum_{j=0}^{\infty}\|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(0)}\right\}^{1/q(0)}\\=:&\ \mathrm{E}_{31}+\mathrm{E}_{32}.\end{aligned}$$

Similar to the estimate for E_{11} and E_{12}, we can get

$$\|\mu_{\Omega,b}^{\theta}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

For E_{31}. Since $\alpha(0)+\frac{n}{p(0)}>0$, $j,k<0$ and $j\ge k+2$, then by Lemma 2.3 we have

$$\mathrm{E}_{31}\le C\left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{-1}2^{(k-j)(\alpha(0)+\frac{n}{p(0)})}2^{j\alpha(0)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(0)}\right\}$$

$$
\begin{aligned}
&\leq C\Bigg\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(0)}\|f\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q(0)}\Bigg\}^{1/q(0)}\\
&= C\|f\|_{\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

For E_{32}. Since $k<0\leq j$, then by Lemma 2.6 we have

$$
\begin{aligned}
\|\mu_\Omega^b(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C2^{\frac{kn}{p(0)}-\frac{jn}{p_\infty}}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{\frac{kn}{p(0)}}2^{-j(\alpha_\infty+\frac{n}{p_\infty})}2^{j\alpha_\infty}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Noting that $\frac{n}{p(0)},\alpha_\infty+\frac{n}{p_\infty}>0$, it follows from the Hölder's inequality in l^1 that

$$
\begin{aligned}
\mathrm{E}_{32} &\leq C\Bigg\{\sum_{k=-\infty}^{-1}2^{\frac{kn}{p(0)}}\Bigg(\sum_{j=0}^{\infty}2^{j\alpha_\infty}2^{-j(\alpha_\infty+\frac{n}{p_\infty})}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{q(0)}\Bigg\}^{1/q(0)}\\
&\leq C\Bigg(\sum_{k=-\infty}^{-1}2^{\frac{kn}{p(0)}q(0)}\Bigg)^{1/q(0)}\Bigg(\sum_{j=0}^{\infty}2^{-j(\alpha_\infty+\frac{n}{p_\infty})q'_\infty}\Bigg)^{1/q'_\infty}\Bigg(\sum_{j=0}^{\infty}2^{j\alpha_\infty q_\infty}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_\infty}\Bigg)^{1/q_\infty}\\
&\leq C\Bigg\{\sum_{j=0}^{\infty}2^{j\alpha_\infty q_\infty}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_\infty}\Bigg\}^{1/q_\infty}\\
&= C\|f\|_{\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Finally, let's estimate E_6, since j,k are non-negative integers and $j\geq k+2$, by Lemma 2.6, we have

$$
\|\mu_{\Omega,b}^\theta(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leq C2^{(k-j)\frac{n}{p_\infty}}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
$$

So, we can obtain

$$
\mathrm{E}_6\leq C\Bigg\{\sum_{k=0}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_\infty+\frac{n}{p_\infty})}2^{j\alpha_\infty}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)^{q_\infty}\Bigg\}^{1/q_\infty}.
$$

Since $\alpha_\infty+\frac{n}{p_\infty}>0$, then by Lemma 2.3 and Proposition 1.1, we get

$$
\begin{aligned}
\mathrm{E}_6 &\leq C\Bigg\{\sum_{j=0}^{\infty}2^{j\alpha_\infty q_\infty}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_\infty}\Bigg\}^{1/q_\infty}\\
&= C\|f\|_{\dot{\mathcal{K}}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Combining with above estimates, we complete the proof of Theorem 1.3. □

Proof of Theorem 1.5. Let $f\in H\dot{\mathcal{K}}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(\mathbb{R}^n)$. By Lemma 2.8, for each $j\in\mathbb{Z}$, we have

$$
f=\sum_{j=-\infty}^{\infty}\lambda_j a_j.
$$

Where $\lambda_j \geq 0$ and a is a $(\alpha(\cdot), p_1(\cdot))$-atom. Using Proposition 1.1 we have

$$\begin{aligned}
\|\mu^{\theta}_{\Omega,b}(f)\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)} &\approx \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \|\mu^{\theta}_{\Omega,b}(f)\chi_k\|^{q_2(0)}_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right\}^{1/q_2(0)} \\
&\quad + \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q_{2\infty}} \|\mu^{\theta}_{\Omega,b}(f)\chi_k\|^{q_{2\infty}}_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right\}^{1/q_{2\infty}} \\
&\leq \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|\mu^{\theta}_{\Omega,b}(a_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_2(0)} \right\}^{1/q_2(0)} \\
&\quad + \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|\mu^{\theta}_{\Omega,b}(a_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_2(0)} \right\}^{1/q_2(0)} \\
&\quad + \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q_{2\infty}} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|\mu^{\theta}_{\Omega,b}(a_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_{2\infty}} \right\}^{1/q_{2\infty}} \\
&\quad + \left\{ \sum_{k=0}^{\infty} 2^{k\alpha_\infty q_{2\infty}} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|\mu^{\theta}_{\Omega,b}(a_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_{2\infty}} \right\}^{1/q_{2\infty}} \\
&=: \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3 + \mathrm{F}_4.
\end{aligned}$$

Now estimate F_1. By the Minkowski inequality

$$\begin{aligned}
|\mu^{\theta}_{\Omega,b}(a_j)(x)| &\leq \left(\int_0^{|x|} \left| \int_{|x-y|\leq t} \frac{\Omega(x,x-y)}{|x-y|^{n-\theta}} [b(x)-b(y)]^m a_j(y)dy \right|^2 \frac{dt}{t^{2\theta+1}} \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{|x|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x,x-y)}{|x-y|^{n-\theta}} [b(x)-b(y)]^m a_j(y)dy \right|^2 \frac{dt}{t^{2\theta+1}} \right)^{\frac{1}{2}} \\
&=: \mathrm{F}_{11} + \mathrm{F}_{12}.
\end{aligned}$$

Note that $x \in A_k,\ y \in A_j,\ j \leq k-2$. So we have $|x-y| \sim |x| \sim 2^k$,

$$\left| \frac{1}{|x-y|^{2\theta}} - \frac{1}{|x|^{2\theta}} \right| \leq C \frac{|y|}{|x-y|^{2\theta+1}},$$

then

$$\begin{aligned}
\mathrm{F}_{11} &\leq C \int_{\mathbb{R}^n} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\theta}} |b(x)-b(y)|^m |a_j(y)| \left(\int_{|x-y|}^{|x|} \frac{dt}{t^{2\theta+1}} \right)^{\frac{1}{2}} dy \\
&\leq C \int_{B_j} \frac{|\Omega(x,x-y)|}{|x-y|^{n-\theta}} |b(x)-b(y)|^m |a_j(y)| \frac{|y|^{1/2}}{|x-y|^{\theta+\frac{1}{2}}} dy.
\end{aligned}$$

By the $\tau-$order vanishing moments of a_j with $\tau \geq [\alpha_+ - n(1-\frac{1}{p_{1-}})]$, subtract the Taylor expansion of $\frac{1}{|x-y|^{n-\frac{1}{2}}}$ at x, we have

$$\mathrm{F}_{11} \leq C \int_{B_j} \frac{|y|^{1+\tau}}{|x|^{n+\frac{1}{2}+\tau}} |a_j(y)||\Omega(x,x-y)||b(x)-b(y)|^m$$

$$
\begin{aligned}
&\leq C2^{-k(n+\tau+\frac{1}{2})+j(\tau+1)}\int_{B_j}|a_j(y)||\Omega(x,x-y)||b(x)-b(y)|^m dy\\
&\leq C2^{-k(n+\tau+\frac{1}{2})+j(\tau+1)}\|a_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\Big\{|b(x)-b_{B_j}|^m\|\Omega(x,x-\cdot)\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\quad+\|\Omega(x,x-\cdot)(b_{B_j}-b(\cdot))^m\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\Big\}\\
&\leq C2^{-k(n+\tau+\frac{1}{2})+j(\tau+1)}2^{jn}\|b\|_*^m\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})}\frac{\|a_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\leq C2^{-k(n+\tau+\frac{1}{2})+j(n+\tau+1)}\frac{\|a_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}.
\end{aligned}
$$

Similarly to F_{12}, we also have

$$
|\mu^\theta_{\Omega,b}(a_j)(x)|\leq C2^{-k(n+\tau+\frac{1}{2})+j(n+\tau+1)}\frac{\|a_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}.
$$

Which gives

$$
\begin{aligned}
\|\mu^\theta_{\Omega,b}(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}&\leq C2^{-k(n+\tau+\frac{1}{2})+j(n+\tau+1)}\|a_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\leq C2^{(j-k)(n+\tau)}2^{j-\frac{k}{2}}\|a_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}.
\end{aligned}
$$

In view of $\frac{1}{p_1(\cdot)}-\frac{1}{p_2(\cdot)}=\frac{1}{2n}$, it follows from Lemma 2.6 that

$$
\|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\approx|B_k|^{\frac{1}{p_2(\cdot)}}\approx|B_k|^{\frac{1}{p_1(\cdot)}-\frac{1}{2n}}\approx 2^{-\frac{k}{2}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
$$

So we have

$$
\begin{aligned}
\|\mu^\theta_{\Omega,b}(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}&\leq C2^{(j-k)(n+\tau)}2^{j-k}\|a_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\leq C2^{(j-k)(n+\tau)}\|a_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\leq C2^{(j-k)(n+\tau-\frac{n}{p_1(0)})}\|a_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

by Definition 1.4 and Lemma 2.4, we can get

$$
\begin{aligned}
\mathrm{F}_1&=\left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|\mu^\theta_{\Omega,b}(a_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\
&\leq\left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{(j-k)(n+\tau-\frac{n}{p_1(0)}-\alpha(0))}\right)^{q_2(0)}\right\}^{1/q_2(0)}.
\end{aligned}
$$

We can choose τ large enough such that $\tau - \alpha_+ + n(1 - \frac{1}{(p_1)_-}) > 0$, by Lemma 2.8, we obtain

$$\mathrm{F}_1 \leq C\bigg(\sum_{k=-\infty}^{-1} |\lambda_k|^{q_2(0)}\bigg)^{1/q_2(0)} \leq C\bigg(\sum_{k=-\infty}^{-1} |\lambda_k|^{q_1(0)}\bigg)^{1/q_1(0)} \leq C\|f\|_{H\dot{\mathcal{K}}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Let us estimate F_2. Applying (2) and (3) in Definition 1.4 and Lemma 2.4, we have

$$\begin{aligned}
\mathrm{F}_2 &= \bigg\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\bigg(\sum_{j=k-2}^{\infty} |\lambda_j|\|\mu^{\theta}_{\Omega,b}(a_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\bigg)^{q_2(0)}\bigg\}^{1/q_2(0)} \\
&\leq \bigg\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\bigg(\sum_{j=k-2}^{\infty} |\lambda_j|\|a_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\bigg)^{q_2(0)}\bigg\}^{1/q_2(0)} \\
&\leq \bigg\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\bigg(\sum_{j=k-2}^{-1} |\lambda_j|\|a_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\bigg)^{q_2(0)}\bigg\}^{1/q_2(0)} \\
&\quad + \bigg\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\bigg(\sum_{j=0}^{\infty} |\lambda_j|\|a_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\bigg)^{q_2(0)}\bigg\}^{1/q_2(0)} \\
&\leq C\bigg\{\sum_{k=-\infty}^{-1} \bigg(\sum_{j=k-2}^{-1} |\lambda_j| 2^{(k-j)\alpha(0)}\bigg)^{q_2(0)}\bigg\}^{1/q_2(0)} \\
&\quad + C\bigg\{\sum_{k=-\infty}^{-1} \bigg(\sum_{j=0}^{\infty} |\lambda_j| 2^{(k-j)\alpha_- + j(\alpha(0)-\alpha_-) + k(\alpha_- - \alpha_\infty)}\bigg)^{q_2(0)}\bigg\}^{1/q_2(0)}.
\end{aligned}$$

Since $\alpha_- \leq \min(\alpha(0), \alpha_\infty)$, we have

$$j(\alpha(0) - \alpha_-) + k(\alpha_- - \alpha_\infty) \leq 0.$$

By Lemma 2.8 and the fact of $j \geq 0 > k$, we can obtain

$$\mathrm{F}_2 \leq C\bigg(\sum_{k=-\infty}^{-1} |\lambda_k|^{q_2(0)}\bigg)^{1/q_2(0)} \leq C\bigg(\sum_{k=-\infty}^{-1} |\lambda_k|^{q_1(0)}\bigg)^{1/q_1(0)} \leq C\|f\|_{H\dot{\mathcal{K}}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Similar to the estimation of F_1 and F_2, we can also estimate F_3 and F_4, in fact, we just need to replace $\alpha(0), p_2(0)$ and $q_2(0)$ with $\alpha_\infty, (p_2)_\infty$ and $(q_2)_\infty$ respectively.

Combining with the estimates of F_1, F_2, F_3 and F_4, we have

$$\|\mu^{\theta}_{\Omega,b}(f)\|_{\dot{\mathcal{K}}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{H\dot{\mathcal{K}}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

The proof of Theorem 1.5 is completed. □

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