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THIRD-ORDER ACCURATE IMPLICIT-EXPLICIT TIME-STEPPING SCHEME FOR A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH FRACTIONAL ORDER $\alpha \in (1, 2)$

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Abstract This study focuses on designing a high-order accurate method to solve the nonlinear fractional differential equation with the Caputo derivative of fractional order $\alpha \in (1, 2)$. First, we design two third-order accurate numerical approximations using the graded meshes for the fractional integral, and then employ them to establish the implicit-explicit time-stepping scheme for the equivalent Volterra integral form of the original equation. The error estimates of the two third-order numerical approximations, as well as the stability and the error estimates of the proposed time-stepping scheme, are presented rigorously. This method is further applied to solve a class of multi-term differential equations as well as a nonlinear time-fractional diffusion–wave equation. Numerical examples are extensively provided to demonstrate the effectiveness of the proposed schemes.

Keywords Third-order accurate, nonlinear fractional differential equation, implicit-explicit time-stepping scheme, stability, error estimate.

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1. Introduction

In this paper, we are interested in the construction of a high-order time-stepping scheme for the following nonlinear fractional differential equation:

$$ {}_{C}D^{\alpha}_{0,t}y(t) = -\lambda y(t) + f(t, y(t)), \text{ in } (0, T], \tag{1.1}$$

where T is a fixed time and $\lambda > 0$. The corresponding initial conditions of equation (1.1) are $y^{(k)}(0) = y_k, k = 0, 1$. The Caputo derivative ${}_{C}D^{\alpha}_{0,t}u(t)$ with $\alpha \in (1, 2)$ is defined by

$$ {}_{C}D^{\alpha}_{0,t}v(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} v''(s)\mathrm{d}s.$$

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Typically, we require the nonlinear function f to be continuous and to satisfy a Lipschitz condition regarding its second argument, with a Lipschitz constant L over an appropriate set G [5].

It is well recognized that fractional differential equations can be used to establish effective modeling in science and engineering because of their great ability to capture nonlocal memory effects and intricate dynamic behaviors. There exist numerous applications in various disciplines, such as anomalous diffusion [19], transport in inhomogeneous media (e.g., subsurface flow [9]), biofluid dynamics (e.g., viscoelastic blood flow models [22]), and financial option pricing [8], etc.

In recent years, researchers have developed various effective numerical methods for fractional differential equations and have formulated their related numerical theories; see [3,4,6,12,25,27, 28,30]. Specifically, for nonlinear fractional differential equations, employing implicit time-stepping directly results in nonlinear discrete systems. Such systems always require iterative methods, which leads to significant computational costs. To address this problem, several numerical techniques have been introduced, such as the predictor-corrector method [7,20,24,29], exponential time differencing methods [23], block fractional generalized Adams method [10], spectrally accurate step-by-step method [1], and implicit-explicit time-stepping method [2], etc. In addition, designing the high-order accurate method for fractional differential equations is not an easy job due to the initially weak singularity appearing in the solution of the problem considered; see [4,12,16,25]. We may refer to the book [15] and the two review papers [11,13] for further discussion.

To address the challenges of computational efficiency and the accuracy impact caused by the nonlinear term and initial weak singularity, significant efforts have been made; see [2,12,14,17, 26,28], just to name a few. In [2], the authors employed the adding correction term method and the implicit-explicit technique to design stable second-order accurate numerical schemes. In [14], the authors developed efficient predictor-corrector Adams methods using nonuniform meshes and established the general Gronwall inequality to facilitate the numerical analysis of these schemes. Liu et al. designed predictor-corrector Adams methods using graded meshes, demonstrating that these methods achieve $(1+\alpha)$-order accuracy for $\alpha \in (0,1)$ and second-order accuracy for $\alpha \in (1,2)$, provided appropriate mesh grading is applied [17]. It is noted that Lyu and Vong constructed a temporal third-order accurate difference scheme for solving the time-fractional Benjamin–Bona–Mahony equation utilizing graded meshes [18]. They mentioned the applicability of their method to nonlinear fractional differential equations; however, a detailed numerical analysis was not included and also the study was limited to $\alpha \in (0,1)$. The extension of their method to $\alpha \in (1,2)$ remains to be investigated. So, to the best of our knowledge, there is still a scarcity of numerical studies focused on effective numerical schemes achieving higher than second-order accuracy for solving fractional differential equations, particularly for fractional orders $\alpha \in (1,2)$.

This paper focuses on developing a stable, high-order accurate numerical method for solving fractional differential equations with a fractional order $\alpha \in (1,2)$. First, we transform the original problem into an equivalent Volterra integral form, followed by the development of two third-order accuracy formulas using graded meshes to approximate linear and nonlinear terms. Using these formulas, we successfully establish a third-order accurate implicit-explicit time-stepping scheme. The error estimates for both third-order numerical approximations are rigorously established. Subsequently, using the Gronwall inequality as presented in [14], we systematically establish the stability and error estimates of the implicit-explicit time-stepping scheme. To further demonstrate the effectiveness of the proposed method, we employ it to solve a class of multi-term fractional differential equations and a nonlinear time-fractional diffusion–wave equation. Finally, extensive numerical examples are provided to verify the accuracy and efficiency of the proposed schemes.

The outline of this paper is listed below. In Section 2, we introduce two third-order numerical approaches for fractional integrals, later utilized to formulate an implicit-explicit time-stepping scheme. Additionally, we provide error estimates for these methods and demonstrate the stability and error estimate of the time-stepping scheme. Sections 3 and 4 include a further discussion as well as numerical examples to substantiate our theoretical results.

2. Derivation of the scheme

Note that the identity $D_{0,t}^{-\alpha}{}_{C}D_{0,t}^{\alpha}v(t) = v(t) - v(0) - tv'(0)$ is valid; see [15, Proposition 1.1.5]. Here, the fractional integral operator is given by

$$D_{0,t}^{-\alpha}v(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)\mathrm{d}s.$$

So, by operating the operator $D_{0,t}^{-\alpha}$ on both sides of equation (1.1), we can obtain the following equivalent integral equation:

$$y(t) = y_0 + ty_1 + D_{0,t}^{-\alpha}(-\lambda y(t) + f(t, y(t))). \tag{2.1}$$

For ease of discussion, we assume that the solution y in (1.1) satisfies the regularity [17, (1.4)]:

$$|w'(t)| \le ct^{\upsilon-1}, \quad |w''(t)| \le ct^{\upsilon-2}, \quad |w'''(t)| \le ct^{\upsilon-3}, \tag{2.2}$$

where $w(t) := {}_{C}D_{0,t}^{\alpha}y(t) \in C^3(0,T]$ and $\upsilon \in (0,1)$.

Since the solution of the fractional differential equation (1.1) is not always sufficiently smooth at the initial time layer, we adopt nonuniform meshes $\{t_k := T(\frac{k}{N})^{\gamma}\}_{k=0}^{N}$ in time. Here, N is a positive integer and γ is the grading parameter, which is greater than one. Denote the time stepsize $\tau_k = t_k - t_{k-1}$.

Consider equation (2.1) at $t = t_n$ for $1 \le n \le N$, we have

$$y(t_n) = y_0 + t_n y_1 + G_1(t_n) + G_2(t_n), \tag{2.3}$$

where $G_1(t_n) = -D_{0,t}^{-\alpha}\lambda y(t)|_{t=t_n}$ and $G_2(t_n) = D_{0,t}^{-\alpha}f(t,y)|_{t=t_n}$. It is essential to derive efficient numerical approximations in time for the two terms: G_1 and G_2. So, in the following part we introduce two third-order accurate approximations for the fractional integral with $\alpha \in (1,2)$.

2.1. Third-order accurate approximations for fractional integral

Without loss of generality, we consider the construction for the high-order numerical approximation of the fractional integral $D_{0,t}^{-\alpha}w(t)$ at $t = t_n$ where the fractional order is restricted at $\alpha \in (1,2)$. Note that the extension to the other case of α is straightforward.

Using the linear interpolation for the first interval $[0, t_1]$ and the quadratic interpolation based on the three points t_{k-2}, t_{k-1} and t_k for the subsequent interval $[t_{k-1}, t_k](k \ge 2)$, we can derive the following numerical approximation for $D_{0,t}^{-\alpha}w(t)$ at $t = t_n$.

$$D_{0,t}^{-\alpha}w(t)|_{t=t_n} \approx \sum_{k=0}^{n} A_{k,n}w(t_k) =: D_{\tau_n}^{-\alpha}w(t_n), \quad 1 \le n \le N, \tag{2.4}$$

where the coefficients $A_{k,n}(k \geq 0)$ are defined as $P_{k,n}$ given in [18, (2.8)], the details of which are omitted here for the sake of brevity.

In order to derive the implicit-explicit time-stepping scheme for the fractional model (1.1), we construct another numerical approximation for the fractional integral $D_{0,t}^{-\alpha} w(t)$ at $t = t_n$ without using the current time level $t = t_n$. So, we approximate $w(t)$ by its initial value $w(t_0)$ when $t \in (0, t_1)$ and by linear interpolation based on the two points t_0 and t_1 when $t \in (t_1, t_2)$, while the other intervals are dealt with similarly to that of [18, (2.8)], we therefore have

$$
\begin{aligned}
D_{0,t}^{-\alpha} w(t)|_{t=t_n} \approx & \frac{1}{\Gamma(\alpha)} \Big(\int_0^{t_1} + \int_{t_1}^{t_2} \Big) (t_n - s)^{\alpha-1} \tilde{\Pi}_{1,1} w(s)\, ds \\
& + \frac{1}{\Gamma(\alpha)} \sum_{k=3}^{n-1} \int_{t_{k-1}}^{t_k} (t_n - s)^{\alpha-1} \Pi_{2,k} w(s)\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\alpha-1} \tilde{\Pi}_{2,n} w(s))\, ds
\end{aligned}
$$

for $n > 1$. It should be noted that the final term in the above approximation becomes zero for $n = 2$; its inclusion serves purely for simplification purposes. Here,

$$
\begin{aligned}
\tilde{\Pi}_{1,1} w(t) &= w(t_0) \frac{t_1 - t}{t_1 - t_0} + w(t_1) \frac{t - t_0}{t_1 - t_0}, \quad t \in (t_0, t_2), \\
\Pi_{2,k} w(t) &= w(t_{k-2}) \frac{(t - t_{k-1})(t - t_k)}{(t_{k-2} - t_{k-1})(t_{k-2} - t_k)} \\
&\quad + w(t_{k-1}) \frac{(t - t_{k-2})(t - t_k)}{(t_{k-1} - t_{k-2})(t_{k-1} - t_k)} \\
&\quad + w(t_k) \frac{(t - t_{k-2})(t - t_{k-1})}{(t_k - t_{k-2})(t_k - t_{k-1})}, \quad t \in (t_{k-1}, t_k), \\
\tilde{\Pi}_{2,n} w(t) &= w(t_{n-3}) \frac{(t - t_{n-2})(t - t_{n-1})}{(t_{n-3} - t_{n-2})(t_{n-3} - t_{n-1})} \\
&\quad + w(t_{n-2}) \frac{(t - t_{n-3})(t - t_{n-1})}{(t_{n-2} - t_{n-3})(t_{n-2} - t_{n-1})} \\
&\quad + w(t_{n-1}) \frac{(t - t_{n-3})(t - t_{n-2})}{(t_{n-1} - t_{n-3})(t_{n-1} - t_{n-2})}, \quad t \in (t_{n-1}, t_n).
\end{aligned}
$$

We approximate the fractional integral at $t = t_1$ as follows.

$$
D_{0,t}^{-\alpha} w(t)|_{t=t_1} \approx \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_n - s)^{\alpha-1} w(t_0)\, ds.
$$

After a tedious computation, we obtain the numerical approximation for the fractional integral $D_{0,t}^{-\alpha} w(t)$ at $t = t_n$:

$$
D_{0,t}^{-\alpha} w(t)|_{t=t_n} \approx \sum_{k=0}^{n-1} \tilde{A}_{k,n} w(t_k) =: \tilde{D}_{\tau_n}^{-\alpha} w(t_n). \tag{2.5}
$$

To simplify the notation, the subscript vector of the coefficients is used to denote the corresponding vectors, i.e., $\tilde{A}_{0:k,n}$ denotes $[\tilde{A}_{0,n}, \tilde{A}_{1,n}, \cdots, \tilde{A}_{k,n}]$. So, the coefficients are presented in (2.5) is given by

$$\tilde{A}_{0:n-1,n} = \begin{cases} l_0^n, & \text{if } n=1, \\ 2[l_1^n, l_2^n], & \text{if } n=2, \\ [l_1^n + c_{2,n} + l_{-3}^n, l_2^n + d_{2,n} + l_{-2}^n, e_{2,n} + l_{-1}^n], & \text{if } n=3, \\ [l_1^n + c_{2,n}, l_2^n + c_{3,n} + d_{2,n} + l_{-3}^n, d_{3,n} + e_{2,n} + l_{-2}^n, e_{3,n} + l_{-1}^n], & \text{if } n=4, \\ [l_1^n + c_{2,n}, l_2^n + c_{3,n} + d_{2,n}, & \\ c_{4,n} + d_{3,n} + e_{2,n} + l_{-3}^n, d_{4,n} + e_{3,n} + l_{-2}^n, e_{4,n} + l_{-1}^n], & \text{if } n=5, \\ [l_1^n + l_1^n + c_{2,n}, l_2^n + c_{3,n} + d_{2,n}, c_{4:n-2,n} + d_{3:n-3,n} + e_{2:n-4,n}, & \\ c_{n-1,n} + d_{n-2,n} + e_{n-3,n} + l_{-3}^n, d_{n-1,n} + e_{n-2,n} + l_{-2}^n, & \\ e_{n-1,n} + l_{-1}^n], & \text{if } n \geq 6. \end{cases}$$

Here, the coefficients $l_j^n (j=0,1,2)$ are defined as

$$\begin{aligned} l_0^n &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_n - s)^{\alpha-1}\, ds = \frac{t_1^\alpha}{\Gamma(\alpha+1)}, \\ l_1^n &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_n - s)^{\alpha-1} \frac{t_1 - s}{t_1 - t_0}\, ds \\ &= \frac{(t_n - t_2)^\alpha (t_2 - t_1)}{t_1 \Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+2) t_1} \left[(t_n - t_2)^{\alpha+1} - (t_n - t_1)^{\alpha+1} \right], \\ l_2^n &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_n - s)^{\alpha-1} \frac{s - t_0}{t_1 - t_0}\, ds \\ &= \frac{1}{t_1 \Gamma(\alpha+1)} \left[(t_n - t_1)^\alpha t_1 - (t_n - t_2)^\alpha t_2 \right] + \frac{1}{t_1 \Gamma(\alpha+2)} \left[(t_n - t_1)^{\alpha+1} - (t_n - t_2)^{\alpha+1} \right]. \end{aligned}$$

The other coefficients $l_j^n (j=-1,-2,-3)$ and $c_{k,n}, d_{k,n}, e_{k,n}$ are given as the same as in [18, (2.9)].

Remark 2.1. We remark that we expand the approach for numerical approximation of the fractional integral, initially described for $\alpha \in (0,1)$ in [18], to include the case where $\alpha \in (1,2)$. In addition, instead of using a constant function to approximate $w(t)$ in the interval $[t_1, t_2]$ [18], we employ linear interpolation, which makes our method fundamentally different from [18]. The resulting method significantly improves accuracy and also helps in the error estimate for the resulting numerical approximation for the fractional integral where $\alpha \in (1,2)$.

Next, we present some useful properties for the weights $A_{k,n}$ and $\tilde{A}_{k,n}$, and then illustrate the error estimates for the difference operators $D_{\tau_n}^{-\alpha}$ and $\tilde{D}_{\tau_n}^{-\alpha}$.

Using a similar idea as that present in Lemma 2.1 of [18], the following properties can be readily obtained for the weights $A_{k,n}$ and $\tilde{A}_{k,n}$. The corresponding proof is omitted for the sake of brevity.

Lemma 2.1. *The coefficients $A_{k,n}$ and $\tilde{A}_{k,n}$ satisfy the following properties.*

$$|A_{k,n}| \leq \begin{cases} C\tau_{k+1}(t_n - t_k)^{\alpha-1}, & 0 \leq k \leq n-1, \\ \dfrac{\tau_n^\alpha}{\Gamma(\alpha+2)}, & k = n, \end{cases} \quad \textit{for} \quad n \geq 1, \tag{2.6}$$

and

$$\left| \tilde{A}_{k,n} \right| \leq C\tau_{k+1}(t_n - t_k)^{\alpha-1}, \quad 0 \leq k \leq n-1.$$

Moreover,

$$\sum_{k=0}^{n}|A_{k,n}| \leq Ct_n^\alpha \quad \textit{and} \quad \sum_{k=0}^{n-1}\left|\tilde{A}_{k,n}\right| \leq Ct_n^\alpha, \quad n \geq 1.$$

We present the following error estimates for the difference operators $D_{\tau_n}^{-\alpha}$ and $\tilde{D}_{\tau_n}^{-\alpha}$.

Lemma 2.2. *Suppose that the function $w(t)$ satisfies the conditions: $|w'(t)| \leq Ct^{\sigma-1}$, $|w''(t)| \leq Ct^{\sigma-2}$, and $|w'''(t)| \leq Ct^{\sigma-3}$ for $\sigma \in (0,2)$, then it holds that*

$$\left|D_{0,t}^{-\alpha}w(t)|_{t=t_n} - \overline{D}_{\tau_n}^{-\alpha}w(t_n)\right| \leq C\varrho(N,\gamma,\sigma+1),$$

for $n = 1,2,\cdots,N$. Here, the difference operator $\overline{D}_{\tau_n}$ represents $D_{\tau_n}^{-\alpha}$ or $\tilde{D}_{\tau_n}^{-\alpha}$, the notation ϱ is given by

$$\varrho(N,\gamma,\varsigma) = \begin{cases} N^{-\gamma\varsigma}, & \textit{if } \gamma\varsigma < 3, \\ N^{-3}\ln(N), & \textit{if } \gamma\varsigma = 3, \\ N^{-3}, & \textit{if } \gamma\varsigma > 3. \end{cases}$$

Especially, if we take $\sigma \in (2,3)$, then

$$\left|D_{0,t}^{-\alpha}w(t)|_{t=t_n} - \overline{D}_{\tau_n}^{-\alpha}w(t_n)\right| \leq CN^{-3},$$

under the uniform meshes.

Proof. We only consider the case $\sigma \in (0,2)$ since the case $\sigma \in (2,3)$ is trivial in view of the classical interpolation theory. We first consider the estimate for $D_{\tau_n}^{-\alpha}$. Since

$$\begin{aligned}\left|D_{0,t}^{-\alpha}w(t)|_{t=t_n} - D_{\tau_n}^{-\alpha}w(t_n)\right| &= \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_n-s)^{\alpha-1}\left[w(s)-\Pi_{1,1}w(s)\right]\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}(t_n-s)^{\alpha-1}\left[w(s)-\Pi_{2,k}w(s)\right]\mathrm{d}s \\ &=: P_1 + P_2.\end{aligned}$$

Here, the interpolation operator $\Pi_{1,1}$ are defined as

$$\Pi_{1,1}w(t) = w(t_0)\frac{t_1-t}{t_1-t_0} + w(t_1)\frac{t-t_0}{t_1-t_0}, \quad t \in (t_0,t_1).$$

In view of the estimate of I_1 with $\alpha \in (1,2)$ in [17, Lemma 2.1], we readily have $|P_1| \leq CN^{-\gamma(\sigma+1)}$. It remains to estimate the term P_2. By similar argument with I_2 in [18, Lemma 3.1], we can obtain

$$\begin{aligned}|P_2| &\leq C\sum_{k=2}^{\lfloor n/2\rfloor}\tau_k^3 t_{k-1}^{\sigma-3}\int_{t_{k-1}}^{t_k}(t_n-s)^{\alpha-1}\mathrm{d}s + C\sum_{k=\lfloor n/2\rfloor+1}^{n}\tau_k^3 t_{k-1}^{\sigma-3}\int_{t_{k-1}}^{t_k}(t_n-s)^{\alpha-1}\mathrm{d}s \\ &=: P_{21} + P_{22}.\end{aligned} \tag{2.7}$$

Note that $\alpha \in (1,2)$, we derive that

$$|P_{21}| \leq C\sum_{k=2}^{\lfloor n/2\rfloor}\tau_k^4 t_{k-1}^{\sigma-3}(t_n-t_{k-1})^{\alpha-1}$$

$$
\begin{aligned}
&\le C \sum_{k=2}^{\lfloor n/2 \rfloor} \tau_k^4 t_{k-1}^{\sigma-3} t_n^{\alpha-1} \\
&\le C \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{k^{4(\gamma-1)}}{N^{4\gamma}} \frac{(k-1)^{\gamma(\sigma-3)}}{N^{\gamma(\sigma-3)}} \frac{n^{\gamma(\alpha-1)}}{N^{\gamma(\alpha-1)}} \\
&\le C \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{k^{4(\gamma-1)}}{N^{4\gamma}} \frac{(k-1)^{\gamma(\sigma-3)}}{N^{\gamma(\sigma-3)}} \\
&\le C N^{-\gamma(\sigma+1)} \sum_{k=2}^{\lfloor n/2 \rfloor} k^{\gamma(\sigma+1)-4}.
\end{aligned}
$$

So, if $\gamma(\sigma+1) < 3$, one has

$$
|P_{21}| \le C N^{-\gamma(\sigma+1)} \sum_{k=2}^{\lfloor n/2 \rfloor} k^{\gamma(\sigma+1)-4} \le C N^{-\gamma(\sigma+1)}.
$$

When $\gamma(\sigma+1) = 3$, the term P_{21} has the estimate:

$$
|P_{21}| \le C N^{-3} \sum_{k=2}^{\lfloor n/2 \rfloor} k^{-1} \le C N^{-3} \ln(N).
$$

If $\gamma(\sigma+1) > 3$, we have

$$
|P_{21}| \le C N^{-\gamma(\sigma+1)} n^{\gamma(\sigma+1)-3} = C \left(\frac{n}{N}\right)^{\gamma(\sigma+1)-3} N^{-3} \le C N^{-3}.
$$

Hence, we can obtain the estimate for P_{21} as follows.

$$
|P_{21}| \le \begin{cases} C N^{-\gamma(1+\sigma)}, & \gamma(\sigma+1) - 3 < 0, \\ C N^{-3} \ln(N), & \gamma(\sigma+1) - 3 = 0, \\ C N^{-3}, & \gamma(\sigma+1) - 3 > 0. \end{cases}
$$

It remains to consider the estimate for P_{22}. In fact, one can derive that

$$
\begin{aligned}
|P_{22}| &\le C \tau_n^3 t_{[n/2]}^{\sigma-3} \int_{t_{\lfloor n/2 \rfloor}}^{t_n} (t_n - s)^{\alpha-1} \, \mathrm{ds} \\
&\le C \left(\frac{n^{\gamma-1}}{N^\gamma}\right)^3 \left(\frac{n}{N}\right)^{\gamma(\sigma-3)} \left(\frac{n}{N}\right)^{\gamma\alpha} \\
&\le C N^{-\gamma(\sigma+\alpha)} n^{\gamma(\sigma+\alpha)-3}.
\end{aligned}
$$

So, one can obtain

$$
|P_{22}| \le \begin{cases} C N^{-\gamma(\sigma+\alpha)}, & \gamma(\sigma+\alpha) - 3 < 0, \\ C N^{-3}, & \gamma(\sigma+\alpha) - 3 \ge 0. \end{cases}
$$

Putting the two estimates for P_1 and P_2 together, and noting that $\sigma + \alpha > \sigma + 1$, we obtain the desired result.

Now, we turn to present the proof of the estimate for $\tilde{D}_{\tau_n}^{-\alpha}$. For $n \geq 3$, in $\tilde{D}_{\tau_n}^{-\alpha}$, we can obtain

$$\begin{aligned}\left|D_{0,t}^{-\alpha} w(t)|_{t=t_n} - \tilde{D}_{\tau_n}^{-\alpha} w(t_n)\right| \leq & \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_n-s)^{\alpha-1}[w(s)-\tilde{\Pi}_{1,1}w(s)]\,\mathrm{d}s\right| \\ &+\frac{1}{\Gamma(\alpha)}\left|\sum_{k=2}^{n-1}\int_{t_{k-1}}^{t_k}(t_n-s)^{\alpha-1}\left[w(s)-\Pi_{2,k}w(s)\right]\mathrm{d}s\right| \\ &+\frac{1}{\Gamma(\alpha)}\left|\int_{t_{n-1}}^{t_n}(t_n-s)^{\alpha-1}\left[w(s)-\tilde{\Pi}_{2,n}w(s)\right]\mathrm{d}s\right| \\ =:& \tilde{P}_0+\tilde{P}_1+\tilde{P}_2.\end{aligned}$$

For the first term $\tilde{P}_0$, we directly have $|\tilde{P}_0| \leq CN^{-\gamma(\sigma+1)}$ since this term is the same with P_1. For the term $\tilde{P}_1$, applying similar idea with that in the estimate of $\tilde{D}_{\tau_n}^{-\alpha}$, we readily have

$$\tilde{P}_1 \leq C\sum_{k=3}^{n-1}\tau_k^3 t_{k-1}^{\sigma-3}\int_{t_{k-1}}^{t_k}(t_n-s)^{\alpha-1}\,ds \leq \begin{cases} CN^{-\gamma(\sigma+1)}, & \gamma(\sigma+1)-3<0, \\ CN^{-3}\ln(N), & \gamma(\sigma+1)-3=0, \\ CN^{-3}, & \gamma(\sigma+1)-3>0. \end{cases}$$

For the estimate of the last term $\tilde{P}_2$, we can derive that

$$\begin{aligned}|\tilde{P}_2| &\leq C\tau_n^3 t_{n-1}^{\sigma-3}\int_{t_{n-1}}^{t_n}(t_n-s)^{\alpha-1}\,ds \\ &\leq C\tau_n^{3+\alpha}t_{n-1}^{\sigma-3} \\ &\leq Cn^{\gamma(\sigma+\alpha)-(3+\alpha)}N^{-\gamma(\sigma+\alpha)} \\ &\leq CN^{-\min\{\gamma(\sigma+\alpha),3+\alpha\}}.\end{aligned}$$

Now, we consider the two special cases $n=1$ and $n=2$. For $n=1$, we have

$$\begin{aligned}\left|D_{0,t}^{-\alpha} w(t)|_{t=t_n} - \tilde{D}_{\tau_n}^{-\alpha} w(t_n)\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_n-s)^{\alpha-1}(w(s)-w(0))\,\mathrm{d}s\right| \\ &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_n-s)^{\alpha-1}\int_0^s w'(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\right| \\ &\leq C\int_0^{t_1}(t_n-s)^{\alpha-1}s^\sigma\,\mathrm{d}s \\ &\leq CN^{-\gamma(\sigma+1)},\end{aligned}$$

where we have used the monotonically decreasing property of the function $(t_n-s)^{\alpha-1}$ for the last inequality. For $n=2$, one readily has

$$\begin{aligned}\left|D_{0,t}^{-\alpha} w(t)|_{t=t_n} - \tilde{D}_{\tau_n}^{-\alpha} w(t_n)\right| &\leq \frac{2}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_n-s)^{\alpha-1}\left[w(s)-\tilde{\Pi}_{1,1}w(s)\right]\mathrm{d}s\right| \\ &\leq Ct_2^{\sigma+1}t_n^{\alpha-1} \\ &\leq CN^{-\gamma(\sigma+1)}.\end{aligned}$$

So, combining the above estimates for the three terms $\tilde{P}_j (j=0,1,2)$ for the case $n \geq 3$ and the two cases $n=1$ and $n=2$, and noting that $\sigma+\alpha > \sigma+1$, one can easily obtain the error bound for $\tilde{D}_{\tau_n}^{-\alpha}$. □

2.2. The scheme and its numerical analysis

Using the two approximations (2.4) and (2.5) for the terms G_1 and G_2 in (2.3), we obtain the following linearized implicit-explicit time-stepping scheme:

$$y^n = y_0 + t_n y_1 - \sum_{k=0}^{n} A_{k,n}\lambda y^k + \sum_{k=0}^{n-1} \tilde{A}_{k,n} f(t_k, y^k), \tag{2.8}$$

where y^n is the approximation of $y(t_n)$.

Next, we present the corresponding numerical theory for the linearized implicit-explicit time-stepping scheme (2.8). Before presenting the stability result, it is essential to demonstrate the Gronwall inequality.

Lemma 2.3. [14] *Assume that α, C, $T > 0$ and $b_{j,k} \le C\tau_{j+1}(t_k - t_j)^{\alpha-1}$ for $j = 0, 1, \dots, k-1$ for $0 = t_0 < t_1 < \cdots < t_N = T$, $k = 1, 2, \dots, N$ where N is a positive integer and $\tau_{j+1} = t_{j+1} - t_j$. Let g_0 be positive and the sequence $\{\psi_k\}$ satisfy*

$$\begin{cases} \psi_0 \le g_0, \\ \psi_k \le \displaystyle\sum_{j=0}^{k-1} b_{j,k}\psi_j + g_0, \end{cases}$$

then

$$\psi_k \le C g_0, \quad k = 1, 2, \dots, N.$$

Denote $\tilde{y}^n$ as the approximate solution of y^n and the perturbation $\varepsilon^n = \tilde{y}^n - y^n$ for $n \ge 1$. So, by the equation (2.8), we can obtain the perturbation equation:

$$\varepsilon^n = \varepsilon_0 + t_n \varepsilon_1 - \sum_{k=0}^{n} A_{k,n}\lambda \varepsilon^k + \sum_{k=0}^{n-1} \tilde{A}_{k,n}(f(t_k, \tilde{y}^k) - f(t_k, y^k)), \tag{2.9}$$

where ε_0 and ε_1 are the corresponding perturbations of y_0 and y_1.

We are now in a position to present the stability result.

Theorem 2.1. *The implicit-explicit time-stepping scheme* (2.8) *is stable in the sense that*

$$|\varepsilon^n| \le C(|\varepsilon_0| + t_n|\varepsilon_1|)$$

for $N^\alpha \ge 2\lambda \frac{\gamma^\alpha T^\alpha}{\Gamma(\alpha+2)}$, that is, the small perturbations that occur in the initial time do not cause a large error in the numerical solution.

Proof. From equation (2.9), we have

$$\begin{aligned} |\varepsilon^n| &= |\varepsilon_0 + t_n\varepsilon_1 - \sum_{k=0}^{n} A_{k,n}\lambda\varepsilon^k + \sum_{k=0}^{n-1} \tilde{A}_{k,n}(f(t_k, \tilde{y}^k) - f(t_k, y^k))| \\ &\le |\varepsilon_0| + |t_n\varepsilon_1| + \lambda\sum_{k=0}^{n} |A_{k,n}||\varepsilon^k| + \sum_{k=0}^{n-1} |\tilde{A}_{k,n}||(f(t_k, \tilde{y}^k) - f(t_k, y^k))| \\ &\le |\varepsilon_0| + |t_n\varepsilon_1| + \lambda\sum_{k=0}^{n} |A_{k,n}||\varepsilon^k| + L\sum_{k=0}^{n-1} |\tilde{A}_{k,n}||\varepsilon^k|, \end{aligned}$$

from which yields

$$\Big(1-\lambda|A_{n,n}|\Big)|\varepsilon^n| \le |\varepsilon_0| + t_n|\varepsilon_1| + \sum_{k=0}^{n-1}|A_{k,n}||\varepsilon^k| + L\sum_{k=0}^{n-1}|\tilde{A}_{k,n}||\varepsilon^k|,$$

where we have used the Lipschitz condition for the nonlinear term $f(t,y)$.

Note that $1-\lambda|A_{n,n}| \ge 1-\lambda\frac{\tau_n^\alpha}{\Gamma(\alpha+2)}$ in view of Lemma 2.1. Besides, for the graded meshes, it holds that $\tau_n \le \gamma\tau t_n^{1-1/\gamma}$ with $\tau = T^{1/\gamma}/N$ [21, (7)], so, we have

$$1-\lambda|A_{n,n}| \ge 1-\lambda\frac{(\gamma\tau t_n^{1-1/\gamma})^\alpha}{\Gamma(\alpha+2)} \ge 1-\lambda\frac{(\gamma\tau T^{1-1/\gamma})^\alpha}{\Gamma(\alpha+2)} = 1-\lambda\frac{(\gamma T)^\alpha}{N^\alpha\Gamma(\alpha+2)}.$$

Choosing $N^\alpha \ge 2\lambda\frac{\gamma^\alpha T^\alpha}{\Gamma(\alpha+2)}$, we have $1-\lambda|A_{n,n}| \ge \frac{1}{2}$. So,

$$|\varepsilon^n| \le C\Big(|\varepsilon_0| + t_n|\varepsilon_1| + \sum_{k=0}^{n-1}|A_{k,n}||\varepsilon^k| + L\sum_{k=0}^{n-1}|\tilde{A}_{k,n}||\varepsilon^k|\Big),$$

from which the desired result can be obtained by using Lemma 2.1 and Grönwall inequality (see Lemma 2.3). Thus, the proof is completed. □

Finally, we present the error estimate for the implicit-explicit time-stepping scheme (2.8).

Theorem 2.2. *Assume that the solution regularity* (2.2) *of the nonlinear fractional differential equation* (1.1) *is satisfied. When* $N^\alpha \ge 2\lambda\frac{\gamma^\alpha T^\alpha}{\Gamma(\alpha+2)}$, *the implicit-explicit time-stepping scheme* (2.8) *has the following error estimate.*

$$|y(t_n) - y^n| \le C\varrho(N,\gamma,\upsilon+1).$$

Proof. Denote $e^n = y(t_n) - y^n$. Based on Equations (2.3) and (2.8), we derive the following error equation.

$$e^n = -\sum_{k=0}^{n} A_{k,n}\lambda e^k + \sum_{k=0}^{n-1}\tilde{A}_{k,n}(f(t_k,y(t_k)) - f(t_k,y^k)) + R_1^n + R_2^n, \tag{2.10}$$

where $R_1^n = D_{\tau_n}^{-\alpha}\lambda y(t_n) - D_{0,t}^{-\alpha}\lambda y(t)|_{t=t_n}$ and $R_2^n = D_{0,t}^{-\alpha}f(t,y(t))|_{t=t_n} - \tilde{D}_{\tau_n}^{-\alpha}f(t_n,y(t_n))$. Under the regularity assumption (2.2), we know that the solution $y(t)$ of equation (1.1) behaves like $t^{\upsilon+\alpha}$; thus, by virtue of Lemma 2.2, we obtain $|R_1^n| \le C\varrho(N,\gamma,\upsilon+1)$. Similarly, combining the equality $w(t) := f(t,y(t)) = {}_CD_{0,t}^\alpha y(t) + \lambda y(t)$ and Lemma 2.2, we can derive $|R_2^n| \le C\varrho(N,\gamma,\upsilon+1)$.

To derive the error estimate for the linearized implicit-explicit time-stepping scheme (2.8), we first prove that the following expression holds by mathematical induction.

$$|e^n| \le C\sum_{k=0}^{n-1}(|A_{k,n}| + |\tilde{A}_{k,n}|)|e^k| + C\varrho(N,\gamma,\upsilon+1), \quad 1 \le n \le N. \tag{2.11}$$

From the error equation (2.10), we obtain

$$|e^n| \le \lambda\sum_{k=0}^{n}|A_{k,n}||e^k| + \sum_{k=0}^{n-1}|\tilde{A}_{k,n}||f(t_k,y(t_k)) - f(t_k,y^k)| + |R_1^n + R_2^n|. \tag{2.12}$$

For $n = 1$, based on $e^0 = 0$ with the Lipschitz condition of f, we can deduce that

$$|e^1| \le \lambda |A_{1,1}| \cdot |e^1| + |R_1^1 + R_2^1|. \tag{2.13}$$

Combining Lemma 2.1 and the equality $\tau_1 = t_1 = TN^{-\gamma}$, it follows that

$$1 - \lambda |A_{1,1}| \ge 1 - \lambda \frac{\tau_1^\alpha}{\Gamma(\alpha+2)} = 1 - \lambda \frac{T^\alpha N^{-\gamma\alpha}}{\Gamma(\alpha+2)} \ge \frac{1}{2},$$

where we choose $N^{\gamma\alpha} \ge \frac{2\lambda T^\alpha}{\Gamma(\alpha+2)}$. Substituting the above estimation into Inequality (2.13) we get

$$|e^1| \le C|R_1^1 + R_2^1| \le C\varrho(N, \gamma, \upsilon + 1).$$

Assume equation (2.11) holds for $1 \le n \le m-1 (2 < m < N)$. For equation (2.12), the case $n = m$ leads to

$$|e^m| \le \lambda \sum_{k=0}^{m} |A_{k,m}||e^k| + L \sum_{k=0}^{m-1} |\tilde{A}_{k,m}||e^k| + |R_1^m + R_2^m|,$$

where we have used the Lipschitz condition of f. Further, we obtain

$$(1 - \lambda |A_{m,m}|)|e^m| \le C \sum_{k=0}^{m-1} (|A_{k,m}| + |\tilde{A}_{k,m}|)|e^k| + C\varrho(N, \gamma, \upsilon + 1). \tag{2.14}$$

Similarly, we have $1 - \lambda |A_{m,m}| \ge \frac{1}{2}$ when $N^\alpha \ge 2\lambda \frac{\gamma^\alpha T^\alpha}{\Gamma(\alpha+2)}$. Substituting the above expression into Inequality (2.14), we have

$$|e^m| \le C \sum_{k=0}^{m-1} (|A_{k,m}| + |\tilde{A}_{k,m}|)|e^k| + C\varrho(N, \gamma, \upsilon + 1), \tag{2.15}$$

which implies that equation (2.11) holds.

Thus, the use of Lemma 2.3 to equation (2.11) yields the desired result. All this completes the proof. □

3. Further discussions on the capability of the proposed method

In this part, we extend our method to solve a class of multi-term fractional differential equations and a one-dimensional nonlinear time-fractional diffusion-wave equation to show the broad capability of our method.

3.1. Two-term fractional differential equation

For ease of discussion, we take the following two-term fractional differential equation as an example.

$$_C D_{0,t}^\alpha y(t) + {}_C D_{0,t}^\beta y(t) = -\lambda y(t) + f(t, y(t)), \tag{3.1}$$

with $1 < \beta < \alpha < 2$. The solution y in (3.1) is assumed to satisfy the regularity:

$$|\tilde{w}'(t)| \le ct^{\upsilon-1}, \quad |\tilde{w}''(t)| \le ct^{\upsilon-2}, \quad |\tilde{w}'''(t)| \le ct^{\upsilon-3}, \tag{3.2}$$

where $\tilde{w}(t) := {}_C D^{\alpha}_{0,t} y(t) + {}_C D^{\beta}_{0,t} y(t) \in C^3(0,T]$ and $\upsilon \in (0,1)$. The other necessary settings, such as the initial value conditions, are the same as in the one-term case (1.1).

Note that the following identity holds.

$$D^{-\alpha}_{0,t} {}_C D^{\beta}_{0,t} y(t) = D^{-(\alpha-\beta)}_{0,t} y(t) - \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} y_0 - \frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} y_1.$$

So, operating the fractional integral operator $D^{-\alpha}_{0,t}$ on both sides of the two-term fractional differential equation (3.1), one has

$$\begin{aligned} y(t) = &\Big(1 + \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\Big) y_0 + \Big(t + \frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\Big) y_1 \\ &- D^{-(\alpha-\beta)}_{0,t} y(t) - \lambda D^{-\alpha}_{0,t} y(t) + D^{-\alpha}_{0,t} f(t, y(t)). \end{aligned}$$

Applying the two approximations (2.4) and (2.5), we obtain the following efficient linearized implicit-explicit time-stepping scheme for solving (3.1).

$$\begin{aligned} y^n = &\Big(1 + \frac{t_n^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\Big) y_0 + \Big(t_n + \frac{t_n^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\Big) y_1 \\ &- D^{-(\alpha-\beta)}_{\tau_n} y^n - \lambda D^{-\alpha}_{\tau_n} y^n + \tilde{D}^{-\alpha}_{\tau_n} f(t_n, y^n). \end{aligned} \tag{3.3}$$

Here, we use the notation $D^{-(\alpha-\beta)}_{\tau_n}$ to denote the approximation of the fractional integral operator $D^{-(\alpha-\beta)}_{0,t}$ with $0 < \alpha - \beta < 1$, which is the same as in the case $1 < \alpha < 2$ (2.4). So, one can see that the proposed time-stepping scheme (3.3) is stable with accuracy $O(\varrho(N, \gamma, \upsilon+1))$ when $N^{\alpha} \geq 2(\lambda+1)\frac{\gamma^{\alpha} T^{\alpha}}{\Gamma(\alpha+2)}$. We omit the details since the corresponding proof can be obtained using an argument similar to Theorems 2.1 and 2.2.

Remark 3.1. We remark that, when the fractional order is $\zeta \in (0,1)$, one can use the idea of Lemma 2.2 to derive the error estimate

$$|D^{-\zeta}_{0,t} w(t) - D^{-\zeta}_{\tau_n} w(t_n)| \leq C \varrho(N, \gamma, \sigma + \zeta),$$

providing that $w(t)$ satisfies $|w'(t)| \leq C t^{\sigma-1}$, $|w''(t)| \leq C t^{\sigma-2}$, and $|w'''(t)| \leq C t^{\sigma-3}$ with $\sigma \in (0,2)$. If the regularity parameter $\sigma = \zeta$, then the result reduces to the special case

$$|D^{-\zeta}_{0,t} w(t) - D^{-\zeta}_{\tau_n} w(t_n)| \leq C \varrho(N, \gamma, 2\zeta),$$

which is discussed in [18, Lemma 3.1].

3.2. Nonlinear time-fractional diffusion-wave equation

We extend the derived scheme (2.8) to solve the nonlinear time-fractional diffusion-wave equation:

$${}_C D^{\alpha}_{0,t} u(x,t) = \partial_x^2 u(x,t) + f(x,t,u), \quad \text{in } (0,L) \times (0,T], \tag{3.4}$$

subject to the initial and boundary conditions: $u(x,0) = \phi_0(x), \partial_t u(x,0) = \phi_1(x)$, and $u(0,t) = u(L,t) = 0$. Here, the two functions $\phi_0(x)$ and $\phi_1(x)$ are suitable smooth. We assume that the regularity of the solution satisfies $\left|\frac{\partial^k u}{\partial x^k}\right| \leq C$ for $k = 3,4,5,6$, and $\left|\frac{\partial^l w}{\partial t^l}\right| \leq C(1 + t^{\upsilon-l})$ for $w(x,t) = {}_C D^{\alpha}_{0,t} u(x,t)$ and $l = 1,2,3$.

Combining the compact difference method in space with the scheme (2.8), we readily obtain the fully discrete linearized compact difference scheme to solve equation (3.4):

$$\mathcal{A}u_i^n = \mathcal{A}(\phi_0)_i + t_n\mathcal{A}(\phi_1)_i + \sum_{k=0}^{n} A_{k,n}\delta_x^2 u_i^k + \sum_{k=0}^{n-1} \tilde{A}_{k,n}\mathcal{A}f(x_i, t_k, u_i^k), \tag{3.5}$$

for $n \geq 1$ and $i \in \Omega_h = \{1, 2, \cdots, M\}$ where M is a positive integer. Here, the solution u_i^n is a numerical approximation of $u(x_i, t_n)$. In addition, $(\phi_0)_i = \phi_0(x_i)$ and $(\phi_1)_i = \phi_1(x_i)$. $\mathcal{A}$ is a compact difference operator defined by

$$\mathcal{A}w_i = (I + \frac{h^2}{12}\delta_x^2)w_i, \quad i \in \Omega_h,$$

with identity I and difference operator $\delta_x^2 w_i = (w_{i+1} - 2w_i + w_{i-1})/h^2$. The spatial stepsize is given by $h = L/M$. The stability and error estimate of the scheme (3.5) is beyond the scope of this paper, so we limit ourselves to the corresponding numerical verification. We expect that the scheme (3.5) is stable with accuracy of $O(\varrho(N, \gamma, \upsilon + 1) + h^4)$ for a sufficiently small temporal step size.

4. Numerical examples

In this section, we present numerical examples to verify the efficiency and accuracy of the proposed numerical approximations (2.4) and (2.5), as well as the linearized implicit-explicit time-stepping scheme (2.8). Additionally, we apply the scheme (2.8) to solve a nonlinear time-fractional diffusion-wave equation. We take $T = 1$ and $\lambda = 1$ in the nonlinear differential equation (1.1). Unless otherwise noted, we measure errors in the infinity norm, that is, Error := $\max_{0\leq n\leq N} |y(t_n) - y^n|$.

Example 4.1 (Accuracy of the two numerical approximations)**.** Utilizing $w(t) = t^\sigma$ in approximations (2.4) and (2.5), we present the numerical findings in Tables 1-5 for $\sigma = 0.8, 1.8,$ and 2.1.

It can be seen that for $\sigma < 2$, the third-order convergence order can be achieved by appropriately selecting the grading parameter γ, specifically $\gamma = 3/\sigma$ for (2.4) and $\gamma = 3/(1+\sigma)$ for (2.5), as shown in Tables 1-4. For $\sigma > 2$, that is, $\sigma = 2.1$ in Table 5, both approximations maintain third-order convergence on uniform meshes. These results confirm the correctness of the error estimates stated in Lemma 2.2.

Example 4.2. Consider the fractional differential equation (1.1) with an exact solution $y(t) = t^\beta$ ($\beta \in (1, 2)$ and $\beta > \alpha$) and a nonlinear term y^3. Thus, the source term can be given by

$$f(t, y) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}t^{\beta-\alpha} + \lambda t^\beta - y^3 + t^{3\beta}.$$

It can be seen that ${}_CD_{0,t}^\alpha y(t) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}t^{\beta-\alpha}$. So, we let $\upsilon = \beta - \alpha$ in Theorem 2.2. Using the implicit-explicit time-stepping scheme (2.8) for the different parameters $\beta = \upsilon + \alpha$ with a fixed $\upsilon = 0.4$, we obtain the corresponding numerical results, which are shown in Table 6.

From Table 6, we can see that the convergence order is only 1.4 when using the uniform meshes (i.e., $\gamma = 1$), while one can recover the third-order convergence order by setting the

Table 1. The L^∞-norm errors of for Example 4.1 with $\sigma = 0.8$ by applying (2.4).

Method(2.4)	N	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
		Error	Rate	Error	Rate	Error	Rate
$\gamma = 1$	8	1.05E-03	-	1.07E-03	-	9.88E-04	-
	16	2.82E-04	1.90	3.02E-04	1.82	2.80E-04	1.82
	32	7.97E-05	1.82	8.60E-05	1.81	7.95E-05	1.81
	64	2.28E-05	1.80	2.46E-05	1.81	2.27E-05	1.81
	128	6.55E-06	1.80	7.04E-06	1.80	6.50E-06	1.80
$\gamma = \frac{3}{1+\sigma}$	8	3.12E-04	-	2.86E-04	-	2.32E-04	-
	16	5.11E-05	2.61	4.94E-05	2.53	4.20E-05	2.46
	32	7.83E-06	2.71	7.77E-06	2.67	6.75E-06	2.64
	64	1.15E-06	2.76	1.16E-06	2.74	1.02E-06	2.72
	128	1.66E-07	2.80	1.69E-07	2.78	1.50E-07	2.77
$\gamma = \frac{3}{\sigma}$	8	9.54E-04	-	7.54E-04	-	5.48E-04	-
	16	1.09E-04	3.13	8.59E-05	3.13	6.26E-05	3.13
	32	1.29E-05	3.07	1.02E-05	3.07	7.43E-06	3.07
	64	1.58E-06	3.04	1.24E-06	3.04	9.05E-07	3.04
	128	1.90E-07	3.05	1.51E-07	3.04	1.10E-07	3.04

Table 2. The L^∞-norm errors for Example 4.1 with $\sigma = 0.8$ by applying (2.5).

Method(2.5)	N	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
		Error	Rate	Error	Rate	Error	Rate
$\gamma = 1$	8	2.32E-02	-	2.21E-02	-	1.80E-02	-
	16	6.78E-03	1.78	6.88E-03	1.68	5.99E-03	1.59
	32	1.96E-03	1.79	2.05E-03	1.75	1.84E-03	1.71
	64	5.65E-04	1.79	5.98E-04	1.78	5.44E-04	1.76
	128	1.62E-04	1.80	1.73E-04	1.79	1.59E-04	1.78
$\gamma = \frac{3}{1+\sigma}$	8	8.36E-03	-	8.78E-03	-	7.84E-03	-
	16	1.08E-03	2.96	1.15E-03	2.94	1.04E-03	2.91
	32	1.37E-04	2.98	1.46E-04	2.97	1.34E-04	2.96
	64	1.73E-05	2.98	1.85E-05	2.98	1.70E-05	2.98
	128	2.19E-06	2.98	2.34E-06	2.98	2.15E-06	2.98

grading parameter to $\gamma = 3/(1+\upsilon)$. So, this phenomenon confirms the error estimate stated in Theorem 2.2.

We further compare the derived scheme (2.8) with the predictor-corrector Adams method presented in [17] for Example 4.2 with $\sigma = 0.9$. The grading parameter is set to $\gamma = 3/(1+\upsilon)$ for scheme (2.8) and $2/(1+\upsilon)$ for the predictor-corrector Adams method. The corresponding numerical results are given in Table 7.

Table 3. The L^∞-norm errors for Example 4.1 with $\sigma = 1.8$ by applying (2.4).

Method(2.4)	N	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
		Error	Rate	Error	Rate	Error	Rate
$\gamma = 1$	8	3.62E-04	-	3.92E-04	-	3.61E-04	-
	16	4.97E-05	2.86	5.45E-05	2.85	5.07E-05	2.83
	32	6.85E-06	2.86	7.53E-06	2.85	7.05E-06	2.85
	64	9.47E-07	2.85	1.04E-06	2.85	9.76E-07	2.85
	128	1.31E-07	2.85	1.44E-07	2.85	1.35E-07	2.85
$\gamma = \frac{3}{1+\sigma}$	8	2.20E-04	-	2.36E-04	-	2.22E-04	-
	16	2.56E-05	3.11	2.69E-05	3.13	2.57E-05	3.11
	32	2.97E-06	3.11	3.00E-06	3.16	2.90E-06	3.15
	64	3.45E-07	3.11	3.29E-07	3.19	3.21E-07	3.18
	128	4.00E-08	3.11	3.53E-08	3.22	3.48E-08	3.21
$\gamma = \frac{3}{\sigma}$	8	9.67E-05	-	7.24E-05	-	4.99E-05	-
	16	1.29E-05	2.91	1.00E-05	2.85	7.20E-06	2.79
	32	1.63E-06	2.98	1.28E-06	2.97	9.29E-07	2.95
	64	2.04E-07	3.00	1.60E-07	3.00	1.17E-07	2.99
	128	2.54E-08	3.00	2.00E-08	3.00	1.46E-08	3.00

Table 4. The L^∞-norm errors for Example 4.1 with $\sigma = 1.8$ by applying (2.5).

Method(2.5)	N	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
		Error	Rate	Error	Rate	Error	Rate
$\gamma = 1$	8	3.41E-03	-	3.30E-03	-	2.74E-03	-
	16	4.92E-04	2.79	5.04E-04	2.71	4.43E-04	2.63
	32	7.07E-05	2.80	7.42E-05	2.76	6.69E-05	2.73
	64	1.01E-05	2.80	1.08E-05	2.78	9.83E-06	2.77
	128	1.45E-06	2.80	1.55E-06	2.79	1.43E-06	2.79
$\gamma = \frac{3}{1+\sigma}$	8	2.67E-03	-	2.61E-03	-	2.19E-03	-
	16	3.32E-04	3.01	3.43E-04	2.93	3.04E-04	2.85
	32	4.13E-05	3.01	4.36E-05	2.97	3.95E-05	2.94
	64	5.12E-06	3.01	5.47E-06	2.99	5.01E-06	2.98
	128	6.35E-07	3.01	6.82E-07	3.00	6.28E-07	3.00

As shown in Table 7, it is evident that the computational efficiency of the derived scheme (2.8) is clearly superior to that of the predictor-corrector scheme presented in [17], with reduced computing time and a higher convergence order.

Example 4.3. Consider the fractional differential equation (1.1) with exact solution $y(t) = t^\beta$ ($\beta \in (1, 2)$ and $\beta > \alpha$) and nonlinear term $\sin(y)$. Thus, the source term can be given by

Table 5. The L^∞-norm errors for Example 4.1 with $\sigma = 2.1$ on the uniform meshes.

Method	N	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
		Error	Rate	Error	Rate	Error	Rate
(2.4)	8	3.35E-04	-	3.43E-04	-	3.03E-04	-
	16	4.09E-05	3.04	4.23E-05	3.02	3.78E-05	3.00
	32	4.98E-06	3.04	5.18E-06	3.03	4.66E-06	3.02
	64	6.07E-07	3.04	6.33E-07	3.03	5.71E-07	3.03
	128	7.41E-08	3.04	7.72E-08	3.03	6.97E-08	3.03
(2.5)	8	1.94E-03	-	1.89E-03	-	1.58E-03	-
	16	2.32E-04	3.06	2.37E-04	2.99	2.08E-04	2.92
	32	2.75E-05	3.08	2.87E-05	3.05	2.58E-05	3.01
	64	3.24E-06	3.08	3.43E-06	3.07	3.11E-06	3.05
	128	3.82E-07	3.09	4.06E-07	3.08	3.71E-07	3.07

Table 6. The L^∞ errors of implicit-explicit time-stepping scheme (2.8) for Example 4.2 with $\upsilon = 0.4$.

Scheme	N	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
		Error	Rate	Error	Rate	Error	Rate
$\gamma = 1$	20	1.61E-02	-	1.80E-02	-	2.78E-02	-
	40	5.84E-03	1.47	6.84E-03	1.40	1.09E-02	1.35
	80	2.16E-03	1.43	2.60E-03	1.40	4.18E-03	1.38
	160	8.12E-04	1.42	9.86E-04	1.40	1.60E-03	1.39
	320	3.06E-04	1.41	3.74E-04	1.40	6.07E-04	1.39
$\gamma = \frac{3}{1+\upsilon}$	20	1.46E-03	-	1.87E-03	-	3.04E-03	-
	40	1.89E-04	2.95	2.40E-04	2.96	3.90E-04	2.96
	80	2.44E-05	2.96	3.06E-05	2.97	4.99E-05	2.97
	160	3.12E-06	2.96	3.91E-06	2.97	6.37E-06	2.97
	320	3.99E-07	2.97	5.00E-07	2.97	8.14E-07	2.97

$$f(t,y) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} + \lambda t^{\beta} - \sin(y) + \sin(t^{\beta}).$$

Here, similar to the discussion in Example 4.2, we let $\upsilon (= \beta - \alpha) = 0.6$ in Theorem 2.2. Using the implicit-explicit time-stepping scheme (2.8), the corresponding numerical results can be obtained in Tables 8 and 9. The third-order convergence order and the well performance are also verified by selecting the grading parameter $\gamma = 3/(1+\upsilon)$, which is in line with the error estimate presented in Theorem 2.2.

Example 4.4. Consider the two-term fractional differential equation (3.1) with exact solution $y(t) = t^\eta$ ($\eta \in (1,2)$ and $\eta > \alpha > \beta$) and nonlinear term $\sin(y)$. Thus, the source term can be given by

$$f(t,y) = \frac{\Gamma(1+\eta)}{\Gamma(1+\eta-\alpha)} t^{\eta-\alpha} + \frac{\Gamma(1+\eta)}{\Gamma(1+\eta-\beta)} t^{\eta-\beta} + \lambda t^{\eta} - \sin(y) + \sin(t^{\eta}).$$

Table 7. Numerical comparison of scheme (2.8) and the predictor–corrector scheme [17] for Example 4.2 with $\upsilon = 0.9$.

α	N	Scheme (2.8) Error	Rate	CPU(s)	Scheme [17, (1.6)] Error	Rate	CPU(s)
1.1	40	1.95E-04	-	0.03	2.38E-04	-	0.01
	80	2.47E-05	2.98	0.02	5.18E-05	2.20	0.01
	160	3.11E-06	2.99	0.03	1.32E-05	1.97	0.04
	320	3.91E-07	2.99	0.11	3.41E-06	1.96	0.13
	640	4.92E-08	2.99	0.37	8.80E-07	1.95	0.46
	1280	6.17E-09	2.99	0.95	2.27E-07	1.95	1.28
	2560	7.78E-10	2.99	2.98	5.89E-08	1.95	4.23
1.9	40	3.21E-04	-	0.02	1.69E-04	-	0.01
	80	4.06E-05	2.98	0.02	4.91E-05	1.78	0.01
	160	5.11E-06	2.99	0.03	1.38E-05	1.83	0.04
	320	6.42E-07	2.99	0.11	3.83E-06	1.85	0.14
	640	8.07E-08	2.99	0.39	1.05E-06	1.87	0.51
	1280	1.01E-08	2.99	1.05	2.84E-07	1.88	1.56
	2560	1.27E-09	2.99	3.35	7.65E-08	1.89	5.13

Table 8. The L^∞ errors of implicit-explicit time-stepping scheme (2.8) for Example 4.3 with $\upsilon = 0.6$.

Scheme	N	$\alpha = 1.1$ Error	Rate	$\alpha = 1.5$ Error	Rate	$\alpha = 1.9$ Error	Rate
$\gamma = 1$	20	1.09E-02	-	1.11E-02	-	1.89E-02	-
	40	3.41E-03	1.68	3.57E-03	1.64	6.23E-03	1.60
	80	1.08E-03	1.66	1.17E-03	1.61	2.05E-03	1.60
	160	3.49E-04	1.63	3.84E-04	1.60	6.78E-04	1.60
	320	1.14E-04	1.61	1.27E-04	1.60	2.24E-04	1.60
$\gamma = \dfrac{3}{1+\upsilon}$	20	9.54E-04	-	1.13E-03	-	2.01E-03	-
	40	1.24E-04	2.95	1.44E-04	2.97	2.56E-04	2.97
	80	1.59E-05	2.96	1.83E-05	2.97	3.26E-05	2.97
	160	2.03E-06	2.97	2.33E-06	2.97	4.15E-06	2.97
	320	2.59E-07	2.97	2.97E-07	2.97	5.28E-07	2.97

Here, we let $\upsilon(= \eta - \alpha) = 0.3$ in solution regularity assumption (3.2). Using the time-stepping scheme (3.3), we have the corresponding numerical results in Table 10. We find that, for three various pairs of fractional orders α and β, the observed convergence rates coincide with our theoretical prediction; in particular, third-order accuracy is obtained when the grading parameter is set to $\gamma = 3/(1 + \upsilon)$.

Finally, we use a concrete example to test the numerical performance of the linearized compact difference scheme (3.5).

Table 9. Numerical comparison of scheme (2.8) and the predictor–corrector scheme [17] for Example 4.3 with $\upsilon = 0.8$.

α	N	Scheme (2.8)			Scheme [17, (1.6)]		
		Error	Rate	CPU(s)	Error	Rate	CPU(s)
1.1	40	1.50E-04	-	0.03	2.63E-04	-	0.01
	80	1.89E-05	2.98	0.02	6.90E-05	1.93	0.01
	160	2.39E-06	2.98	0.03	1.80E-05	1.94	0.04
	320	3.02E-07	2.99	0.11	4.70E-06	1.94	0.12
	640	3.82E-08	2.99	0.35	1.22E-06	1.94	0.43
	1280	4.83E-09	2.98	0.92	3.18E-07	1.94	1.19
	2560	6.28E-10	2.94	2.73	8.28E-08	1.94	3.81
1.9	40	3.12E-04	-	0.02	3.18E-04	-	0.01
	80	3.95E-05	2.98	0.02	8.85E-05	1.84	0.01
	160	5.00E-06	2.98	0.03	2.44E-05	1.86	0.04
	320	6.31E-07	2.99	0.11	6.68E-06	1.87	0.12
	640	7.97E-08	2.99	0.36	1.82E-06	1.88	0.43
	1280	1.01E-08	2.99	0.90	4.90E-07	1.89	1.17
	2560	1.28E-09	2.98	2.76	1.32E-07	1.90	3.81

Table 10. The L^∞ errors of implicit-explicit time-stepping scheme (3.3) for Example 4.4 with $\upsilon = 0.3$.

Scheme	N	$(\alpha,\beta) = (1.6, 1.5)$		$(\alpha,\beta) = (1.7, 1.4)$		$(\alpha,\beta) = (1.9, 1.1)$	
		Error	Rate	Error	Rate	Error	Rate
$\gamma = 1$	20	2.25E-02	-	2.04E-02	-	2.03E-02	-
	40	8.76E-03	1.36	7.72E-03	1.40	7.84E-03	1.37
	80	3.44E-03	1.35	2.96E-03	1.38	3.10E-03	1.34
	160	1.35E-03	1.34	1.15E-03	1.37	1.24E-03	1.32
	320	5.35E-04	1.34	4.48E-04	1.36	4.99E-04	1.31
$\gamma = \frac{3}{1+\upsilon}$	20	2.02E-03	-	1.68E-03	-	1.94E-03	-
	40	2.47E-04	3.03	2.07E-04	3.02	2.47E-04	2.97
	80	3.03E-05	3.03	2.58E-05	3.00	3.15E-05	2.97
	160	3.74E-06	3.02	3.24E-06	2.99	4.02E-06	2.97
	320	4.68E-07	3.00	4.21E-07	2.94	5.17E-07	2.96

Example 4.5. Consider the model (3.4) with exact solution $u(x,t) = t^\beta \sin(\pi x)$ and nonlinear term $f(u)$ given below.

$$f(x,t,u) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} \sin(\pi x) + \pi^2 t^\beta \sin(\pi x) - u^3 + (t^\beta \sin(\pi x))^3.$$

Here, we test the L^2-norm errors as

$$E(M,N) = \max_{1\le n\le N} \sqrt{h \sum_{1\le i\le M-1} (u(x_i,t_n) - u_i^n)^2}.$$

In order to verify the optimal error estimate result in time for a less smooth solution (set $\beta = 0.3+\alpha$, thus $\sigma = 0.3$ in Theorem 2.2), we fix $M = 32$ and choose the mesh parameters $\gamma = 1$ and $\gamma = 3/(1+\sigma)$ for the linearized compact difference scheme (3.5). The numerical results are shown in Tables 11 and 12.

Table 11. The L^2-norm errors in time of the scheme (3.5) for Example 4.5 with $\beta = 0.3+\alpha$ and $M = 32$.

Scheme	N	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
		$E(M,N)$	Rate	$E(M,N)$	Rate	$E(M,N)$	Rate
$\gamma = 1$	8	8.68E-02	-	3.40E-02	-	4.06E-02	-
	16	2.63E-02	1.72	1.11E-02	1.62	1.40E-02	1.54
	32	8.51E-03	1.63	4.18E-03	1.41	5.33E-03	1.39
	64	2.93E-03	1.54	1.62E-03	1.37	2.13E-03	1.33
	128	1.05E-03	1.47	6.48E-04	1.32	8.60E-04	1.31
$\gamma = \frac{3}{1+\sigma}$	8	3.76E-02	-	1.53E-02	-	1.58E-02	-
	16	3.91E-03	3.27	1.48E-03	3.37	1.96E-03	3.00
	32	4.13E-04	3.24	1.87E-04	2.99	2.52E-04	2.96
	64	4.40E-05	3.23	2.39E-05	2.97	3.23E-05	2.96
	128	4.68E-06	3.23	3.08E-06	2.96	4.23E-06	2.93

Table 12. The L^2-norm errors in space of the scheme (3.5) for Example 4.5 with $\beta = 0.3+\alpha$ and $N = 512$.

Scheme	M	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
		$E(M,N)$	Rate	$E(M,N)$	Rate	$E(M,N)$	Rate
$\gamma = 1$	3	2.77E-03	-	2.67E-03	-	2.07E-03	-
	5	3.49E-04	4.05	3.21E-04	4.15	2.33E-04	4.28
	6	1.67E-04	4.03	1.44E-04	4.38	1.48E-04	2.47
	7	1.49E-04	0.74	1.07E-04	1.95	1.45E-04	0.15
$\gamma = \frac{3}{1+\sigma}$	3	2.76E-03	-	2.68E-03	-	2.09E-03	-
	5	3.48E-04	4.05	3.38E-04	4.05	2.64E-04	4.05
	6	1.67E-04	4.03	1.62E-04	4.03	1.27E-04	4.03
	7	8.99E-05	4.02	8.74E-05	4.02	6.82E-05	4.02

One can see that the temporal and spatial convergence orders recover third-order and fourth-order with an appropriate grading parameter $\gamma = 1/(1+\sigma)$ when $\alpha = 1.1, 1.5$, and 1.9, see Tables 11 and 12. This observation satisfies our prediction. Although we obtain the desired convergence orders in our numerical tests, it should be noted, however, that the stability of the linearized difference scheme (3.5), the numerical results of which are not presented here for the sake of simplicity, still suffers from the boundedness of the ratio of temporal and spatial stepsizes. Further investigation is needed to determine the stability condition.

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